Radiation condition for a non smooth interface between a dielectric and a metamaterial

Waves 2011

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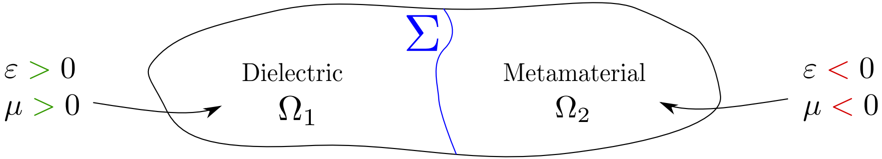
Introduction: setting of the problem

Time-harmonic problem in electromagnetism (at a given frequency) set in a heterogeneous bounded domain:



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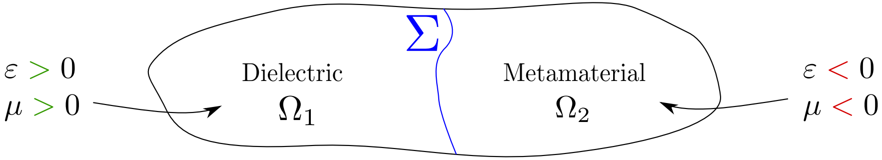

Negative metamaterial
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Structure with negative permittivity ε and permeability μ

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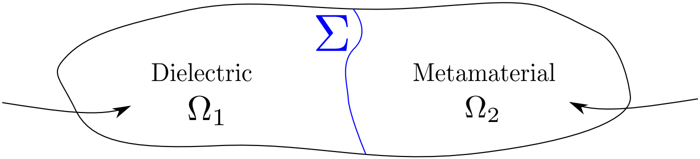

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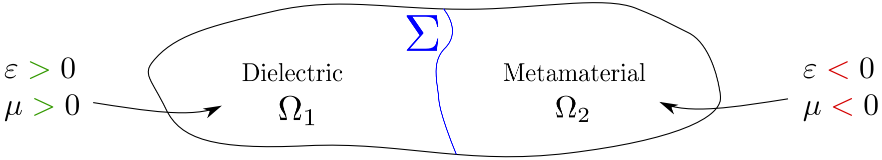

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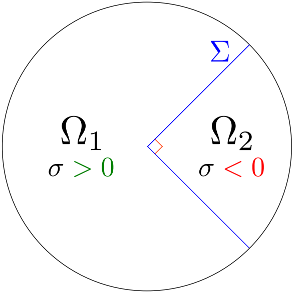
Unusual transmission problem because the sign of the coefficients ε and μ changes.

Introduction: modelization of the problem

Difficulty of the scalar problem concentrated in the study of the problem:

$$(\mathscr{P}) \quad \left| \begin{array}{l} \text{Find } u \in H^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\sigma\,\nabla u) = f \quad \text{in } \Omega. \end{array} \right.$$



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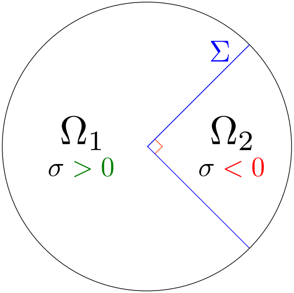

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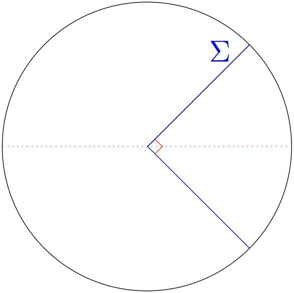

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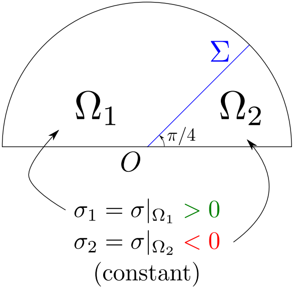

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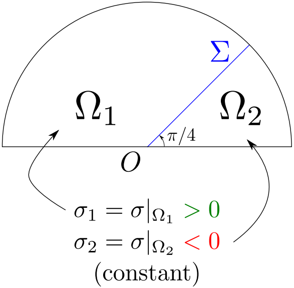

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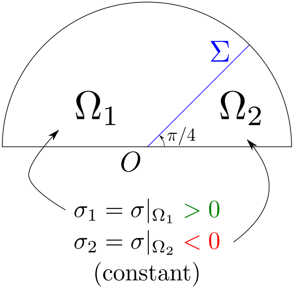

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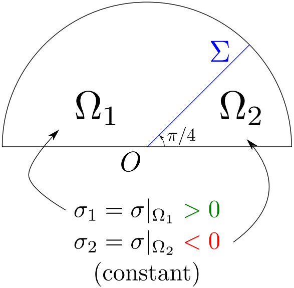

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Definition. We will say that the problem $(\mathscr{P})$ is well-posed if the operator $A = \mathrm{div}\,(\sigma\nabla\cdot)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

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  - For a symmetric domain (w.r.t. Σ) with $\sigma_2 = -\sigma_1$, we can build a kernel of infinite dimension.

Outline of the talk

1) A presentation of the T-coercivity method to find a criterion on σ to ensure that problem $(\mathscr{P})$ is well-posed in $H_0^1(\Omega)$.

1. A variational technique: the T-coercivity approach

Outline of the talk

1) A presentation of the T-coercivity method to find a criterion on σ to ensure that problem $(\mathscr{P})$ is well-posed in $H_0^1(\Omega)$.

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3) An approximation of the solution in the new functional framework using PML in the neighbourhood of the corner.

1. A variational technique: the T-coercivity approach
2. A new functional framework in the critical interval
   - Analogy with a waveguide problem
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Idea of the T-coercivity 1/2

Let T be an isomorphism of $H_0^1(\Omega)$.

$$(\mathscr{P}) \Leftrightarrow (\mathscr{P}_V) \left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ such that:} \\ a(u,v) = l(v),\ \forall v \in H_0^1(\Omega). \end{array} \right.$$

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Goal: Find $\mathtt{T}$ such that a is $\mathtt{T}$-coercive: $\displaystyle\int_\Omega \sigma \nabla u \cdot \nabla(\mathtt{T}u) \geq C\, \|u\|^2_{H_0^1(\Omega)}$.

In this case, Lax-Milgram $\Rightarrow (\mathscr{P}_V^{\mathtt{T}})$ (and so $(\mathscr{P}_V)$) is well-posed.

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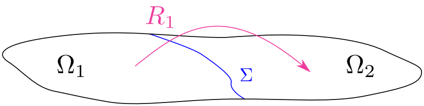

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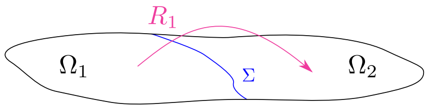

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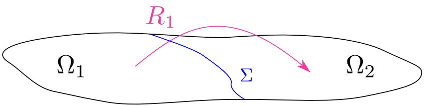

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2. $\mathtt{T}_1 \circ \mathtt{T}_1 = Id$ so $\mathtt{T}_1$ is an isomorphism of $H^1_0(\Omega)$

Idea of the $\mathtt{T}$-coercivity 2/2

3. One has $a(u, \mathtt{T}_1 u) = \displaystyle\int_{\Omega} |\sigma||\nabla u|^2 - 2\int_{\Omega_2} \sigma_2 \nabla u \cdot \nabla(R_1\, u_1)$

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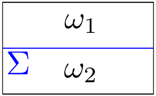
5. Conclusion:

Theorem. If the contrast $\kappa_\sigma = \sigma_2/\sigma_1 \notin [-\|R_2\|^2; -1/\|R_1\|^2]$ (critical interval) then $\operatorname{div}(\sigma \nabla \cdot)$ is an isomorphism from $H^1_0(\Omega)$ to $H^{-1}(\Omega)$.

Choice of R_1,R_2?

- A simple case: symmetric domain



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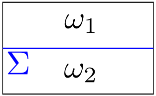

symmetry w.r.t. Σ

$R_1 = S_\Sigma$ and $R_2 = S_\Sigma$

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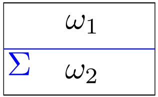

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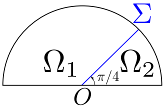

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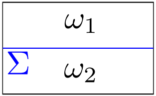

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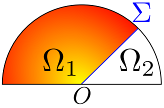

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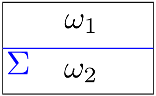

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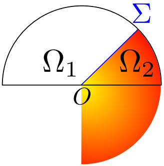
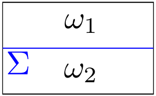


Action of R_1: symmetry w.r.t θ

Choice of R_1,R_2?

- A simple case: symmetric domain



symmetry w.r.t. Σ
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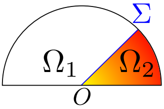

Action of R_1: symmetry + dilatation w.r.t θ

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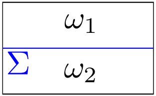

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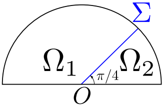

R_1: symmetry + dilatation w.r.t θ
R_2: symmetry w.r.t θ + extension by 0

Choice of R_1,R_2?

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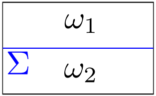

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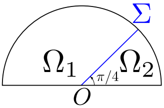

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Proposition. If the contrast $\kappa_\sigma = \sigma_2/\sigma_1 \notin [-1, -1/3]$ (critical interval) then the problem $(\mathscr{P})$ is well-posed.

Choice of R_1,R_2?

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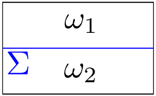

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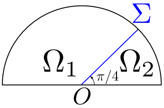

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Key remark. For a general curvilinear polygonal interface, the critical interval reduces to $\{-1\}$ if and only if there is no corner in Σ.

Transition: from variational methods to Fourier/Mellin techniques

What happens in the critical interval, i.e. for $\kappa_\sigma \in [-1, -1/3]$???

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$\Rightarrow$ Fourier/Mellin tool (Dauge-Texier 97, Nazarov)

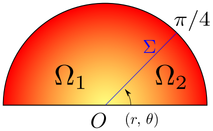
1. A variational technique: the T-coercivity approach

2. A new functional framework in the critical interval
   - Analogy with a waveguide problem
   - Statement of the result

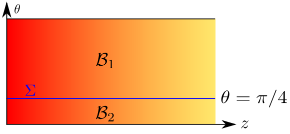
3. Numerical approximation

Analogy with a waveguide problem

- Bounded sector Ω



- Half-strip $\mathcal{B}$



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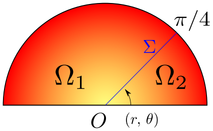

$$(z, \theta) = (-\ln r, \theta)$$

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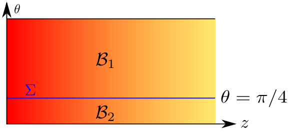

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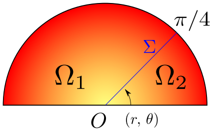

- Helmholtz equation:

$$\underbrace{-\mathrm{div}(\sigma\,\nabla u)}_{-r^{-2}(\sigma(r\partial_r)^2+\partial_\theta\sigma\partial_\theta)u} = f$$

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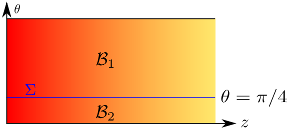

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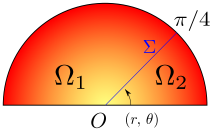

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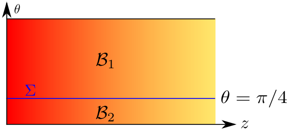

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- Modes in the strip

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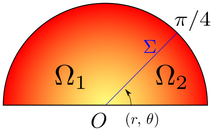

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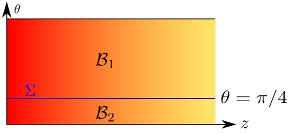

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$s \in H^1(\Omega)$ $\quad\quad \Re e\,\lambda > 0 \quad\quad$ m is evanescent

Analogy with a waveguide problem

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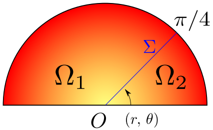

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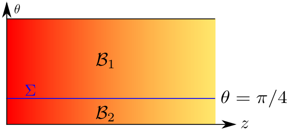

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$(\Re e\,\lambda = a,\ \Im m\,\lambda = b)$

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Analogy with a waveguide problem

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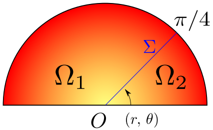

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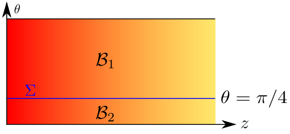

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- This encourages us to use modal decomposition in the half-strip.

Modal analysis in the waveguide

- We look for the solutions of the form $e^{\lambda z}\varphi(\theta)$ to the homogeneous problem (these modes can be computed explicitly for this geometry).

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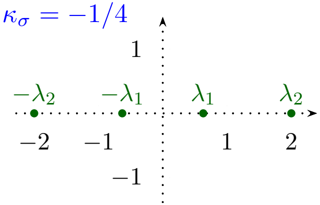

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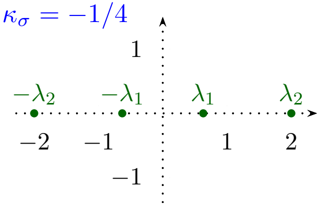

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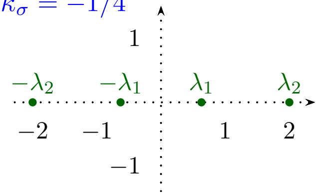
$\kappa_\sigma = -1/4$



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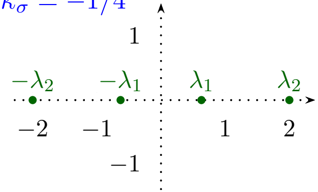

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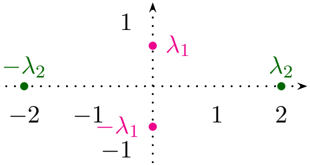

► Inside the critical interval. There are exactly two propagative modes.

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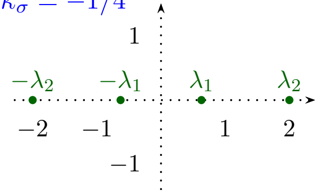

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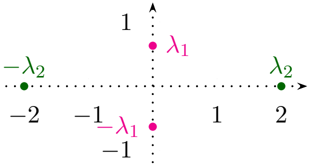

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→ the decomposition on the outgoing modes leads to look for a solution of the form

$$u = \underbrace{c_1\,\varphi_1\,e^{\lambda_1 z}}_{\text{propagative part}} + \underbrace{u_e}_{\text{evanescent part}}.$$

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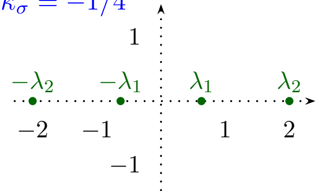

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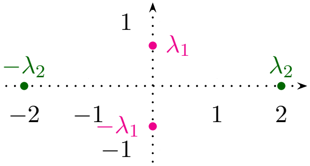

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$\boxed{\text{non } H^1 \text{ framework}}$

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$$W_{-\beta} \quad = \{v \,|\, e^{\beta z} v \in H_0^1(\mathcal{B})\} \qquad \text{space of exponentially decaying functions}$$

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$W_{-\beta}$	$= \{v \,\vert\, e^{\beta z} v \in H_0^1(\mathcal{B})\}$	space of exponentially decaying functions
W^+	$= \mathrm{span}(\zeta \varphi_1 \, e^{\lambda_1 z}) \oplus W_{-\beta}$	propagative part + evanescent part
W_β	$= \{v \,\vert\, e^{-\beta z} v \in H_0^1(\mathcal{B})\}$	space of exponentially growing functions

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Theorem. Let $\kappa_\sigma \in (-1, -1/3)$ and $0 < \beta < 2$. The operator $A^+ : \operatorname{div}(\sigma\nabla\cdot)$ from W^+ to ${W_\beta}^*$ is an isomorphism .

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IDEAS OF THE PROOF:

1. $A_{-\beta} : \text{div}(\sigma \nabla \cdot)$ from $W_{-\beta}$ to ${W_\beta}^*$ is injective but not surjective.

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Ideas of the proof:

1. $A_{-\beta} : \mathrm{div}(\sigma \nabla \cdot)$ from $W_{-\beta}$ to ${W_\beta}^*$ is injective but not surjective.
2. $A_{\beta} : \mathrm{div}(\sigma \nabla \cdot)$ from W_{β} to ${W_{-\beta}}^*$ is surjective but not injective.

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1. $A_{-\beta} : \mathrm{div}(\sigma\,\nabla\cdot)$ from $W_{-\beta}$ to $W_\beta{}^*$ is injective but not surjective.
2. $A_\beta : \mathrm{div}(\sigma\,\nabla\cdot)$ from W_β to $W_{-\beta}{}^*$ is surjective but not injective.
3. The intermediate operator A^+ is injective (energy integral) and surjective (♠ residual theorem).

The new functional framework

Consider $0 < \beta < 2$, ζ a cut-off function (equal to 1 in $+\infty$) and define

$$\begin{array}{lll} W_{-\beta} & = \{v \,|\, e^{\beta z} v \in H_0^1(\mathcal{B})\} & \text{space of exponentially decaying functions} \\ \cap & & \\ W^+ & = \mathrm{span}(\zeta\varphi_1\, e^{\lambda_1 z}) \oplus W_{-\beta} & \text{propagative part + evanescent part} \\ \cap & & \\ W_\beta & = \{v \,|\, e^{-\beta z} v \in H_0^1(\mathcal{B})\} & \text{space of exponentially growing functions} \end{array}$$

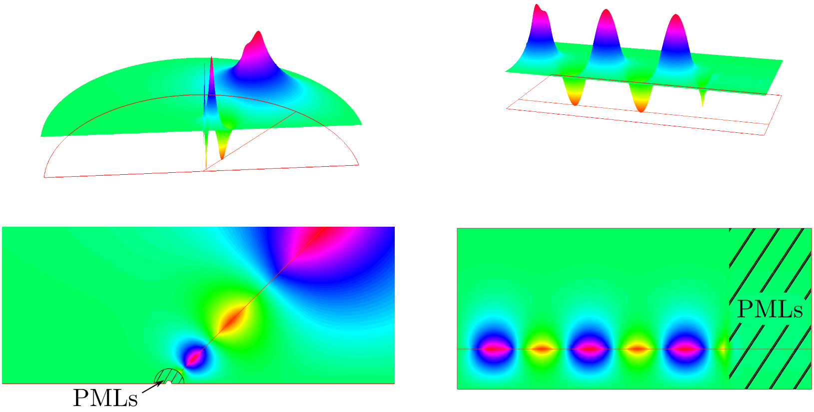
THEOREM. Let $\kappa_\sigma \in (-1, -1/3)$ and $0 < \beta < 2$. The operator $A^+ : \mathrm{div}(\sigma\,\nabla\cdot)$ from W^+ to ${W_\beta}^*$ is an isomorphism .

IDEAS OF THE PROOF:

1. $A_{-\beta} : \mathrm{div}(\sigma\,\nabla\cdot)$ from $W_{-\beta}$ to ${W_\beta}^*$ is injective but not surjective.
2. $A_\beta : \mathrm{div}(\sigma\,\nabla\cdot)$ from W_β to ${W_{-\beta}}^*$ is surjective but not injective.
3. The intermediate operator A^+ is injective (energy integral) and surjective (♠ residual theorem).
4. Limiting absorption principle to select the outgoing mode.

A funny use of PMLs

- We use PMLs (*Perfectly Matched Layers*) to bound the domain $\mathcal{B}$ + finite elements in the truncated strip



$$\kappa_\sigma = 1/1.05$$

A black hole phenomenon in the critical interval

$$\kappa_\sigma = -1/1.3 \in (-1, -1/3)$$

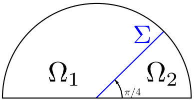
$$(\dots)$$

- Analogous phenomena occur in cuspidal domains in the theory of water-waves and in elasticity.

Conclusion : summary of the results

Problem

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ s. t.:} \\ -\mathrm{div}\,(\sigma\,\nabla u) = f \qquad \text{in } \Omega. \end{array} \right.$$



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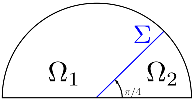
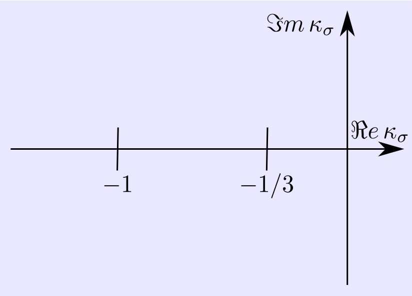


Results

For $\kappa_\sigma \in \mathbb{C}\backslash\mathbb{R}_-$, $(\mathscr{P})$ well-posed in $H_0^1(\Omega)$ (Lax-Milgram)



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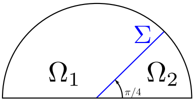
Problem

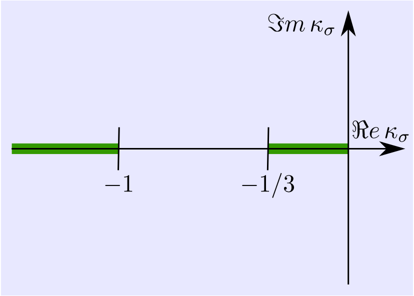
$(\mathscr{P})$ $\left|\begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ s. t.:} \\ -\mathrm{div}\,(\sigma\,\nabla u) = f \quad \text{in } \Omega. \end{array}\right.$



Results

For $\kappa_\sigma \in \mathbb{C}\backslash\mathbb{R}_-$, $(\mathscr{P})$ well-posed in $H_0^1(\Omega)$ (Lax-Milgram)

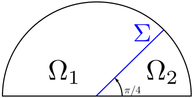
For $\kappa_\sigma \in \mathbb{R}_-^*\backslash[-1,-1/3]$, $(\mathscr{P})$ well-posed in $H_0^1(\Omega)$ (T-coercivity)



Conclusion : summary of the results

Problem

$$(\mathscr{P}) \;\left|\; \begin{array}{l} \text{Find } u \in H^1_0(\Omega) \text{ s. t.:} \\ -\mathrm{div}\,(\sigma\,\nabla u) = f \quad \text{ in } \Omega. \end{array}\right.$$

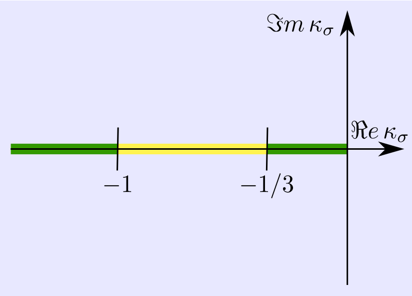


Results

For $\kappa_\sigma \in \mathbb{C}\backslash\mathbb{R}_-$, $(\mathscr{P})$ well-posed in $H^1_0(\Omega)$ (Lax-Milgram)

For $\kappa_\sigma \in \mathbb{R}^*_-\backslash[-1,-1/3]$, $(\mathscr{P})$ well-posed in $H^1_0(\Omega)$ ($\mathtt{T}$-coercivity)

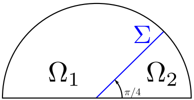
For $\kappa_\sigma \in (-1,-1/3)$, $(\mathscr{P})$ is not well-posed in the Fredholm sense in $H^1_0(\Omega)$ but well-posed in V^+ (PMLs)



Conclusion : summary of the results

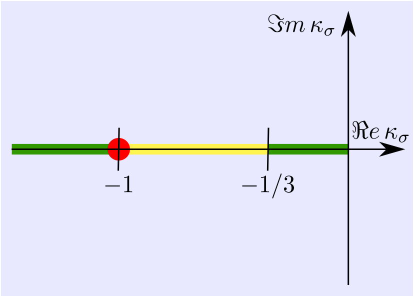
Problem

$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in H_0^1(\Omega) \text{ s. t.:} \\ -\mathrm{div}\,(\sigma \nabla u) = f \quad \text{ in } \Omega. \end{array} \right.$$



Results

- For $\kappa_\sigma \in \mathbb{C}\backslash\mathbb{R}_-$, $(\mathscr{P})$ well-posed in $H_0^1(\Omega)$ (Lax-Milgram)
- For $\kappa_\sigma \in \mathbb{R}_-^*\backslash[-1, -1/3]$, $(\mathscr{P})$ well-posed in $H_0^1(\Omega)$ ($\mathtt{T}$-coercivity)
- For $\kappa_\sigma \in (-1, -1/3)$, $(\mathscr{P})$ is not well-posed in the Fredholm sense in $H_0^1(\Omega)$ but well-posed in V^+ (PMLs)
- $\kappa_\sigma = -1$, $(\mathscr{P})$ ill-posed in $H_0^1(\Omega)$



Generalizations

- ✔ T-coercivity approach can be used for non-constant σ (L^∞) and other problems (Maxwell's equations *(joint work with A.-S. Bonnet-Ben Dhia and P. Ciarlet Jr.)*, the ITEP *(joint work with A.-S. Bonnet-Ben Dhia and H. Haddar)* ...).
- ✔ One can justify convergence of standard finite elements method for simple meshes *(joint work with P. Ciarlet Jr.)*.

Generalizations

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- ✔ One can justify convergence of standard finite elements method for simple meshes *(joint work with P. Ciarlet Jr.)*.

Open problems

- ♠ The case $\kappa_\sigma = -1$ (the most interesting for applications) is not understood yet: there appear singularities all over the interface.
  $\Rightarrow$ Is there a functional framework in which $(\mathscr{P})$ is well-posed?
- ♠ More generally, can we reconsider the homogenization process to take into account interfacial phenomena?
  $\Rightarrow$*METAMATH project (ANR) directed by S. Fliss.*

Thank you for your attention.

A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., *Optimality of T-coercivity for scalar interface problems between dielectrics and metamaterials*, http://hal.archives-ouvertes.fr/hal-00564312_v1/, 2011.

A.-S. Bonnet-Ben Dhia, P. Ciarlet Jr., C.M. Zwölf, *Time harmonic wave diffraction problems in materials with sign-shifting coefficients*, J. Comput. Appl. Math, 234:1912–1919, 2010, Corrigendum *J. Comput. Appl. Math.*, 234:2616, 2010.

M. Dauge, B. Texier, *Problèmes de transmission non coercifs dans des polygones*, http://hal.archives-ouvertes.fr/docs/00/56/23/29/PDF/BenjaminT_arxiv.pdf, 1997.

S.A. Nazarov, J. Taskinen, *Radiation conditions at the top of a rotational cusp in the theory of water-waves*, ESAIM: Mathematical Modelling and Numerical Analysis, 45:947–979, 2011.