A curious instability phenomenon for a rounded corner in presence of a negative material

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Coll. with A.-S. Bonnet-Ben Dhia[2], P. Ciarlet[2], X. Claeys[3] and S.A. Nazarov[4]

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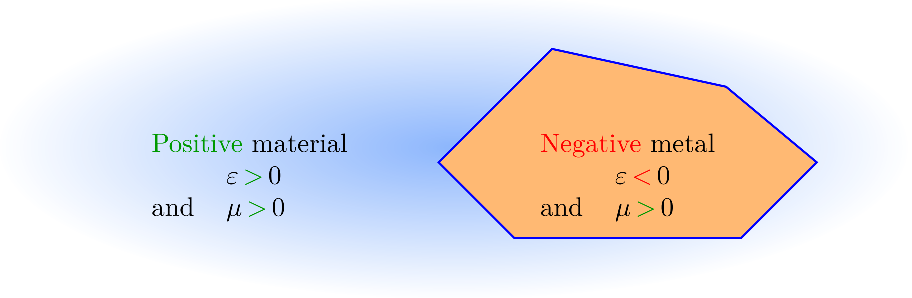
Introduction: general framework

- Scattering by a metal in electromagnetism in time-harmonic regime at optical frequency.
- For metals at optical frequency, $\Re e\,\varepsilon(\omega) < 0$ and $\Im m\,\varepsilon(\omega) << |\Re e\,\varepsilon(\omega)|$.
  $\Rightarrow$ We neglect losses and study the ideal case $\varepsilon(\omega) \in (-\infty; 0)$.



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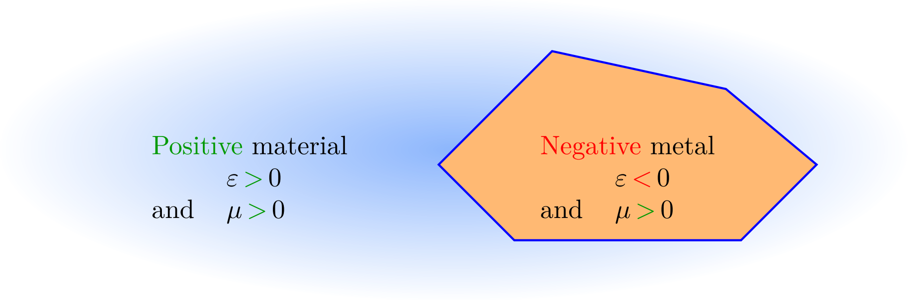
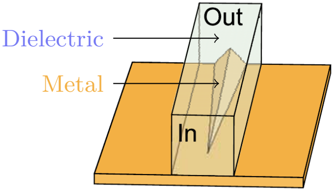
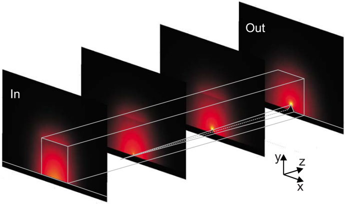


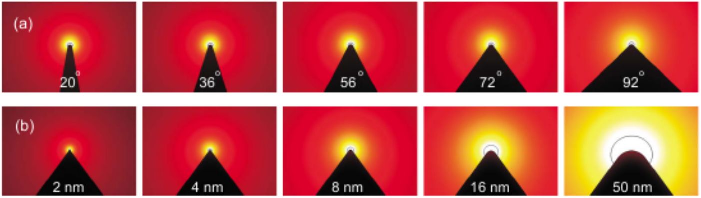
- Waves called Surface Plasmon Polaritons can propagate at the interface between a dielectric and a negative metal.

Introduction: applications

- Surface Plasmons Polaritons can propagate information. Physicists hope to exploit them to reduce the size of computer chips.







Figures from O'Connor *et al.*, Appl. Phys. Lett. 95, 171112 (2009)

- In this context, physicists use singular geometries to focus energy. It allows to stock information.

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- We study a scalar model problem set in a bounded domain $\Omega \subset \mathbb{R}^2$:

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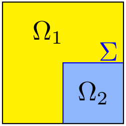

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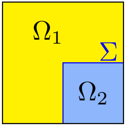

  - $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega) \,|\, \nabla v \in \mathrm{L}^2(\Omega);\ v|_{\partial\Omega} = 0\}$
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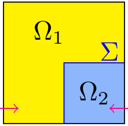

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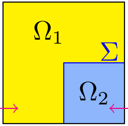

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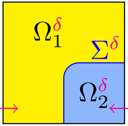

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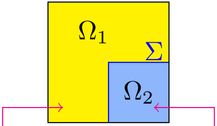

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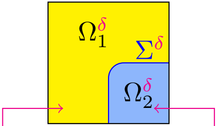

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? What is the behaviour of the sequence $(u^\delta)_\delta$ when δ tends to zero?

Outline of the talk

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We present the properties of the limit problem in the geometry with the real corner ($\delta = 0$). Since σ changes sign, original phenomena appear.

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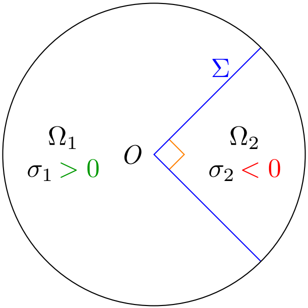
We present the properties of the limit problem in the geometry with the real corner $(\delta = 0)$. Since σ changes sign, original phenomena appear.

3. Asymptotic analysis

We prove a curious instability phenomenon: for certain configurations, $(\mathscr{P}^\delta)$ critically depends on δ.

Setting

- For the numerical experiments, we round the corner in a particular way



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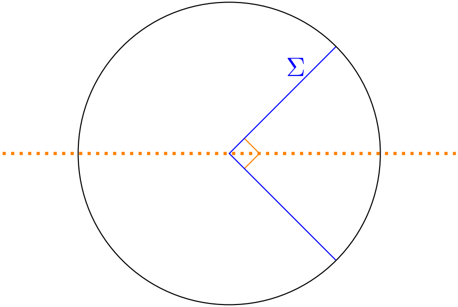

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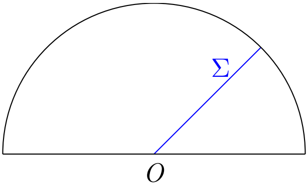

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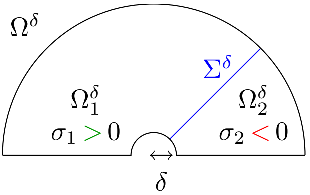

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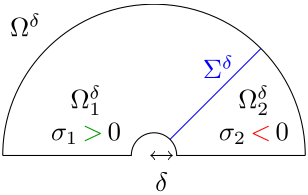

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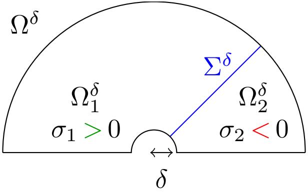

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- Our goal is to study the behaviour of the solution, *if it is well-defined*, of the problem

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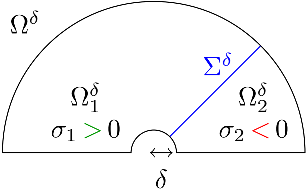

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- We approximate by a usual P1 Finite Element Method this u^δ, *assuming it is well-defined.* The solution of the discretized problem is called u_h^δ.

We display the behaviour of u_h^δ as $\delta \to 0$.

Numerical experiments 1/2

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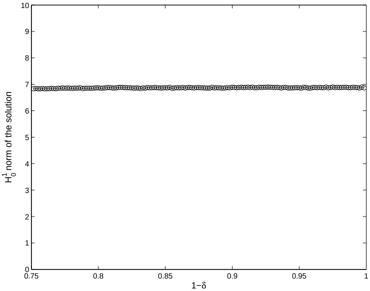

u_h^δ w.r.t. δ

$\|\nabla u_h^\delta\|_{\Omega^\delta}$ w.r.t. $1-\delta$

- For positive materials, it is well-known that $(u^\delta)_\delta$ converges to u, the solution in the limit geometry.
- The rate of convergence depends on the regularity of u.
- To avoid to mesh Ω^δ, we can approximate u^δ by u_h.

Numerical experiments 2/2

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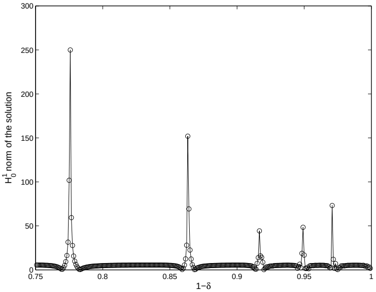
$$\boxed{\sigma_1 = 1 \text{ and } \sigma_2 = -0.9999}$$

$(\ldots)$



u_h^δ w.r.t. δ $\qquad$ $\|\nabla u_h^\delta\|_{\Omega^\delta}$ w.r.t. $1-\delta$

- For this configuration, u_δ seems to depend critically on δ.

In this talk, our goal is to explain this behaviour.

Problems with a sign changing coefficient

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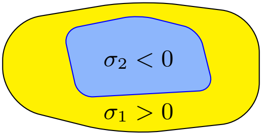
► We have the following properties (see Costabel and Stephan 85, Dauge and Texier 97, Bonnet-Ben Dhia *et al.* 99,10,12,13):

Smooth interface Σ



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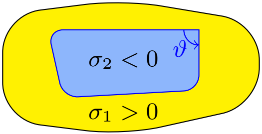
Interface Σ with a corner



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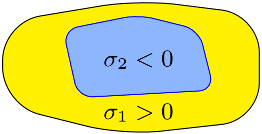

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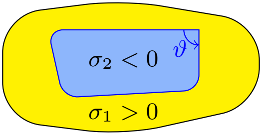

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Well-posedness depends on the smoothness of Σ and on σ.

The problematic of the rounded corner

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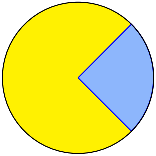

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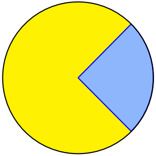

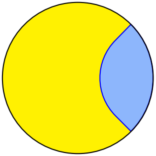

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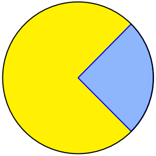

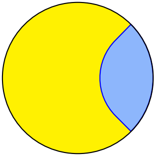

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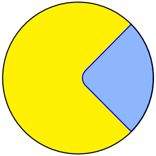

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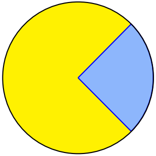

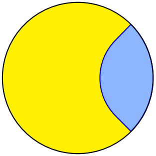

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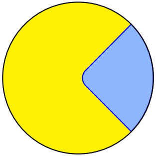

- We need to precise the properties of $(\mathscr{P})$ when the interface has a corner in the case $\kappa_\sigma \in I_c \setminus \{-1\}$.

Properties of the limit problem inside the critical interval

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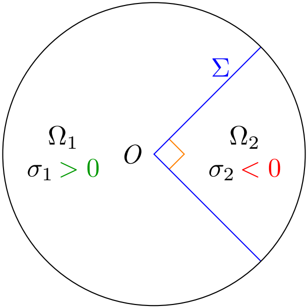

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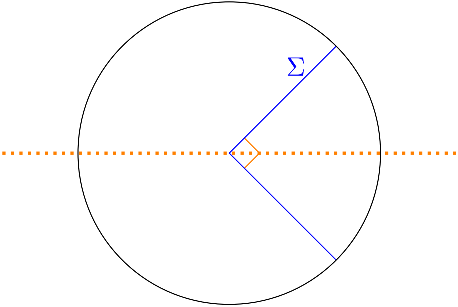

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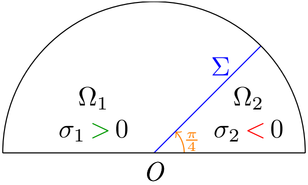

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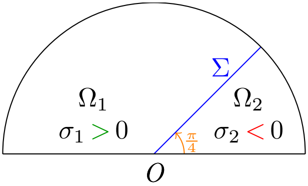

- Using the variational method of the T-coercivity, we prove the

Proposition. The problem $(\mathscr{P})$ is well-posed as soon as the contrast $\kappa_\sigma = \sigma_2/\sigma_1$ satisfies $\kappa_\sigma \notin I_c = [-1; -1/3]$.

Properties of the limit problem inside the critical interval

$$(\mathscr{P}) \quad \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ such that:} \\ -\mathrm{div}(\sigma \nabla u) = f \quad \text{ in } \Omega. \end{array} \right.$$

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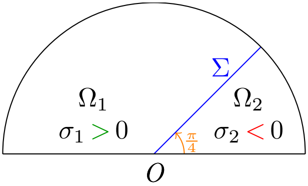

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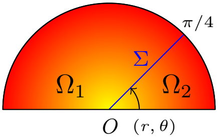
What happens when $\kappa_\sigma \in (-1; -1/3]$?

Analogy with a waveguide problem

- Bounded sector Ω

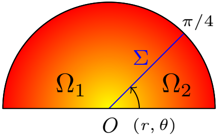


- Equation:

$$\underbrace{-\mathrm{div}(\sigma \nabla u)}_{-r^{-2}(\sigma (r\partial_r)^2 + \partial_\theta \sigma \partial_\theta)u} = f$$

Analogy with a waveguide problem

- Bounded sector Ω



- Equation:

$$\underbrace{-\mathrm{div}(\sigma\nabla u)}_{-r^{-2}(\sigma(r\partial_r)^2+\partial_\theta\sigma\partial_\theta)u} \qquad = f$$

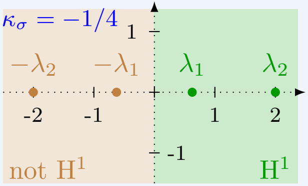
- Singularities in the sector

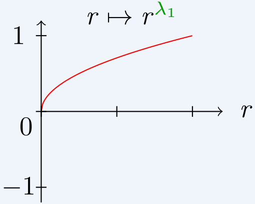
$s(r,\theta) = r^\lambda\varphi(\theta)$

Analogy with a waveguide problem

We compute the singularities $s(r,\theta) = r^{\lambda}\varphi(\theta)$ and we observe two cases:

- Outside the critical interval





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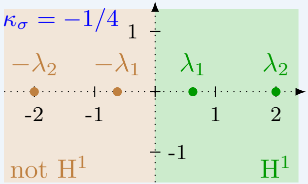
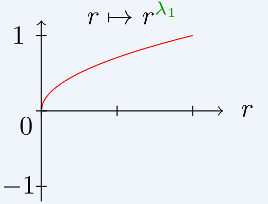
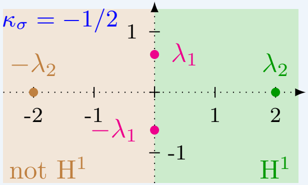
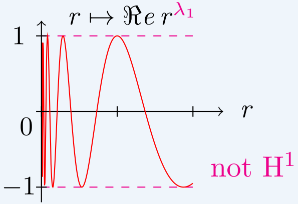




- Inside the critical interval





Analogy with a waveguide problem

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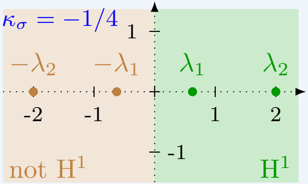

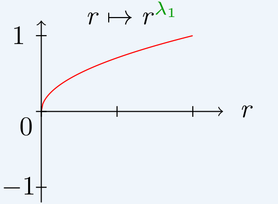

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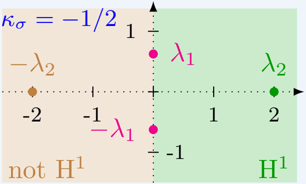
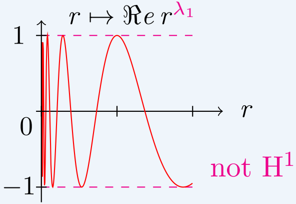
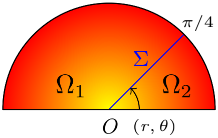




How to deal with the propagative singularities inside the critical interval?

Analogy with a waveguide problem

- Bounded sector Ω



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$$\underbrace{-\mathrm{div}(\sigma\nabla u)}_{-r^{-2}(\sigma(r\partial_r)^2+\partial_\theta\sigma\partial_\theta)u} \qquad = f$$

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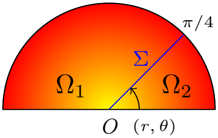
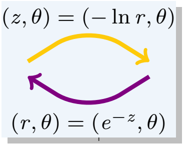
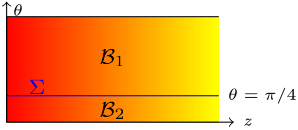
Analogy with a waveguide problem

- Bounded sector Ω





- Half-strip $\mathcal{B}$



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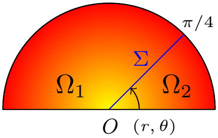

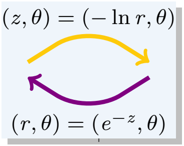

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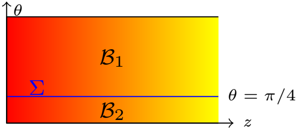

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- Equation:

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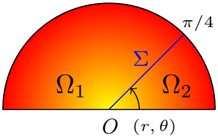
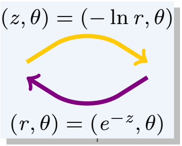
Analogy with a waveguide problem

- Bounded sector Ω





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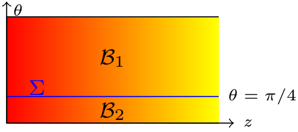

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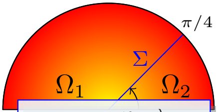
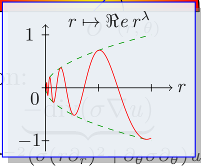
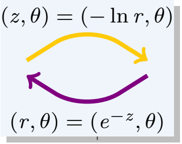
Analogy with a waveguide problem

- Bounded sector Ω







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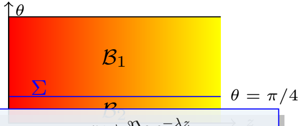

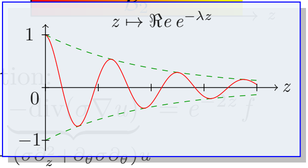

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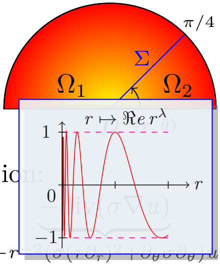
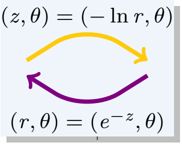
$s \in \mathrm{H}^1(\Omega)$ $\quad \Re e\,\lambda > 0 \quad$ m is evanescent

Analogy with a waveguide problem

- Bounded sector Ω





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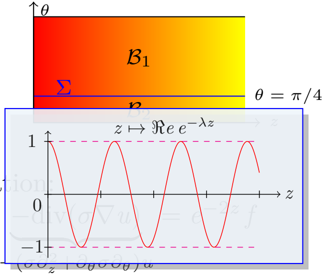

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$(\Re e\,\lambda = a,\ \Im m\,\lambda = b)$

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Analogy with a waveguide problem

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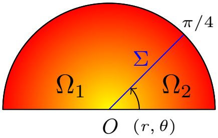

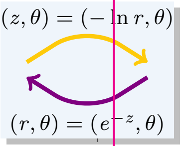

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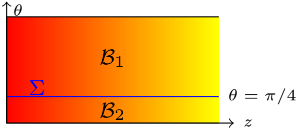

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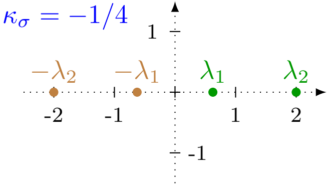
▶ This encourages us to use modal decomposition in the half-strip.

Modal analysis in the waveguide

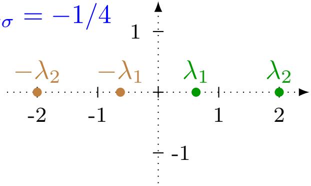


► Outside the critical interval. All the modes are exponentially growing or decaying.

→ We look for an exponentially decaying solution. $\boxed{\text{H}^1 \text{ framework}}$

Modal analysis in the waveguide

$\kappa_\sigma = -1/4$



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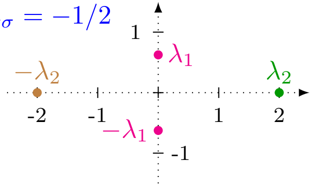
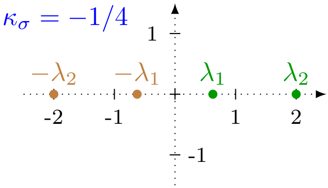


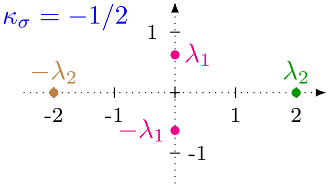
► Inside the critical interval. There are exactly two propagative modes.

Modal analysis in the waveguide



► Outside the critical interval. All the modes are exponentially growing or decaying.

→ We look for an exponentially decaying solution. $\boxed{\mathrm{H}^1 \text{ framework}}$



► Inside the critical interval. There are exactly two propagative modes.

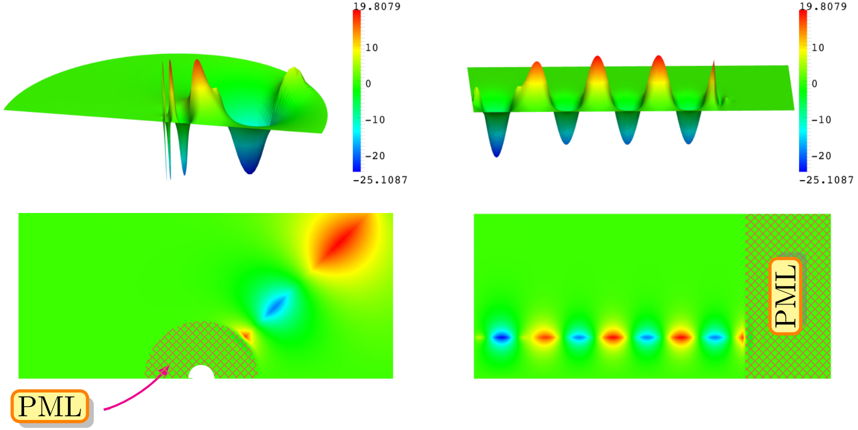
→ The decomposition on the outgoing modes leads to look for a solution of the form

$$u = \underbrace{c_1\, \varphi_1\, e^{\lambda_1 z}}_{\text{propagative part}} + \underbrace{u_e}_{\text{evanescent part}}.$$

$\boxed{\text{non } \mathrm{H}^1 \text{ framework}}$

How to approximate the solution?

- We use a PML (*Perfectly Matched Layer*) to bound the domain $\mathcal{B}$ + finite elements in the truncated strip ($\kappa_\sigma = -0.999 \in (-1; -1/3)$).



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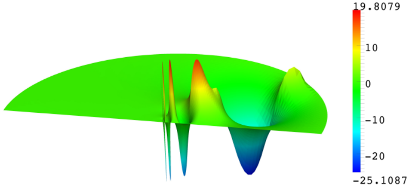
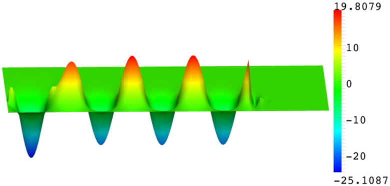
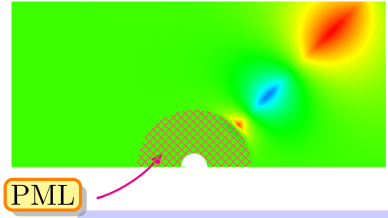
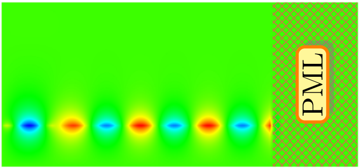


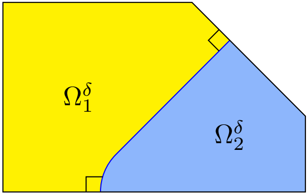


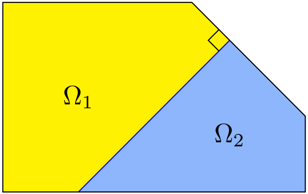




Without the PML, the solution in the truncated strip of length L does not converge when $L \to \infty$. This is what we observed in our numerical experiment for the rounded corner.

Asymptotic analysis





$$(\mathscr{P}^\delta)\ \left|\ \begin{array}{l}\text{Find } u^\delta \in \mathrm{H}_0^1(\Omega) \text{ s.t.:}\\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = f \text{ in } \Omega.\end{array}\right.$$

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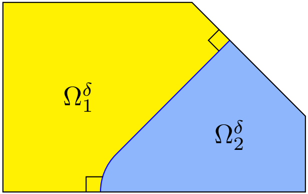
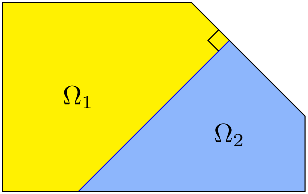
- The behaviour of $(u^\delta)_\delta$ depends on the properties of the limit problem.

Asymptotic analysis





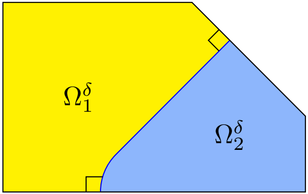
$$(\mathscr{P}^\delta) \left| \begin{array}{l} \text{Find } u^\delta \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = f \text{ in } \Omega. \end{array} \right.$$

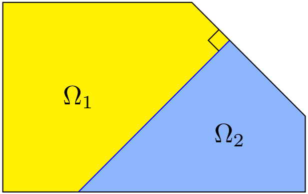
$$(\mathscr{P}) \left| \begin{array}{l} \text{Find } u \in \mathrm{H}^1_0(\Omega) \text{ s.t.:} \\ -\mathrm{div}(\sigma \nabla u) = f \text{ in } \Omega. \end{array} \right.$$

- The behaviour of $(u^\delta)_\delta$ depends on the properties of the limit problem.

If $(\mathscr{P})$ well-posed (in $\mathrm{H}^1_0(\Omega)$), then u^δ is uniquely defined for δ small enough and $(u^\delta)_\delta$ converges to u (as for positive materials).

Asymptotic analysis





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- The behaviour of $(u^\delta)_\delta$ depends on the properties of the limit problem.

If $(\mathscr{P})$ well-posed (in $\mathrm{H}_0^1(\Omega)$), then u^δ is uniquely defined for δ small enough and $(u^\delta)_\delta$ converges to u (as for positive materials).

If the limit problem is well-posed only in the exotic framework, then $(\mathscr{P}^\delta)$ critically depends on the value of the rounding parameter δ.

Asymptotic analysis

IDEA OF THE APPROACH:

1. We prove an *a priori* estimate for u^δ for all δ in some set $\mathscr{S}$ which excludes a discrete set accumulating in zero (♠ hard part of the proof, `S.A. Nazarov's` technique).

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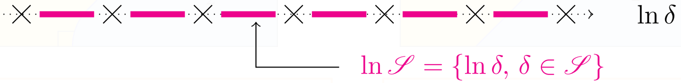

❷ We provide an asymptotic expansion of u^δ, denoted $\hat{u}^\delta$ with the error estimate, for some $\beta > 0$,

$$\|u^\delta - \hat{u}^\delta\|_{\mathrm{H}^1_0(\Omega^\delta)} \leq\ c\,\delta^\beta \|f\|_{\Omega^\delta}, \qquad \forall\delta\in\mathscr{S}.$$

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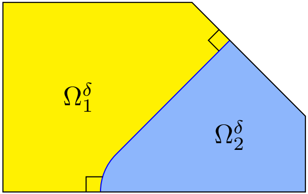
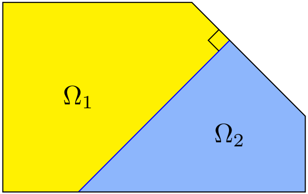
4. Conclusion.

The sequence $(u^\delta)_\delta$ does not converge, even for the L^2-norm!

Asymptotic analysis





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- The behaviour of $(u^\delta)_\delta$ depends on the properties of the limit problem.

If $(\mathscr{P})$ well-posed (in $\mathrm{H}^1_0(\Omega)$), then u^δ is uniquely defined for δ small enough and $(u^\delta)_\delta$ converges to u (as for positive materials).

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1 Numerical experiments

2 Properties of the limit problem

3 Asymptotic analysis

Conclusion

Let us remind the initial question:

What is the behaviour of $(u^\delta)_\delta$ when δ tends to zero?

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What is the behaviour of $(u^\delta)_\delta$ when δ tends to zero?

This depends on the features of the limit problem.

$(\dots)$ $(\dots)$

$\kappa_\sigma = -1.0001 \notin I_c$ $\kappa_\sigma = -0.9999 \in I_c$

When $\kappa_\sigma \in I_c$, $(u^\delta)_\delta$ does not converge, even for the L^2-norm!

In this case, it is impossible to simulate the fields since it is impossible to measure exactly δ. $\Rightarrow$ What happens physically?

Thank you for your attention!!!

A.-S. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet Jr., *T-coercivity for scalar interface problems between dielectrics and metamaterials*, M2AN, 46, 1363–1387, 2012.

A.-S. Bonnet-Ben Dhia, L. Chesnel, X. Claeys, *Radiation condition for a non-smooth interface between a dielectric and a metamaterial*, M3AS, 23, 2013.

L. Chesnel, X. Claeys, S.A. Nazarov, *A curious instability phenomenon for a rounded corner in presence of a negative material*, preprint arXiv:1304.4788, 2013.

Naive approximation

- Let us try a usual Finite Element Method (P1 Lagrange Finite Element). We solve the problem

$$\boxed{\left|\begin{array}{l}\text{Find } u_h \in \mathrm{V}_h \text{ s.t.:}\\ \displaystyle\int_\Omega \sigma \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h, \quad \forall v \in \mathrm{V}_h,\end{array}\right.}$$

where V_h approximates $\mathrm{H}^1_0(\Omega)$ as $h \to 0$ (h is the mesh size).

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THE SEQUENCE (u_h) DOES NOT CONVERGE AS $h \to 0$!!!

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$(\ldots)$ $(\ldots)$

Contrast $\kappa_\sigma = -0.999 \in (-1; -1/3)$.

Naive approximation

- Outside the critical interval, the sequence (u_h) converges.

$(\dots)$ $(\dots)$

Contrast $\kappa_\sigma = -1.001 \notin (-1; -1/3)$.