WAVES 2024

On the breathing of spectral bands in periodic quantum waveguides with inflating resonators

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Collaboration with S.A. Nazarov[2].

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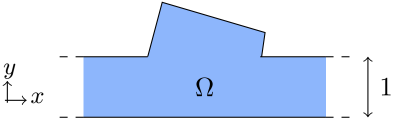
Introduction

- We consider the propagation of waves in a 2D thin periodic quantum waveguide Π^ε.

- Start with some domain $\Omega \subset \mathbb{R}^2$ which coincides with the strip $\mathbb{R} \times (-1/2; 1/2)$ outside of a bounded region (the resonator).



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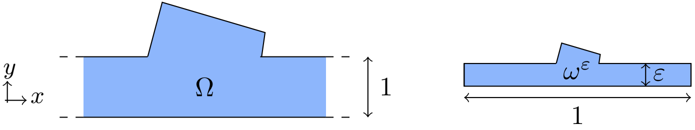

- Shrink Ω by a small factor $\varepsilon > 0$ to create the unit cell

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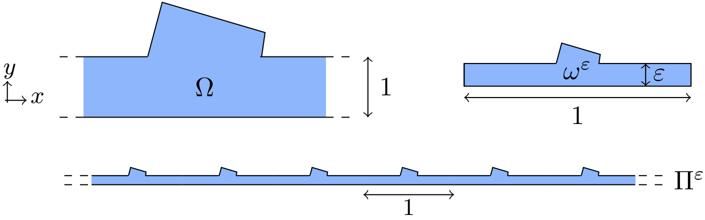

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$$\omega^\varepsilon := \{z = (x, y) \in \mathbb{R}^2 \,|\, z/\varepsilon \in \Omega \text{ and } |x| < 1/2\}$$

- Set $\partial\omega^\varepsilon_\pm := \{\pm 1/2\} \times (-\varepsilon/2; \varepsilon/2)$ and define

$$\boxed{\Pi^\varepsilon := \{z \in \mathbb{R}^2 \,|\, (x - m, y) \in \omega^\varepsilon \cup \partial\omega^\varepsilon_+,\ m \in \mathbb{Z}\}.}$$

- In Π^ε we consider the spectral problem for the Dirichlet Laplacian

$$(\mathscr{P}^\varepsilon)\left|\begin{array}{rcll} -\Delta u^\varepsilon & = & \lambda^\varepsilon\, u^\varepsilon & \text{in } \Pi^\varepsilon \\ u^\varepsilon & = & 0 & \text{on } \partial\Pi^\varepsilon. \end{array}\right.$$

- Denote by A^ε the unbounded operator of $\mathrm{L}^2(\Pi^\varepsilon)$ such that

$$\mathcal{D}(A^\varepsilon) := \{v \in \mathrm{H}^1_0(\Pi^\varepsilon)\,|\,\Delta v \in \mathrm{L}^2(\Pi^\varepsilon)\} \quad \text{and} \quad A^\varepsilon v = -\Delta v.$$

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Goal of the talk

We wish to study the lower part of $\sigma(A^\varepsilon)$, the spectrum of A^ε, as $\varepsilon \to 0$.

Outline of the talk

1 Preparatory work

2 Asymptotic analysis

3 Breathing of spectral bands

Reduction to a problem in the unit cell 1/2

- The Floquet Bloch transform

$$u^\varepsilon(z) \mapsto U^\varepsilon(z,\eta) = \frac{1}{(2\pi)^{1/2}} \sum_{j\in\mathbb{Z}} e^{i\eta j} u^\varepsilon(x+j,y), \quad \eta \in \mathbb{R},$$

converts $(\mathscr{P}^\varepsilon)$ into a spectral problem set in ω^ε with quasi-periodicity boundary conditions at $\partial\omega^\varepsilon_\pm$

$$(\mathscr{P}^\varepsilon(\eta)) \quad \left| \begin{array}{rcll} -\Delta U^\varepsilon(z,\eta) & = & \Lambda^\varepsilon(\eta)\, U^\varepsilon(z,\eta) & z \in \omega^\varepsilon \\ U^\varepsilon(z,\eta) & = & 0 & z \in \partial\omega^\varepsilon \cap \partial\Pi^\varepsilon \\ U^\varepsilon(-1/2,y,\eta) & = & e^{i\eta} U^\varepsilon(+1/2,y,\eta) & y \in (-\varepsilon/2;\varepsilon/2) \\ \partial_x U^\varepsilon(-1/2,y,\eta) & = & e^{i\eta} \partial_x U^\varepsilon(+1/2,y,\eta) & y \in (-\varepsilon/2;\varepsilon/2). \end{array} \right.$$

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- For $\eta \in [0;2\pi)$, the spectrum of $(\mathscr{P}^\varepsilon(\eta))$ is discrete, made of the unbounded sequence of real eigenvalues

$$\boxed{0 < \Lambda_1^\varepsilon(\eta) \le \Lambda_2^\varepsilon(\eta) \le \cdots \le \Lambda_p^\varepsilon(\eta) \le \dots .}$$

Reduction to a problem in the unit cell 2/2

- The functions

$$\eta \mapsto \Lambda_p^\varepsilon(\eta)$$

are continuous so that the spectral bands

$$\boxed{\Upsilon_p^\varepsilon := \{\Lambda_p^\varepsilon(\eta),\ \eta \in [0;2\pi)\}}$$

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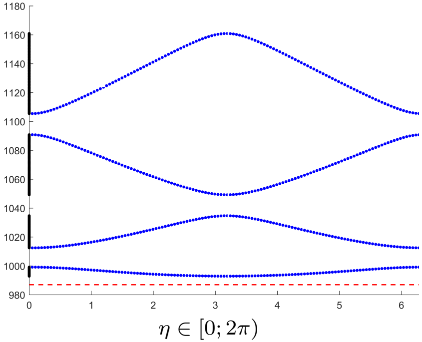

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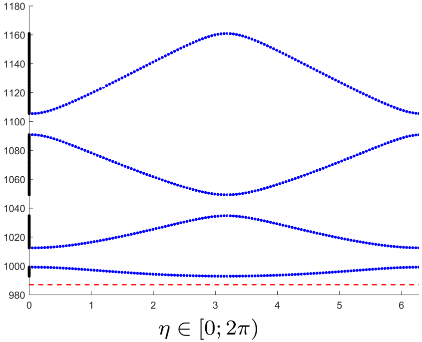


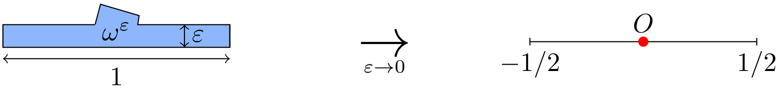
- Finally, we have $\boxed{\sigma(A^\varepsilon) = \bigcup_{p\in\mathbb{N}^* := \{1,2,\dots\}} \Upsilon_p^\varepsilon.}$

To study the behaviour of $\sigma(A^\varepsilon)$ as $\varepsilon \to 0$, we have to consider the asymptotics of $\Lambda_p^\varepsilon(\eta)$ as $\varepsilon \to 0$.

Asymptotic analysis - general picture

- To compute the asymptotics of the almost 1D problem $(\mathscr{P}^\varepsilon(\eta))$, we use techniques of matched asymptotic expansions (see Post 05, Griser 08).

- Roughly speaking, at the limit $\varepsilon \to 0$, we obtain a 1D geometry with a junction point at O



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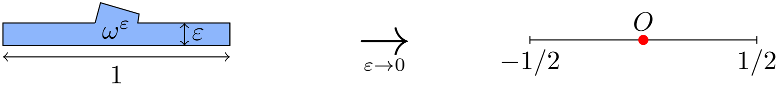
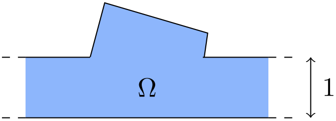


► Classically, we consider different expansions far from O and in a neighbourhood of O that we match in some intermediate regions.

Near field problem

- In the process, the features of the Dirichlet Laplacian in Ω, the near field geometry obtained by zooming at O, play an important role.



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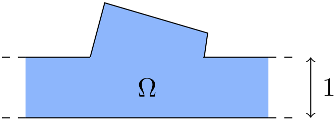

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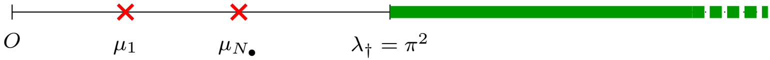
Spectrum of A^Ω:



- The continuous spectrum occupies the ray $[\pi^2; +\infty)$.

- Depending on Ω, A^Ω may have or not discrete spectrum. Assume that A^Ω has exactly $N_\bullet \in \mathbb{N}$ eigenvalues $0 < \mu_1 < \mu_2 \leq \cdots \leq \mu_{N_\bullet} < \pi^2$.

Near field problem at the threshold

- In the sequel, the properties of the problem

$$(\mathscr{P}_\dagger)\left|\begin{array}{rcll}\Delta W+\pi^2 W & = & 0 & \text{in } \Omega\\ W & = & 0 & \text{on } \partial\Omega\end{array}\right.$$

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- $\mathbb{S}$ is a unitary matrix $\Rightarrow$ its 2 eigenvalues lie on the unit circle. In general they are different from -1.

For most Ω, we have $\mathrm{X}_\dagger=\{0\}$.

First main results

For $p \in \mathbb{N}^*$, let $\Upsilon_p^\varepsilon = [a_{p-}^\varepsilon; a_{p+}^\varepsilon]$, with $a_{p-}^\varepsilon \le a_{p+}^\varepsilon$, be the spectral band as introduced before. To simplify, assume absence of trapped modes for $(\mathscr{P}_\dagger)$.

THEOREM: There are constants $c_{p-} < c_{p+}$, $C_p > 0$, $\delta_p > 0$ such that as $\varepsilon \to 0$ we have

For $p = 1, \dots, N_\bullet$:

$$\left| a_{p\pm}^\varepsilon - \left(\varepsilon^{-2}\mu_p + \varepsilon^{-2} e^{-\sqrt{\pi^2-\mu_p}/\varepsilon} c_{p\pm} \right) \right| \le C_p\, e^{-(1+\delta_p)\sqrt{\pi^2-\mu_p}/\varepsilon};$$

For $p = N_\bullet + m$, $m \in \mathbb{N}^*$:

$i)$ if $\mathrm{X}_\dagger = \{0\}$, $\quad \left| a_{p\pm}^\varepsilon - \left(\varepsilon^{-2}\pi^2 + m^2\pi^2 + \varepsilon c_{p\pm} \right) \right| \le C_p\, \varepsilon^{1+\delta_p};$

$ii)$ if $\dim \mathrm{X}_\dagger = 1$, $\quad \left| a_{p\pm}^\varepsilon - \left(\varepsilon^{-2}\pi^2 + c_{p\pm} \right) \right| \le C_p\, \varepsilon^{1+\delta_p};$

$iii)$ if $\dim \mathrm{X}_\dagger = 2$, $\quad \left| a_{p\pm}^\varepsilon - \left(\varepsilon^{-2}\pi^2 + (m-1)^2\pi^2 + \varepsilon c_{p\pm} \right) \right| \le C_p\, \varepsilon^{1+\delta_p}.$

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3. Concerning the next ones, the behaviour depends on $\dim \mathrm{X}_\dagger$:

When $\mathbf{dim\,X_\dagger \neq 1}$, the Υ_p^ε are of length $O(\varepsilon)$. Moreover, between Υ_p^ε and $\Upsilon_{p+1}^\varepsilon$, there is a gap whose length tends to $(2m+1)\pi^2$.

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Generically, the propagation of waves in Π^ε is hampered and occurs only for very narrow intervals of frequencies.

When $\mathbf{dim\, X_\dagger = 1}$, the situation is very different because asymptotically the Υ_p^ε are of length $c_{p+} - c_{p-}$, with in general $c_{p+} > c_{p-}$.

For particular Ω, waves can propagate in Π^ε for much larger intervals of frequencies than above.

Elements of proof – first $N_\bullet$ spectral bands

For $1 \leq p \leq N_\bullet$, let $u^\varepsilon(\cdot, \eta)$ be an eigenfunction associated with $\Lambda_p^\varepsilon(\eta)$.

- As $\varepsilon \to 0$, consider the approximation

$$\Lambda_p^\varepsilon(\eta) = \varepsilon^{-2}\mu_p + \ldots, \qquad u^\varepsilon(z, \eta) = v(z/\varepsilon) + \ldots$$

where $\mu_p \in (0; \pi^2)$, v is an eigenpair of the discrete spectrum of A^Ω.

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- This model is independent of η. We can refine it by constructing corrector terms.

Elements of proof – higher spectral bands

For $p \geq N_\bullet$, let $u^\varepsilon(\cdot, \eta)$ be an eigenfunction associated with $\Lambda_p^\varepsilon(\eta)$.

$\rightarrow$ To simplify, we remove the subscript $_p$ and the dependence on η.

- As $\varepsilon \to 0$, consider the expansions

$$\Lambda^\varepsilon = \varepsilon^{-2}\pi^2 + \nu + \dots, \qquad u^\varepsilon(z) = \gamma^\pm(x)\,\varphi(y/\varepsilon) + \dots \text{ for } \pm x > 0.$$

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Inserting it in $(\mathscr{P}^\varepsilon(\eta))$, we obtain

$$(\mathscr{P}_{1\mathrm{D}}) \left| \begin{array}{rcll} \partial_x^2\gamma^+ + \nu\gamma^+ & = & 0 & \text{in } (0;1/2) \\ \partial_x^2\gamma^- + \nu\gamma^- & = & 0 & \text{in } (-1/2;0) \\ \gamma^-(-1/2) & = & e^{i\eta}\gamma^+(+1/2) & \\ \partial_x\gamma^-(-1/2) & = & e^{i\eta}\partial_x\gamma^+(+1/2). & \end{array} \right.$$

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☛ We must complete this system with conditions at O.

We find them by matching this far field expansion with some inner field expansion of u^ε

$$u^\varepsilon(z) = W(z/\varepsilon) + \ldots.$$

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☛ We must complete this system with conditions at O.

We find them by matching this far field expansion with some inner field expansion of u^ε

$$u^\varepsilon(z) = W(z/\varepsilon) + \dots.$$

We obtain that W must satisfy $(\mathscr{P}_\dagger)$.

Elements of proof – higher spectral bands

Case $\mathbf{X}_\dagger = \{0\}$.

We take $W \equiv 0$ and impose $\gamma^{\pm}(0) = 0$, i.e. Dirichlet conditions at O in $(\mathscr{P}_{1\mathrm{D}})$. Solving $(\mathscr{P}_{1\mathrm{D}})$, we get

$$\left| \begin{array}{l} \Lambda^\varepsilon = \varepsilon^{-2}\pi^2 + (m\pi)^2 + \dots \\ u^\varepsilon(z) = \pm e^{\mp i\eta/2} \sin(m\pi x)\, \varphi(y/\varepsilon) + \dots \ \text{ for } \ \pm x > 0. \end{array} \right.$$

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Case $\dim \mathbf{X}_\dagger = 1$.

We take $W \in \mathrm{X}_\dagger$ and impose

$$\left|\begin{array}{rcl} \cos\theta\,\partial_x\gamma^+(0) & = & \sin\theta\,\partial_x\gamma^-(0) \\ \sin\theta\,\gamma^+(0) & = & \cos\theta\,\gamma^-(0), \end{array}\right.$$

i.e. generalized Kirchoff transmission conditions at O. Here $(\cos\theta, \sin\theta)^\top$ is an eigenvector of $\mathbb{S}$ for the eig. -1. Solving $(\mathscr{P}_{1\mathrm{D}})$, we get

$$\Lambda^\varepsilon = \varepsilon^{-2}\pi^2 + \nu(\eta) + \dots$$

where $\nu(\eta)$ satisfies $\sin(2\theta)\cos\eta = \cos\sqrt{\nu(\eta)}$

Elements of proof – higher spectral bands

Case $\dim \mathbf{X}_\dagger = 2$.

We impose $\partial_x \gamma^{\pm}(0) = 0$, i.e. Neumann conditions at O in $(\mathscr{P}_{1\mathrm{D}})$. Solving $(\mathscr{P}_{1\mathrm{D}})$, we get

$$\left|\begin{array}{l} \Lambda^\varepsilon = \varepsilon^{-2}\pi^2 + (m\pi)^2 + \dots \\ u^\varepsilon(z) = \pm e^{\mp i\eta/2} \cos(m\pi x)\, \varphi(y/\varepsilon) + \dots \ \text{for} \ \pm x > 0. \end{array}\right.$$

Additional comments

- Can we find examples of Ω such that $\dim \mathrm{X}_\dagger = 1$? **Yes!**

$\rightarrow$ The reference strip $\mathbb{R} \times (-1/2; 1/2)$. Indeed in this case $v(x, y) = \cos(\pi y)$ belongs to $\mathrm{X}_\dagger$. We directly compute $\sigma(A^\varepsilon) = [\pi^2/\varepsilon^2; +\infty)$.

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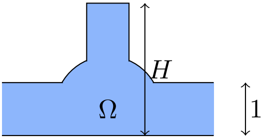

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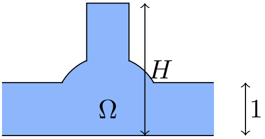

- Can we find Ω such that $\dim \mathrm{X}_\dagger = 2$? **Open question!**

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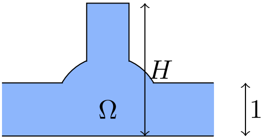


- Can we find Ω such that $\dim \mathrm{X}_\dagger = 2$? **Open question!**

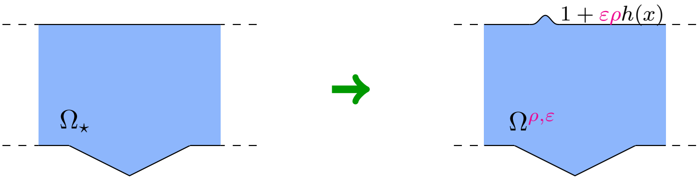
- For the Neumann Laplacian, the analysis is very similar. The near field problem at the threshold simply writes

$$\left|\begin{array}{rcll} \Delta W & = & 0 & \text{in } \Omega \\ \partial_n W & = & 0 & \text{on } \partial\Omega \end{array}\right. \quad \Rightarrow \quad \text{for all } \Omega, \quad \dim \mathrm{X}_\dagger = 1 \ \text{ with } \ \theta = \pi/4$$

$\rightarrow$ Kirchoff trans. condi. at O for the 1D model

Setting

► We wish to describe the change of $\sigma(A^\varepsilon)$ when perturbing the inner field geometry around a particular $\Omega = \Omega_\star$ where $\dim \mathrm{X}_\dagger = 1$.



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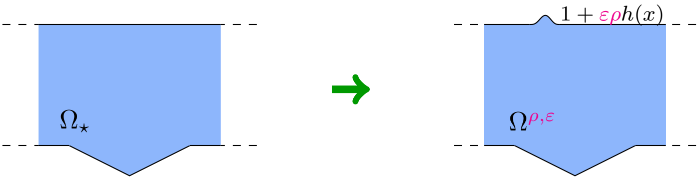

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where $h \in \mathscr{C}_0^\infty(\mathbb{R})$ is a given profile function and ρ a given parameter.

► We denote $\Upsilon_p^{\rho,\varepsilon} := \{\Lambda_p^{\rho,\varepsilon}(\eta),\, \eta \in [0; 2\pi)\}$ the spectral bands of the Dirichlet Laplacian in $\Pi^{\rho,\varepsilon}$, the periodic domain constructed from $\Omega^{\rho,\varepsilon}$.

We emphasize that we make a periodic perturbation of Π^ε.

Second main results

Fix $\rho \in \mathbb{R}$. For $m \in \mathbb{N}^*$ and $p = N_\bullet + m$, let $\Upsilon_p^{\rho,\varepsilon} = [a_{m-}^{\rho,\varepsilon}; a_{m+}^{\rho,\varepsilon}]$, with $a_{m-}^{\rho,\varepsilon} \le a_{m+}^{\rho,\varepsilon}$, be the spectral band as defined above.

Theorem: There are some constants $c_{m-}^{\rho} < c_{m+}^{\rho}$, $C_m > 0$, $\delta_m > 0$ such that as $\varepsilon \to 0$ we have

$$\left| a_{m\pm}^{\rho,\varepsilon} - \left(\varepsilon^{-2}\pi^2 + c_{m\pm}^{\rho} \right) \right| \le C_m \, \varepsilon^{1+\delta_m}.$$

Moreover, we have

$$\lim_{\rho \to -\infty} c_{m\pm}^{\rho} = m^2\pi^2, \qquad c_{1\pm}^{\rho} \underset{\rho \to +\infty}{\sim} -\frac{T^2}{4}\rho^2, \qquad \lim_{\rho \to +\infty} c_{(m+1)\pm}^{\rho} = m^2\pi^2$$

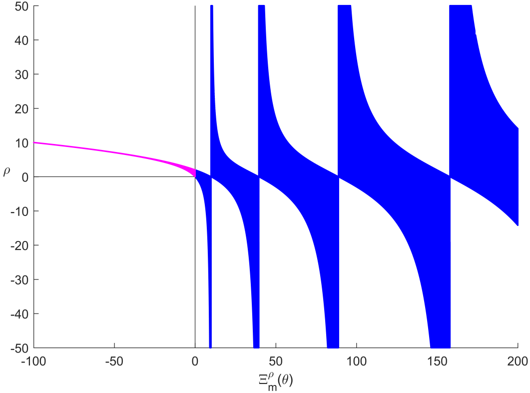
with

$$c_{m+}^{\rho} - c_{m-}^{\rho} \underset{\rho \to \pm\infty}{=} O(1/\rho), \qquad c_{1+}^{\rho} - c_{1-}^{\rho} \underset{\rho \to +\infty}{=} O(e^{-\delta\rho}).$$

Here $T > 0$ is a constant which depends on the profile function h.

Spectral bands of the model

- Spectral bands of the model with respect to ρ (after a shift by $-\pi^2/\varepsilon^2$).



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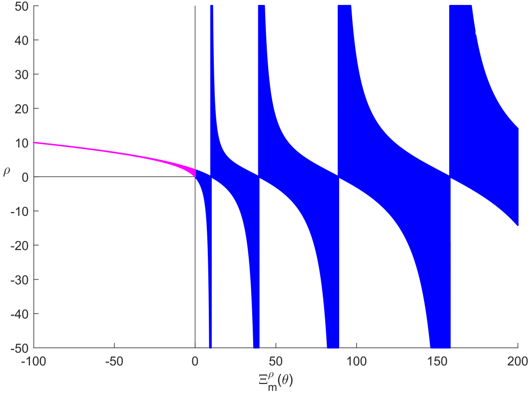

- For ρ running from $-\infty$ to $+\infty$, i.e. when inflating the near field geom. around $\Omega_\star$, the spectral bands expand and shrink $\Rightarrow$ breathing phenomenon of $\sigma(A^\varepsilon)$.

- In the process, a **band** dives below π^2/ε^2, stops breathing and becomes extremely short as $\rho \to +\infty$.

New 1D model with ρ dependence

- We obtain the expansions

$$\Lambda^\varepsilon = \varepsilon^{-2}\pi^2 + \nu + \dots, \qquad u^\varepsilon(z) = \gamma^\pm(x)\,\varphi(y/\varepsilon) + \dots \text{ for } \pm x > 0.$$

where $\gamma^\pm$ satisfy

$$(\mathscr{P}_{1\mathrm{D}})\left|\begin{array}{rcl} \partial_x^2\gamma^+ + \nu\gamma^+ & = & 0 \quad \text{in } (0;1/2) \\ \partial_x^2\gamma^- + \nu\gamma^- & = & 0 \quad \text{in } (-1/2;0) \\ \gamma^-(-1/2) & = & e^{i\eta}\gamma^+(+1/2) \\ \partial_x\gamma^-(-1/2) & = & e^{i\eta}\partial_x\gamma^+(+1/2). \end{array}\right.$$

together with the new transmission conditions

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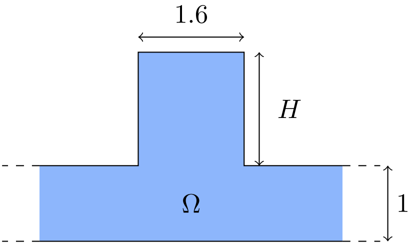
- In particular when $\rho \to \pm\infty$, as expected we get $\gamma^\pm(0) = 0$ (Dirichlet).

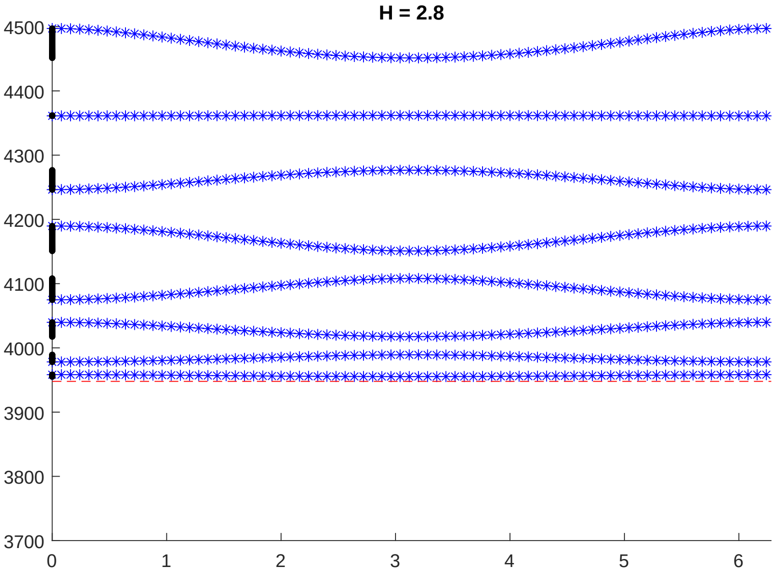
Numerics on the exact problem

- We start from the inner field geometry



- We use `Freefem++` to compute the spectrum of $(\mathscr{P}^{\varepsilon}(\eta))$ in the corresponding unit cell.

Numerics on the exact problem



Conclusion

What we did

- ♠ We studied the asymptotics of the spectrum of the Dirichlet Laplacian in thin periodic waveguides.
  - All bands go to $+\infty$ as $O(\varepsilon^{-2})$;
  - The first bands are extremely short;
  - The length of the next bands depends on the features of the inner field geometry, in particular of $\dim \mathrm{X}_\dagger$.
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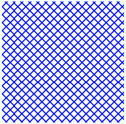
Possible extensions and open questions



1) We could work similarly in other periodic waveguides.
2) Can one find examples of Ω such that $\dim \mathrm{X}_\dagger = 2$?
3) Can one work with other models, i.e. $\Delta\Delta u - k^4 u = 0$+Dirichlet BC?

Thank you!

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