Conference on Mathematics of Wave Phenomena

Non reflection and perfect reflection via Fano resonance in waveguides

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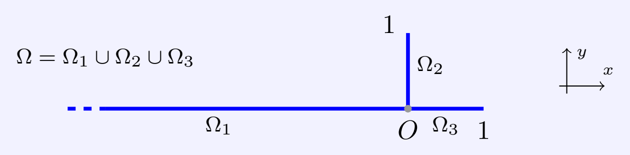
A 1D toy problem

- Fano resonance phenomenon appears in many fields in physics. First, we illustrate it for a simple 1D problem.

A 1D toy problem



- Consider the scattering problem

$$\varphi'' + k^2\varphi = 0 \text{ in } \Omega, \quad \left| \begin{array}{l} \varphi_1 = \varphi_2 = \varphi_3 \text{ at } O \\ \varphi_1' = \varphi_2' + \varphi_3' \text{ at } O \\ \varphi_2' = \varphi_3' = 0 \text{ on } \partial\Omega \end{array} \right. \quad \text{with } \underbrace{\varphi_1 = e^{ikx} + R\, e^{-ikx}}_{\text{radiation condition}}, \; R \in \mathbb{C}.$$

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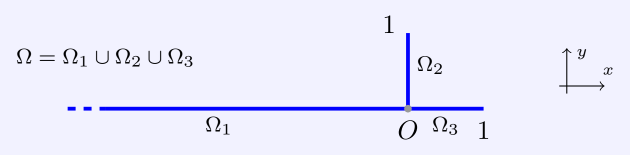

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- Well-posedness $\quad\Leftrightarrow\quad$ invertibility of a 3×3 system $\mathbb{M}\Phi = F$.

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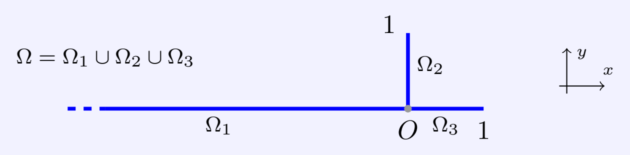

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- Well-posedness $\Leftrightarrow$ invertibility of a 3×3 system $\mathbb{M}\Phi = F$.

- Uniqueness $\Leftrightarrow$ $k \notin (2\mathbb{N}+1)\pi/2$. Existence for all $k \in \mathbb{R}$ $(F \in \ker \mathbb{M}^{\perp})$

$$R = \frac{\cos(k) + 2i\sin(k)}{\cos(k) - 2i\sin(k)}.$$

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- We perturb the geometry: $\Omega^\varepsilon = \Omega_1 \cup \Omega_2 \cup \Omega_3^\varepsilon$ with $\Omega_3^\varepsilon = (0; 1+\varepsilon)$.
  Well-posedness in Ω^ε $\Leftrightarrow$ invertibility of a 3×3 system $\mathbb{M}^\varepsilon \Phi^\varepsilon = F$.

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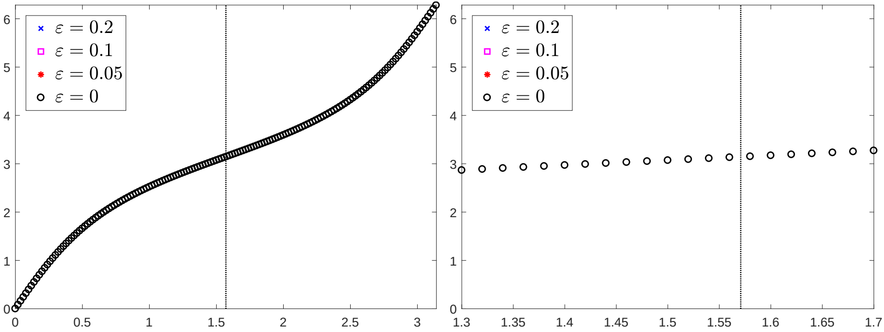

Figure: $k \mapsto \theta^\varepsilon(k)$ *for several* ε *(non uniqueness for* $\varepsilon = 0$, $k = \pi/2$*).*

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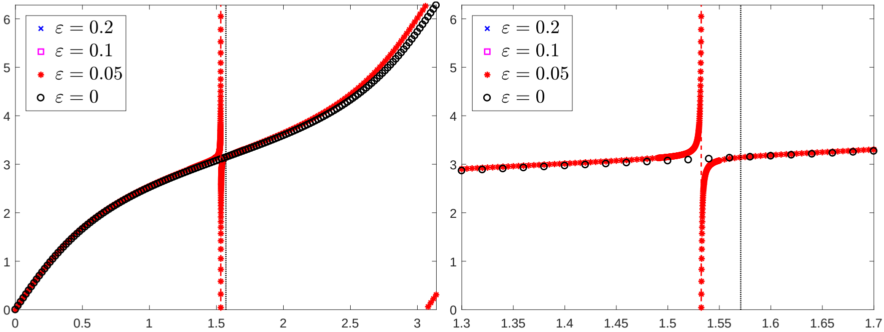

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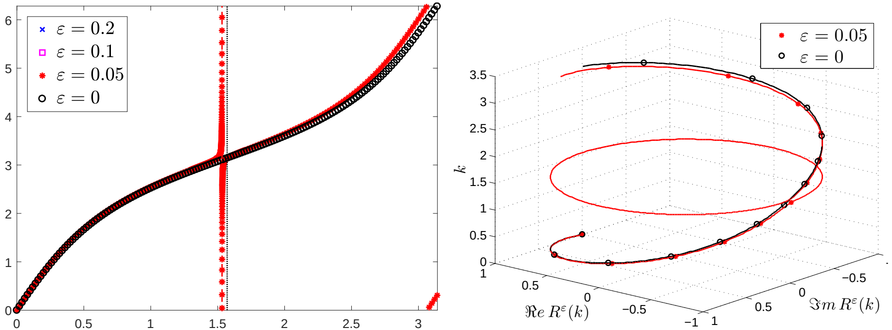

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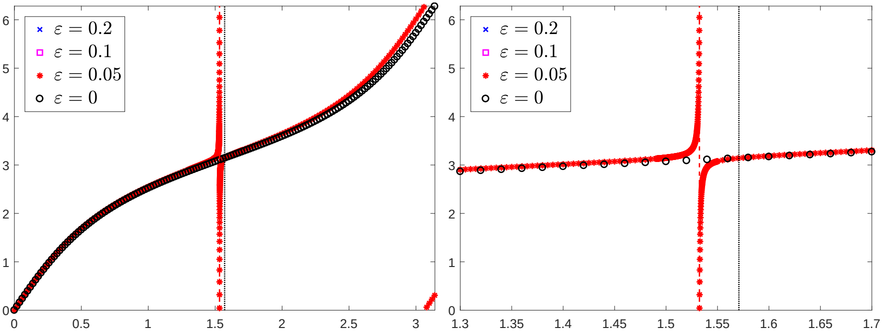

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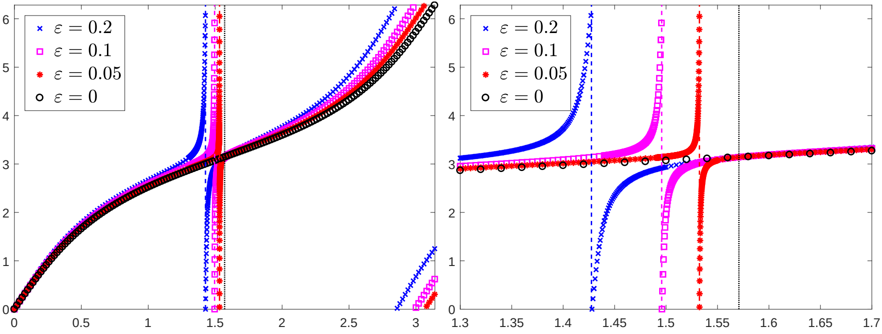

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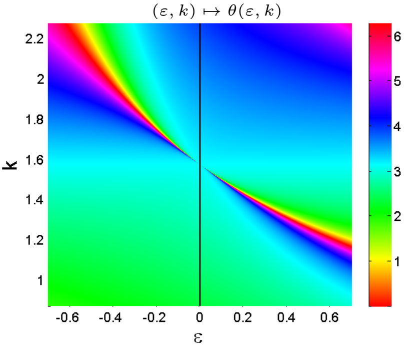
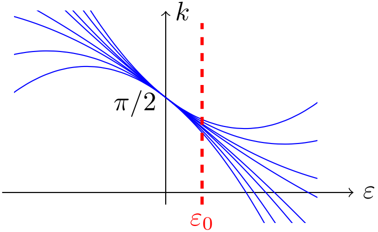
- Set $R(\varepsilon, k) = e^{i\theta(\varepsilon,k)}$ (functions of two variables).





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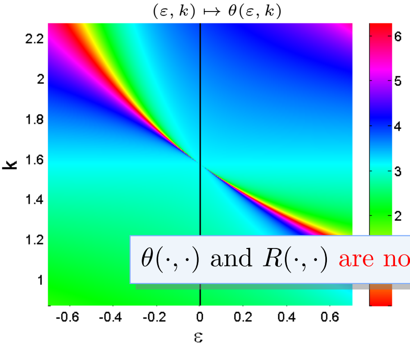

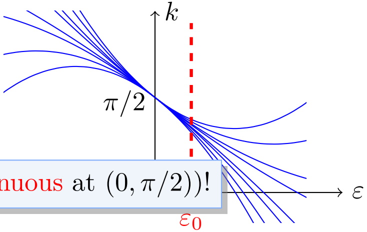

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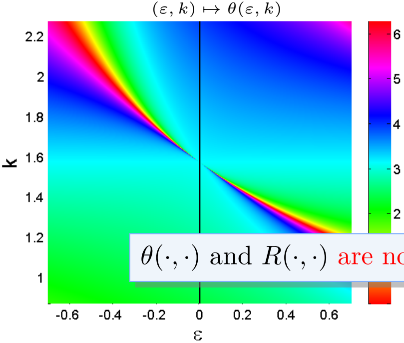

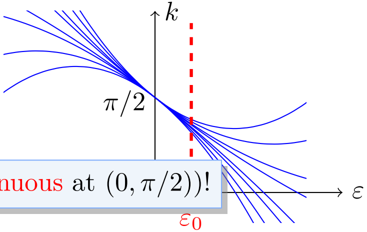

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Goals of the talk

1) Prove a similar Fano resonance phenomenon for a 2D waveguide.
2) Use it to provide examples of non reflection and complete reflection.

→ Similar results in `Shipman et Tu, SIAM Appl. Math, 2012`. We use a different approach and consider a perturbation of the geometry.

Outline of the talk

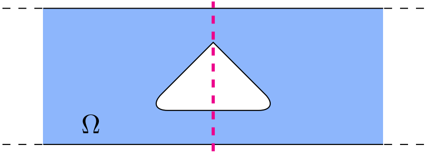
1. The Fano resonance in the 2D waveguide

2. Non reflection and complete reflection

3. Numerical experiments

Setting

► Scattering in time-harmonic regime in a symmetric (to simplify) acoustic waveguide Ω coinciding with $\{(x,y) \in \mathbb{R} \times (0;1)\}$ outside a compact region.



$$(*) \left| \begin{array}{rcll} \Delta v + \lambda v & = & 0 & \text{in } \Omega, \\ \partial_n v & = & 0 & \text{on } \partial\Omega. \end{array} \right.$$

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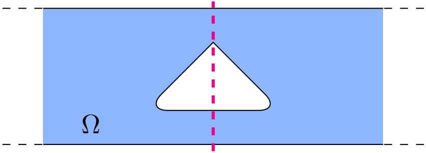

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- We assume that trapped modes exist for $\lambda=\lambda^0\in(0;\pi^2)$:

$$u_{\mathrm{tr}}\in\mathrm{H}^1(\Omega)\setminus\{0\}\text{ satisfies }(*)\text{ for }\lambda=\lambda^0\text{ (non uniqueness).}$$

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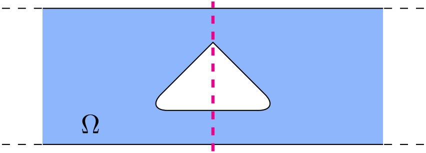

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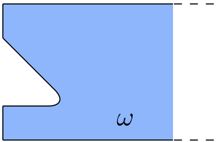
$$u_{\text{tr}} \in \mathrm{H}^1(\Omega) \setminus \{0\} \text{ satisfies } (*) \text{ for } \lambda = \lambda^0 \text{ (non uniqueness).}$$

- Due to symmetry, u_{tr} is also a trapped mode for the half waveguide pb.



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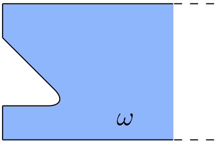
(depends on the sym.)

Scattering problem in the half waveguide



$$
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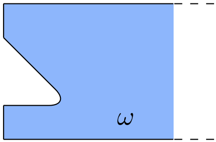

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Propagating	$w_0^\pm(x, y) = e^{\pm ikx}/\sqrt{2k}$,
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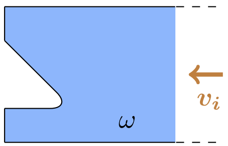

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where $R \in \mathbb{C}$ and $\tilde{v}$ is expo. decaying (uniqueness $\Leftrightarrow$ abs. of trapped modes).

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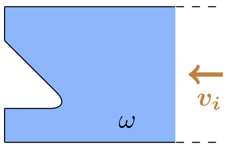

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where $R \in \mathbb{C}$ and $\tilde{v}$ is expo. decaying (uniqueness $\Leftrightarrow$ abs. of trapped modes).

- R is uniquely defined (even for $\lambda = \lambda^0$) and $|R| = 1$ (cons. of energy).

Small perturbation of the geometry

- We perturb slightly ($\varepsilon \geq 0$ is small) the geometry

> Locally $\partial\omega^{\varepsilon}$ coincides with the graph of $x \mapsto 1 + \varepsilon H(x)$,
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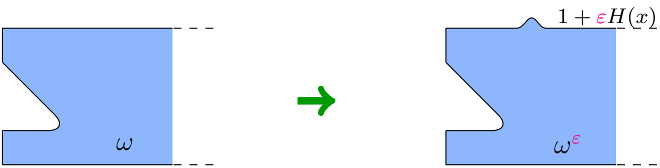

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$\rightarrow$ *We will prove that R is* ***not continuous*** *at $(0, \lambda^0)$ working with the augmented scattering matrix which is* ***continuous*** *at $(0, \lambda^0)$.*

The augmented scattering matrix

- We assume that λ^0 is a simple eigenvalue for $(*)$ and that

$$u_{\mathrm{tr}} = Ke^{-\beta_1 x}\cos(\pi y) + \tilde{u}_{\mathrm{tr}},$$

where $K \neq 0$, $\tilde{u}_{\mathrm{tr}}$ has fast decay.

COMMENTS:

- If $K = 0$, adapt the definition of the augmented scattering matrix.
- This object has been introduced in `Nazarov, Plamenevsky, 1994`.

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$$u_0 = w_0^- + \mathfrak{s}_{00}\, w_0^+ + \mathfrak{s}_{01}\, W_1^+ + \tilde{u}_0, \qquad \text{with } \tilde{u}_0 \text{ fastly expo. decaying}$$
$$u_1 = W_1^- + \mathfrak{s}_{10}\, w_0^+ + \mathfrak{s}_{11}\, W_1^+ + \tilde{u}_1, \qquad \text{with } \tilde{u}_1 \text{ fastly expo. decaying.}$$

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With such radiation conditions, uniqueness holds for $\lambda = \lambda^0$.

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$$u_0 = w_0^- + \mathfrak{s}_{00}\, w_0^+ + \mathfrak{s}_{01}\, W_1^+ + \tilde{u}_0, \qquad \text{with } \tilde{u}_0 \text{ fastly expo. decaying}$$
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- R and $\mathbb{S}$ are related by the formula (valid also when $\mathfrak{s}_{11} = -1$ by cont.):

$$\boxed{R = \mathfrak{s}_{00} - \mathfrak{s}_{01}(1 + \mathfrak{s}_{11})^{-1}\mathfrak{s}_{10}.}$$

Asymptotic analysis for $\mathbb{S}^\varepsilon$

- For $\lambda' \in \mathbb{R}$, set (both the geometry and the frequency are changing)

$$R^\varepsilon = R(\varepsilon, \lambda^0 + \varepsilon\lambda') \qquad \text{and} \qquad \mathbb{S}^\varepsilon = \begin{pmatrix} \mathfrak{s}^\varepsilon_{00} & \mathfrak{s}^\varepsilon_{01} \\ \mathfrak{s}^\varepsilon_{10} & \mathfrak{s}^\varepsilon_{11} \end{pmatrix} = \mathbb{S}(\varepsilon, \lambda^0 + \varepsilon\lambda').$$

Proposition: There is $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0; \varepsilon_0]$,

$$|\mathfrak{s}^\varepsilon_{00} - R| \le C\,\varepsilon, \quad |\mathfrak{s}^\varepsilon_{10} - \varepsilon\mathfrak{s}'_{10}| \le C\,\varepsilon^2, \quad |\mathfrak{s}^\varepsilon_{11} - (-1 + \varepsilon\mathfrak{s}'_{11} + \varepsilon^2\mathfrak{s}''_{11})| \le C\,\varepsilon^3,$$

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INGREDIENTS OF THE PROOF:

- Weighted Sobolev spaces with detached asymptotics.
- Uniqueness for the problem with non standard radiation conditions.
- Rectification of the boundary with “almost identical diffeomorphisms”.
- Theory of perturbations for linear operators (see Kato’s book).

The Fano resonance

- We insert the expansions

$$\mathfrak{s}^\varepsilon_{00} = R + O(\varepsilon), \qquad \mathfrak{s}^\varepsilon_{10} = \varepsilon\mathfrak{s}'_{10} + O(\varepsilon^2), \qquad \mathfrak{s}^\varepsilon_{11} = -1 + \varepsilon\mathfrak{s}'_{11} + \varepsilon^2\mathfrak{s}''_{11} + O(\varepsilon^3)$$

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♣ Case $\mathfrak{s}_{11}' \neq 0 \Leftrightarrow \lambda' \neq \lambda_p'$. We obtain

$$R^{\varepsilon} = R - \varepsilon\frac{\mathfrak{s}_{01}'\mathfrak{s}_{10}'}{\mathfrak{s}_{11}'} + O(\varepsilon^2) \qquad \text{and so} \qquad \lim_{\varepsilon\to 0} R(\varepsilon, \lambda^0 + \varepsilon\lambda') = R.$$

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$$R^{\varepsilon} = R + \frac{\mathfrak{s}_{01}'\mathfrak{s}_{10}'}{i\alpha\mu - \mathfrak{s}_{11}''} + O(\varepsilon), \qquad \text{with } \alpha \in \mathbb{R}.$$

We deduce $$\lim_{\varepsilon \to 0} R(\varepsilon, \lambda^0 + \varepsilon\lambda_p' + \varepsilon^2\mu) = R + \frac{\mathfrak{s}_{01}'\mathfrak{s}_{10}'}{i\alpha\mu - \mathfrak{s}_{11}''}.$$

Comments

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$$\lim_{\varepsilon\to 0} R(\varepsilon, \lambda^0 + \varepsilon\lambda') = R, \qquad \text{for } \lambda' \neq \lambda'_p$$

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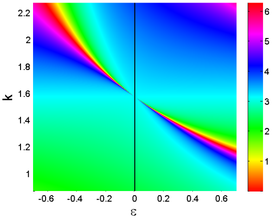
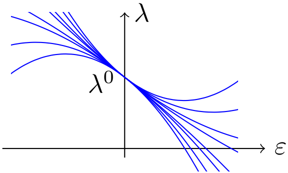
Phase of $R(\cdot,\cdot)$





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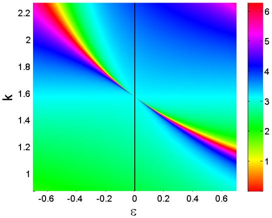

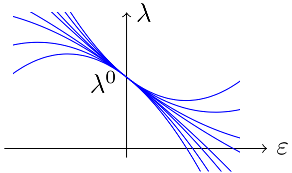

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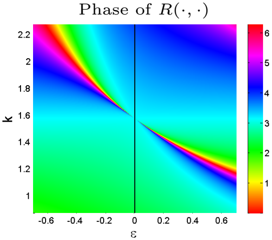

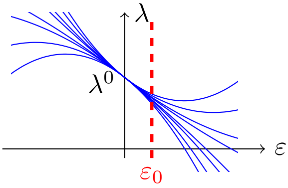

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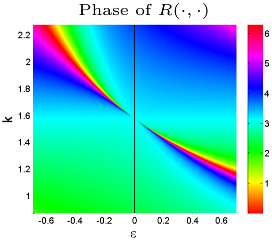
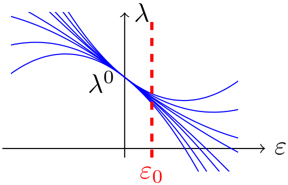
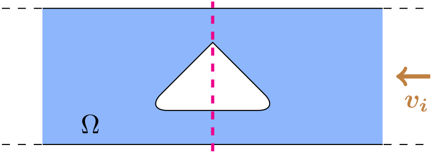




$\to$ For a small given ε_0, the map $\lambda\mapsto R(\varepsilon_0,\lambda)$ exhibits a quick change at $\lambda^0+\varepsilon^0\lambda'_p$. If $\mathfrak{s}''_{11}=0$ and $\mathfrak{s}'''_{11}\neq 0$, the change is even quicker...

Relations for the scattering coefficients

- We come back to the problem in the total waveguide Ω



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$$v = \left| \begin{array}{rl} e^{-ikx} + \mathcal{R}\, e^{+ikx} + \tilde{v}, & x > 0 \qquad \text{(reflection)} \\ \mathcal{T}\, e^{-ikx} + \tilde{v}, & x < 0 \qquad \text{(transmission)} \end{array} \right.$$

with $\mathcal{R}$, $\mathcal{T} \in \mathbb{C}$ and $\tilde{v} \in \mathrm{H}^1(\Omega)$. We have $|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1$.

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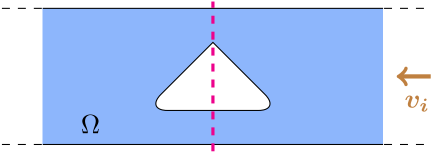

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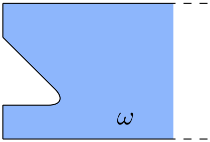
- Introduce the two half-waveguide problems



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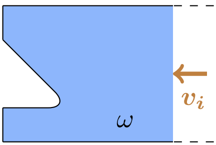
Relations for the scattering coefficients

- Half-waveguide problems admit the solutions

$$\begin{aligned} u &= e^{-ikx} + R_N\, e^{ikx} + \tilde{u}, \qquad && \text{with } \tilde{u} \in \mathrm{H}^1(\omega) \\ U &= e^{-ikx} + R_D\, e^{ikx} + \tilde{U}, \qquad && \text{with } \tilde{U} \in \mathrm{H}^1(\omega). \end{aligned}$$



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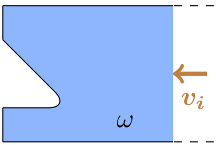

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$$|R_N| = |R_D| = 1.$$

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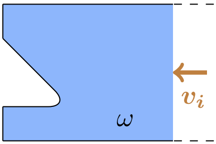

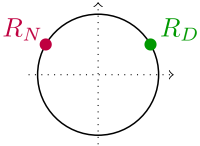

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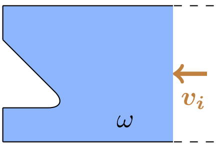

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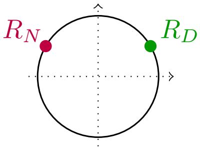

- Using that $v = \dfrac{u+U}{2}$ in ω, $\quad v(x,y) = \dfrac{u(-x,y) - U(-x,y)}{2}$ in $\Omega \setminus \overline{\omega}$,

we deduce that $\boxed{\mathcal{R} = \dfrac{R_N + R_D}{2}}$ and $\boxed{\mathcal{T} = \dfrac{R_N - R_D}{2}.}$

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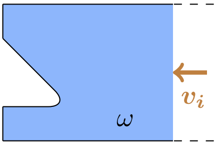

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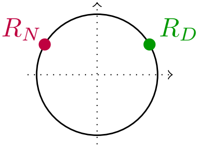

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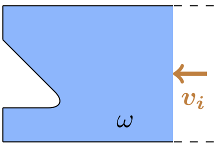
Non reflection $\mathcal{R} = 0$
$\Leftrightarrow \quad R_N = -R_D$

Relations for the scattering coefficients

- Half-waveguide problems admit the solutions



$$u = e^{-ikx} + R_N\, e^{ikx} + \tilde{u}, \qquad \text{with } \tilde{u} \in \mathrm{H}^1(\omega)$$
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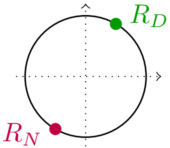

- Due to conservation of energy, one has

$$|R_N| = |R_D| = 1.$$

- Using that $v = \dfrac{u+U}{2}$ in ω, $\quad v(x,y) = \dfrac{u(-x,y) - U(-x,y)}{2}$ in $\Omega \setminus \overline{\omega}$,

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Relations for the scattering coefficients

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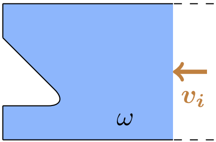
$$\begin{aligned} u &= e^{-ikx} + R_N\, e^{ikx} + \tilde{u}, && \text{with } \tilde{u} \in \mathrm{H}^1(\omega) \\ U &= e^{-ikx} + R_D\, e^{ikx} + \tilde{U}, && \text{with } \tilde{U} \in \mathrm{H}^1(\omega). \end{aligned}$$



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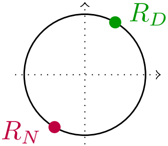

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Non reflection $\mathcal{R} = 0$
$\Leftrightarrow \quad R_N = -R_D$

Perfect reflection $\mathcal{T} = 0$
$\Leftrightarrow \quad R_N = R_D$

Non reflection and perfect reflection

$$\boxed{\mathcal{R} = \frac{R_N + R_D}{2}} \qquad \boxed{\mathcal{T} = \frac{R_N - R_D}{2}}$$

- To set ideas, we assume that u_{tr} is symmetric w.r.t. (Oy).
  $\Rightarrow u_{\mathrm{tr}}$ is a trapped mode for the pb with Neumann B.Cs.

i) No trapped modes for the Dirichlet pb at $\lambda = \lambda^0$. This implies

$$|R_D(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu) - R_D(0, \lambda^0)| \le C\,\varepsilon, \qquad \forall \varepsilon \in (0; \varepsilon_0],\ \mu \in [-c\varepsilon^{-1}; c\varepsilon].$$

ii) $\mu \mapsto R_N(\varepsilon, \lambda^0 + \varepsilon\lambda'_p + \varepsilon^2\mu)$ rushes on the unit circle for $\mu \in [-c\varepsilon^{-1}; c\varepsilon]$.

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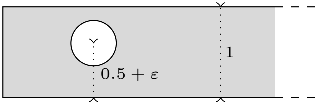
Proposition:

$\exists \lambda_\varepsilon$, with $\lambda_\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $\mathcal{R}(\varepsilon, \lambda_\varepsilon) = 0$ (non reflection).

$\exists \tilde{\lambda}_\varepsilon$, with $\tilde{\lambda}_\varepsilon - \lambda^0 = O(\varepsilon)$, s.t. for ε small, $\mathcal{T}(\varepsilon, \tilde{\lambda}_\varepsilon) = 0$ (perfect reflection).

The Fano resonance

- Numerics using FE methods (`Freefem++`) with DtN maps or PMLs.
- Left: domain ω^ε. Right: u_{tr} (trapped mode) for $\varepsilon = 0$.



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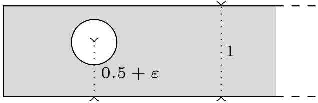


- Since $|R^\varepsilon| = 1$ (conservation of energy), $\exists \theta^\varepsilon \in]-\pi;\pi]$ s.t. $R^\varepsilon = e^{i\theta^\varepsilon}$.

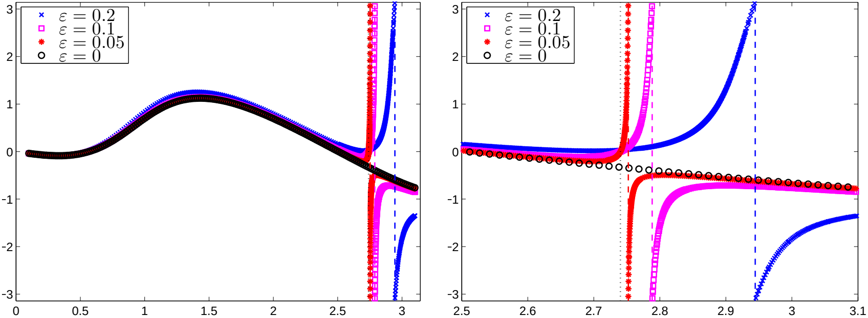
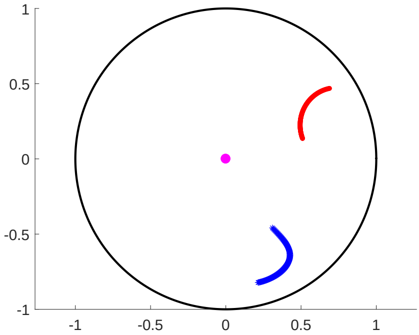


Figure: $k \mapsto \theta^\varepsilon(k)$ *for several* ε *(non uniqueness for* $\varepsilon = 0$, $k = 2.7403$*).*

Non reflection/perfect reflection

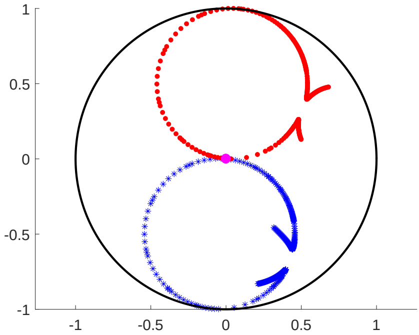
- **Scattering coefficients** for $k \in (2.5; 3.1)$.

No shift $(\boldsymbol{\varepsilon} = \mathbf{0})$



$k \mapsto \mathcal{R}(0, k)$ $k \mapsto \mathcal{T}(0, k)$

Small shift $(\boldsymbol{\varepsilon} > \mathbf{0})$



$k \mapsto \mathcal{R}(0.05, k)$ $k \mapsto \mathcal{T}(0.05, k)$

Non reflection/perfect reflection

- Example of setting where $\mathcal{R}(\varepsilon, \lambda^{\varepsilon}) = 0$ (non reflection).

$\Re e\, v$

$\Re e\, (v - v_i)$

- Example of setting where $\mathcal{T}(\varepsilon, \lambda^{\varepsilon}) = 0$ (perfect reflection).

$\Re e\, v$

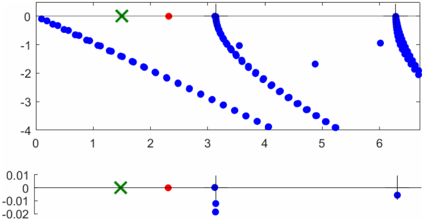
Frequency behaviour

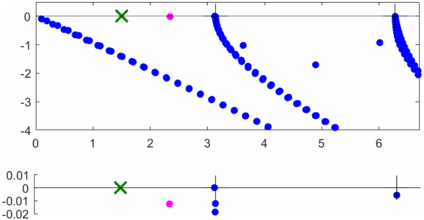
No shift $(\varepsilon = \mathbf{0})$ | Small shift $(\varepsilon > \mathbf{0})$

- $k \mapsto \Re e\, v(k)$

- **Complex spectrum** computed with PMLs (we zoom at the real axis).

• Trapped mode • Complex resonance





Conclusion

What we did

- We proved the Fano resonance phenomenon in a 2D waveguide.

  If trapped modes exist for $(\varepsilon, \lambda) = (0, \lambda^0)$*, then for* $\varepsilon > 0$ *small,* $\lambda \mapsto R(\varepsilon, \lambda)$ *has a quick variation at* λ^0*.* `Symmetry is not needed.`

- We use it to show examples of non reflection and perfect reflection. `Symmetry is essential.`

- The technique works with other B.C. (Dirichlet, ...), other kinds of perturbation (penetrable obstacles, ...), in any dim..

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Future work

1) Without symmetry, how to show that $\mathcal{T}$ still passes through zero?
2) Is there non reflection/perfect reflection for $k > \pi$ (monomode regime was essential in the mechanism)?
3) What happens if λ^0 is not a simple eigenvalue?

Thank you for your attention!