Waves 2015

Far field invisibility for a finite set of incident/scattering directions

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General setting

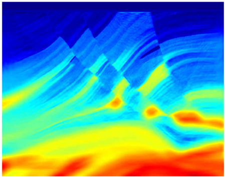
► We are interested in methods based on the propagation of waves to determine the shape, the physical properties of objects, in an exact or qualitative manner, from given measurements.

► General principle of the methods:

i) send waves in the medium;

ii) measure the scattered field;

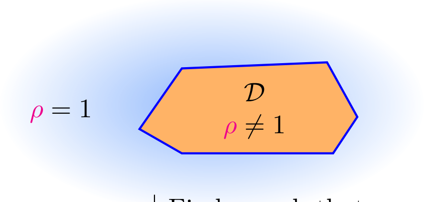
iii) deduce information on the structure.



- Many techniques: Xray, ultrasound imaging, seismic tomography, ...
- Many applications: biomedical imaging, non destructive testing of materials, geophysics, ...

Model problem

- Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\mathbb{R}^2$.

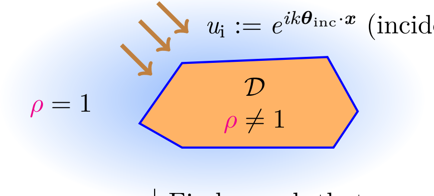


$$\left|\begin{array}{l} \text{Find } u \text{ such that} \\ \begin{array}{rcll} -\Delta u & = & k^2 \rho\, u & \text{in } \mathbb{R}^2, \\ u & = & u_{\mathrm{i}} + u_{\mathrm{s}} & \text{in } \mathbb{R}^2, \end{array} \\ \displaystyle\lim_{r \to +\infty} \sqrt{r} \left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}} \right) = 0. \end{array}\right. \qquad (1)$$

Model problem

▶ Scattering in time-harmonic regime of an incident plane wave by a bounded penetrable inclusion $\mathcal{D}$ (coefficients ρ) in $\mathbb{R}^2$.

$u_{\mathrm{i}} := e^{ik\boldsymbol{\theta}_{\mathrm{inc}}\cdot\boldsymbol{x}}$ (incident dir. $\boldsymbol{\theta}_{\mathrm{inc}} \in \mathbb{S}^1$)



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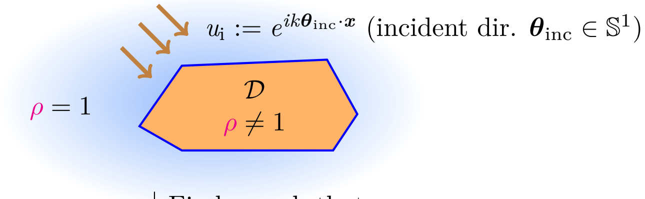

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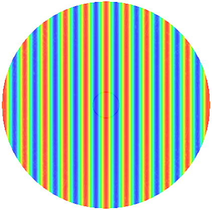
DEFINITION:
- u_i = incident field (data)
- u = total field (uniquely defined by (1))
- u_s = scattered field (uniquely defined by (1)).

Far field pattern

- Numerical approximation of the solution to the previous problem:

$\Re e\, u_{\mathrm{i}}$



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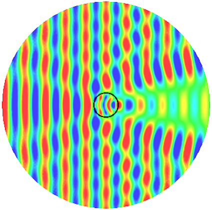

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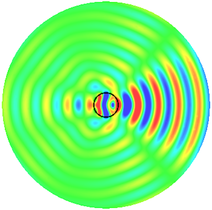

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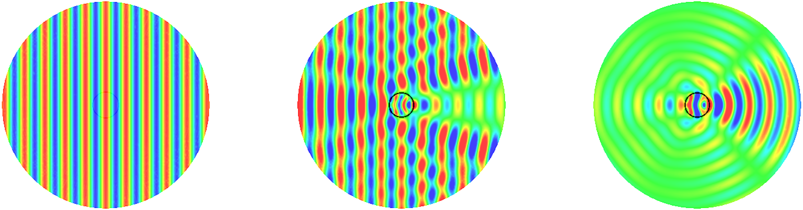

- The scattered field of an incident plane wave of direction $\boldsymbol{\theta}_{\mathrm{inc}}$ behaves in each direction like a cylindrical wave at infinity:

$$u_{\mathrm{s}}(\boldsymbol{x}, \boldsymbol{\theta}_{\mathrm{inc}}) = \frac{e^{ikr}}{\sqrt{r}} \Big(\; u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_{\mathrm{sca}}, \boldsymbol{\theta}_{\mathrm{inc}}) \; + O(1/r) \Big).$$

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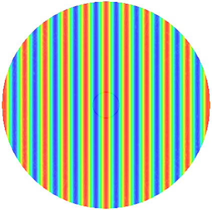

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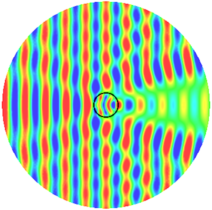

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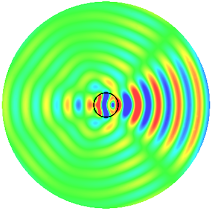

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DEFINITION: The map $\boxed{u_{\mathrm{s}}^{\infty}(\cdot,\cdot)} : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{C}$ is called the far field pattern.

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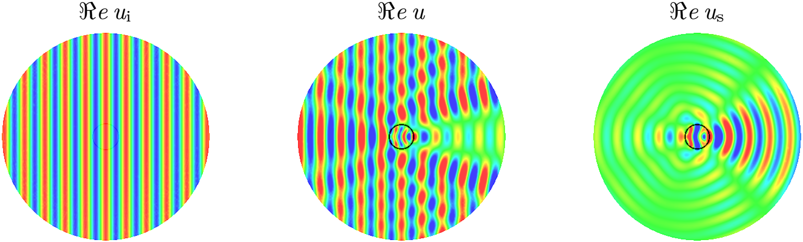

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The far field pattern is the quantity one can measure at infinity (the other terms are too small).

Goal of the talk

- The goal of imaging techniques is to find features of the inclusion from the knowledge of $u_s^\infty(\cdot,\cdot)$ on a subset of $\mathbb{S}^1 \times \mathbb{S}^1$.
  - In literature, most of the techniques require a continuum of data.
  - In practice, one has a finite number of emitters and receivers.

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– In practice, one has a finite number of emitters and receivers.

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► At least two reasons to study invisibility questions:

1) We can wish to hide objects (cloaking like in Andrew Norris's talk).

2) It allows to understand limits of imaging techniques.

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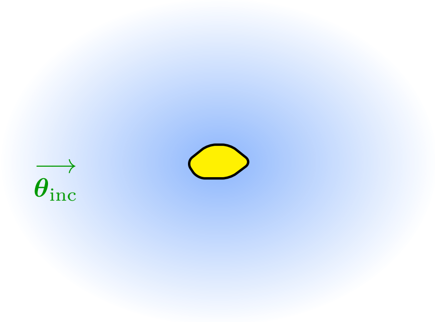
- To simplify the presentation, only one incident direction $\boldsymbol{\theta}_{\text{inc}}$ and N scattering directions $\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N$ (given).



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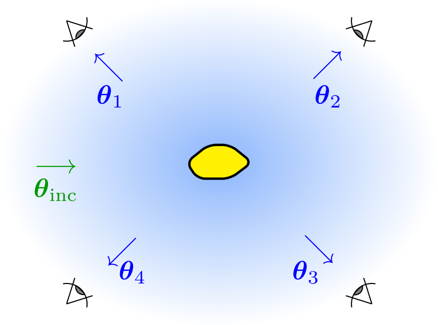

$\rightarrow$ We measure $u_s^\infty(\boldsymbol{\theta}_1), \ldots, u_s^\infty(\boldsymbol{\theta}_N)$.

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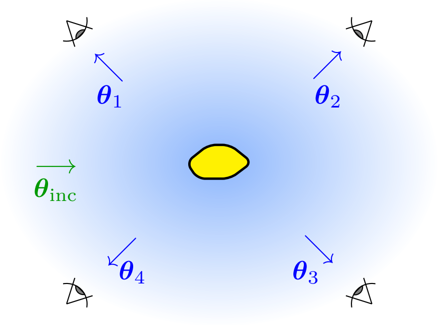

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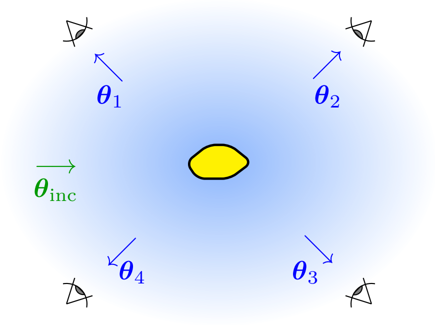

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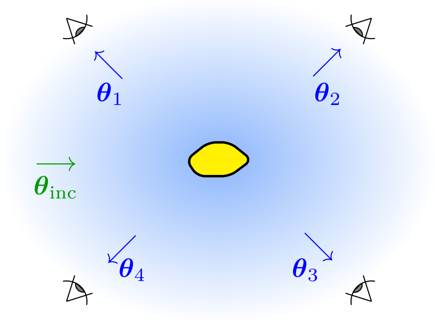

In this talk, we explain how to construct inclusions such that

$$u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1) = \cdots = u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N) = 0.$$

- These inclusions cannot be detected from far field measurements.
- We assume that k and the support of the inclusion $\overline{\mathcal{D}}$ are given.

Goal of the talk

Find a real valued function $\rho \not\equiv 1$, with $\rho - 1$ supported in $\overline{\mathcal{D}}$, such that the solution to the problem

$$
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\qquad \displaystyle\lim_{r\to+\infty} \sqrt{r}\left(\frac{\partial u_{\mathrm{s}}}{\partial r} - iku_{\mathrm{s}}\right) = 0
\end{array}
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$$

verifies $u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1) = \cdots = u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N) = 0$.

Outline of the talk

1. General scheme
2. The forbidden case
3. Numerical experiments

1 General scheme

2 The forbidden case

3 Numerical experiments

Origin of the method

- We will work as in the proof of the implicit functions theorem.

• This idea was used in `Nazarov 11` to construct waveguides for which there are embedded eigenvalues in the continuous spectrum.

• It has been adapted in `Bonnet-Ben Dhia & Nazarov 13` to build invisible perturbations of waveguides (see also `Bonnet-Ben Dhia, Nazarov & Taskinen 14` for an application to a water-wave problem).

• In `Chesnel, Hyvönen & Staboulis 15` it has been used to construct invisible conductivity perturbation for the point electrode model in inverse impedance tomography.

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• In Chesnel, Hyvönen & Staboulis 15 it has been used to construct invisible conductivity perturbation for the point electrode model in inverse impedance tomography.

- Connections with the ongoing work of Arens & Sylvester?

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$
(
N complex measurements $\Rightarrow$ $2N$ real measurements)

Sketch of the method

- Define $\sigma = \rho - 1$ and gather the measurements in the vector
$$F(\sigma) = (F_1(\sigma), \ldots, F_{2N}(\sigma))^\top \in \mathbb{R}^{2N}.$$
- No obstacle leads to null measurements $\Rightarrow$ $F(0) = 0$.

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- We look for small perturbations of the reference medium: $\sigma = \varepsilon\mu$ where $\varepsilon > 0$ is a small parameter and where μ has be to determined.

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If G^ε is a contraction, the fixed-point equation has a unique solution $\vec{\tau}^{\,\mathrm{sol}}$.
Set $\sigma^{\mathrm{sol}} := \varepsilon\mu^{\mathrm{sol}}$. We have $F(\sigma^{\mathrm{sol}}) = 0$ (invisible inclusion).

Calculus of $dF(0)$ 1/2

- For our problem, we have ($\sigma = \rho - 1$)

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- We obtain the expansion (Born approx.), for small ε

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$$\begin{aligned} dF(0)(\mu) = \Big(&\int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \cos(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \\ &\int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_1)\cdot\boldsymbol{x})\, d\boldsymbol{x}, \dots, \int_{\mathcal{D}} \mu \sin(k(\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_N)\cdot\boldsymbol{x})\, d\boldsymbol{x}\Big) \end{aligned}$$

Calculus of $dF(0)$ 2/2

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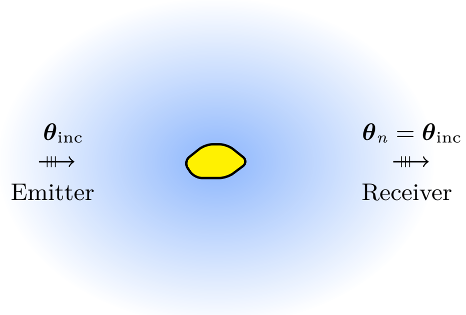
Is $dF(0) : \mathrm{L}^\infty(\mathcal{D}) \to \mathbb{R}^{2N}$ onto ?

- Clearly, we need to avoid the configuration $\boldsymbol{\theta}_{\text{inc}} - \boldsymbol{\theta}_n = 0$.



Construction of the shape functions

1. $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\text{inc}}$ for $n = 1, \ldots, N$

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2. We need to construct some $\mu_0 \in \ker dF(0)$, *i.e.* some μ_0 satisfying

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❷ We take

$$\boxed{\mu_0 = \mu_0^{\#} - \sum_{m=1}^{N} \left(\int_{\mathcal{D}} \mu_{1,m}\, \mu_0^{\#}\, d\boldsymbol{x} \right) \mu_{1,m} - \sum_{m=1}^{N} \left(\int_{\mathcal{D}} \mu_{2,m}\, \mu_0^{\#}\, d\boldsymbol{x} \right) \mu_{2,m}}$$

where $\mu_0^{\#} \notin \operatorname{span}\{\mu_{1,1}, \ldots, \mu_{1,N}, \mu_{2,1}, \ldots, \mu_{2,N}\}$.

Main result

Proposition: Assume that $\boldsymbol{\theta}_n \neq \boldsymbol{\theta}_{\mathrm{inc}}$ for $n = 1, \dots, N$. For $\boldsymbol{\varepsilon}$ **small enough**, define $\rho^{\mathrm{sol}} = 1 + \varepsilon\mu^{\mathrm{sol}}$ with

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verifies $u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_1) = \cdots = u_{\mathrm{s}}^{\infty}(\boldsymbol{\theta}_N) = 0$.

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→ Existence of invisible inclusions may appear not so surprising since there are $2N$ measurements and $\rho \in \mathrm{L}^\infty(\mathcal{D})$. *Let us see the case* $\boldsymbol{\theta}_n = \boldsymbol{\theta}_{\text{inc}}$...

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The case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$

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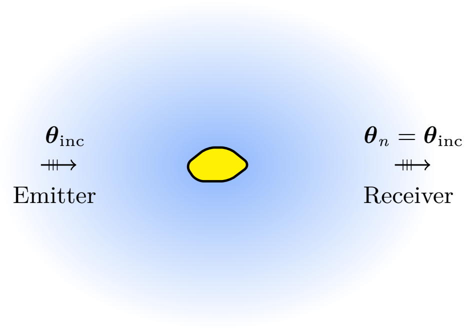

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- No solution if $\mathcal{D}$ has corners and under certain assumptions on ρ.

- `Corners always scatter, E. Blåsten, L. Päivärinta, J. Sylvester, 2014`
- `Corners and edges always scatter, J. Elschner, G. Hu, 2015`

- And if $\mathcal{D}$ is smooth? $\Rightarrow$ The problem seems open.

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Data and algorithm

► We can solve the fixed point problem using an iterative procedure: we set $\vec{\tau}^{\,0} = (0, \ldots, 0)^\top$ then define

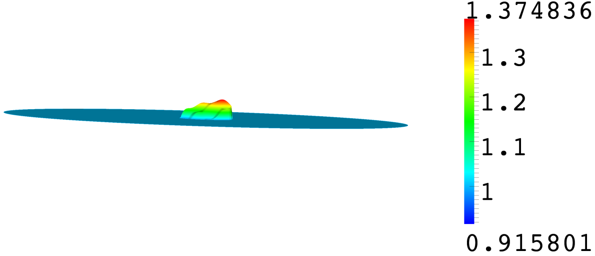
$$\boxed{\vec{\tau}^{\,n+1} = G^{\varepsilon}(\vec{\tau}^{\,n}).}$$

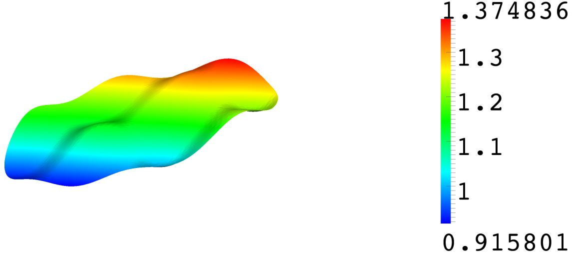
► At each step, we solve a scattering problem. We use a P2 finite element method set on the ball B_8. On $\partial\mathrm{B}_8$, a truncated Dirichlet-to-Neumann map with 13 harmonics serves as a transparent boundary condition.

► For the numerical experiments, we take $\mathcal{D} = \mathrm{B}_1$, $M = 3$ (3 directions of observation) and

$$\left|\begin{array}{ll} \boldsymbol{\theta}_{\text{inc}} = (\cos(\psi_{\text{inc}}), \sin(\psi_{\text{inc}})), & \psi_{\text{inc}} = 0^\circ \\ \boldsymbol{\theta}_1 = (\cos(\psi_1), \sin(\psi_1)), & \psi_1 = 90^\circ \\ \boldsymbol{\theta}_2 = (\cos(\psi_2), \sin(\psi_2)), & \psi_2 = 180^\circ \\ \boldsymbol{\theta}_3 = (\cos(\psi_3), \sin(\psi_3)), & \psi_3 = 225^\circ \end{array}\right.$$

Results: coefficient ρ at the end of the process





Results: scattered field

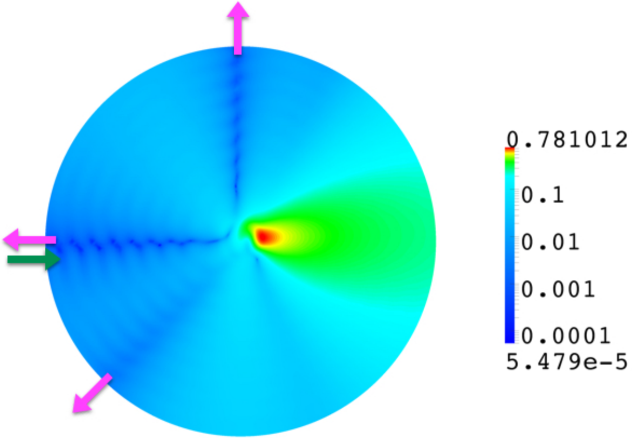


Figure: $|u_s|$ at the end of the fixed point procedure in logarithmic scale. As desired, we see it is very small far from $\mathcal{D}$ in the directions corresponding to the angles 90°, 180° and 225°. The domain is equal to B_8.

Results: far field pattern

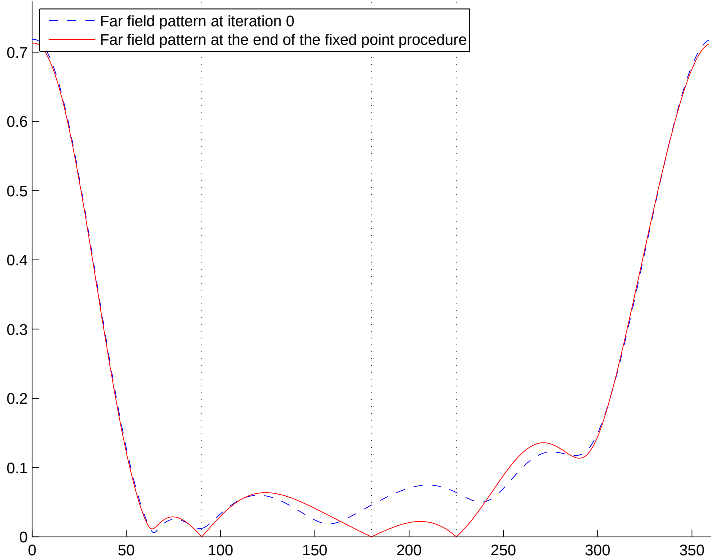


Figure: The dotted lines show the directions where we want u_s^∞ to vanish.

1. General scheme
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Conclusion

What we did

- ♠ We explained how to construct invisible inclusions in a setting with a finite number of incident/scattering directions.
- ♠ We need to avoid the case $\boldsymbol{\theta}_{\text{inc}} = \boldsymbol{\theta}_n$.

$\rightarrow$ The approach also works when there are several incident directions.

Future work

1) Can we reiterate the process to construct larger invisible perturbations of the reference medium?
2) Can we construct invisible inclusions for other models (Maxwell, elasticity,...)?
3) Can we hide flies (small Dirichlet obstacles)? *Work in progress...*
4) Similar questions in waveguides (finite number of propagative waves $\Leftrightarrow$ finite number of directions). How to achieve transmission invisibility?

Thank you for your attention!!!