INTERIOR TRANSMISSION EIGENVALUE PROBLEM AND T-COERCIVITY

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ABSTRACT

We study the so-called interior transmission problem using the T-coercivity approach. In particular, we prove that this problem, which appears when one is interested in the reconstruction of the support of an inclusion embedded in a homogeneous medium, is of Fredholm type and that so-called transmission eigenvalues form at most a discrete set. The simple technique we propose allows to treat cases, which were not covered by existing methods, where the difference between the inclusion index and the background index changes sign.

1. INTRODUCTION

The term "interior transmission eigenvalue problem" refers to a family of spectral problems which appear in scattering theory for inhomogeneous medium. In particular, they arise when one is interested in the reconstruction of an inclusion embedded in a homogeneous medium from multi-static measurements of diffracted fields at a given frequency. In this case, to implement any method, it is natural to require that, for the given frequency, there are no waves which do not scatter. Mathematically, this last property boils down to say that the frequency is not a transmission eigenvalue, that is, an eigenvalue of the interior transmission problem (see (2) where u denotes the total field and w, the incident field). This explains why it is crucial to prove that transmission eigenvalues form at most a discrete set with infinity as the only accumulation point.

In this talk, we focus on the scalar case of a heterogeneous and possibly anisotropic medium for which the contrast in the scattering medium occurs in two independent functions A and n (see (2)) which are respectively equal to Id and 1 for the reference medium. From a technical point of view, the sesquilinear form associated with the natural variational formulation of this interior transmission problem, defined in (3), exhibits a sign-changing in its principal part. Consequently, the associated operator is not strongly elliptic and its study is not standard. One observes an equivalent difficulty in the study of the transmission problem between a dielectric and a negative metamaterial in harmonic regime. To tackle it, we can use the T-coercivity technique [1]. The idea consists in testing, in variational formulations, not directly against the field, but against a simple geometrical transformation of the field, in order to restore some properties of positivity for the associated operators. In [2], thanks to this simple approach, we have been able to extend the results of [3]: only the values of $A - Id$ in a neighbourhood of the boundary actually matter for determining whether or not the problem is of Fredholm type.

2. SETTING OF THE PROBLEM

Consider D a bounded domain of $\mathbb{R}^3$, with Lipschitz boundary ∂D and denote ν the outward unit normal. Let $A \in \mathrm{L}^\infty(D, \mathbb{R}^{3\times 3})$ be a matrix valued function such that $A(x)$ is symmetric for almost all $x \in D$. The function $n \in \mathrm{L}^\infty(D, \mathbb{R})$ will be scalar real valued. We suppose that

$$\begin{array}{l} A_- := \inf\limits_{x\in D} \inf\limits_{\xi\in\mathbb{R}^3, |\xi|=1} (\xi \cdot A(x)\xi) > 0\ ; \\ A_+ := \sup\limits_{x\in D} \sup\limits_{\xi\in\mathbb{R}^3, |\xi|=1} (\xi \cdot A(x)\xi) < \infty\ ; \\ n_- := \inf\limits_{x\in D} n(x) > 0 \text{ and } n_+ := \sup\limits_{x\in D} n(x) < \infty. \end{array} \tag{1}$$

The transmission eigenvalue problem reads :

$$\left|\begin{array}{lcll} \multicolumn{4}{l}{\text{Find } (u,w) \in \mathrm{H}^1(D)\times\mathrm{H}^1(D) \text{ such that :}} \\ \mathrm{div}\,(A\nabla u) + k^2 n u & = & 0 & \text{in } D \\ \Delta w + k^2 w & = & 0 & \text{in } D \\ u - w & = & 0 & \text{on } \partial D \\ \nu\cdot A\nabla u - \nu\cdot\nabla w & = & 0 & \text{on } \partial D. \end{array}\right. \tag{2}$$

Values of $k \in \mathbb{C}$ for which problem (2) has a nontrivial solution (u, w) are called transmission eigenvalues. If $\mathcal{O}$ is an open subset of $\mathbb{R}^3$, we denote $(\cdot,\cdot)_{\mathcal{O}}$ the Hermitian scalar products of $\mathrm{L}^2(\mathcal{O})$ and $(\mathrm{L}^2(\mathcal{O}))^3$. The pair (u, w) satisfies problem (2) if and only if (u, w) satisfies the problem

$$\left| \begin{array}{l} \text{Find } (u,w) \in \mathrm{X} \text{ such that, for all } (u',w') \in \mathrm{X}, \\ a_k((u,w),(u',w')) = 0, \end{array} \right. \quad (3)$$

with $a_k((u,w),(u',w')) = (A\nabla u, \nabla u')_D - (\nabla w, \nabla w')_D - k^2\left((nu,u')_D - (w,w')_D\right)$ and $\mathrm{X} = \{(u,w) \in \mathrm{H}^1(D) \times \mathrm{H}^1(D) \,|\, u - w \in \mathrm{H}^1_0(D)\}$. Define the operator $\mathscr{A}_k$ from X to X such that, for all $((u,w),(u',w')) \in \mathrm{X} \times \mathrm{X}$, $(\mathscr{A}_k(u,w),(u',w'))_{\mathrm{H}^1(D)\times\mathrm{H}^1(D)} = a_k((u,w),(u',w'))$. This eigenvalue problem differs from classical ones because a_k is not coercive on X neither “coercive+compact”.

3. THE T-COERCIVITY METHOD

For the sake of clarity, we present the technique in the simple case: $A_+ < 1$ and $n_+ < 1$. The idea is to consider an equivalent formulation of (3) where a_k is replaced by $a_k^{\mathtt{T}}$ defined by

$$a_k^{\mathtt{T}}((u,w),(u',w')) := a_k((u,w),\mathtt{T}(u',w')), \quad (4)$$

$\mathtt{T}$ being an *ad hoc* isomorphism of X. Indeed, $(u,w) \in \mathrm{X}$ satisfies $a_k((u,w),(u',w')) = 0$ for all $(u',w') \in \mathrm{X}$ if, and only if, it satisfies $a_k^{\mathtt{T}}((u,w),(u',w')) = 0$ for all $(u',w') \in \mathrm{X}$. In the present case, let us take $\mathtt{T}(u,w) := (u - 2w, -w)$ ($\mathtt{T}$ is an isomorphism since $\mathtt{T}^2 = Id$). Using Young's inequality, one has for $k = i\kappa$ with $\kappa \in \mathbb{R}^*$, $\forall \alpha, \beta > 0$, $\forall (u,w) \in \mathrm{X}$,

$$\begin{array}{rl} & |a_k^{\mathtt{T}}((u,w),(u,w))| \\ = & |(A\nabla u,\nabla u)_D + (\nabla w,\nabla w)_D - 2(A\nabla u,\nabla w)_D \\ & +\kappa^2\left((nu,u)_D + (w,w)_D - 2(nu,w)_D\right)\big| \\ \geq & (A\nabla u,\nabla u)_D + (\nabla w,\nabla w)_D \\ & +\kappa^2\left((nu,u)_D + (w,w)_D\right) \\ & -2\,|(A\nabla u,\nabla w)_D| - 2\kappa^2\,|(nu,w)_D| \\ \geq & ((1-\alpha)A\nabla u,\nabla u)_D + ((1-\alpha^{-1}A_+)\nabla w,\nabla w)_D \\ & +\kappa^2\left(((1-\beta)nu,u)_D + ((1-\beta^{-1}n_+)w,w)_D\right). \end{array}$$

Taking α and β such that $A_+ < \alpha < 1$ and $n_+ < \beta < 1$, this estimate proves that $a_k^{\mathtt{T}}$ is coercive over X. Using Lax-Milgram theorem and since $\mathtt{T}$ is an isomorphism of X, one deduces that $\mathscr{A}_k$ is an isomorphism of X for $k = i\kappa$ with $\kappa \in \mathbb{R}^*$. Besides, for a general $k \in \mathbb{C}$, the operator $\mathscr{A}_k$ is a compact perturbation of an isomorphism of X since the embedding of X in $\mathrm{L}^2(D) \times \mathrm{L}^2(D)$ is compact.
Using a localization argument and applying the analytic Fredholm theorem, finally, we can prove the

Theorem 1 • *If $A(x) \leq A^\star Id < Id$ or if $Id < A_\star Id \leq A(x)$ a.e. in a neighbourhood of ∂D, then for all $k \in \mathbb{C}$, the operator $\mathscr{A}_k$ is Fredholm from* X *to* X.
• *If $A(x) \leq A^\star Id < Id$ and $n(x) \leq n^\star < 1$ or if $Id < A_\star Id \leq A(x)$ and $1 < n_\star \leq n(x)$ a.e. in a neighbourhood of ∂D, then the set of transmission eigenvalues is discrete. Moreover, there exist two positive constants ρ and δ such that if $k \in \mathbb{C}$ satisfies $|k| > \rho$ and $|\Re e\, k| < \delta\,|\Im m\, k|$, then k is not a transmission eigenvalue.*

This result is optimal in the sense that, when the sign of $A - Id$ changes (or worse, when $A - Id$ reduces to zero) in a neighbourhood of the boundary, there are geometries and values of A for which the interior transmission problem is not Fredholm in H^1 because of the apparition of “strong” singularities (see [1] for the transmission problem between a dielectric and a negative metamaterial).

4. DISCUSSIONS

The $\mathtt{T}$-coercivity approach can be used to deal with the interior transmission problem for Maxwell's equations ([4]). However, up to now, the question of existence of real transmission eigenvalues when $A - Id$ or $n - 1$ change sign is still an open question. Indeed, the equivalent formulation of problem (3) we consider, which presents a useful property of positivity, is no longer symmetric: this prevents using the nice *min-max* arguments of [5].

5. REFERENCES

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