Discretization of the Ergodic Functional Central Limit Theorem

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Abstract

In this paper, we study how discretization affects the Functional Central Limit Theorem (FCLT) established by Bhattachatya for stationnary and ergodic Markov processes $(X_t)_{t\geqslant 0}$ with unique invariant measure μ and infinitesimal generator A [2] that states: $n^{1/2}\frac{1}{n}\int_0^n Af(X_t)dt$ converges in Law towards the centered Wiener distribution with variance $-2\langle f, Af\rangle_\nu$. In particular we show that standard method such as Riemman, Trapezoïd or Simpsons (respectively of order 1, 2 and 3) can be applied to approximate the integral and still obtain a FCLT. In this case the convergence does not happen with rate $n^{1/2}$ but $n^{q/(2q+1)}$ where q is the order of the method applied to discretize the integral. Moreover, our results remain valid when X is replaced by a q-weak order approximation (not necessarily stationnary) as soon as X admits a unique invariant distribution. We propose applications concerning first order FCLT for the approximation of Markov Brownian diffusion stationary regimes with Euler scheme (where we recover existing results from literature) and second order FCLT for the approximation of Brownian diffusion stationary regimes using Talay [24] scheme of weak order two.
Keywords : Ergodic theory, Markov processes, Invariant measures, Central Limit Theorem, Stochastic approximation.
AMS MSC 2010: 60G10, 47A35, 60F05, 60J25, 60J35, 65C20.

1 Introduction

In this paper, we study the rate of convergence fo the FCLT satisfied by a random weighted empirical measure built using a recursive algorithm introduced in [18] and inspired by [10] for the approximation of the invariant distribution (denoted ν) of a Feller processes $(X_t)_{t\geqslant 0}$. In particular, we establish discretizated versions of the FCLT presented in [2] where the time integral is replaced by (weighted)-empirical measures of the Markov process or of one of its weak-order approximation. The weights applied to compute empirical measures and the weak-order of the approximation of the Markov process are both crucial to derive our FCLT, also

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monitoring its rate.

Invariant distributions are crucial in the study of the long term behavior of stochastic differential systems (see [9] and [5] for an overview of the subject.) and their computatio has already been widely explored in the literature. In [23], explicit exact expressions of the invariant density distribution for some solutions of Stochastic Differential Equations are given.

However, in many cases there is no explicit formula for ν. A first approach consists in studying the convergence, as t tends to infinity, of the semigroup $(P_t)_{t\geqslant 0}$ of the Markov process $(X_t)_{t\geqslant 0}$ with infinitesimal generator A towards the invariant measure ν. This is done *e.g.* in [7] for the total variation topology which is thus adapted when the simulation of P_T is possible for T large enough.

As soon as $(X_t)_{t\geqslant 0}$ can be simulated, we can device Monte Carlo method to estimate $(P_t)_{t\geqslant 0}$, *i.e.* $\mathbb{E}[f(X_t)]$, $t \geqslant 0$, injecting a second term in the error analysis. When $(X_t)_{t\geqslant 0}$ cannot be simulated at a reasonable cost, a solution is to simulate an approximation of $(X_t)_{t\geqslant 0}$ *i.e.* numerical scheme $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ built with transition functions $(\mathcal{Q}_{\gamma_n})_{n\in\mathbb{N}^*}$ (given a step sequence $(\gamma_n)_{n\in\mathbb{N}}$, $\Gamma_0 = 0$ and $\Gamma_n = \gamma_1 + .. + \gamma_n$). If the process $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ weakly converges towards $(X_t)_{t\geqslant 0}$, a construction relies on numerical homogeneous schemes ($(\gamma_n)_{n\in\mathbb{N}}$ is constant, $\gamma_n = \gamma_1 > 0$, for every $n \in \mathbb{N}^*$). This approach induces two more terms to control in the approximation of ν in addition to the error between P_T and ν for a large enough fixed $T > 0$, such that there exists $n(T) \in \mathbb{N}^*$,with $T = n(T)\gamma_1$: The first one is due to the weak approximation of $\mathbb{E}[f(X_T)]$ by $\mathbb{E}[f(\overline{X}^{\gamma_1}_T)]$ and the second one is due to the Monte Carlo error resulting from the computation of $\mathbb{E}[f(\overline{X}^{\gamma_1}_T]$.

Such an approach does not benefit from the ergodic feature of $(X_t)_{t\geqslant 0}$. In fact, as investigated in [24] for Brownian diffusions, the ergodic (or positive recurrence) property of $(X_t)_{t\geqslant 0}$ is also satisfied by its approximation $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ at least for small enough time steps $\gamma_n = \gamma_1, n \in \mathbb{N}^*$. Then $(\overline{X}^{\gamma_1}_{\Gamma_n})_{n\in\mathbb{N}}$ has an invariant distribution ν^{γ_1} (supposed to be unique for simplicity) and the sequence of empirical measures

$$\nu^{\gamma_1}_n(dx) = \frac{1}{\Gamma_n}\sum_{k=1}^{n}\gamma_1\delta_{\overline{X}^{\gamma_1}_{\Gamma_{k-1}}}(dx), \qquad \Gamma_n = n\gamma_1.$$

almost surely weakly converges to ν^{γ_1}. With this last result makes it is possible to compute by simulation, arbitrarily accurate approximations of $\nu^{\gamma_1}(f)$ using only one simulated path of $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$. It is an ergodic - or Langevin - simulation of $\nu^{\gamma_1}(f)$. At this point, it remains to establish at least that $\nu^{\gamma_1}(f)$ converges to $\nu(f)$ when γ_1 converges to zero and, if possible, at which rate. In [24] this rate was shown to depend closely on the weak order of the scheme. Notice that the rate of convergence of $(\nu^{\gamma_1}_n)_{n\in\mathbb{N}^*}$ to ν^{γ_1} is not established in this paper. Another approach was proposed in [1], still for Brownian diffusions, which avoids the asymptotic analysis between ν^{γ_1} and ν. The authors directly prove that the discrete time Markov process $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$, with step sequence $\gamma = (\gamma_n)_{n\in\mathbb{N}}$ vanishing to 0, weakly converges toward ν. Therefore, the resulting error is made of two terms. The first one is due to this weak convergence and the second one to the Monte Carlo error involved in the computation of the law of $\overline{X}^{\gamma}_{\Gamma_n}$, for n large enough. The reader may notice that in mentioned approaches, strong ergodicity assumptions are required for the process with infinitesimal generator A.

In [10], these two ideas are combined to design a Langevin Euler Monte Carlo recursive algorithm with decreasing steps which *a.s.* weakly converges to the right target ν. This paper treats the case where $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ is a (inhomogeneous) Euler scheme with decreasing steps associated to a strongly mean reverting Brownian diffusion process. The sequence $(\nu^{\gamma}_n)_{n\in\mathbb{N}^*}$ is defined as the weighted empirical measures of the path of $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ (which is the procedure that is used in every work we mention from now on and which is also the one we use in this paper). In particular, the *a.s.* weak convergence of

$$\nu^{\gamma}_n(dx) = \frac{1}{\Gamma_n}\sum_{k=1}^{n}\gamma_k\delta_{\overline{X}^{\gamma}_{\Gamma_{k-1}}}(dx), \qquad \Gamma_n = \sum_{k=1}^{n}\gamma_k, \tag{1}$$

toward the (non-empty) set $\mathcal{V}$ of the invariant distributions of the underlying Brownian diffusion is established. Notice also that, this approach does not require that the invariant measure ν is unique by contrast with the results obtained in [24] and [1] or in [4] where the authors study of the total variation convergence for the Euler scheme with decreasing steps of the over-damped Langevin diffusion under strong ergodicity assumptions. Moreover, when the invariant measure ν is unique, it is proved that $\lim_{n\to+\infty}\nu^{\gamma}_n f = \nu f$ *a.s.* for a larger class of test functions than $\mathcal{C}^0$ which contains $\nu - a.s.$ continuous functions with polynomial growth *i.e.* convergence for the Wasserstein distance.

In the spirit of [2], which states that $n^{1/2}\frac{1}{n}\int_0^n Af(X_t)dt$ converges in distribution towards the centered Wiener distribution with variance $-2\langle f, Af\rangle_{\nu}$ (where $(X_t)_{t\geqslant 0}$ is supposed to be stationnary and ergodic), a FCLT with given rate (referred to as first one) is also established for the empirical measures of the Euler scheme. More specificaly, it is shown that the convergence in distribution of $(\nu^{\gamma}_n(f_A))_{n\in\mathbb{N}^*}$ to zero for test functions f which can be written $f_A = Af$ happens with rate $n^{1/3}$ for a well choosen step sequence. This whole study is made under strongly mean reverting setting and the extension to the weakly mean reverting setting has been realized first in [20].

Concerning the study of the almost sure convergence, this first paper gave rise to many generalizations and extensions. In [11], the initial result is extended to the case of Euler scheme of Brownian diffusions with weakly mean reverting properties. Thereafter, in [12], the class of test functions for which we have $\lim_{n\to+\infty}\nu^{\gamma}_n f = \nu f$ *a.s.* (when the invariant distribution is unique) is extended to include functions with exponential growth. Finally, in [21], the results concerning the polynomial case are shown to hold for the computation of invariant measures for weakly mean reverting Levy driven diffusion processes, still using the algorithm from [10]. For a more complete overview of the studies concerning (1) for the Euler scheme, the reader can also refer to [15], [13], [20], [16], [17] or [14].

Those results are extended in [18] and generalized to the case where $(\mathcal{Q}_{\gamma})_{\gamma>0}$ is not specified explicitly, to approximate invariant, not necessarily unique, distributions for general Feller processes. In [18], an abstract framework, that can be used to prove every mentioned existing result, is developed which suggests various applications beyond the Euler scheme of Levy processes. See for instance [19]

In this paper, we extend the abstract framework introduced in [18] in order to study rate of convergence of empirical measures $(\nu_n)_{n\in\mathbb{N}^*}$ to ν, supposed to be unique, in the FCLT for test functions with form $f_A = Af$ and under weakly mean-reverting framework. In particular we establish an abstract first order FCLT (see Theorem 3.2) which enables to obtain a discretized version of [2] and recover every existing results concerning rates of convergence ([10], [12], [20] or [14]). Convergence and rate of convergence results for the Euler scheme are given as example in the end of Section 3. Moreover, we improve this first result and establish an abstract second order FCLT (see Theorem 3.3). This last result relies both on the second weak order of the stochastic approximation $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ and on a generalization of (1), considering

$$\nu^{\eta}_n(dx) = \frac{1}{H_n}\sum_{k=1}^{n}\eta_k\delta_{\overline{X}^{\gamma}_{\Gamma_{k-1}}}(dx), \qquad H_n = \sum_{k=1}^{n}\eta_k, \tag{2}$$

with $(\eta_n)_{n\in\mathbb{N}^*}$ a well chosen weight sequence, namely $\eta_{2,n+1} = C_{\gamma,\eta_2}(\gamma_n+\gamma_{n+1})/2, C_{\gamma,\eta_2} > 0$ (with $\gamma_0 = 0$) for every $n \in \mathbb{N}$. The third order FCLT is then established in Theorem 3.4 and requires the step-weight assumption $\eta_{3,2n+1} := C_{\gamma,\eta_3}(\gamma_{2n+1}+\gamma_{2n-1})/3$, $\eta_{3,2n+2} := 4C_{\gamma,\eta_3}\gamma_{2n+1}/3$, $\gamma_{2n+1} = \gamma_{2n+2}, C_{\gamma,\eta_3} > 0$, $n \in \mathbb{N}$. Notice that those choices of weights appears as extension of the standards the Riemann, Trapezoïdal or Simpson's homogeneous approximations of integrals. Up to our knowledge no second or higher order FCLT had been derived in any situation so far in the literature.

Then, we apply those results to the second weak order scheme of Talay for Brownian diffusion processes introduced in [24]. In particular in Theorem 4.1, we establish the convergence of the empirical measures for some L^p-Wasserstein distances, $p > 0$. We also establish a first order FCLT for $(\nu^{\gamma}_n)_{n\in\mathbb{N}^*}$. In this case the convergence has the same rate as for the Euler scheme. Finally we establish the second order FCLT for $(\nu^{\eta_2}_n)_{n\in\mathbb{N}^*}$. This last result can not be obtained for the Euler scheme as it is simply a first weak order scheme. From a practical viewpoint, one may consider a weak q-order scheme $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ for $q = 1, 2, 3$. Then, considering a step sequence of the form $\gamma_n = 1/n^{1/(2q+1)}$ (only when n even if $q = 3$), $n \in \mathbb{N}^*$, we can achieve the rate $n^{q/(2q+1)}$ for the the FCLT of order q satisfied by $(\nu^{\eta_q}_n)_{n\in\mathbb{N}^*}$. One may refer to Remark 4.1. We think that combining our techniques we could build q-order method for $q > 3$ with FCLT at rate $n^{q/(2q+1)}$. This is left to further research as the identification of the limit law in the FCLT growth in difficulty as q goes high.

2 Convergence to invariant distributions - A general approach

In this section, we present the abstract framework from [18] to show the convergence of weighted empirical measures defined in a similar way as in (2) and built from an approximation $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ of a Feller process $(X_t)_{t\geqslant 0}$ (which are not specified explicitly). Given that the step sequence $(\gamma_n)_{n\in\mathbb{N}^*} \underset{n\to+\infty}{\to} 0$, it *a.s.* weakly converges to the set $\mathcal{V}$, of the invariant distributions of $(X_t)_{t\geqslant 0}$. This framework is based on as weak as possible mean reverting assumptions on the pseudo-generator of $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ on the one hand and appropriate rate conditions on the step sequence $(\gamma_n)_{n\in\mathbb{N}^*}$ on the other hand.

2.1 Presentation of the abstract framework

2.1.1 Notations

Let $(E, |.|)$ be a locally compact separable metric space, we denote $\mathcal{C}(E)$ the set of continuous functions on E and $\mathcal{C}_0(E)$ the set of continuous functions that vanish a infinity. We equip this space with the sup norm $\|f\|_\infty = \sup_{x\in E}|f(x)|$ so that $(\mathcal{C}_0(E), \|.\|_\infty)$ is a Banach space. We will denote $\mathcal{B}(E)$ the σ-algebra of Borel subsets of E and $\mathcal{P}(E)$ the family of Borel probability

measures on E. We will denote by $\mathcal{K}_E$ the set of compact subsets of E.
Finally, for every Borel function $f : E \to \mathbb{R}$, and every $l_\infty \in \mathbb{R} \cup \{-\infty, +\infty\}$, $\lim\limits_{x\to\infty} f(x) = l_\infty$ if and only if for every $\epsilon > 0$, there exists a compact $K_\epsilon \subset \mathcal{K}_E$ such that $\sup_{x\in K_\epsilon^c} |f(x) - l_\infty| < \epsilon$ if $l_\infty \in \mathbb{R}$, $\inf_{x\in K_\epsilon^c} f(x) > 1/\epsilon$ if $l_\infty = +\infty$, and $\sup\limits_{x\in K_\epsilon^c} f(x) < -1/\epsilon$ if $l_\infty = -\infty$ with $K_\epsilon^c = E\backslash K_\epsilon$.

2.1.2 Construction of the random measures

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space. We consider a Feller process $(X_t)_{t\geqslant 0}$ (see [6] for details) on $(\Omega, \mathcal{G}, \mathbb{P})$ taking values in a locally compact and separable metric space E. We denote by $(P_t)_{t\geqslant 0}$ the Feller semigroup (see [22]) of this process. We recall that $(P_t)_{t\geqslant 0}$ is a family of linear operators from $\mathcal{C}_0(E)$ to itself such that $P_0 f = f$, $P_{t+s} f = P_t P_s f$, $t, s \geqslant 0$ (semigroup property) and $\lim\limits_{t\to 0} \|P_t f - f\|_\infty = 0$ (Feller property). Using this semigroup, we can introduce the infinitesimal generator of $(X_t)_{t\geqslant 0}$ as a linear operator A defined on a subspace $\mathcal{D}(A)$ of $\mathcal{C}_0(E)$, satisfying: For every $f \in \mathcal{D}(A)$,

$$Af = \lim_{t\to 0} \frac{P_t f - f}{t}$$

exists for the $\|.\|_\infty$-norm. The operator $A : \mathcal{D}(A) \to \mathcal{C}_0(E)$ is thus well defined and $\mathcal{D}(A)$ is called the domain of A. As a consequence of the Echeverria Weiss theorem, the set of invariant distributions for $(X_t)_{t\geqslant 0}$ can be characterized in the following way:

$$\mathcal{V} = \{\nu \in \mathcal{P}(E), \forall t \geqslant 0, P_t \nu = \nu\} = \{\nu \in \mathcal{P}(E), \forall f \in \mathcal{D}(A), \nu(Af) = 0\}.$$

The starting point of our reasoning is thus to consider an approximation of A. First, we introduce the family of transition kernels $(\mathcal{Q}_\gamma)_{\gamma>0}$ from $\mathcal{C}_0(E)$ to itself. Now, let us define the family of linear operators $\widetilde{A} := (\widetilde{A}_\gamma)_{\gamma>0}$ from $\mathcal{C}_0(E)$ into itself, as follows

$$\forall f \in \mathcal{C}_0(E), \quad \gamma > 0, \qquad \widetilde{A}_\gamma f = \frac{\mathcal{Q}_\gamma f - f}{\gamma}.$$

The family $\widetilde{A}$ is usually called the pseudo-generator of the transition kernels $(\mathcal{Q}_\gamma)_{\gamma>0}$ and is an approximation of A as γ tends to zero. From a practical viewpoint, the main interest of our approach is that it is reasonnable to assume that there exists $\overline{\gamma} > 0$ such that for every $x \in E$ and every $\gamma \in [0, \overline{\gamma}]$, $\mathcal{Q}_\gamma(x, dy)$ is simulable at a reasonable computational cost. The family $(\mathcal{Q}_\gamma)_{\gamma>0}$ is used to build $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ (this notation replaces $(\overline{X}^{\gamma}_{\Gamma_n})_{n\in\mathbb{N}}$ from now for clarity in the writing) as the non-homogeneous Markov approximation of the Feller process $(X_t)_{t\geqslant 0}$. It is defined on the time grid $\{\Gamma_n = \sum\limits_{k=1}^{n} \gamma_k, n \in \mathbb{N}\}$ with the time-step sequence $\gamma := (\gamma_n)_{n\in\mathbb{N}^*}$ satisfying

$$\forall n \in \mathbb{N}^*, \quad 0 < \gamma_n \leqslant \overline{\gamma} := \sup_{n\in\mathbb{N}^*} \gamma_n < +\infty, \qquad \lim_{n\to+\infty} \gamma_n = 0 \quad \text{and} \qquad \lim_{n\to+\infty} \Gamma_n = +\infty.$$

Notice that we will sometimes use the notation γ_{-m} for $m \in \mathbb{N}$. In this case we will always use the convention $\gamma_{-m} = 0$. The transition probability distributions of $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ are given by $\mathcal{Q}_{\gamma_n}(x, dy), n \in \mathbb{N}^*$, $x \in E$, *i.e.* :

$$\mathbb{P}(\overline{X}_{\Gamma_{n+1}} \in dy | \overline{X}_{\Gamma_n}) = \mathcal{Q}_{\gamma_{n+1}}(\overline{X}_{\Gamma_n}, dy), \quad n \in \mathbb{N}.$$

We can canonically extend $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ into a *càdlàg* process by setting $\overline{X}(t,\omega) = \overline{X}_{\Gamma_{n(t)}}(\omega)$ with $n(t) = \inf\{n \in \mathbb{N}, \Gamma_{n+1} > t\}$. Then $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ is a simulable (as soon as $\overline{X}_0$ is) non-homogeneous Markov chain with transitions

$$\forall m \leqslant n, \qquad \overline{P}_{\Gamma_m,\Gamma_n}(x,dy) = \mathcal{Q}_{\gamma_{m+1}} \circ \cdots \circ \mathcal{Q}_{\gamma_n}(x,dy),$$

and law

$$\mathcal{L}(\overline{X}_{\Gamma_n}|\overline{X}_0 = x) = \overline{P}_{\Gamma_n}(x,dy) = \mathcal{Q}_{\gamma_1} \circ \cdots \circ \mathcal{Q}_{\gamma_n}(x,dy).$$

We use $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ to design a Langevin Monte Carlo algorithm. Notice that this approach is generic since the approximation transition kernels $(\mathcal{Q}_\gamma)_{\gamma>0}$ are not explicitly specified and then, it can be used in many different configurations including among others, weak numerical schemes or exact simulation *i.e.* $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}} = (X_{\Gamma_n})_{n\in\mathbb{N}}$. This is of main interest in this paper as we show later that using high weak order schemes of $(X_t)_{t\geqslant 0}$ leads to higher rates of convergence in the FCLT satisfied by the weighted empirical measures. Notice that weighted empirical measures are built in a quite more general way than in (1) as we consider some general weights which are not necessarily equal to the time steps. We define this weight sequence. Let $\eta := (\eta_n)_{n\in\mathbb{N}^*}$ be such that

$$\forall n \in \mathbb{N}^*, \quad \eta_n \geqslant 0, \quad \lim_{n\to+\infty} H_n = +\infty, \qquad \text{with} \qquad H_n := H_{\eta,n} = \sum_{k=1}^{n} \eta_k. \tag{3}$$

Now we present our algorithm introduced in [18] and adapted from the one introduced in [10] designed with a Euler scheme with decreasing steps $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ of a Brownian diffusion process $(X_t)_{t\geqslant 0}$. For $x \in E$, let δ_x denote the Dirac mass at point x. For every $n \in \mathbb{N}^*$, we define the random weighted empirical random measures as follows

$$\nu_n^\eta(dx) = \frac{1}{H_n}\sum_{k=1}^{n} \eta_k \delta_{\overline{X}_{\Gamma_{k-1}}}(dx). \tag{4}$$

This section of the paper is dedicated to present how to prove that *a.s.* every weak limiting distribution of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ belongs to $\mathcal{V}$. In particular when the invariant measure of $(X_t)_{t\geqslant 0}$ is unique, *i.e.* $\mathcal{V} = \{\nu\}$, then $\mathbb{P} - a.s.$ $\lim_{n\to+\infty} \nu_n^\eta f = \nu f$, for a generic class of continuous test functions f. The approach consists in two steps. First, we establish a tightness property to obtain existence of at least one weak limiting distribution for $(\nu_n^\eta)_{n\in\mathbb{N}^*}$. Then, in a second step, we identify everyone of these limiting distributions with an invariant distributions of the Feller process $(X_t)_{t\geqslant 0}$.

2.1.3 Assumptions on the random measures

In this part, we present the necessary assumptions on the pseudo-generator $\widetilde{A} = (\widetilde{A}_\gamma)_{\gamma>0}$ in order to prove the convergence of the empirical measures $(\nu_n^\eta)_{n\in\mathbb{N}^*}$.

Mean reverting recursive control

In this framework, we introduce a well suited assumption, referred to as the mean reverting recursive control of the pseudo-generator $\widetilde{A}$. This assumption leads to a tightness property on $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ from which follows the existence (in weak sense) of a limiting distribution for $(\nu_n^\eta)_{n\in\mathbb{N}^*}$. A supplementary interest of this approach is that it is designed to obtain the *a.s.* convergence of $(\nu_n^\eta(f))_{n\in\mathbb{N}^*}$ for a generic class of continuous test functions f which is larger then $\mathcal{C}_b(E)$.

To do so, we introduce a Lyapunov function V related to $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$. Assume that V a Borel function such that

$$\mathrm{L}_V \quad \equiv \quad V:(E\to[v_*,+\infty), v_*>0 \quad \text{and} \quad \lim_{x\to\infty} V(x)=+\infty. \tag{5}$$

We now relate V to $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ introducing its mean reversion Lyapunov property. Let $\psi,\phi:[v_*,\infty)\to(0,+\infty)$ some Borel functions such that $\widetilde{A}_\gamma\psi\circ V$ exists for every $\gamma\in(0,\overline{\gamma}]$. Let $\alpha>0$ and $\beta\in\mathbb{R}$. We assume

$$\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta) \quad \equiv$$
$$\begin{cases} (i) & \exists n_0\in\mathbb{N}^*,\forall n\geqslant n_0, x\in E, \quad \widetilde{A}_{\gamma_n}\psi\circ V(x)\leqslant \frac{\psi\circ V(x)}{V(x)}(\beta-\alpha\phi\circ V(x)). \\ (ii) & \liminf_{y\to+\infty}\phi(y)>\beta/\alpha. \end{cases} \tag{6}$$

$\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta)$ is called the weakly mean reverting recursive control assumption of the pseudo generator for Lyapunov function V.

Lyapunov functions are usually used to show the existence and sometimes the uniqueness of the invariant measure of Feller processes. In particular, when $p=1$, the condition $\mathcal{RC}_{Q,V}(I_d,I_d,\alpha,\beta)(i)$ appears as the discrete version of $AV\leqslant\beta-\alpha V$, which is used in that interest for instance in [9], [5], [1] or[15].

The condition $\mathcal{RC}_{Q,V}(V^p,I_d,\alpha,\beta)(i)$, $p\geqslant 1$, is studied in the seminal paper [10] (and then in [11] with $\phi(y)=y^a, a\in(0,1]$,$y\in[v_*,\infty)$) concerning the Wasserstein convergence of the weighted empirical measures of the Euler scheme with decreasing steps of a Brownian diffusions. When $\phi=I_d$, the Euler scheme is also studied for markov switching Brownian diffusions in [14]. Notice also that $\mathcal{RC}_{Q,V}(I_d,\phi,\alpha,\beta)(i)$ with ϕ concave appears in [3] to prove sub-geometrical ergodicity of Markov chains. In [12], a similar hypothesis to $\mathcal{RC}_{Q,V}(I_d,\phi,\alpha,\beta)(i)$ (with ϕ not necessarily concave and $\widetilde{A}_{\gamma_n}$ replaced by A), is also used to study the Wasserstein but also exponential convergence of the weighted empirical measures (4) for the Euler scheme of a Brownian diffusions. Finally in [21] similar properties as $\mathcal{RC}_{Q,V}(V^p,V^a,\alpha,\beta)(i)$, $a\in(0,1]$, $p>0$, are developped in the study of the Euler scheme for Levy processes.

On the one hand, the function ϕ controls the mean reverting property. In particular, we call strongly mean reverting property when $\phi=I_d$ and weakly mean reverting property when $\lim_{y\to+\infty}\phi(y)/y=0$, for instance $\phi(y)=y^a$, $a\in(0,1)$ for every $y\in[v_*,\infty)$. On the other hand, the function ψ is closely related to the identification of the set of test functions f for which we have $\lim_{n\to+\infty}\nu_n^\eta(f)=\nu(f)$ $a.s.$, when ν is the unique invariant distribution of the underlying Feller process.

To this end, for $s\geqslant 1$, which is related to step weight assumption, we introduce the sets of test functions for which we will show the $a.s.$ convergence of the weighted empirical measures (4):

$$\mathcal{C}_{\tilde{V}_{\psi,\phi,s}}(E)=\{f\in\mathcal{C}(E),|f(x)|=\underset{x\to\infty}{o}(\tilde{V}_{\psi,\phi,s}(x))\}, \tag{7}$$
$$\text{with} \quad \tilde{V}_{\psi,\phi,s}:E\to\mathbb{R}_+,x\mapsto\tilde{V}_{\psi,\phi,s}(x):=\frac{\phi\circ V(x)\psi\circ V(x)^{1/s}}{V(x)}.$$

Notice that our approach benefits from providing generic results because we consider general Feller processes and approximations but also because the functions ϕ and ψ are not specified explicitly.

Infinitesimal generator approximation

This section presents the assumption that enables to characterize the limiting distributions of the $a.s.$ tight sequence $(\nu_n^{\eta}(dx,\omega))_{n\in\mathbb{N}^*}$. We aim to estimate the distance between $\mathcal{V}$ and ν_n^{η} (see (4)) for n large enough. We thus introduce an hypothesis concerning the distance between $(\widetilde{A}_{\gamma})_{\gamma>0}$, the pseudo-generator of $(\mathcal{Q}_{\gamma})_{\gamma>0}$, and A, the infinitesimal generator of $(P_t)_{t\geqslant 0}$. We assume that there exists $\mathcal{D}(A)_0 \subset \mathcal{D}(A)$ with $\mathcal{D}(A)_0$ dense in $\mathcal{C}_0(E)$ such that:

$$\mathcal{E}(\widetilde{A},A,\mathcal{D}(A)_0) \quad \equiv \quad \forall \gamma \in (0,\overline{\gamma}], \forall f \in \mathcal{D}(A)_0, \forall x \in E, \qquad |\widetilde{A}_{\gamma} f(x) - Af(x)| \leqslant \Lambda_f(x,\gamma), \tag{8}$$

where $\Lambda_f : E \times \mathbb{R}_+ \to \mathbb{R}_+$ can be represented in the following way: Let $(\tilde{\Omega},\tilde{\mathcal{G}},\tilde{\mathbb{P}})$ be a probability space. Let $g : E \to \mathbb{R}_+^q$, $q \in \mathbb{N}$, be a locally bounded Borel measurable function and let $\tilde{\Lambda}_f : (E \times \mathbb{R}_+ \times \tilde{\Omega}, \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \tilde{\mathcal{G}}) \to \mathbb{R}_+^q$ be a measurable function such that $\sup_{i\in\{1,\ldots,q\}} \tilde{\mathbb{E}}[\sup_{x\in E} \sup_{\gamma\in(0,\overline{\gamma}]} \tilde{\Lambda}_{f,i}(x,\gamma,\tilde{\omega})] < +\infty$ and that we have the representation

$$\forall x \in E, \forall \gamma \in (0,\overline{\gamma}], \qquad \Lambda_f(x,\gamma) = \langle g(x), \tilde{\mathbb{E}}[\tilde{\Lambda}_f(x,\gamma,\tilde{\omega})] \rangle_{\mathbb{R}^q}$$

Moreover, we assume that for every $i \in \{1,\ldots,q\}$, $\sup_{n\in\mathbb{N}^*} \nu_n^{\eta}(g_i,\omega) < +\infty$, $\mathbb{P}(d\omega) - a.s.$, and that $\tilde{\Lambda}_{f,i}$ satisfies one of the following two properties:
There exists a measurable function $\underline{\gamma} : (\tilde{\Omega},\tilde{\mathcal{G}}) \to ((0,\overline{\gamma}],\mathcal{B}((0,\overline{\gamma}]))$ such that:

$$\textbf{I)}\ \tilde{\mathbb{P}}(d\tilde{\omega}) - a.s \qquad \begin{cases} (i) & \forall K \in \mathcal{K}_E, \quad \lim\limits_{\gamma\to 0} \sup\limits_{x\in K} \tilde{\Lambda}_{f,i}(x,\gamma,\tilde{\omega}) = 0, \\ (ii) & \lim\limits_{x\to\infty} \sup\limits_{\gamma\in(0,\underline{\gamma}(\tilde{\omega})]} \tilde{\Lambda}_{f,i}(x,\gamma,\tilde{\omega}) = 0, \end{cases} \tag{9}$$

or

$$\textbf{II)}\ \tilde{\mathbb{P}}(d\tilde{\omega}) - a.s \qquad \lim_{\gamma\to 0} \sup_{x\in E} \tilde{\Lambda}_{f,i}(x,\gamma,\tilde{\omega}) g_i(x) = 0. \tag{10}$$

Remark 2.1. *Let $(F,\mathcal{F},\lambda)$ be a measurable space. Using the exact same approach, the results we obtain hold when we replace the probability space $(\tilde{\Omega},\tilde{\mathcal{G}},\tilde{\mathbb{P}})$ by the product measurable space $(\tilde{\Omega}\times F, \tilde{\mathcal{G}}\otimes\mathcal{F}, \tilde{\mathbb{P}}\otimes\lambda)$ in the representation of Λ_f and in (9) and (10) but we restrict to that case for sake of clarity in the writing. This observation can be useful when we study jump process where λ can stand for the jump intensity.*

This representation assumption benefits from the fact that the transition functions $(\mathcal{Q}_{\gamma}(x,dy))_{\gamma\in(0,\overline{\gamma}]}$, $x \in E$, can be represented using distributions of random variables which are involved in the computation of $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}^*}$. In particular, this approach is well adapted to stochastic approximations associated to a time grid such as numerical schemes for stochastic differential equations with a Brownian part or/and a jump part.

Growth control and Step Weight assumptions

We conclude with hypothesis concerning the control of the martingale part of one step of our approximation. Let $\rho \in [1,2]$ and let $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function. For $F \subset \{f, f : (E,\mathcal{B}(E)) \to (\mathbb{R},\mathcal{B}(\mathbb{R}))\}$ and $g : E \to \mathbb{R}_+$ a Borel function, we assume that, for every $n \in \mathbb{N}$,

$$\mathcal{GC}_Q(F,g,\rho,\epsilon_{\mathcal{I}}) \equiv \quad \mathbb{P}-a.s. \quad \forall f \in F, \qquad \mathbb{E}[|f(\overline{X}_{\Gamma_{n+1}}) - \mathcal{Q}_{\gamma_{n+1}} f(\overline{X}_{\Gamma_n})|^{\rho} | \overline{X}_{\Gamma_n}] \leqslant C_f \epsilon_{\mathcal{I}}(\gamma_{n+1}) g(\overline{X}_{\Gamma_n}), \tag{11}$$

with $C_f > 0$ a finite constant which may depend on f.

Remark 2.2. *The reader may notice that $\mathcal{GC}_Q(F,g,\rho,\epsilon_{\mathcal{I}})$ holds as soon as (11) is satisfied with $\mathcal{Q}_{\gamma_{n+1}}f(\overline{X}_{\Gamma_n})$, $n\in\mathbb{N}^*$, replaced by a $\mathcal{F}_n^{\overline{X}} := \sigma(\overline{X}_{\Gamma_k}, k\leqslant n)$- progressively measurable process $(\mathfrak{X}_n)_{n\in\mathbb{N}^*}$ since we have $\mathcal{Q}_{\gamma_{n+1}}f(\overline{X}_{\Gamma_n}) = \mathbb{E}[f(\overline{X}_{\Gamma_{n+1}})|\overline{X}_{\Gamma_n}]$ and $\mathbb{E}[|f(\overline{X}_{\Gamma_{n+1}}) - \mathcal{Q}_{\gamma_{n+1}}f(\overline{X}_{\Gamma_n})|^{\rho}|\overline{X}_{\Gamma_n}] \leqslant 2^{\rho}\mathbb{E}[|f(\overline{X}_{\Gamma_{n+1}}) - \mathfrak{X}_n|^{\rho}|\overline{X}_{\Gamma_n}]$ for every $\mathfrak{X}_n\in \mathrm{L}^2(\mathcal{F}_n^{\overline{X}})$.*

We will combine this first assumption with the following step weight related ones:

$$\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}}) \quad\equiv\quad \mathbb{P}-a.s.\quad \sum_{n=1}^{\infty}\Big|\frac{\eta_n}{H_n\gamma_n}\Big|^{\rho}\epsilon_{\mathcal{I}}(\gamma_n)g(\overline{X}_{\Gamma_n})<+\infty, \tag{12}$$

and

$$\mathcal{SW}_{\mathcal{II},\gamma,\eta}(F) \quad\equiv\quad \mathbb{P}-a.s.\quad \forall f\in F,$$
$$\sum_{n=0}^{\infty}\frac{(\eta_{n+1}/\gamma_{n+1}-\eta_n/\gamma_n)_+}{H_{n+1}}|f(\overline{X}_{\Gamma_n})|<+\infty, \tag{13}$$

with the convention $\eta_0/\gamma_0=1$. Notice that this last assumption holds as soon as the sequence $(\eta_n/\gamma_n)_{n\in\mathbb{N}^*}$ is non-increasing.

At this point we can focus now on the main results concerning this general approach.

2.1.4 Almost sure tightness

From the recursive control assumption, Theorem 2.1 establishes the *a.s.* tightness of the sequence $(\nu_n^{\eta})_{n\in\mathbb{N}^*}$ and also provides a uniform control of $(\nu_n^{\eta})_{n\in\mathbb{N}^*}$ on a generic class of test functions.

Theorem 2.1. *Let $s\geqslant 1$, $\rho\in[1,2]$, $v_*>0$, and let us consider the Borel functions $V:E\to[v_*,\infty)$, $g:E\to\mathbb{R}_+$, $\psi:[v_*,\infty)\to\mathbb{R}_+$ and $\epsilon_{\mathcal{I}}:\mathbb{R}_+\to\mathbb{R}_+$ an increasing function. We have the following properties:*

A. *Assume that $\widetilde{A}_{\gamma_n}(\psi\circ V)^{1/s}$ exists for every $n\in\mathbb{N}^*$, and that $\mathcal{GC}_Q((\psi\circ V)^{1/s},g,\rho,\epsilon_{\mathcal{I}})$ (see (11)), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}})$ (see (12)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}((\psi\circ V)^{1/s})$ (see (13) hold. Then*

$$\mathbb{P}\text{-}a.s.\quad \sup_{n\in\mathbb{N}^*} -\frac{1}{H_n}\sum_{k=1}^{n}\eta_k\widetilde{A}_{\gamma_k}(\psi\circ V)^{1/s}(\overline{X}_{\Gamma_{k-1}})<+\infty. \tag{14}$$

B. *Let $\alpha>0$ and $\beta\in\mathbb{R}$. Let $\phi:[v_*,\infty)\to\mathbb{R}_+^*$ be a continuous function such that $C_\phi := \sup_{y\in[v_*,\infty)}\phi(y)/y<\infty$. Assume that (14) holds and*

- ***i.*** *$\mathcal{RC}_{Q,V}(\psi,\phi,\alpha,\beta)$ (see (6)) holds.*
- ***ii.*** *L_V (see (5)) holds and $\lim\limits_{y\to+\infty}\frac{\phi(y)\psi(y)^{1/s}}{y}=+\infty$.*

Then,

$$\mathbb{P}\text{-}a.s.\quad \sup_{n\in\mathbb{N}^*}\nu_n^{\eta}(\tilde{V}_{\psi,\phi,s})<+\infty. \tag{15}$$

with $\tilde{V}_{\psi,\phi,s}$ defined in (7). Therefore, the sequence $(\nu_n^{\eta})_{n\in\mathbb{N}^*}$ is $\mathbb{P}-a.s.$ tight.

2.1.5 Identification of the limit

In Theorem 2.1, the tightness - and then existenve of a weak limiting distribution - of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is established. From Theorem 2.2, it follows that every limiting point of this sequence is an invariant distribution of the Feller process with infinitesimal generator A.

Theorem 2.2. *Let $\rho \in [1,2]$. We have the following properties:*

A. *Let $\mathcal{D}(A)_0 \subset \mathcal{D}(A)$, with $\mathcal{D}(A)_0$ dense in $\mathcal{C}_0(E)$. We assume that $\widetilde{A}_{\gamma_n} f$ exists for every $f \in \mathcal{D}(A)_0$ and every $n \in \mathbb{N}^*$. Also assume that there exists $g : E \to \mathbb{R}_+$ a Borel function and $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing function such that $\mathcal{GC}_Q(\mathcal{D}(A)_0, g, \rho, \epsilon_{\mathcal{I}})$ (see (11)) and $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}})$ (see (12)) hold and that*

$$\lim_{n\to+\infty} \frac{1}{H_n} \sum_{k=1}^{n} |\eta_{k+1}/\gamma_{k+1} - \eta_k/\gamma_k| = 0. \tag{16}$$

Then

$$\mathbb{P}\text{-}a.s. \quad \forall f \in \mathcal{D}(A)_0, \qquad \lim_{n\to+\infty} \frac{1}{H_n} \sum_{k=1}^{n} \eta_k \widetilde{A}_{\gamma_k} f(\overline{X}_{\Gamma_{k-1}}) = 0. \tag{17}$$

B. *We assume that (17) and $\mathcal{E}(\widetilde{A}, A, \mathcal{D}(A)_0)$ (see (8)) hold. Then*

$$\mathbb{P}\text{-}a.s. \quad \forall f \in \mathcal{D}(A)_0, \qquad \lim_{n\to+\infty} \nu_n^\eta(Af) = 0.$$

*It follows that, $\mathbb{P}-a.s.$, every weak limiting distribution ν_∞^η of the sequence $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ belongs to $\mathcal{V}$, the set of the invariant distributions of $(X_t)_{t\geqslant 0}$. Finally, if the hypothesis from Theorem 2.1 point **B.** hold and $(X_t)_{t\geqslant 0}$ has a unique invariant distribution, i.e. $\mathcal{V} = \{\nu\}$, then*

$$\mathbb{P}\text{-}a.s. \quad \forall f \in \mathcal{C}_{\tilde{V}_{\psi,\phi,s}}(E), \qquad \lim_{n\to+\infty} \nu_n^\eta(f) = \nu(f), \tag{18}$$

with $\mathcal{C}_{\tilde{V}_{\psi,\phi,s}}(E)$ defined in (7).

In the particular case where the function ψ is polynomial, (18) also reads as the $a.s.$ convergence of the empirical measures for some L^p-Wasserstein distances, $p > 0$, that we will study further in this paper for some numerical schemes of some diffusion processes. From the liberty granted by the choice of ψ in this abstract framework, where only a recursive control with mean reverting is required, we will also propose an application for functions ψ with exponential growth.

2.2 About Growth control and Step Weight assumptions

The following Lemma presents a L_1-finiteness property that we can obtain under recursive control hypothesis and strongly mean reverting assumptions ($\phi = I_d$). This result is thus useful to prove $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g,\rho,\epsilon_{\mathcal{I}})$ (see (12)) or $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(F)$ (see (13)) for well chosen F and g in this specific situation.

Lemma 2.1. *Let $v_* > 0$, $V : E \to [v_*, \infty)$, $\psi : [v_*, \infty) \to \mathbb{R}_+$, such that $\widetilde{A}_{\gamma_n} \psi \circ V$ exists for every $n \in \mathbb{N}^*$. Let $\alpha > 0$ and $\beta \in \mathbb{R}$. We assume that $\mathcal{RC}_{Q,V}(\psi, I_d, \alpha, \beta)$ (see (6)) holds and that $\mathbb{E}[\psi \circ V(\overline{X}_{\Gamma_{n_0}})] < +\infty$ for every $n_0 \in \mathbb{N}^*$. Then*

$$\sup_{n\in\mathbb{N}} \mathbb{E}[\psi \circ V(\overline{X}_{\Gamma_n})] < +\infty \tag{19}$$

In particular, let $\rho \in [1,2]$ *and* $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \to \mathbb{R}_+$*, an increasing function. It follows that if* $\sum_{n=1}^{\infty} \left|\frac{\eta_n}{H_n\gamma_n}\right|^{\rho} \epsilon_{\mathcal{I}}(\gamma_n) < +\infty$*, then* $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\psi \circ V, \rho, \epsilon_{\mathcal{I}})$ *holds and if* $\sum_{n=0}^{\infty} \frac{(\eta_{n+1}/\gamma_{n+1} - \eta_n/\gamma_n)_+}{H_{n+1}} < +\infty$*, then* $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(\psi \circ V)$ *is satisfied*

Now, we provide a general way to obtain $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(g, \rho, \epsilon_{\mathcal{I}})$ and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(F)$ for some specific g and F as soon as a recursive control with weakly mean reversion assumption holds.

Lemma 2.2. *Let* $v_* > 0$*,* $V : E \to [v_*, \infty)$*,* $\psi, \phi : [v_*, \infty) \to \mathbb{R}_+$*, such that* $\widetilde{A}_{\gamma_n} \psi \circ V$ *exists for every* $n \in \mathbb{N}^*$*. Let* $\alpha > 0$ *and* $\beta \in \mathbb{R}$*. We also introduce the non-increasing sequence* $(\theta_n)_{n \in \mathbb{N}^*}$ *such that* $\sum_{n \geqslant 1} \theta_n \gamma_n < +\infty$*. We assume that* $\mathcal{RC}_{Q,V}(\psi, \phi, \alpha, \beta)$ *(see (6)) holds and that* $\mathbb{E}[\psi \circ V(\overline{X}_{\Gamma_{n_0}})] < +\infty$ *for every* $n_0 \in \mathbb{N}^*$*. Then*

$$\sum_{n=1}^{\infty} \theta_n \gamma_n \mathbb{E}[\tilde{V}_{\psi,\phi,1}(\overline{X}_{\Gamma_{n-1}})] < +\infty$$

with $\tilde{V}_{\psi,\phi,1}$ *defined in (7). In particular, let* $\rho \in [1,2]$ *and* $\epsilon_{\mathcal{I}} : \mathbb{R}_+ \to \mathbb{R}_+$*, an increasing function. If we also assume*

$$\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho, \epsilon_{\mathcal{I}}) \quad \equiv \quad \Big(\gamma_n^{-1} \epsilon_{\mathcal{I}}(\gamma_n) \big(\frac{\eta_n}{H_n \gamma_n}\big)^{\rho}\Big)_{n \in \mathbb{N}^*} \text{ is non-increasing and}$$
$$\sum_{n=1}^{\infty} \Big(\frac{\eta_n}{H_n \gamma_n}\Big)^{\rho} \epsilon_{\mathcal{I}}(\gamma_n) < +\infty, \tag{20}$$

then we have $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\tilde{V}_{\psi,\phi,1}, \rho, \epsilon_{\mathcal{I}})$ *(see (12)). Finally,if*

$$\mathcal{SW}_{\mathcal{II},\gamma,\eta} \quad \equiv \quad \Big(\frac{\big(\frac{\eta_{n+1}}{\gamma_{n+1}} - \frac{\eta_n}{\gamma_n}\big)_+}{\gamma_n H_n}\Big)_{n \in \mathbb{N}^*} \text{ is non-increasing and}$$
$$\sum_{n=1}^{\infty} \frac{(\eta_{n+1}/\gamma_{n+1} - \eta_n/\gamma_n)_+}{H_n} < +\infty, \tag{21}$$

then we have $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(\tilde{V}_{\psi,\phi,1})$ *(see (13)).*

3 Rate of convergence - A general approach

In this section, we extend the abstract framework from Section 2 to establish the rate of convergence of the empirical measures (4) to ν supposed to be unique. The approach we propose consists in two part. First we give appropriate weak error estimations and on the other hand we give suitable step weight assumptions to control the martingale part of the empirical measures. Notice that, together with the choice of weights, the weak error estimation is the crucial tool to obtain high rate of convergence of the weighted empirical measures.

3.1 Assumption on the random measures

Weak approximation assumption

Let $F \subset \{f, f : (E, \mathcal{B}(E)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))\}$. Let $q \in \mathbb{N}$ be the weak order of the approximation. We consider the linear operator $\mathfrak{M}_q$ defined on F. Let $\tilde{\eta}_q : \mathbb{R}_+ \times \{0,1\} \to \mathbb{R}_+$ such that the weight sequence $(\tilde{\eta}_{q,n})_{n \in \mathbb{N}^*} = (\tilde{\eta}_q(\gamma_n, n \mod 2))_{n \in \mathbb{N}^*}$ is decreasing and satisfies (3) and such that $\mathbb{P} - a.s.$, $\lim_{n \to \infty} \nu_n^{\tilde{\eta}_q}(\mathfrak{M}_q f) = \nu(\mathfrak{M}_q f)$ for every $f \in F$. We suppose that

$$\mathcal{E}_q(F,\tilde{A},A,\mathfrak{M},\tilde{\eta}_q) \equiv \forall f\in F, \forall x\in E, \forall \gamma\in(0,\overline{\gamma}], \forall e\in\{0,1\}, \quad \Big|\mathcal{R}_q f(x,\gamma,e)-\tilde{\eta}_q(\gamma,e)\mathfrak{M}_q f(x)\Big| \leqslant \tilde{\eta}_q(\gamma,e)\Lambda_{f,q}(x,\gamma), \tag{22}$$

with

$$\mathcal{R}_1 f(x,\gamma,e)=\mathcal{R}_1 f(x,\gamma) := -\tilde{\mathcal{R}}_1 f(x,\gamma)$$
$$\mathcal{R}_2 f(x,\gamma,e)=\mathcal{R}_2 f(x,\gamma) := \frac{\gamma}{2}\tilde{\mathcal{R}}_1 A f(x,\gamma)-\tilde{\mathcal{R}}_2 f(x,\gamma)$$
$$\mathcal{R}_3 f(x,\gamma,e) := -\frac{\gamma^2}{2+e}\tilde{\mathcal{R}}_1 A^2 f(x,\gamma)+\frac{(2-e)\gamma^2}{3}\tilde{\mathcal{R}}_1 A^2 f(x,\gamma)-\tilde{\mathcal{R}}_3 f(x,\gamma)$$

with, for every $m\in\{0,\ldots,q-1\}$, the measurable functions

$$\begin{array}{rcl}\tilde{\mathcal{R}}_m f \;: E\times\mathbb{R}_+ & \to & \mathbb{R}\\ (x,\gamma) & \mapsto & \gamma\tilde{A}_\gamma f(x)-\sum_{i=1}^m \frac{\gamma^i}{i!}A^i f(x),\end{array}$$

which are supposed to be well defined for every $f\in F$. In addition, we also assume that $\Lambda_{f,q}: E\times\mathbb{R}_+\to\mathbb{R}_+$ can be represented in the following way: Let $(\tilde{\Omega},\tilde{\mathcal{G}},\tilde{\mathbb{P}})$ be a probability space. Let $g: E\to\mathbb{R}_+^l$, $l\in\mathbb{N}^*$, be a locally bounded Borel measurable function and let $\tilde{\Lambda}_{f,q}:(E\times\mathbb{R}_+\times\tilde{\Omega},\mathcal{B}(E)\otimes\mathcal{B}(\mathbb{R}_+)\otimes\tilde{\mathcal{G}})\to\mathbb{R}_+^l$ be a measurable function such that

$$\sup_{i\in\{1,\ldots,l\}}\tilde{\mathbb{E}}[\sup_{x\in E}\sup_{\gamma\in(0,\overline{\gamma}]}\tilde{\Lambda}_{f,q,i}(x,\gamma,\tilde{\omega})]<+\infty \tag{23}$$

and that the following representation assumption holds

$$\forall x\in E, \forall\gamma\in(0,\overline{\gamma}], \qquad \Lambda_{f,q}(x,\gamma)=\langle g(x),\tilde{\mathbb{E}}[\tilde{\Lambda}_{f,q}(x,\gamma,\tilde{\omega})]\rangle_{\mathbb{R}^l}.$$

Moreover, we assume that for every $i\in\{1,\ldots,l\}$, $\sup_{n\in\mathbb{N}^*}\nu_n^{\tilde{\eta}_q}(g_i,\omega)<+\infty$, $\mathbb{P}-a.s.$, and that $\tilde{\Lambda}_{f,i}$ satisfies one of the two following properties.

There exists a measurable function $\underline{\gamma}:(\tilde{\Omega},\tilde{\mathcal{G}})\to((0,\overline{\gamma}],\mathcal{B}((0,\overline{\gamma}]))$ such that:

$$\textbf{I)}\ \tilde{\mathbb{P}}(d\tilde{\omega})-a.s \quad \begin{cases}(i) & \forall K\in\mathcal{K}_E, \quad \lim_{\gamma\to 0}\sup_{x\in K}\tilde{\Lambda}_{f,q,i}(x,\gamma,\tilde{\omega})=0,\\ (ii) & \lim_{|x|\to\infty}\sup_{\gamma\in(0,\underline{\gamma}(\tilde{\omega})]}\tilde{\Lambda}_{f,q,i}(x,\gamma,\tilde{\omega})=0,\end{cases} \tag{24}$$

or

$$\textbf{II)}\ \tilde{\mathbb{P}}(d\tilde{\omega})-a.s \qquad \lim_{\gamma\to 0}\sup_{x\in E}\tilde{\Lambda}_{f,q,i}(x,\gamma,\tilde{\omega})g_i(x)=0. \tag{25}$$

Remark 3.1. *Let $(F,\mathcal{F},\lambda)$ be a measurable space. Using the exact same approach, the results we obtain hold when we replace the probability space $(\tilde{\Omega},\tilde{\mathcal{G}},\tilde{\mathbb{P}})$ by the product measurable space $(\tilde{\Omega}\times F,\tilde{\mathcal{G}}\otimes\mathcal{F},\tilde{\mathbb{P}}\otimes\lambda)$ in the representation of $\Lambda_{f,q}$ and in (24) and (25). It is a similar observation as in the study of the convergence as pointed out in Remark 2.1.*

Growth assumption

We denote by $\mathcal{P}_{\overline{X},2}$ the set of $\mathcal{F}_n^{\overline{X}} := \sigma(\overline{X}_{\Gamma_k}, k \leqslant n)$- progressively measurable processes $(\mathfrak{X}_n)_{n\in\mathbb{N}^*}$ with $\mathfrak{X}_{n+1} \in \mathrm{L}^2(\mathcal{F}_n^{\overline{X}})$ and $\mathbb{E}[\mathfrak{X}_{n+1}|\overline{X}_{\Gamma_n}] = 0$ for every $n \in \mathbb{N}$. Let $\rho \in [1, 2]$ and let $\epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}} : \mathbb{R}_+ \to \mathbb{R}_+$ be two increasing functions such that the weight sequence $(\epsilon_{\mathfrak{X},n})_{n\in\mathbb{N}^*} = (\epsilon_{\mathfrak{X}}(\gamma_n))_{n\in\mathbb{N}^*}$ satisfies (3). Let $F \subset \{f, f : (E, \mathcal{B}(E)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))\}$ and $g : E \to \mathbb{R}_+$ be a Borel measurable function. We consider the linear operator $\mathfrak{V}$ defined on F and we assume that $A^m f$ is well defined for every $m \in \{0, \ldots, q-1\}$ and every $f \in F$ and that

$$\begin{aligned}\mathcal{GC}_{Q,q}(F, g, \rho, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}}, \mathfrak{V}) \quad &\equiv \quad \mathbb{P}-a.s. \quad \forall f \in F, \exists \mathfrak{X}_f \in \mathcal{P}_{\overline{X},2} \\ \mathbb{E}\Big[\Big|B_q f(\overline{X}_{\Gamma_n}, \overline{X}_{\Gamma_{n+1}}, \gamma_{n+1}, n \mod 2) - \mathfrak{X}_{f,n+1}&\Big|^\rho \Big|\overline{X}_{\Gamma_n}\Big] \leqslant C_f \epsilon_{\mathcal{GC}}(\gamma_{n+1}) g(\overline{X}_{\Gamma_n}). \qquad (26)\end{aligned}$$

with, for every $x, y \in \mathbb{E}$, $\gamma \geqslant 0$, $e \in \{0, 1\}$,

$$\begin{aligned}B_1 f(x, y, \gamma, e) = B_1 f(x, y, \gamma) &:= \mathcal{Q}_\gamma f(x) - f(y),\\ B_2 f(x, y, \gamma, e) = B_2 f(x, y, \gamma) &:= B_1 f(x, y, \gamma) - \frac{\gamma}{2} B_1 A f(x, y, \gamma),\\ B_3 f(x, y, \gamma, e) &:= B_1 f(x, y, \gamma) - \frac{(2-e)\gamma}{3} B_2 A f(x, y, \gamma) - \frac{\gamma^2}{6} B_1 A^2 f(x, y, \gamma)\\ &= B_1 f(x, y, \gamma) - \frac{(2-e)\gamma}{3} B_1 A f(x, y, \gamma) + \frac{\gamma^2(1-e)}{6} B_1 A^2 f(x, y, \gamma)\end{aligned}$$

and also $\mathbb{E}[|\mathfrak{X}_{f,n+1}|^2|\overline{X}_{\Gamma_n}] = \epsilon_{\mathfrak{X}}(\gamma_{n+1})\mathfrak{V}f(\overline{X}_{\Gamma_n})$ with for every $f \in F$, $\lim_{n\in\mathbb{N}^*} \nu_n^{\epsilon_{\mathfrak{X}}}(\mathfrak{V}f, \omega) = \nu(\mathfrak{V}f)$, $\mathbb{P}-a.s.$, and

$$\forall \mathfrak{E} > 0, \quad \lim_{n\to\infty} \frac{1}{H_{\epsilon_{\mathfrak{X}},n}} \sum_{k=0}^{n-1} \mathbb{E}[|\mathfrak{X}_{f,k+1}|^2 \mathbb{1}_{|\mathfrak{X}_{f,k+1}| > \sqrt{H_{\epsilon_{\mathfrak{X}},n}}\mathfrak{E}} |\overline{X}_{\Gamma_k}] \overset{\mathbb{P}}{=} 0. \qquad (27)$$

Remark 3.2. *The reader may notice that* $\mathcal{GC}_{Q,q}(F, q, \rho, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}}, \mathfrak{V})$ *holds as soon as (11) is satisfied with* $\mathcal{Q}_{\gamma_{n+1}} A^m f(\overline{X}_{\Gamma_n})$, $n \in \mathbb{N}^*$, $m \in \mathbb{N}^*$ *replaced by a* $\mathcal{F}_n^{\overline{X}} := \sigma(\overline{X}_{\Gamma_k}, k \leqslant n)$*- progressively measurable process* $(\mathfrak{X}_{m,n})_{n\in\mathbb{N}^*}$*, since* $\rho \in [1, 2]$ *and we have* $\mathcal{Q}_{\gamma_{n+1}} A^m f(\overline{X}_{\Gamma_n}) = \mathbb{E}[A^m f(\overline{X}_{\Gamma_{n+1}})|\overline{X}_{\Gamma_n}]$ *and* $\mathbb{E}[\mathfrak{X}_{f,n+1}|\overline{X}_{\Gamma_n}] = 0$.

In the following we will combine this assumption with

$$\mathcal{SW}_{\mathcal{GC},\gamma}(g, \rho, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}}) \quad \equiv \quad \mathbb{P}-a.s. \quad \sum_{n=1}^{\infty} \frac{\epsilon_{\mathcal{GC}}(\gamma_n)}{H_{\epsilon_{\mathfrak{X}},n}^{\rho/2}} g(\overline{X}_{\Gamma_n}) < +\infty. \qquad (28)$$

Notice that, as a consequence of Lemma 2.2, if we suppose that $\mathcal{RC}_{Q,V}(\psi, \phi, \alpha, \beta)$ (see (6)) holds, that $\mathbb{E}[\psi \circ V(\overline{X}_{\Gamma_{n_0}})] < +\infty$ for every $n_0 \in \mathbb{N}^*$ and that

$$\mathcal{SW}_{\mathcal{GC},\gamma}(\rho, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}}) \quad \equiv \quad \Big(\frac{\epsilon_{\mathcal{GC}}(\gamma_n)}{\gamma_n H_{\epsilon_{\mathfrak{X}},n}^{\rho/2}}\Big)_{n\in\mathbb{N}^*} \text{ is nonincreasing and } \sum_{n=1}^{\infty} \frac{\epsilon_{\mathcal{GC}}(\gamma_n)}{H_{\epsilon_{\mathfrak{X}},n}^{\rho/2}} < +\infty, \qquad (29)$$

holds, then we have $\mathcal{SW}_{\mathcal{GC},\gamma}(\tilde{V}_{\psi,\phi,1}, \rho, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}})$ (see (28)) with $\tilde{V}_{\psi,\phi,1}$ defined in (7).

3.2 Convergence rate results

We begin with some preliminary results.

Lemma 3.1. ***(Kronecker).*** *Let $(a_n)_{n\in\mathbb{N}^*}$ and $(b_n)_{n\in\mathbb{N}^*}$ be two sequences of real numbers. If $(b_n)_{n\in\mathbb{N}^*}$ is non-decreasing, strictly positive, with $\lim_{n\to+\infty} b_n = +\infty$ and $\sum_{n\geqslant 1} a_n/b_n$ converges in $\mathbb{R}$, then*

$$\lim_{n\to+\infty} \frac{1}{b_n}\sum_{k=1}^{n} a_k = 0.$$

Theorem 3.1. ***(Chow** (see [8], Theorem 2.17)).* *Let $(M_n)_{n\in\mathbb{N}^*}$ be a real valued martingale with respect to some filtration $\mathcal{F} = (\mathcal{F}_n)_{n\in\mathbb{N}}$. Then*

$$\lim_{n\to+\infty} M_n = M_\infty \in \mathbb{R} \quad a.s. \quad \text{on the event}$$

$$\bigcup_{r\in[0,1]} \Big\{ \sum_{n=1}^{\infty} \mathbb{E}[|M_n - M_{n-1}|^{1+r}|\mathcal{F}_{n-1}] < +\infty \Big\}.$$

Now, we give a general CLT result from [8] (Corollary 3.1) which applies to martingale arrays.

Proposition 3.1. *Let $(\tilde{M}_{k,n})_{k\in\{1,..,n\},n\in\mathbb{N}}$ be a $\mathbb{R}$-valued martingale array and define $\mathcal{F}^{\tilde{M}}_{k,n} = \sigma(\tilde{M}_{i,n}, i \in \{0,\dots,k\})$. We assume that $(\tilde{M}_n)_{n\in\mathbb{N}}$ satisfies the Lindeberg condition:*

$$\forall \mathfrak{E} > 0, \quad \lim_{n\to\infty} \sum_{k=0}^{n-1} \mathbb{E}[|\tilde{M}_{k+1,n} - \tilde{M}_{k,n}|^2 \mathbb{1}_{|\tilde{M}_{k+1,n}-\tilde{M}_{k,n}|>\mathfrak{E}} |\mathcal{F}^{\tilde{M}}_{k,n}] \overset{\mathbb{P}}{=} 0 \tag{30}$$

and that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \mathbb{E}[|\tilde{M}_{k+1,n} - \tilde{M}_{k,n}|^2 |\mathcal{F}^{\tilde{M}}_{k,n}] \overset{\mathbb{P}}{=} \zeta^2_{\tilde{M}} \tag{31}$$

with $\zeta^2_{\tilde{M}}$ an almost sure finite random variable. Then

$$\lim_{n\to\infty} \tilde{M}_{n,n} \overset{law}{=} \tilde{\mathcal{N}}(\zeta^2_{\tilde{M}}), \tag{32}$$

where $\tilde{\mathcal{N}}(\zeta^2_{\tilde{M}})$ is a random variable with Laplace transform $\mathbb{E}[\exp(v\tilde{\mathcal{N}}(\zeta^2_{\tilde{M}})] = \mathbb{E}[\exp(v^2\zeta^2_{\tilde{M}}/2))]$ for every $v \in \mathbb{R}$.

3.2.1 The first order FCLT

Considering the first order weak approximation $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ of a Feller process $(X_t)_{t\geqslant 0}$, we establish a first order FCLT satisfied by its empirical measures. In particular this result can be used when $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ is the Euler scheme with decreasing steps of $(X_t)_{t\geqslant 0}$ and recover results from [10], [12], [20] or [14].

Theorem 3.2. *Let* $F \subset \{f, f : (E, \mathcal{B}(E)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})), Af \in \mathcal{C}_b(E)\}$, $g : E \to \mathbb{R}_+$ *a Borel function, let* $\tilde{\eta}_1, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}} : \mathbb{R}_+ \to \mathbb{R}_+$ *be three increasing functions and let* $\mathfrak{M}_1$ *and* $\mathfrak{V}$ *be two linear operators defined on* F. *Finally let* $\eta_n := C_{\gamma,\eta}\gamma_n, C_{\gamma,\eta} > 0$, $n \in \mathbb{N}^*$ *be the weight sequence.*

Assume that $\mathcal{E}_1(F, \tilde{A}, A, \mathfrak{M}_1, \tilde{\eta}_1)$ *(see (22)),* $\mathcal{GC}_{Q,1}(F, g, \rho, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}}, \mathfrak{V})$ *(see (11) and (30)) and* $\mathcal{SW}_{\mathcal{GC},\gamma}(g, \rho, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}})$ *(see (28)) hold.*

Then, for every $f \in F$ *we have the following properties:*

A. *If* $\lim_{n\to\infty} \sqrt{H_{\epsilon_{\mathfrak{X}},n}}/H_{\tilde{\eta}_1,n} = +\infty$, *then*

$$\lim_{n\to\infty} \frac{H_n}{C_{\gamma,\eta}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}\nu_n^{\eta}(Af) \overset{law}{=} \mathcal{N}(0, \nu(\mathfrak{V}f)). \tag{33}$$

B. *If* $\lim_{n\to\infty} \sqrt{H_{\epsilon_{\mathfrak{X}},n}}/H_{\tilde{\eta}_1,n} = \hat{l} \in \mathbb{R}_+^*$, *then*

$$\lim_{n\to\infty} \frac{H_n}{C_{\gamma,\eta}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}\nu_n^{\eta}(Af) \overset{law}{=} \mathcal{N}(\hat{l}^{-1}\nu(\mathfrak{M}_1 f), \nu(\mathfrak{V}f)). \tag{34}$$

C. *If* $\lim_{n\to\infty} \sqrt{H_{\epsilon_{\mathfrak{X}},n}}/H_{\tilde{\eta}_1,n} = 0$, *then*

$$\lim_{n\to\infty} \frac{H_n}{C_{\gamma,\eta}H_{\tilde{\eta}_1,n}}\nu_n^{\eta}(Af) \overset{\mathbb{P}}{=} \nu(\mathfrak{M}_1 f) \tag{35}$$

Moreover, when $\mathfrak{V} = 0$ *this convergence is almost sure.*

Te proof of this result is similar but simpler than for the second order case so we invite the reader to refer to Theorem 3.3 and its proof thereafter.

3.2.2 The second order FCLT

When we consider the second order weak approximation $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ of a Feller process $(X_t)_{t\geqslant 0}$, it is possible to obtain convergence of some weighted empirical measures at a better rate using the following result. A crucial point to obtain this result is to consider a specific weight sequence when we build the weighted empirical measures (4). In this case the weight sequence is inspired by the trapezoidal approximation of the integral while in the first order FCLT the choice $\eta = C_{\gamma,\eta}\gamma$ can be seen as a non homogeneous version of the Riemann approximation of the standard integral.

Theorem 3.3. *Let* $F \subset \{f, f : (E, \mathcal{B}(E)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})), Af \in \mathcal{C}_b(E)\}$, $g : E \to \mathbb{R}_+$ *a Borel function,* $\tilde{\eta}_2, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}} : \mathbb{R}_+ \to \mathbb{R}_+$ *be three increasing functions and let* $\mathfrak{M}_2$ *and* $\mathfrak{V}$ *be two linear operators defined on* F. *Finally let* $\eta_{n+1} := C_{\gamma,\eta}(\gamma_n + \gamma_{n+1})/2, C_{\gamma,\eta} > 0$, $n \in \mathbb{N}$, *be the weight sequence.*

Assume that $\mathcal{E}_2(F, \tilde{A}, A, \mathfrak{M}_2, \tilde{\eta}_2)$ *(see (22)),* $\mathcal{GC}_{Q,2}(F, g, \rho, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}}, \mathfrak{V})$ *(see (11) and (30)) and* $\mathcal{SW}_{\mathcal{GC},\gamma}(g, \rho, \epsilon_{\mathfrak{X}}, \epsilon_{\mathcal{GC}})$ *(see (28)) hold.*

Then, for every $f \in F$, *we have the following properties:*

A. *If* $\lim_{n\to\infty} \sqrt{H_{\epsilon_{\mathfrak{X}},n}}/H_{\tilde{\eta}_2,n} = +\infty$, *then*

$$\lim_{n\to\infty} \frac{H_n}{C_{\gamma,\eta}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}\nu_n^{\eta}(Af) \overset{law}{=} \mathcal{N}(0, \nu(\mathfrak{V}f)). \tag{36}$$

B. *If* $\lim_{n\to\infty}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}/H_{\tilde{\eta}_2,n}=\hat{l}\in\mathbb{R}_+^*$, *then*

$$\lim_{n\to\infty}\frac{H_n}{C_{\gamma,\eta}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}\nu_n^\eta(Af)\overset{law}{=}\mathcal{N}(\hat{l}^{-1}\nu(\mathfrak{M}_2f),\nu(\mathfrak{V}f)). \tag{37}$$

C. *If* $\lim_{n\to\infty}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}/H_{\tilde{\eta}_2,n}=0$, *then*

$$\lim_{n\to\infty}\frac{H_n}{C_{\gamma,\eta}H_{\tilde{\eta}_2,n}}\nu_n^\eta(Af)\overset{\mathbb{P}}{=}\nu(\mathfrak{M}_2f) \tag{38}$$

Moreover, when $\mathfrak{V}=0$ *this convergence is almost sure.*

Proof. Let $n\in\mathbb{N}$. We begin by noticing that the following decomposition holds

$$\begin{aligned}\nu_n^\eta(Af)=\frac{1}{H_n}\sum_{k=1}^n\eta_kAf(\overline{X}_{\Gamma_{k-1}})=&\frac{C_{\gamma,\eta}}{H_n}\sum_{k=1}^n\frac{\gamma_k}{2}(Af(\overline{X}_{\Gamma_k})+Af(\overline{X}_{\Gamma_{k-1}}))\\&-\frac{C_{\gamma,\eta}\gamma_n}{2H_n}Af(\overline{X}_{\Gamma_n}).\end{aligned}$$

Since Af is a bounded function, the second term of the *r.h.s.* of the above equation mulyiplied by $\frac{H_n}{C_{\gamma,\eta}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}$ or $\frac{H_n}{C_{\gamma,\eta}H_{\tilde{\eta}_2,n}}$ converges to zero. We study the first term of the *r.h.s.* of the above equation. The first step consists in showing that, for every $n\in\mathbb{N}$, we have

$$\begin{aligned}\frac{\gamma_{n+1}}{2}(Af(\overline{X}_{\Gamma_{n+1}})+Af(\overline{X}_{\Gamma_n}))=&f(\overline{X}_{\Gamma_{n+1}})-f(\overline{X}_{\Gamma_n})\\&-f(\overline{X}_{\Gamma_{n+1}})+\mathcal{Q}_{\gamma_{n+1}}f(\overline{X}_{\Gamma_n})+\frac{\gamma_{n+1}}{2}(Af(\overline{X}_{\Gamma_{n+1}})-\mathcal{Q}_{\gamma_{n+1}}Af(\overline{X}_{\Gamma_n}))\\&+\frac{\gamma_{n+1}}{2}\tilde{\mathcal{R}}_1Af(\overline{X}_{\Gamma_n},\gamma_{n+1})-\tilde{\mathcal{R}}_2f(\overline{X}_{\Gamma_n},\gamma_{n+1}),\\=&f(\overline{X}_{\Gamma_{n+1}})-f(\overline{X}_{\Gamma_n})+B_2f(\overline{X}_{\Gamma_n},\overline{X}_{\Gamma_{n+1}},\gamma_{n+1})+\mathcal{R}_2f(\overline{X}_{\Gamma_n},\gamma_{n+1})\end{aligned} \tag{39}$$

with the notation from $\mathcal{E}_2(F,\tilde{A},A,\mathfrak{M}_2,\tilde{\eta}_2)$ (see (22)). To prove this relationship, we introduce the following decomposition

$$f(\overline{X}_{\Gamma_{n+1}})-f(\overline{X}_{\Gamma_n})=f(\overline{X}_{\Gamma_{n+1}})-\mathcal{Q}_{\gamma_{n+1}}f(\overline{X}_{\Gamma_n})+\mathcal{Q}_{\gamma_{n+1}}f(\overline{X}_{\Gamma_n})-f(\overline{X}_{\Gamma_n})$$

Still with notations from $\mathcal{E}_2(F,\tilde{A},A,\mathfrak{M}_2,\tilde{\eta}_2)$ (see (22)) (when $m=0$), we recall that

$$\mathcal{Q}_{\gamma_{n+1}}f(\overline{X}_{\Gamma_n})-f(\overline{X}_{\Gamma_n})=\sum_{i=1}^2\frac{\gamma_{n+1}^i}{i!}A^if(\overline{X}_{\Gamma_n})+\tilde{\mathcal{R}}_2f(\overline{X}_{\Gamma_n},\gamma_{n+1}),$$

which yields

$$\begin{aligned}\gamma_{n+1}Af(\overline{X}_{\Gamma_n})=&f(\overline{X}_{\Gamma_{n+1}})-f(\overline{X}_{\Gamma_n})+B_1f(\overline{X}_{\Gamma_n},\overline{X}_{\Gamma_{n+1}},\gamma_{n+1})\\&-\frac{\gamma_{n+1}^2}{2}A^2f(\overline{X}_{\Gamma_n})-\tilde{\mathcal{R}}_2f(\overline{X}_{\Gamma_n},\gamma_{n+1})\end{aligned}$$

Now, following the same approach, still using the notation from $\mathcal{E}_2(F,\tilde{A},A,\mathfrak{M}_2,\tilde{\eta}_2)$ (see (22)) (when f is replaced by Af and $m=1$), gives

$$\begin{aligned}\frac{\gamma_{n+1}^2}{2}A^2f(\overline{X}_{\Gamma_n}) =&\frac{\gamma_{n+1}}{2}(Af(\overline{X}_{\Gamma_{n+1}})-Af(\overline{X}_{\Gamma_n}))+\frac{\gamma_{n+1}}{2}B_1Af(\overline{X}_{\Gamma_n},\overline{X}_{\Gamma_{n+1}},\gamma_{n+1})\\ &-\frac{\gamma_{n+1}}{2}\tilde{\mathcal{R}}_1Af(\overline{X}_{\Gamma_n},\gamma_{n+1}).\end{aligned}$$

Injecting this decomposition in the expansion of $\gamma_{n+1}Af(\overline{X}_{\Gamma_n})$ and rearranging the terms gives (39). In a first step, we show that

$$\lim_{n\to\infty}\frac{1}{\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}\sum_{k=1}^n \mathfrak{Q}_{\gamma_k}f(\overline{X}_{\Gamma_{k-1}})-f(\overline{X}_{\Gamma_k})+\frac{\gamma_k}{2}(Af(\overline{X}_{\Gamma_k})-\mathfrak{Q}_{\gamma_k}Af(\overline{X}_{\Gamma_{k-1}}))\overset{law}{=}\mathcal{N}(0,\nu(\mathfrak{V}f)).$$

From Proposition 3.1, since (27) holds and $\lim_{n\in\mathbb{N}^*}\nu_n^{\epsilon_{\mathfrak{X}}}(\mathfrak{V}f,\omega)=\nu(\mathfrak{V}f)$, $\mathbb{P}-a.s.$, we have

$$\lim_{n\to\infty}\frac{1}{\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}\sum_{k=1}^n\mathfrak{X}_{f,k}\overset{law}{=}\mathcal{N}(0,\nu(\mathfrak{V}f))$$

Notice that when $\mathfrak{V}=0$ the $l.h.s.$ of the above equation is $\mathbb{P}-a.s.$ equal to zero for every $f\in F$. Now, to obtain the convergence in law, we are going to show that $\mathbb{P}-a.s$, for every $f\in F$,

$$.\quad\lim_{n\to+\infty}\frac{1}{\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}\sum_{k=1}^n B_2f(\overline{X}_{\Gamma_n},\overline{X}_{\Gamma_{n+1}},\gamma_{n+1})-\mathfrak{X}_{f,k}=0.$$

This last result is a consequence of Kronecker's Lemma as soon as we prove the $a.s.$ convergence of the martingale $(M_n)_{n\in\mathbb{N}^*}$ defined by $M_0:=0$ and

$$M_n:=\sum_{k=1}^n\frac{1}{\sqrt{H_{\epsilon_{\mathfrak{X}},k}}}B_2f(\overline{X}_{\Gamma_n},\overline{X}_{\Gamma_{n+1}},\gamma_{n+1})-\mathfrak{X}_{f,k}.$$

From the Chow's theorem (see Theorem 3.1), this $a.s.$ convergence is a direct consequence of the $a.s.$ finiteness of the series

$$\sum_{k=1}^n\mathbb{E}[|M_k-M_{k-1}|^\rho|\overline{X}_{\Gamma_{k-1}}],$$

which follows from $\mathcal{GC}_{Q,2}(F,g,\rho,\epsilon_{\mathfrak{X}},\epsilon_{\mathcal{GC}},\mathfrak{V})$ (see (11)) together with $\mathcal{SW}_{\mathcal{GC},\gamma}(g,\rho,\epsilon_{\mathfrak{X}},\epsilon_{\mathcal{GC}})$ (see (28))). To complete the proof, let us show now that

$$\mathbb{P}-a.s.\quad\forall f\in F\quad\lim_{n\to\infty}\frac{1}{H_{\tilde{\epsilon}_2,n}}\sum_{k=1}^n\mathcal{R}_2f(\overline{X}_{\Gamma_{k-1}},\gamma_k)=\nu(\mathfrak{M}_2f).$$

As a direct consequence of $\mathcal{E}_2(F,\tilde{A},A,\mathfrak{M}_2,\tilde{\eta}_2)$ (see (22)), since $\mathbb{P}-a.s.$, $\lim_{n\to\infty}\nu_n^{\tilde{\eta}_2}(\mathfrak{M}_2f)=\nu(\mathfrak{M}_2f)$ for every $f\in F$, we only have to prove that

$$\mathbb{P}-a.s.\quad\forall f\in F\quad\lim_{n\to\infty}\frac{1}{H_{\tilde{\epsilon}_2,n}}\sum_{k=1}^n\mathcal{R}_2f(\overline{X}_{\Gamma_{k-1}},\gamma_k)-\tilde{\eta}_{2,k}\mathfrak{M}_2f(\overline{X}_{\Gamma_{k-1}})=0,$$

which holds as soon as

$$\mathbb{P}-a.s. \quad \forall f \in F \quad \lim_{n\to\infty} \frac{1}{H_{\tilde{\eta}_2,n}} \sum_{k=1}^{n} \tilde{\eta}_{2,k} \Lambda_{f,2}(\overline{X}_{\Gamma_{k-1}}, \gamma_k) = 0, \tag{40}$$

We recall that we have the following decomposition

$$\forall f \in F, \forall x \in E, \forall \gamma \in [0, \overline{\gamma}], \qquad \Lambda_{f,2}(x,\gamma) = \langle g(x), \tilde{\mathbb{E}}[\tilde{\Lambda}_{f,2}(x,\gamma)] \rangle_{\mathbb{R}^l}$$

with $g : (E, \mathcal{B}(E)) \to \mathbb{R}_+^l$, $l \in \mathbb{N}^*$, a locally bounded Borel measurable function and $\tilde{\Lambda}_{f,2} : (E \times \mathbb{R}_+ \times \tilde{\Omega}, \mathcal{B}(E) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \tilde{\mathcal{G}}) \to \mathbb{R}_+^l$ a measurable function such that $\sup_{i \in \{1,\ldots,l\}, x \in E, \gamma \in (0,\overline{\gamma}]} \tilde{\mathbb{E}}[\tilde{\Lambda}_{f,2,i}(x,\gamma)] < +\infty$. Since for every $i \in \{1, \ldots, l\}$, $\sup_{n \in \mathbb{N}^*} \nu_n^{\tilde{\eta}_2}(g_i, \omega) < +\infty$, $\mathbb{P}(d\omega) - a.s.$, (40) follows from the following result:

Let $(\overline{x}_n)_{n\in\mathbb{N}} \in E^{\otimes \mathbb{N}}$. Assume that $\sup_{i \in \{1,\ldots,l\}} \sup_{n \in \mathbb{N}^*} \frac{1}{H_{\tilde{\eta}_2,n}} \sum_{k=1}^{n} \tilde{\eta}_{2,k} g_i(\overline{x}_{k-1}) < +\infty$, then, for every $f \in F$,

$$\lim_{n\to\infty} \frac{1}{H_{\tilde{\eta}_2,n}} \sum_{k=1}^{n} \tilde{\eta}_{2,k} \Lambda_{f,2}(\overline{x}_{k-1}, \gamma_k) = 0.$$

In order to obtain this result, we are going to show that, for every $f \in F$, every $i \in \{1, \ldots, l\}$, and every $(\overline{x}_n)_{n\in\mathbb{N}} \in E^{\otimes \mathbb{N}}$, then

$$\tilde{\mathbb{P}}(d\tilde{\omega}) - a.s. \quad \lim_{n\to\infty} \frac{1}{H_{\tilde{\eta}_2,n}} \sum_{k=1}^{n} \tilde{\eta}_{2,k} \tilde{\Lambda}_{f,2,i}(\overline{x}_{k-1}, \gamma_k, \tilde{\omega}) g_i(\overline{x}_{k-1}) = 0,$$

and the result will follow from the Dominated Convergence theorem since for every $n \in \mathbb{N}^*$,

$$\frac{1}{H_{\tilde{\eta}_2,n}} \sum_{k=1}^{n} \tilde{\eta}_{2,k} \tilde{\Lambda}_{f,2,i}(\overline{x}_{k-1}, \gamma_k, \omega) g_i(\overline{x}_{k-1}) \leqslant \sup_{x \in E} \sup_{\gamma \in (0,\overline{\gamma}]} \tilde{\Lambda}_{f,2,i}(x,\gamma,\tilde{\omega}) \sup_{n \in \mathbb{N}^*} \frac{1}{H_{\tilde{\eta}_2,n}} \sum_{k=1}^{n} \tilde{\eta}_{2,k} g_i(\overline{x}_{k-1}) < +\infty.$$

with $\tilde{\mathbb{E}}[\sup_{x\in E} \sup_{\gamma \in (0,\overline{\gamma}]} \tilde{\Lambda}_{f,2,i}(x,\gamma,\tilde{\omega})] < +\infty$ and $\sup_{n \in \mathbb{N}^*} \frac{1}{H_{\tilde{\eta}_2,n}} \sum_{k=1}^{n} \tilde{\eta}_{2,k} g_i(\overline{x}_{k-1}) < +\infty$. We fix $f \in F$, $i \in \{1, .., N\}$ and $(\overline{x}_n)_{N\in\mathbb{N}} \in E^{\otimes \mathbb{N}}$ and we assume that $\mathcal{E}_2(\tilde{A}, A, \mathfrak{M}_2, \tilde{\eta}_2)$ **I)** (see (24)) holds for $\tilde{\Lambda}_{f,i}$. If instead $\mathcal{E}_2(\tilde{A}, A, \mathfrak{M})$ **II)** (see (25)) is satisfied, the proof is similar but simpler so we leave it to the reader.

Let $\underline{n}(\tilde{\omega}) := \inf\{n \in \mathbb{N}^*, \sup_{k \geqslant n} \gamma_k \leqslant \underline{\gamma}(\tilde{\omega})\}$. By assumption $\mathcal{E}_2(F, \tilde{A}, A, \mathfrak{M}_2, \tilde{\eta}_2)$ **I)** (ii)(see (25)), $\tilde{\mathbb{P}}(d\tilde{\omega}) - a.s$, for every $R > 0$, there exists $K_R(\tilde{\omega}) \in \mathcal{K}_E$ such that

$$\sup_{x \in K_R^c(\tilde{\omega})} \sup_{\gamma \in (0,\underline{\gamma}(\tilde{\omega})]} \tilde{\Lambda}_{f,2,i}(x,\gamma,\tilde{\omega}) < 1/R.$$

Moreover,

$$\sup_{n \geqslant \underline{n}(\tilde{\omega})} \frac{1}{H_{\tilde{\eta}_2,n}} \sum_{k=\underline{n}(\tilde{\omega})}^{n} \tilde{\eta}_{2,k} \tilde{\Lambda}_{f,2,i}(\overline{x}_{k-1}, \gamma_k, \tilde{\omega}) g(\overline{x}_{k-1}) \mathbb{1}_{K_R^c(\tilde{\omega})}(\overline{x}_{k-1})$$

$$\leqslant \sup_{x \in K_R^c(\tilde{\omega})} \sup_{\gamma \in (0,\underline{\gamma}(\tilde{\omega})]} \tilde{\Lambda}_{f,2,i}(x,\gamma,\tilde{\omega}) \sup_{n \in \mathbb{N}^*} \frac{1}{H_{\tilde{\eta}_2,n}} \sum_{k=1}^{n} \tilde{\eta}_{2,k} g_i(\overline{x}_{k-1}).$$

We let R tends to infinity and since $\sup_{n\in\mathbb{N}^*}\frac{1}{H_{\tilde{\eta}_2,n}}\sum_{k=1}^{n}\tilde{\eta}_{2,k}g_i(\overline{x}_{k-1})<+\infty$, the *l.h.s.* of the above equation converges $\tilde{\mathbb{P}}(d\tilde{\omega})-a.s.$ to 0. Finally, since $\underline{n}(\tilde{\omega})$ is $\tilde{\mathbb{P}}(d\tilde{\omega})-a.s.$ finite, we also have

$$\tilde{\mathbb{P}}(d\tilde{\omega})-a.s.\quad \forall R>0,\quad \lim_{n\to\infty}\frac{1}{H_{\tilde{\eta}_2,n}}\sum_{k=1}^{\underline{n}(\tilde{\omega})-1}\tilde{\eta}_{2,k}\tilde{\Lambda}_{f,2,i}(\overline{x}_{k-1},\gamma_k,\tilde{\omega})g(\overline{x}_{k-1})\mathbb{1}_{K^c_R(\tilde{\omega})}(\overline{x}_{k-1})=0.$$

Moreover, from $\mathcal{E}_2(F,\tilde{A},A,\mathfrak{M}_2,\tilde{\eta}_2)$ **I)** (i)(see (24)), we derive that, $\tilde{\mathbb{P}}(d\tilde{\omega})-a.s.$, for every $R>0$, $\lim_{n\to\infty}\tilde{\Lambda}_{f,2,i}(\overline{x}_{n-1},\gamma_n,\tilde{\omega})\mathbb{1}_{K_R(\tilde{\omega})}(\overline{x}_{k-1})=0$, Then, since g_i is a locally bounded function, as an immediate consequence of the Cesaro's lemma, we obtain

$$\tilde{\mathbb{P}}(d\tilde{\omega})\quad \forall R>0,\quad \lim_{n\to\infty}\frac{1}{H_{\tilde{\eta}_2,n}}\sum_{k=1}^{n}\tilde{\eta}_{2,k}\tilde{\Lambda}_{f,2,i}(\overline{x}_{k-1},\gamma_k,\tilde{\omega})g_i(\overline{x}_{k-1})\mathbb{1}_{K_R(\tilde{\omega})}(\overline{x}_{k-1})=0$$

Applying the same approach for every $i\in\{1,\dots,q\}$, the Dominated Convergence Theorem yields:

$$\forall(\overline{x}_n)_{n\in\mathbb{N}}\in E^{\otimes\mathbb{N}},\forall f\in F,\qquad \lim_{n\to\infty}\frac{1}{H_{\tilde{\eta}_2,n}}\sum_{k=1}^{n}\Lambda_{f,2}(\overline{x}_{k-1},\gamma_k)=0.$$

Finally, since for every $i\in\{1,..,q\}$, $\sup_{n\in\mathbb{N}^*}\nu_n^{\tilde{\eta}_2}(g_i,\omega)<+\infty$, $\mathbb{P}-a.s.$, then (40) follows. We gather all the terms together and the proof is completed.

□

3.2.3 The third order FCLT

Using higher order weak approximation $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ of a Feller process $(X_t)_{t\geqslant 0}$, it is possible to derive a third order FCLT. As for the second order case, the crucial point to obtain this result is to consider a specific weight sequence which is, in this case, inspired from the standard Simpson's approximation of the integral.

Theorem 3.4. *Let $F\subset\{f,f:(E,\mathcal{B}(E))\to(\mathbb{R},\mathcal{B}(\mathbb{R})),Af\in\mathcal{C}_b(E)\}$, $g:E\to\mathbb{R}_+$ a Borel function, $\epsilon_{\mathfrak{X}},\epsilon_{\mathcal{GC}}:\mathbb{R}_+\to\mathbb{R}_+$ be two increasing functions, $\tilde{\eta}_3:\mathbb{R}_+\times\{0,1\}\to\mathbb{R}$, and let $\mathfrak{M}_3$ and $\mathfrak{V}$ be two linear operators defined on F.*

Finally, let $\eta_{2n+1}:=C_{\gamma,\eta}(\gamma_{2n+1}+\gamma_{2n-1})/3$, $\eta_{2n+2}:=4C_{\gamma,\eta}\gamma_{2n+1}/3, C_{\gamma,\eta}>0$, $n\in\mathbb{N}$, be the weight sequence and suppose that the step sequence satisfies $\gamma_{2n+1}=\gamma_{2n+2}$ for every $n\in\mathbb{N}$.

Assume that $\mathcal{E}_3(F,\tilde{A},A,\mathfrak{M}_3,\tilde{\eta}_3)$ (see (22)), $\mathcal{GC}_{Q,3}(F,g,\rho,\epsilon_{\mathfrak{X}},\epsilon_{\mathcal{GC}},\mathfrak{V})$ (see (11) and (30)) and $\mathcal{SW}_{\mathcal{GC},\gamma}(g,\rho,\epsilon_{\mathfrak{X}},\epsilon_{\mathcal{GC}})$ (see (28)) hold.

Then, for every $f\in F$, we have the following properties:

A. *If $\lim_{n\to\infty}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}/H_{\tilde{\eta}_3,n}=+\infty$, then*

$$\lim_{n\to\infty}\frac{H_n}{C_{\gamma,\eta}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}\nu_n^{\eta}(Af)\overset{law}{=}\mathcal{N}(0,\nu(\mathfrak{V}f)).\tag{41}$$

B. *If $\lim_{n\to\infty}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}/H_{\tilde{\eta}_3,n}=\hat{l}\in\mathbb{R}_+^*$, then*

$$\lim_{n\to\infty}\frac{H_n}{C_{\gamma,\eta}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}\nu_n^{\eta}(Af)\overset{law}{=}\mathcal{N}(\hat{l}^{-1}\nu(\mathfrak{M}_3f),\nu(\mathfrak{V}f)).\tag{42}$$

C. *If* $\lim_{n\to\infty}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}/H_{\tilde{\eta}_3,n}=0$*, then*

$$\lim_{n\to\infty}\frac{H_n}{C_{\gamma,\eta}H_{\tilde{\eta}_3,n}}\nu_n^\eta(Af)\overset{\mathbb{P}}{=}\nu(\mathfrak{M}_3f) \tag{43}$$

Moreover, when $\mathfrak{V}=0$ *this convergence is almost sure.*

Proof. Let $n\in\mathbb{N}$. We begin by noticing that the following decomposition holds

$$\begin{aligned}\nu_n^\eta(Af)=&\frac{1}{H_n}\sum_{k=1}^n\eta_kAf(\overline{X}_{\Gamma_{k-1}})\\ =&\frac{C_{\gamma,\eta}}{H_n}\sum_{k=0}^{\lfloor(n-1)/2\rfloor-1}\frac{\gamma_{2k+1}}{3}Af(\overline{X}_{\Gamma_{2k}})+\frac{4\gamma_{2k+1}}{3}Af(\overline{X}_{\Gamma_{2k+1}})+\frac{\gamma_{2k+1}}{3}Af(\overline{X}_{\Gamma_{2k+2}})\\ &+\frac{C_{\gamma,\eta}(\eta_{2\lfloor(n-1)/2\rfloor+1}-\gamma_{2\lfloor(n-1)/2\rfloor+1}/3)}{H_n}Af(\overline{X}_{\Gamma_{2\lfloor(n-1)/2\rfloor}})\\ &+(1-n\mod 2)\frac{C_{\gamma,\eta}\eta_{2\lfloor(n-1)/2\rfloor+2}}{H_n}Af(\overline{X}_{\Gamma_{2\lfloor(n-1)/2\rfloor+1}})\end{aligned}$$

Since Af is a bounded function, the second and third terms of the *r.h.s.* of the above equation mulyiplied by $\frac{H_n}{C_{\gamma,\eta}\sqrt{H_{\epsilon_{\mathfrak{X}},n}}}$ or $\frac{H_n}{C_{\gamma,\eta}H_{\tilde{\eta}_3,n}}$ converge to zero. We study the first term of the *r.h.s.* of the above equation. The crucial point of the proof is too show that, for every $n\in\mathbb{N}$, we have

$$\begin{aligned}&\frac{\gamma_{2n+1}}{3}Af(\overline{X}_{\Gamma_{2n}})+\frac{4\gamma_{2n+1}}{3}Af(\overline{X}_{\Gamma_{2n+1}}))\frac{\gamma_{2n+1}}{3}Af(\overline{X}_{\Gamma_{2n+2}})\\ &=f(\overline{X}_{\Gamma_{2n+1}})-f(\overline{X}_{\Gamma_{2n}})+B_3f(\overline{X}_{\Gamma_{2n}},\overline{X}_{\Gamma_{2n+1}},\gamma_{2n+1},0)+\mathcal{R}_3f(\overline{X}_{\Gamma_{2n}},\gamma_{2n+1},0)\\ &\quad+f(\overline{X}_{\Gamma_{2n+2}})-f(\overline{X}_{\Gamma_{2n+1}})+B_3f(\overline{X}_{\Gamma_{2n+1}},\overline{X}_{\Gamma_{2n+2}},\gamma_{2n+2},1)+\mathcal{R}_3f(\overline{X}_{\Gamma_{2n+1}},\gamma_{2n+2},1)\end{aligned} \tag{44}$$

with the notation from $\mathcal{E}_3(F,\tilde{A},A,\mathfrak{M}_3,\tilde{\eta}_3)$ (see (22)). Notice that once this expansion is obtained, the rest of the proof follows similar arguments as for the second order case and won't be detailed in this one.

Similarly as for the second order, we first introduce the following decomposition

$$f(\overline{X}_{\Gamma_{n+1}})-f(\overline{X}_{\Gamma_n})=f(\overline{X}_{\Gamma_{n+1}})-\mathcal{Q}_{\gamma_{n+1}}f(\overline{X}_{\Gamma_n})+\mathcal{Q}_{\gamma_{n+1}}f(\overline{X}_{\Gamma_n})-f(\overline{X}_{\Gamma_n})$$

Still with notations from $\mathcal{E}_3(F,\tilde{A},A,\mathfrak{M}_3,\tilde{\eta}_3)$ (see (22)), we notice that, for $k\in\mathbb{N}$,

$$\mathcal{Q}_{\gamma_{k+1}}f(\overline{X}_{\Gamma_k})-f(\overline{X}_{\Gamma_k})=\sum_{i=1}^3\frac{\gamma_{k+1}^i}{i!}A^if(\overline{X}_{\Gamma_k})+\tilde{\mathcal{R}}_3f(\overline{X}_{\Gamma_k},\gamma_{k+1}),$$

Following the same reasonning as for the second order case, we apply the above expansion to A^2 with remainder of order one and obtain

$$\begin{aligned}\frac{\gamma_{k+1}^3}{6}A^3f(\overline{X}_{\Gamma_k})=&\frac{\gamma_{k+1}^2}{6}(A^2f(\overline{X}_{\Gamma_{k+1}})-A^2f(\overline{X}_{\Gamma_k}))+\frac{\gamma_{k+1}^2}{6}B_1A^2f(\overline{X}_{\Gamma_k},\overline{X}_{\Gamma_{k+1}},\gamma_{k+1})\\ &-\frac{\gamma_{k+1}^2}{6}\tilde{\mathcal{R}}_1A^2f(\overline{X}_{\Gamma_k},\gamma_{k+1}).\end{aligned}$$

Gathering the two expansions above yields

$$
\begin{aligned}
\gamma_{k+1} Af(\overline{X}_{\Gamma_k}) =& f(\overline{X}_{\Gamma_{k+1}}) - f(\overline{X}_{\Gamma_k}) \\
&= B_1(\overline{X}_{\Gamma_k}, \overline{X}_{\Gamma_{k+1}}, \gamma_{k+1}) - \frac{\gamma_{k+1}^2}{6} B_1 A^2 f(\overline{X}_{\Gamma_k}, \overline{X}_{\Gamma_{k+1}}, \gamma_{k+1}) \\
&- \frac{\gamma_{k+1}^2}{6} A^2 f(\overline{X}_{\Gamma_{k+1}}) - \frac{\gamma_{k+1}^2}{3} A^2 f(\overline{X}_{\Gamma_k}) + \tilde{\mathcal{R}}_1 A^2 f(\overline{X}_{\Gamma_k}, \gamma_{k+1}) - \tilde{\mathcal{R}}_3 f(\overline{X}_{\Gamma_k}, \gamma_{k+1})
\end{aligned}
$$

This last expansion being valid for every $k \in \mathbb{N}$, we apply it to $k = 2n$ and $k = 2n+1$ to obtain (since $\gamma_{2n+1} = \gamma_{2n+2}$)

$$
\begin{aligned}
\gamma_{2n+1}(Af(\overline{X}_{\Gamma_{2n}}) + Af(\overline{X}_{\Gamma_{2n+1}})) =& f(\overline{X}_{\Gamma_{2n+1}}) - f(\overline{X}_{\Gamma_{2n}}) + f(\overline{X}_{\Gamma_{2n+2}}) - f(\overline{X}_{\Gamma_{2n+1}}) \\
&+ B_1(\overline{X}_{\Gamma_{2n}}, \overline{X}_{\Gamma_{2n+1}}, \gamma_{2n+1}) - \frac{\gamma_{2n+1}^2}{6} B_1 A^2 f(\overline{X}_{\Gamma_{2n}}, \overline{X}_{\Gamma_{2n+1}}, \gamma_{2n+1}) \\
&+ B_1(\overline{X}_{\Gamma_{2n+1}}, \overline{X}_{\Gamma_{2n+2}}, \gamma_{2n+2}) - \frac{\gamma_{2n+2}^2}{6} B_1 A^2 f(\overline{X}_{\Gamma_{2n+1}}, \overline{X}_{\Gamma_{2n+2}}, \gamma_{2n+2}) \\
&- \frac{\gamma_{2n+1}^2}{3} (A^2 f(\overline{X}_{\Gamma_{2n+1}}) + A^2 f(\overline{X}_{\Gamma_{2n}})) \\
&- \frac{\gamma_{2n+1}^2}{6} (A^2 f(\overline{X}_{\Gamma_{2n+2}}) + A^2 f(\overline{X}_{\Gamma_{2n+1}})) \\
&+ \tilde{\mathcal{R}}_1 A^2 f(\overline{X}_{\Gamma_{2n}}, \gamma_{2n+1}) - \tilde{\mathcal{R}}_3 f(\overline{X}_{\Gamma_{2n}}, \gamma_{2n+1}) \\
&+ \tilde{\mathcal{R}}_1 A^2 f(\overline{X}_{\Gamma_{2n+1}}, \gamma_{2n+2}) - \tilde{\mathcal{R}}_3 f(\overline{X}_{\Gamma_{2n+1}}, \gamma_{2n+2}).
\end{aligned}
$$

In order to obtain (44), we now apply (39) with f replaced by Af and n replaced by $2n$ and by $2n + 1$.

□

3.3 Example - The Euler scheme

Using this abstract approach, we recover the results obtained in [10] or [20] concerning the study of the Euler scheme of a d-dimensional Brownian diffusion under weakly mean reverting properties. We consider a N-dimensional Brownian motion $(W_t)_{t \geqslant 0}$. We are interested in the strong solution - assumed to exist and to be unique - of the d-dimensional stochastic equation

$$
X_t = x + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dW_s \tag{45}
$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times N}$. Let $V : \mathbb{R} \to [1, +\infty)$, the Lyapunov function of this system such that L_V (see (5)) holds with $E = \mathbb{R}^d$, and

$$
|\nabla V|^2 \leqslant C_V V, \qquad \|D^2 V\|_\infty < +\infty.
$$

Moreover, we assume that for every $x \in \mathbb{R}$, $|b(x)|^2 + \mathrm{Tr}[\sigma\sigma^*(x)] \leqslant V^a(x)$ for some $a \in (0, 1]$. Finally, for $p \geqslant 1$, we introduce the following L_p-mean reverting property of V,

$$
\begin{aligned}
&\exists \alpha > 0, \beta \in \mathbb{R}, \forall x \in \mathbb{R}, \\
&\qquad \langle \nabla V(x), b(x) \rangle + \frac{1}{2} \|\lambda_p\|_\infty 2^{(2p-3)_+} \mathrm{Tr}[\sigma\sigma^*(x)] \leqslant \beta - \alpha V^a(x)
\end{aligned}
$$

with for every $x \in \mathbb{R}^d$, $\lambda_p(x) := \sup\{\lambda_{p,1}(x), \dots, \lambda_{p,d}(x), 0\}$, with $\lambda_{p,i}(x)$ the i-th eigenvalue of the matrix $D^2V(x) + 2(p-1)\nabla V(x)^{\otimes 2}/V(x)$. We now introduce the Euler scheme of $(X_t)_{t\geqslant 0}$. Let $\rho \in [1,2]$ and $\epsilon_{\mathcal{I}}(\gamma) = \gamma^{\rho/2}$ and assume that (16), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\epsilon_{\mathcal{I}})$ (see (20)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)) hold. Let $(U_n)_n$ be a sequence of $\mathbb{R}^N$-valued centered independent and identically distributed random variables with covariance identity and bounded moments of order $2p$. We define the Euler scheme with decreasing steps $(\gamma_n)_{n\in\mathbb{N}^*}$, $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ of $(X_t)_{t\geqslant 0}$ (45) on the time grid $\{\Gamma_n = \sum_{k=1}^n \gamma_k, n \in \mathbb{N}\}$ by

$$\forall n \in \mathbb{N}, \qquad \overline{X}_{\Gamma_{n+1}} = \overline{X}_{\Gamma_n} + \gamma_{n+1} b(\overline{X}_{\Gamma_n}) + \sqrt{\gamma_{n+1}}\sigma(\overline{X}_{\Gamma_n})U_{n+1}, \quad \overline{X}_0 = x.$$

We consider $(\nu_n^\eta(dx,\omega))_{n\in\mathbb{N}^*}$ defined as in (4) with $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ defined above. Now,we specify the measurable functions $\psi,\phi : [1,+\infty) \to [1,+\infty)$ as $\psi_p(y) = y^p$ and $\phi(y) = y^a$. Moreover, let $s \geqslant 1$ such that $ap\rho/s \leqslant p + a - 1$, $p/s + a - 1 > 0$ and $\mathrm{Tr}[\sigma\sigma^*] \leqslant CV^{p/s+a-1}$. Then, it follows from Theorem 2.2 that there exists an invariant distribution ν for $(X_t)_{t\geqslant 0}$. Moreover, $(\nu_n^\eta(dx,\omega))_{n\in\mathbb{N}^*}$ *a.s.* weakly converges toward $\mathcal{V}$, the set of invariant distributions of $(X_t)_{t\geqslant 0}$ and when it is unique *i.e.* $\mathcal{V} = \{\nu\}$, we have

$$\mathbb{P} - a.s. \qquad \lim_{n\to+\infty} \nu_n^\eta(f) = \nu(f),$$

for every $\nu - a.s.$ continuous function $f \in \mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d)$ defined in (7).

In addition to that $\mathbb{P} - a.s.$ Wasserstein converge result we can also establish a first order CLT. Let $\tilde{\rho}_1 \in [1,2]$, let $C_{\gamma,\eta} > 0$ and let us define $\eta_{1,n} = C_{\gamma,\eta}\gamma_n$, $n \in \mathbb{N}^*$ and

$$F_1 = \{f \in \mathcal{C}^4(\mathbb{R}^d;\mathbb{R}), \forall l \in \{2,\dots,4\}, D^l f \in \mathcal{C}_0(\mathbb{R}^d;\mathbb{R})\},$$

and the linear operator $\mathfrak{M}_1$ defined on $\mathcal{C}^4(\mathbb{R}^d;\mathbb{R})$ such that for every $f \in \mathcal{C}^4(\mathbb{R}^d;\mathbb{R})$,

$$\mathfrak{M}_1 f(x) = -\frac{1}{2}\big(D^2 f(x); b(x)^{\otimes 2}\big) - \mathbb{E}\Big[\frac{1}{2}\big(D^3 f(x); (\sigma(x)U)^{\otimes 2} \otimes b(x)\big) + \frac{1}{4!}\big(D^4 f(x); (\sigma(x)U)^{\otimes 4}\big)\Big].$$

Let $\tilde{\eta}_1(\gamma) = \gamma^2$. Assume that (3), (16), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\epsilon_{\mathcal{I}})$ (see (20)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)) hold with η replaced by $\tilde{\eta}_1$ and by γ. Assume that the sequence $(U_n)_{n\in\mathbb{N}^*}$ satisfies $M_{\mathcal{N},3}(U)$ (see (51))) and $M_2(U)$ (see (52)) and that $\mathcal{SW}_{\mathcal{GC},\gamma}(\tilde{\rho}_1,\gamma,\gamma)$ (see (29)) holds.
Also assume that $g_{\sigma,1} \leqslant CV^{p/s+a-1}$, with $g_{\sigma,1} = \mathrm{Tr}[\sigma\sigma^*]^4 + |b|^2$, that $\mathrm{Tr}[\sigma\sigma^*] = o_{|x|\to+\infty}(V^{p/s+a-1})$ and that ν is unique. Finally assume that for every $f \in F_1$, $|\sigma^* Df|^2 \in \mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d)$ and $\mathfrak{M}_1 f \in \mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d)$.

Then, for every $f \in F_1$,

i. If $\lim_{n\to\infty} \sqrt{\Gamma_n}/H_{\tilde{\eta}_1,n} = +\infty$,

$$\lim_{n\to\infty} \sqrt{\Gamma_n}\nu_n^{\eta_1}(Af) \overset{law}{=} \mathcal{N}(0, \nu(|\sigma^* Df|^2)). \tag{46}$$

ii. If $\lim_{n\to\infty} \sqrt{\Gamma_n}/H_{\tilde{\eta}_1,n} = \hat{l} \in \mathbb{R}_+^*$,

$$\lim_{n\to\infty} \sqrt{\Gamma_n}\nu_n^{\eta_1}(Af) \overset{law}{=} \mathcal{N}(\hat{l}^{-1}\nu(\mathfrak{M}_1 f), \nu(|\sigma^* Df|^2)). \tag{47}$$

iii. If $\lim_{n\to\infty} \sqrt{\Gamma_n}/H_{\tilde{\eta}_1,n} = 0$,

$$\lim_{n\to\infty} \frac{H_n}{H_{\tilde{\eta}_1,n}}\nu_n^{\eta_1}(Af) \overset{\mathbb{P}}{=} \nu(\mathfrak{M}_1 f). \tag{48}$$

This result was initially obtained in [10] but under strongly mean reverting assumption *i.e.* $a = 1$. The extension of this result to the weak mean reverting setting was developed in [20].

Remark 3.3. *Notice that if we take $\gamma_n = 1/n^{\xi}$, $\xi \in (0,1/2)$ and $\eta = \gamma$, the mentioned step weight assumptions are satisfied (take $\rho \in (1/(1-\xi),2]$ and $\tilde{\rho}_1 \in (2/(1+\xi),2]$). Then, if we define by*

$$\forall n \in \mathbb{N}^*, \qquad \mathfrak{r}_n = \begin{cases} \sqrt{\Gamma_n} & \text{if} \quad \lim_{n\to\infty} \sqrt{\Gamma_n}/H_{\tilde{\eta}_1,n} = +\infty, \\ \sqrt{\Gamma_n} & \text{if} \quad \lim_{n\to\infty} \sqrt{\Gamma_n}/H_{\tilde{\eta}_1,n} = \hat{l}, \\ \frac{H_n}{H_{\tilde{\eta}_1,n}} & \text{if} \quad \lim_{n\to\infty} \sqrt{\Gamma_n}/H_{\tilde{\eta}_1,n} = 0, \end{cases}$$

the rate of convergence of $(\nu_n^{\eta_1}(Af))_{n\in\mathbb{N}^*}$, we have

$$\mathfrak{r}_n \underset{n\to+\infty}{\sim} Cn^{\xi\wedge(1/2-\xi/2)}.$$

The highest rate of convergence is thus achieved for $\xi = 1/3$ and is given by $\mathfrak{r}_n \underset{n\to+\infty}{\sim} Cn^{1/3}$.

4 Application - The Talay second weak order scheme

Notations.

In the sequel we will use the following notations. First, for $\alpha \in (0,1]$ and f an α-Hölder function we denote $[f]_\alpha = \sup_{x\neq y} |f(y)-f(x)|/|y-x|^\alpha$.
Now, let $d \in \mathbb{N}$. For any $\mathbb{R}^{d\times d}$-valued symmetric matrix S, we define $\lambda_S := \sup\{\lambda_{S,1},\ldots,\lambda_{S,d},0\}$, with $\lambda_{S,i}$ the i-th eigenvalue of S.

Presentation of the main result.

In this section we study the second order convergence of the weighted empirical measures of a scheme designed in [24] and adapted to the case of decreasing time steps. We consider a N-dimensional Brownian motion $(W_t)_{t\geqslant 0}$. We are interested in the solution - assumed to exist and to be unique - of the d-dimensional stochastic equation

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s, \tag{49}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times N}$, are locally bounded functions. The infinitesimal generator of this process is given by

$$Af(x) = \langle b(x), \nabla f(x)\rangle + \frac{1}{2}\sum_{i,j=1}^{d} (\sigma\sigma^*)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) \tag{50}$$

and its domain $\mathcal{D}(A)$ contains $\mathcal{D}(A)_0 = \mathcal{C}^2_K(\mathbb{R}^d)$. Notice that $\mathcal{D}(A)_0$ is dense in $\mathcal{C}_0(E)$. Now, we present the Talay scheme, introduced in [24], of $(X_t)_{t\geqslant 0}$ adapted to the case of decreasing time steps. First, we introduce the random variables that are used to build this scheme. Let $q \in \mathbb{N}^*$, $p \geqslant 0$. Now let $(U_n)_{n\in\mathbb{N}^*}$ be a sequence of $\mathbb{R}^N$-valued independent and identically distributed random variables such that

$$M_{\mathcal{N},q}(U) \quad \equiv \quad \forall n \in \mathbb{N}^*, \forall \tilde{q} \in \{1,\ldots,q\}, \quad \mathbb{E}[(U_n)^{\otimes\tilde{q}}] = \mathbb{E}[(\mathcal{N}(0,I_d))^{\otimes\tilde{q}}], \tag{51}$$

and

$$M_p(U) \qquad \sup_{n\in\mathbb{N}^*} \mathbb{E}[|U_n|^{2p}] < +\infty. \tag{52}$$

Morever, let $(\kappa_n)_{n\in\mathbb{N}^*}$ be a sequence of $\mathbb{R}^{N\times N}$-valued independent and identically distributed random variables such that for every $n \in \mathbb{N}^*$, κ_n is made of $N \times N$ independent components and for every $(i,j) \in \{1,\dots,N\}^2$, $\mathbb{P}(\kappa_n^{i,j} = -1/2) = \mathbb{P}(\kappa_n^{i,j} = 1/2) = 1/2$. At this point we define the sequence $(\mathcal{W}_n)_{n\in\mathbb{N}^*}$ of $\mathbb{R}^{N\times N}$-valued random variables such that for every $n \in \mathbb{N}^*$,

$$\forall i,j \in \{1,\dots,N\}, \qquad \mathcal{W}_n^{i,i} = |U_n^i|^2 - 1 \quad \text{and} \quad \mathcal{W}_n^{i,j} = U_n^i U_n^j - \kappa_n^{i\wedge j, i\vee j} \quad \text{for } i \neq j. \tag{53}$$

For every $n \in \mathbb{N}$, the Talay scheme with decreasing steps is defined by

$$\begin{aligned} \overline{X}_{\Gamma_{n+1}} =& \overline{X}_{\Gamma_n} + \sqrt{\gamma_{n+1}}\sigma(\overline{X}_{\Gamma_n})U_{n+1} + \gamma_{n+1}\Big(b(\overline{X}_{\Gamma_n}) + (D\sigma(\overline{X}_{\Gamma_n}); \sigma(\overline{X}_{\Gamma_n})\mathcal{W}_{n+1}^*)\Big) \\ &+ \gamma_{n+1}^{3/2}\tilde{\sigma}(\overline{X}_{\Gamma_n})U_{n+1} + \gamma_{n+1}^2 Ab(\overline{X}_{\Gamma_n}), \end{aligned} \tag{54}$$

with, for every $i \in \{1,\dots,N\}$, and $j \in \{1,\dots,d\}$, $\tilde{\sigma}_{j,i} = (\tilde{\sigma}_i)_j$ where

$$\begin{aligned} \tilde{\sigma}_i \ : \mathbb{R}^d &\to \mathbb{R}^d \\ x &\mapsto \sum_{l=1}^d \Big(\partial_{x_l} b(x)\sigma_{l,i}(x) + \partial_{x_l}\sigma_{l,i}(x)b(x) + \sum_{j=1}^d (\sigma\sigma^*)_{l,j}(x)\frac{\partial^2 \sigma_i}{\partial x_l \partial x_j}(x)\Big). \end{aligned}$$

with, for every $i \in \{1,\dots,N\}$, $\sigma_i : \mathbb{R}^d \to \mathbb{R}^d, x \mapsto \sigma_i(x) = (\sigma_{1,i}(x),\dots,\sigma_{d,i}(x))$
We will also denote $\Delta\overline{X}_{n+1} = \overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}$ and

$$\Delta\overline{X}^1_{n+1} = \gamma_{n+1}^{1/2}\sigma(\overline{X}_{\Gamma_n})U_{n+1} = \gamma_{n+1}^{1/2}\sum_{i=1}^N \sigma_i(\overline{X}_{\Gamma_n})U^i_{n+1}, \quad \Delta\overline{X}^2_{n+1} = \gamma_{n+1}b(\overline{X}_{\Gamma_n}), \tag{55}$$

$$\Delta\overline{X}^3_{n+1} = (D\sigma(\overline{X}_{\Gamma_n}); \sigma(\overline{X}_{\Gamma_n})\mathcal{W}^*_{n+1}) = \gamma_{n+1}\sum_{i,j=1}^N\sum_{l=1}^d \partial_{x_l}\sigma_i(\overline{X}_{\Gamma_n})\sigma_{l,j}(\overline{X}_{\Gamma_n})\mathcal{W}^{i,j}_{n+1},$$

$$\Delta\overline{X}^4_{n+1} = \gamma_{n+1}^{3/2}\tilde{\sigma}(\overline{X}_{\Gamma_n})U_{n+1} = \gamma_{n+1}^{3/2}\sum_{i=1}^N \tilde{\sigma}_i(\overline{X}_{\Gamma_n})U^i_{n+1}$$

$$\Delta\overline{X}^5_{n+1} = \gamma_{n+1}^2 Ab(\overline{X}_{\Gamma_n})$$

and $\overline{X}^i_{\Gamma_{n+1}} = \overline{X}_{\Gamma_n} + \sum_{j=1}^i \Delta\overline{X}^i_{n+1}$. Now, we assume the existence of a Lyapunov function $V : \mathbb{R}^d \to [v_*,\infty)$, $v_* > 0$, satisfying L_V (see (5)) and which is essentially quadratic:

$$|\nabla V|^2 \leqslant C_V V, \qquad \sup_{x\in\mathbb{R}^d}|D^2V(x)| < +\infty \tag{56}$$

It remains to introudce the mean-reverting property of V. We define

$$\forall x \in \mathbb{R}^d, \quad \lambda_\psi(x) := \lambda_{D^2V(x)+2\nabla V(x)^{\otimes 2}\psi''\circ V(x)\psi'\circ V(x)^{-1}}. \tag{57}$$

When $\psi(y) = \psi_p(y) = y^p$, we will also use the notation λ_p instead of λ_ψ. Now, let $\phi : [v_*,+\infty) \to \mathbb{R}_+$, and assume that for every $x \in \mathbb{R}^d$,

$$\mathfrak{B}(\phi) \quad \equiv \quad |b(x)|^2 + \mathrm{Tr}[\sigma\sigma^*(x)] + |D\sigma(x)|^2\,\mathrm{Tr}[\sigma\sigma^*(x)] + |\tilde{\sigma}(x)|^2 + |Ab(x)|^2 \leqslant C\phi\circ V(x). \tag{58}$$

We are now able to introduce the L_p mean-reverting property of V. Let $p \geqslant 0$. Let $\beta \in \mathbb{R}$, $\alpha > 0$. We assume that $\liminf_{y\to\infty} \phi(y) > \beta/\alpha$ and

$$\mathcal{R}_p(\alpha,\beta,\phi,V) \quad \equiv \quad \forall x \in \mathbb{R}^d, \quad \langle \nabla V(x), b(x)\rangle + \frac{1}{2}\chi_p(x) \leqslant \beta - \alpha\phi\circ V(x), \tag{59}$$

with

$$\chi_p(x) = \begin{cases} \|\lambda_1\|_\infty \mathrm{Tr}[\sigma\sigma^*(x)] & \text{if } p \leqslant 1 \\ \|\lambda_p\|_\infty 2^{(2p-3)_+}\mathrm{Tr}[\sigma\sigma^*(x)] & \text{if } p > 1. \end{cases} \tag{60}$$

Finally we consider the linear operator $\mathfrak{M}_1$ defined on $\mathcal{C}^4(\mathbb{R}^d;\mathbb{R})$ such that for every $f \in \mathcal{C}^4(\mathbb{R}^d;\mathbb{R})$,

$$\begin{aligned}\mathfrak{M}_1 f(x) = &- \big(Df(x); Ab(x)\big) \\ &- \mathbb{E}\Big[\frac{1}{2}\big(D^2 f(x); b(x)^{\otimes 2} + 2b(x)\otimes(D\sigma(x);\sigma(x)W^*) + (D\sigma(x);\sigma(x)W^*)^{\otimes 2}\big) \\ &+ \frac{1}{2}\big(D^3 f(x); (\sigma(x)U)^{\otimes 2}\otimes(b(x) + (D\sigma(x);\sigma(x)W^*)) + (\sigma(x)U)\otimes(\tilde{\sigma}(x)U)\big) \\ &+ \frac{1}{4!}\big(D^4 f(x); (\sigma(x)U)^{\otimes 4}\big)\Big].\end{aligned} \tag{61}$$

We also consider the linear operator $\mathfrak{M}_2$ defined on $\mathcal{C}^6(\mathbb{R}^d;\mathbb{R})$ such that for every $f \in \mathcal{C}^6(\mathbb{R}^d;\mathbb{R})$, $\mathfrak{M}_2 f = \tilde{\mathfrak{M}}_2 f - \frac{1}{2}\mathfrak{M}_1 A f$ with

$$\begin{aligned}\tilde{\mathfrak{M}}_2 f(x) = &- \mathbb{E}\Big[\big(D^2 f(x); \frac{1}{2}(\tilde{\sigma}(x)U)^{\otimes 2}) + b(x)\otimes Ab(x)\big) \\ &+ \frac{1}{2}\big(D^3 f(x); \frac{1}{3}(D\sigma(x);\sigma(x)W^*)^{\otimes 3} + b(x)^{\otimes 2}\otimes(D\sigma(x);\sigma(x)W^*) + (\sigma(x)U)^{\otimes 2}\otimes Ab(x) \\ &\qquad + (\sigma(x)U)\otimes(b(x) + (D\sigma(x);\sigma(x)W^*))\otimes(\tilde{\sigma}(x)U) + \frac{1}{3}b(x)^{\otimes 3}\big) \\ &+ \frac{1}{2}\big(D^4 f(x); \frac{1}{2}(\sigma(x)U)^{\otimes 2}\otimes(b(x)^{\otimes 2} + 2b(x)\otimes(D\sigma(x);\sigma(x)W^*) + (D\sigma(x);\sigma(x)W^*)^{\otimes 2}) \\ &\qquad + \frac{1}{3}(\sigma(x)U)^{\otimes 3}\otimes(\tilde{\sigma}(x)U)\big) \\ &+ \frac{1}{4!}\big(D^5 f(x); (\sigma(x)U)^{\otimes 4}\otimes(b(x) + (D\sigma(x);\sigma(x)W^*))\big) \\ &+ \frac{1}{6!}\big(D^6 f(x); (\sigma(x)U)^{\otimes 6}\big)\Big].\end{aligned} \tag{62}$$

We are now in a position to provide our main result concerning convergence of weighted empirical measures of the Talay scheme. This first part of this result concerns the $\mathbb{P}-a.s.$ Wasserstein convergence while the second part establishes first and second order CLT.

Theorem 4.1. *Let $p > 0, a \in (0,1]$, $s \geqslant 1, \rho \in [1,2]$ and, $\psi_p(y) = y^p$, $\phi(y) = y^a$ and $\epsilon_{\mathcal{I}}(\gamma) = \gamma^{\rho/2}$. Let $\alpha > 0$ and $\beta \in \mathbb{R}$.*

- ***A.*** *Assume that the sequence $(U_n)_{n\in\mathbb{N}^*}$ satisfies $M_{\mathcal{N},2}(U)$ (see (51)) and $M_{(2p)\vee(2p\rho/s)\vee 2}(U)$ (see (52)). Also assume that (56), $\mathfrak{B}(\phi)$ (see (58)), $\mathcal{R}_p(\alpha,\beta,\phi,V)$ (see (59)), L_V (see (5), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\epsilon_{\mathcal{I}})$ (see (20)), $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{p/s})$ (see (13)) and (16) also hold and that*

$ap\rho/s \leqslant p+a-1$.

Then, if $p/s+a-1>0$, $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is $\mathbb{P}-a.s.$ tight and

$$\mathbb{P}\text{-}a.s. \quad \sup_{n\in\mathbb{N}^*} \nu_n^\eta(V^{p/s+a-1}) < +\infty. \tag{63}$$

Moreover, assume also that b, σ, $|D\sigma|\operatorname{Tr}[\sigma\sigma^*]^{1/2}$, $\tilde{\sigma}$ and Ab have sublinear growth and that $g_\sigma \leqslant CV^{p/s+a-1}$, with $g_\sigma = \operatorname{Tr}[\sigma\sigma^*] + |D\sigma|\operatorname{Tr}[\sigma\sigma^*]^{1/2} + \operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*]^{1/2}$. Then, every weak limiting distribution ν of $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is an invariant distribution of $(X_t)_{t\geqslant 0}$ and when ν is unique, we have

$$\mathbb{P}\text{-}a.s. \quad \forall f \in \mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d), \qquad \lim_{n\to+\infty} \nu_n^\eta(f) = \nu(f), \tag{64}$$

with $\mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d)$ defined in (7). Notice that when $p/s \leqslant p\vee 1+a-1$, the assumption $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{p/s})$ (see (13)) can be replaced by $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)).

B. *Let $q\in\{1,2\}$, let $\tilde{\rho}_q \in [1,2]$, let $C_{\gamma,\eta}>0$ and let us define $\eta_{1,n} = C_{\gamma,\eta}\gamma_n$, $\eta_{2,n+1} = C_{\gamma,\eta}(\gamma_n+\gamma_{n+1})/2$, $n\in\mathbb{N}^*$ (with $\gamma_0=0$) and*

$$F_q = \{f\in\mathcal{C}^{2(q+1)}(\mathbb{R}^d;\mathbb{R}), \forall l\in\{1,\ldots,2(q+1)\}, D^l f\in\mathcal{C}_0(\mathbb{R}^d;\mathbb{R}), Af\in F_1 \textit{ if } q=2\}.$$

Finally let $\tilde{\eta}_q(\gamma) = \gamma^{q+1}$.

Assume that the sequence $(U_n)_{n\in\mathbb{N}^*}$ satisfies $M_{\mathcal{N},2q+1}(U)$ (see (51))) and $M_{q+1}(U)$ (see (52)) and that $\mathcal{SW}_{\mathcal{GC},\gamma}(\tilde{\rho}_q,\gamma,\gamma)$ (see (29)) holds.
*Also assume that $g_{\sigma,q} \leqslant CV^{p/s+a-1}$, with $g_{\sigma,q} = \operatorname{Tr}[\sigma\sigma^*]^{2(q+1)} + |b|^{q+1} + |D\sigma|^{q+1}\operatorname{Tr}[\sigma\sigma^*]^{(q+1)/2} + \operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*] + |Ab|^q$, that $\operatorname{Tr}[\sigma\sigma^*] = o_{|x|\to+\infty}(V^{p/s+a-1})$, that ν is unique and that (3) and the hypothesis from point **A.** hold with η replaced by $\tilde{\eta}_q$ and by γ. Finally assume that for every $f\in F_q$, $|\sigma^* Df|^2 \in \mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d)$ and $\mathfrak{M}_q f \in \mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d)$.*

Then, for every $f\in F_q$, we have

i. *If $\lim_{n\to\infty}\sqrt{\Gamma_n}/H_{\tilde{\eta}_q,n} = +\infty$, ,*

$$\lim_{n\to\infty} \sqrt{\Gamma_n}\,\nu_n^{\eta_q}(Af) \overset{law}{=} \mathcal{N}(0,\nu(|\sigma^* Df|^2)). \tag{65}$$

ii. *If $\lim_{n\to\infty}\sqrt{\Gamma_n}/H_{\tilde{\eta}_q,n} = \hat{l}\in\mathbb{R}_+^*$,*

$$\lim_{n\to\infty} \sqrt{\Gamma_n}\,\nu_n^{\eta_q}(Af) \overset{law}{=} \mathcal{N}(\hat{l}^{-1}\nu(\mathfrak{M}_q f),\nu(|\sigma^* Df|^2)). \tag{66}$$

iii. *If $\lim_{n\to\infty}\sqrt{\Gamma_n}/H_{\tilde{\eta}_q,n} = 0$,*

$$\lim_{n\to\infty} \frac{H_n}{H_{\tilde{\eta}_q,n}}\nu_n^{\eta_q}(Af) \overset{\mathbb{P}}{=} \nu(\mathfrak{M}_q f) \tag{67}$$

Remark 4.1. *Notice that if we take $\gamma_n = 1/n^\xi$, $\xi\in(0,1/(q+1))$, the mentioned step weight assumptions of Theorem 4.1 point **B.** are satisfied (take $\rho\in(1/(1-\xi),2]$ and $\tilde{\rho}_q\in(2/(1+\xi),2]$). Then, if we define by*

$$\forall n \in \mathbb{N}^*, \qquad \mathfrak{r}_{q,n} = \begin{cases} \sqrt{\Gamma_n} & \textit{if} \quad \lim_{n\to\infty} \sqrt{\Gamma_n}/H_{\tilde{\eta}_q,n} = +\infty, \\ \sqrt{\Gamma_n} & \textit{if} \quad \lim_{n\to\infty} \sqrt{\Gamma_n}/H_{\tilde{\eta}_q,n} = \hat{l}, \\ \frac{H_n}{H_{\tilde{\eta}_q,n}} & \textit{if} \quad \lim_{n\to\infty} \sqrt{\Gamma_n}/H_{\tilde{\eta}_q,n} = 0, \end{cases}$$

the rate of convergence of $(\nu_n^{\eta_q}(Af))_{n\in\mathbb{N}^*}$*, we have*

$$\mathfrak{r}_{q,n} \underset{n\to+\infty}{\sim} Cn^{(q\xi)\wedge(1/2-\xi/2)}.$$

The highest rate of convergence is thus achieved for $\xi = 1/(2q+1)$ *and is given by* $\mathfrak{r}_{q,n} \underset{n\to+\infty}{\sim} Cn^{q/(2q+1)}$*. In particular in the first order case* $(q=1)$ *we have* $\mathfrak{r}_{1,n} \underset{n\to+\infty}{\sim} Cn^{1/3}$ *which is, as expected, the same rate as for the Euler scheme (see Remark 3.3). However, for the second order case* $(q=2)$ *we obtain a faster rate of convergence since* $\mathfrak{r}_{2,n} \underset{n\to+\infty}{\sim} Cn^{2/5}$*. This rate can be achieved because* $(\overline{X}_{\Gamma_n})_{n\in\mathbb{N}}$ *is a second weak order scheme but also because the step sequence* $(\eta_{2,n})_{n\in\mathbb{N}^*}$ *is well chosen.*
Notice that if we had considered a third order scheme and step sequence η_3 *defined as in Theorem 3.4, then the same conclusion about rate of the third FCLT - that could be established - would have remained valid with* q *replaced by* 3.

The next part of this Section is dedicated to the proof of Theorem 4.1.

4.1 Recursive control

Proposition 4.1. *Let* $v_* > 0$*, and let* $\phi : [v_*, \infty) \to \mathbb{R}_+$ *be a continuous function such that* $C_\phi := \sup_{y\in[v_*,\infty)} \phi(y)/y < +\infty$*. Now let* $p > 0$ *and define* $\psi_p(y) = y^p$*. Let* $\alpha > 0$ *and* $\beta \in \mathbb{R}$*.*

Assume that $(U_n)_{n\in\mathbb{N}^*}$ *is a sequence of independent random variables such that* U *satisfies* $M_{\mathcal{N},2}(U)$ *(see (51)) and* $M_{(2p)\vee 2}(U)$ *(see (52)).*
Also assume that (56), $\mathfrak{B}(\phi)$ *(see (58)),* $\mathcal{R}_p(\alpha,\beta,\phi,V)$ *(see (59)), are satisfied.*

Then, for every $\tilde{\alpha} \in (0,\alpha)$*, there exists* $n_0 \in \mathbb{N}^*$*, such that*

$$\forall n \geqslant n_0, \forall x \in \mathbb{R}^d, \quad \tilde{A}_{\gamma_n}\psi \circ V(x) \leqslant \frac{\psi_p \circ V(x)}{V(x)} p(\beta - \tilde{\alpha}\phi \circ V(x)). \tag{68}$$

Then $\mathcal{RC}_{Q,V}(\psi,\phi,p\tilde{\alpha},p\beta)$ *(see (6)) holds for every* $\tilde{\alpha} \in (0,\alpha)$ *such that* $\liminf_{y\to+\infty} \phi(y) > \beta/\tilde{\alpha}$*. Moreover, when* $\phi = Id$ *we have*

$$\sup_{n\in\mathbb{N}} \mathbb{E}[V^p(\overline{X}_{\Gamma_n})] < +\infty. \tag{69}$$

Proof. We distinguish the cases $p \geqslant 1$ and $p \in (0,1)$.

Case $p \geqslant 1$. First ,we focus on the case $p \geqslant 1$. From the Taylor's formula and the definition of $\lambda_{\psi_p} = \lambda_p$ (see (57)), we have

$$\begin{aligned}
\psi_p \circ V(\overline{X}_{\Gamma_{n+1}}) =& \psi_p \circ V(\overline{X}_{\Gamma_n}) + \langle \overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}, \nabla V(\overline{X}_{\Gamma_n})\rangle \psi_p' \circ V(\overline{X}_{\Gamma_n}) \\
&+ \frac{1}{2}\big(D^2V(\Upsilon_{n+1})\psi_p' \circ V(\Upsilon_{n+1}) + \nabla V(\Upsilon_{n+1})^{\otimes 2}\psi_p'' \circ V(\Upsilon_{n+1}); (\overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n})^{\otimes 2}\big) \\
\leqslant & \psi_p \circ V(\overline{X}_{\Gamma_n}) + \langle \overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}, \nabla V(\overline{X}_{\Gamma_n})\rangle \psi_p' \circ V(\overline{X}_{\Gamma_n}) \\
&+ \frac{1}{2}\lambda_p(\Upsilon_{n+1})\psi_p' \circ V(\Upsilon_{n+1})|\overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}|^2. \qquad (70)
\end{aligned}$$

with $\Upsilon_{n+1} \in (\overline{X}_{\Gamma_n}, \overline{X}_{\Gamma_{n+1}})$. First, from (56), we have $\sup_{x\in\mathbb{R}^d} \lambda_p(x) < +\infty$.

Since U and $\mathcal{W}$ are made of centered random variables, we deduce from $M_{\mathcal{N},2}(U)$ (see (51)) and $M_4(U)$ (see (52)) that

$$\begin{aligned}
&\mathbb{E}[\overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}|\overline{X}_{\Gamma_n}] = \gamma_{n+1}b(\overline{X}_{\Gamma_n}) + \gamma_{n+1}^2 Ab(\overline{X}_{\Gamma_n}) \\
&\mathbb{E}[|\overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}|^2|\overline{X}_{\Gamma_n}] \leqslant \gamma_{n+1}\mathrm{Tr}[\sigma\sigma^*(\overline{X}_{\Gamma_n})] + \gamma_{n+1}^{3/2}\mathfrak{C}\Big(\mathrm{Tr}[\sigma\sigma^*(\overline{X}_{\Gamma_n})] + |b(\overline{X}_{\Gamma_n})|^2 \\
&\qquad + |D\sigma(\overline{X}_{\Gamma_n})|^2 \,\mathrm{Tr}[\sigma\sigma^*(\overline{X}_{\Gamma_n})] + |\tilde{\sigma}(x)|^2 + |Ab(x)|^2\Big)
\end{aligned}$$

with $\mathfrak{C}$ a positive constant. Assume first that $p = 1$. Using $\mathfrak{B}(\phi)$ (see (58)), for every $\tilde{\alpha} \in (0, \alpha)$, there exists $n_0(\tilde{\alpha})$ such that for every $n \geqslant n_0(\tilde{\alpha})$,

$$\begin{aligned}
\gamma_{n+1}^2 Ab(\overline{X}_{\Gamma_n}) + &\frac{1}{2}\|\lambda_1\|_\infty \gamma_{n+1}^{3/2}\mathfrak{C}\Big(\mathrm{Tr}[\sigma\sigma^*(\overline{X}_{\Gamma_n})] + |b(\overline{X}_{\Gamma_n})|^2 \qquad (71)\\
&+ |D\sigma(\overline{X}_{\Gamma_n})|^2 \,\mathrm{Tr}[\sigma\sigma^*(\overline{X}_{\Gamma_n})] + |\tilde{\sigma}(x)|^2 + |Ab(x)|^2\Big) \leqslant \gamma_{n+1}(\alpha - \tilde{\alpha})\phi \circ V(\overline{X}_{\Gamma_n}).
\end{aligned}$$

From assumption $\mathcal{R}_p(\alpha, \beta, \phi, V)$ (see (59) and (60)), we conclude that

$$\tilde{A}_{\gamma_n}\psi_1 \circ V(x) \leqslant \beta - \tilde{\alpha}\phi \circ V(x)$$

Assume now that $p > 1$.Since $|\nabla V| \leqslant C_V V$ (see (56)), then $\sqrt{V}$ is Lipschitz. Now, we use the following inequality: Let $l \in \mathbb{N}^*$. We have

$$\forall \alpha > 0, \forall u_i \in \mathbb{R}^d, i = 1, \ldots, l, \qquad \Big|\sum_{i=1}^{l} u_i\Big|^\alpha \leqslant l^{(\alpha-1)_+}\sum_{i=1}^{l}|u_i|^\alpha. \qquad (72)$$

$$\begin{aligned}
V^{p-1}(\Upsilon_{n+1}) \leqslant& \big(\sqrt{V}(\overline{X}_{\Gamma_n}) + [\sqrt{V}]_1|\overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}|\big)^{2p-2} \\
\leqslant& 2^{(2p-3)_+}(V^{p-1}(\overline{X}_{\Gamma_n}) + [\sqrt{V}]_1^{2p-2}|\overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}|^{2p-2})
\end{aligned}$$

To study the 'remainder' of (70), we multiply the above inequality by $|\overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}|^2$. First, we study the second term which appears in the *r.h.s.* and using $\mathfrak{B}(\phi)$ (see (58)), for everyy $p \geqslant 1$,

$$|\overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}|^{2p} \leqslant C\gamma_{n+1}^p \phi \circ V(\overline{X}_{\Gamma_n})^p(1 + |U_{n+1}|^{4p}).$$

Let $\hat{\alpha} \in (0, \alpha)$. Then, we deduce from $M_{2p}(U)$ (see (52)) that there exists $n_0(\hat{\alpha}) \in \mathbb{N}$ such that for any $n \geqslant n_0(\hat{\alpha})$, we have

$$\mathbb{E}[|\overline{X}_{\Gamma_{n+1}} - \overline{X}_{\Gamma_n}|^{2p}|\overline{X}_{\Gamma_n}] \leqslant \gamma_{n+1}\phi \circ V(\overline{X}_{\Gamma_n})^p \frac{\alpha - \hat{\alpha}}{\|\phi/I_d\|_\infty^{p-1}\|\lambda_p\|_\infty 2^{(2p-3)_+}[\sqrt{V}]_1^{2p-2}}$$

To treat the other term of the 'remainder' of (70) we proceed as in (71) with $\|\lambda_1\|_\infty$ replaced by $\|\lambda_p\|_\infty 2^{2p-3}[\sqrt{V}]_1^{2p-2}$, α replace by $\hat{\alpha}$ and $\tilde{\alpha} \in (0, \hat{\alpha})$. We gather all the terms together and using (60), for every $n \geqslant n_0(\tilde{\alpha}) \vee n_0(\hat{\alpha})$, we obtain

$$\begin{aligned}\mathbb{E}[V^p(\overline{X}_{\Gamma_{n+1}}) - V^p(\overline{X}_{\Gamma_n})|\overline{X}_{\Gamma_n}] \leqslant & \gamma_{n+1} p V^{p-1}(\overline{X}_{\Gamma_n})(\beta - \alpha\phi \circ V(\overline{X}_{\Gamma_n})) \\ & + \gamma_{n+1} p V^{p-1}(\overline{X}_{\Gamma_n})\Big(\phi \circ V(\overline{X}_{\Gamma_n})(\hat{\alpha} - \tilde{\alpha}) \\ & \qquad + (\alpha - \hat{\alpha})\frac{V^{1-p}(\overline{X}_{\Gamma_n})\phi \circ V(\overline{X}_{\Gamma_n})^p}{\|\phi/I_d\|_\infty^{p-1}}\Big) \\ \leqslant & \gamma_{n+1} V^{p-1}(\overline{X}_{\Gamma_n})(\beta p - \tilde{\alpha} p \phi \circ V(\overline{X}_{\Gamma_n})).\end{aligned}$$

which is exactly the recursive control for $p > 1$.

Case $p \in (0, 1)$. Now, let $p \in (0, 1)$ so that $x \mapsto x^p$ is concave. it follows that

$$V^p(\overline{X}_{\Gamma_{n+1}}) - V^p(\overline{X}_{\Gamma_n}) \leqslant p V^{p-1}(\overline{X}_{\Gamma_n})(V(\overline{X}_{\Gamma_{n+1}}) - V(\overline{X}_{\Gamma_n}))$$

We have just proved that we have the recursive control $\mathcal{RC}_{Q,V}(\psi, \phi, \alpha, \beta)$ holds for $\psi = I_d$ (with α replaced by $\tilde{\alpha} > 0$), and since V takes positive values, we obtain

$$\begin{aligned}\mathbb{E}[V^p(\overline{X}_{\Gamma_{n+1}}) - V^p(\overline{X}_{\Gamma_n})|\overline{X}_{\Gamma_n}] \leqslant & p V^{p-1}(\overline{X}_{\Gamma_n})\mathbb{E}[V(\overline{X}_{\Gamma_{n+1}}) - V(\overline{X}_{\Gamma_n})|\overline{X}_{\Gamma_n}] \\ \leqslant & V^{p-1}(\overline{X}_{\Gamma_n})(p\beta - p\tilde{\alpha}\phi \circ V(\overline{X}_{\Gamma_n})),\end{aligned}$$

which completes the proof of (68). The proof of (69) is an immediate application of Lemma 2.1 as soon as we notice that the increments of the Talay scheme have finite polynomial moments which implies (19).

□

4.2 Infinitesimal approximation

Proposition 4.2. *Assume that b, σ, $|D\sigma| \operatorname{Tr}[\sigma\sigma^*]^{1/2}$, $\tilde{\sigma}$ and Ab have sublinear growth. We have the following properties:*

A. *Assume that the sequence $(U_n)_{n \in \mathbb{N}^*}$ satisfies $M_{\mathcal{N},2}(U)$ (see (51)) and that $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(\operatorname{Tr}[\sigma\sigma^*]) < +\infty$, $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(|D\sigma| \operatorname{Tr}[\sigma\sigma^*]^{1/2}) < +\infty$ and $\sup_{n \in \mathbb{N}^*} \nu_n^\eta(\operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*]^{1/2}) < +\infty$.*

Then, $\mathcal{E}(\tilde{A}, A, \mathcal{D}(A)_0)$ (see (8)) is satisfied.

B. *Let $F_1 = \{f \in \mathcal{C}^4(\mathbb{R}^d; \mathbb{R}), \forall q \in \{1, \ldots, 4\}, D^q f \in \mathcal{C}_0(\mathbb{R}^d; \mathbb{R})\}$, let $\mathfrak{M}_1$ defined in (61) and let $\tilde{\eta}_1(\gamma) = \gamma^2$.*

Assume that the sequence $(U_n)_{n\in\mathbb{N}^*}$ *satisfies* $M_{\mathcal{N},3}(U)$ *(see (51))) and* $M_2(U)$ *(see (52)) and that* $\sup_{n\in\mathbb{N}^*}\nu_n^{\tilde{\eta}_1}(g_1)<+\infty$, *with* $g_1:\mathbb{R}^d\to\mathbb{R}$ *such that for every* $x\in\mathbb{R}^d$, $g_1(x)=\mathrm{Tr}[\sigma\sigma^*(x)]^2+|b(x)|^2+|D\sigma(x)|^2\,\mathrm{Tr}[\sigma\sigma^*(x)]+\mathrm{Tr}[\tilde{\sigma}\tilde{\sigma}^*(x)]+|Ab(x)|$. *Finally assume that* $\mathbb{P}-a.s.$, *for every* $f\in F_1$, $\lim_{n\to\infty}\nu_n^{\tilde{\eta}_1,n}(\mathfrak{M}_1 f)=\nu(\mathfrak{M}_1 f)$.

Then $\mathcal{E}_1(F_1,\tilde{A},A,\mathfrak{M}_1,\tilde{\eta}_1)$ *(see (22)) is satisfied.*

C. *Let* $F_2=\{f\in\mathcal{C}^6(\mathbb{R}^d;\mathbb{R}),\forall q\in\{2,\ldots,6\},D^qf\in\mathcal{C}_0(\mathbb{R}^d;\mathbb{R}),Af\in F_1\}$, *let* $\mathfrak{M}_2$ *defined in (62) and let* $\tilde{\eta}_2(\gamma)=\gamma^3$.

Assume that the sequence $(U_n)_{n\in\mathbb{N}^*}$ *satisfies* $M_{\mathcal{N},5}(U)$ *(see (51))) and* $M_3(U)$ *(see (52)) and that* $\sup_{n\in\mathbb{N}^*}\nu_n^{\tilde{\eta}_2}(g_2)<+\infty$ *with* $g_2:\mathbb{R}^d\to\mathbb{R}$ *such that for every* $x\in\mathbb{R}^d$, $g_2(x)=\mathrm{Tr}[\sigma\sigma^*(x)]^3+|b(x)|^3+|D\sigma(x)|^3\,\mathrm{Tr}[\tilde{\sigma}\tilde{\sigma}^*(x)]^{3/2}+\mathrm{Tr}[\tilde{\sigma}\tilde{\sigma}^*(x)]+|Ab(x)|^2$. *Finally assume that* $\mathbb{P}-a.s.$, *for every* $f\in F_2$, $\lim_{n\to\infty}\nu_n^{\tilde{\eta}_2,n}(\mathfrak{M}_2 f)=\nu(\mathfrak{M}_2 f)$.

Then $\mathcal{E}_2(F_2,\tilde{A},A,\mathfrak{M}_2,\tilde{\eta}_2)$ *(see (22)) is satisfied.*

Proof. The proof of point **A.** is very similar to the proof of point **B.** and point **C.** but simpler and thus left to the reader. The proof of point **B.** and point **C.** is a direct consequence of the following Lemma.

Lemma 4.1. *Assume that* b, σ, $|D\sigma|\,\mathrm{Tr}[\sigma\sigma^*]^{1/2}$, $\tilde{\sigma}$ *and* Ab *have sublinear growth. We have the following properties:*

A. *Assume that the sequence* $(U_n)_{n\in\mathbb{N}^*}$ *satisfies* $M_{\mathcal{N},3}(U)$ *(see (51)) and* $M_2(U)$ *(see (52)).*

Then, for every $f\in\mathcal{C}^4(\mathbb{R}^d;\mathbb{R})$ *such that* $D^qf\in\mathcal{C}_0(\mathbb{R}^d;\mathbb{R})$ *for* $q\in\{1,\ldots,4\}$, *then*

$$\Big|\mathbb{E}[f(\overline{X}_{\Gamma_{n+1}})-f(\overline{X}_{\Gamma_n})|\overline{X}_{\Gamma_n}]-\gamma_{n+1}Af(\overline{X}_{\Gamma_n})+\gamma_{n+1}^2\mathfrak{M}_1 f(\overline{X}_{\Gamma_n})\Big| \leqslant \gamma_{n+1}^2\Lambda_{f,1}(\overline{X}_{\Gamma_n},\gamma_{n+1}),$$

with, given $l\in\mathbb{N}^*$ *and a probability space* $(\tilde{\Omega},\tilde{\mathcal{G}},\tilde{\mathbb{P}})$,

$$\forall x\in\mathbb{R}^d,\forall\gamma\in(0,\overline{\gamma}],\qquad \Lambda_{f,1}(x,\gamma)=\langle g_1(x),\tilde{\mathbb{E}}[\tilde{\Lambda}_{f,1}(x,\gamma,\tilde{\omega})]\rangle_{\mathbb{R}^l},$$

with $\tilde{\Lambda}_{f,1}$ *satisfying (23) and (24),* $\mathfrak{M}_1$ *defined in (61) and* $g_1:\mathbb{R}^d\to\mathbb{R}^l$, *such that for every* $x\in\mathbb{R}^d$, $|g_1(x)|\leqslant 1+\mathrm{Tr}[\sigma\sigma^*(x)]^2+|b(x)|^2+|D\sigma(x)|^2\,\mathrm{Tr}[\sigma\sigma^*(x)]+\mathrm{Tr}[\tilde{\sigma}\tilde{\sigma}^*(x)]+|Ab(x)|$.

B. *Assume that the sequence* $(U_n)_{n\in\mathbb{N}^*}$ *satisfies* $M_{\mathcal{N},5}(U)$ *(see (51)) and* $M_3(U)$ *(see (52)).*

Then, for every $f\in\mathcal{C}^6(\mathbb{R}^d;\mathbb{R})$ *such that* $D^qf\in\mathcal{C}_0(\mathbb{R}^d;\mathbb{R})$ *for* $q\in\{2,\ldots,6\}$, *then*

$$\Big|\mathbb{E}[f(\overline{X}_{\Gamma_{n+1}})-f(\overline{X}_{\Gamma_n})|\overline{X}_{\Gamma_n}]-\gamma_{n+1}Af(\overline{X}_{\Gamma_n})-\frac{\gamma_{n+1}^2}{2}A^2f(\overline{X}_{\Gamma_n})+\gamma_{n+1}^3\tilde{\mathfrak{M}}_2 f(\overline{X}_{\Gamma_n})\Big| \leqslant \gamma_{n+1}^3\Lambda_{f,2}(\overline{X}_{\Gamma_n},\gamma_{n+1}),$$

with, given $l\in\mathbb{N}^*$ *and a probability space* $(\tilde{\Omega},\tilde{\mathcal{G}},\tilde{\mathbb{P}})$,

$$\forall x\in\mathbb{R}^d,\forall\gamma\in(0,\overline{\gamma}],\qquad \Lambda_{f,2}(x,\gamma)=\langle g_2(x),\tilde{\mathbb{E}}[\tilde{\Lambda}_{f,2}(x,\gamma,\tilde{\omega})]\rangle_{\mathbb{R}^l},$$

with $\tilde{\Lambda}_{f,2}$ satisfying (23) and (24) and $\tilde{\mathfrak{M}}_2$ defined in (62) and $g_2 : \mathbb{R}^d \to \mathbb{R}^l$, such that for every $x \in \mathbb{R}^d$, $|g_2(x)| \leqslant 1 + \mathrm{Tr}[\sigma\sigma^*(x)]^3 + |b(x)|^3 + |D\sigma(x)|^3 \,\mathrm{Tr}[\sigma\sigma^*(x)]^{3/2} + \mathrm{Tr}[\tilde{\sigma}\tilde{\sigma}^*(x)] + |Ab(x)|^2$.

Notice that to obtain Proposition 4.2 point **B.**, we use Lemma 4.1 point **A.** and to obtain Proposition 4.2 point **C.**, we combine Lemma 4.1 point **A.** (with f replaced by Af) and Lemma 4.1 point **B.**

Proof of Lemma 4.1. We simply prove point point **B.**. The proof of point point **A.** is similar but simpler. The first step consists in writing the following decomposition

$$f(\overline{X}_{\Gamma_{n+1}}) - f(\overline{X}_{\Gamma_n}) = \sum_{j=0}^{4} f(\overline{X}^j_{\Gamma_n}) - f(\overline{X}^{j-1}_{\Gamma_n})$$

with notations (55) and $\overline{X}^0_{\Gamma_n} = \overline{X}_{\Gamma_n}$. At this point it remains to study each term of the sum of the *r.h.s.* of the above equation. For $j = 1$, we use Taylor expansion at order 6 and it follows that

$$\begin{aligned}|\mathbb{E}[f(\overline{X}^1_{\Gamma_n})|\overline{X}_{\Gamma_n}] - f(\overline{X}_{\Gamma_n})| \leqslant & \sum_{i=1}^{6} \frac{\gamma_{n+1}^{i/2}(D^i f(\overline{X}_{\Gamma_n}); \mathbb{E}[(\sigma(\overline{X}_{\Gamma_n})U_{n+1})^{\otimes i})|\overline{X}_{\Gamma_n}])}{i!} \\ & + \gamma^3_{n+1}\Lambda_{f,2,1}(\overline{X}_{\Gamma_n}, \gamma_{n+1})\end{aligned}$$

with $\Lambda_{f,2,1}(x,\gamma) = g_{2,1}(x)\tilde{\mathbb{E}}[\tilde{\Lambda}_{f,2,1}(x,z,\gamma)]$ where $\tilde{\Lambda}_{f,2,1}(x,\gamma) = \tilde{\mathcal{R}}_{f,2,1}(x,z,\gamma,U,\Theta)$ with $U \sim \mathbb{P}_U$, $\Theta \sim \mathcal{U}_{[0,1]}$ under $\tilde{\mathbb{P}}$, $g_{2,1}(x) = \mathrm{Tr}[\sigma\sigma^*(x)]^3$ and

$$\begin{aligned}\tilde{\mathcal{R}}_{f,2,1} \;:\; \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^N \times [0,1] &\to \mathbb{R}_+ \\ (x,\gamma,u,\theta) &\mapsto \tilde{\mathcal{R}}_{f,2,1}(x,\gamma,u,\theta),\end{aligned}$$

with

$$\tilde{\mathcal{R}}_{f,2,1}(x,\gamma,u,\theta) = \frac{|u|^6}{5!}(1-\theta)^5|D^6 f(x+\theta\sqrt{\gamma}\sigma(x)u) - D^6 f(x)|.$$

We are going to prove that $\tilde{\Lambda}_{f,2,1}$ satisfies (24). We fix $u \in \mathbb{R}^N$ and $\theta \in [0,1]$. Now, since the function σ has sublinear growth, there exists $C_\sigma \geqslant 0$ such that $|\sigma(x)| \leqslant C_\sigma(1+|x|)$ for every $x \in \mathbb{R}^d$. Therefore, since f has compact support, there exists $\underline{\gamma}(u,\theta) > 0$ and $R > 0$ such that

$$\sup_{|x|>R} \sup_{\gamma \leqslant \underline{\gamma}(u,\theta)} \tilde{\mathcal{R}}_{f,2,1}(x,\gamma,u,\theta) = 0.$$

It follows that (24) (ii) holds. Moreover since $D^6 f$ is bounded, and $M_3(U)$ (see (52)) holds, $\tilde{\Lambda}_{f,2}$ also satisfies (23).

The rest of the proof is completely similar and involves heavy calculus so we just give the sketch to follow for $j = 2$ and invite the reader to follow the same line for $j \in \{3,4,5\}$. For $j = 2$, we use Taylor expansion at order 3 and it follows that

$$\begin{aligned}|\mathbb{E}[f(\overline{X}^2_{\Gamma_n})|\overline{X}_{\Gamma_n}] - f(\overline{X}^1_{\Gamma_n})| \leqslant & \sum_{i=1}^{3} \frac{\gamma^i_{n+1}\mathbb{E}[(D^i f(\overline{X}^1_{\Gamma_n}); (b(\overline{X}_{\Gamma_n}))^{\otimes i})|\overline{X}_{\Gamma_n}]}{i!} \\ & + \frac{\gamma^3_{n+1}}{2}(D^3 f(\overline{X}_{\Gamma_n}); b(\overline{X}_{\Gamma_n})^{\otimes 3}) + \gamma^3_{n+1}\Lambda_{f,2,2}(\overline{X}_{\Gamma_n}, \gamma_{n+1})\end{aligned}$$

with $\Lambda_{f,2,2}(x,\gamma) = g_{2,2}(x)\tilde{\mathbb{E}}[\tilde{\Lambda}_{f,2,2}(x,z,\gamma)]$ where $\tilde{\Lambda}_{f,2,2}(x,\gamma) = \tilde{\mathcal{R}}_{f,2,2}(x,z,\gamma,U,\Theta)$ with $U \sim \mathbb{P}_U$, $\Theta \sim \mathcal{U}_{[0,1]}$ under $\tilde{\mathbb{P}}$, $g_{2,2}(x) = |b(x)|^3$ and

$$\begin{array}{rrcl} \tilde{\mathcal{R}}_{f,2,1} & : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^N \times [0,1] & \to & \mathbb{R}_+ \\ & (x,\gamma,u,\theta) & \mapsto & \tilde{\mathcal{R}}_{f,2,1}(x,\gamma,u,\theta), \end{array}$$

with

$$\tilde{\mathcal{R}}_{f,2,2}(x,\gamma,u,\theta) = \frac{1}{2}(1-\theta)^2 |D^3 f(x+\sqrt{\gamma}\sigma(x)u+\theta\gamma b(x)) - D^3 f(x)|.$$

Following the same approach as for the case $j=1$ we can show that $\tilde{\Lambda}_{f,2,2}$ satisfies (24) and (23).

To complete the study for $j=1$, we replace $D^i f(\overline{X}^1_{\Gamma_n}$, $i \in \{1,2\}$ by an upper bound of their Taylor expansion at order $2(3-i)$ and at point $\overline{X}^{j-1}_{\Gamma_n} = \overline{X}_{\Gamma_n}$, that is

$$\begin{aligned} |\mathbb{E}[D^i f(\overline{X}^1_{\Gamma_n})|\overline{X}_{\Gamma_n}] - D^i f(\overline{X}_{\Gamma_n})| \leqslant & \sum_{\bar{i}=1}^{2(3-i)} \frac{\gamma_{n+1}^{\bar{i}/2}(D^{\bar{i}+i} f(\overline{X}_{\Gamma_n}); \mathbb{E}[(\sigma(\overline{X}_{\Gamma_n})U_{n+1})^{\otimes \bar{i}})|\overline{X}_{\Gamma_n}])}{\bar{i}!} \\ & + \gamma_{n+1}^{3-i}\Lambda_{D^i f,2,1}(\overline{X}_{\Gamma_n},\gamma_{n+1}) \end{aligned}$$

with $\Lambda_{D^i f,2,2}(x,\gamma) = \mathrm{Tr}[\sigma\sigma^*(x)]^{3-i}\tilde{\mathbb{E}}[\tilde{\Lambda}_{D^i f,2,2}(x,z,\gamma)]$ where $\tilde{\Lambda}_{D^i f,2,2}(x,\gamma) = \tilde{\mathcal{R}}_{D^i f,2,2}(x,z,\gamma,U,\Theta)$ with $U \sim \mathbb{P}_U$, $\Theta \sim \mathcal{U}_{[0,1]}$ under $\tilde{\mathbb{P}}$, and

$$\begin{array}{rrcl} \tilde{\mathcal{R}}_{D^i f,2,2} & : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^N \times [0,1] & \to & \mathbb{R}_+ \\ & (x,\gamma,u,\theta) & \mapsto & \tilde{\mathcal{R}}_{D^i f,2,2}(x,\gamma,u,\theta), \end{array}$$

with

$$\tilde{\mathcal{R}}_{D^i f,2,2}(x,\gamma,u,\theta) = \frac{|u|^{2(3-i)}}{(5-2i)!}(1-\theta)^{5-2i}|D^{3-i}f(x+\theta\sqrt{\gamma}\sigma(x)u) - D^{3-i}f(x)|.$$

Following the same approach as for the case $j=1$ we can show that $\tilde{\Lambda}_{D^i f,2,2}$ satisfies (24) and (23). We do not detail the rest of the proof which is similar but simply describe the approach we use. For $j=\{3,4,5\}$ we apply the same method as for $j=2$: We first use the Taylor expansion at point $\overline{X}^{j-1}_{\Gamma_n}$ such that the remainder has the form $\gamma_{n+1}^3\Lambda_{f,2,j}$. Then we develop each term of this expansion at point $\overline{X}^{j-2}_{\Gamma_n}$ at a well chosen order such that the global remainder is still of the form $\gamma_{n+1}^3\Lambda_{f,2,j}$ ($\Lambda_{f,2,j}$ is obviously changed). We iterate the method until we use the Taylor expansion at point $\overline{X}_{\Gamma_n}$. Then, the final remainder $\Lambda_{f,2}$ has the expected form and the term which appears in the expansion can be identified with $\gamma_{n+1}Af(\overline{X}_{\Gamma_n}) + \frac{\gamma_{n+1}^2}{2}A^2 f(\overline{X}_{\Gamma_n}) + \gamma_{n+1}^3\tilde{\mathfrak{M}}_2 f(\overline{X}_{\Gamma_n})$. To complete the proof we notice that for every $f \in \mathcal{C}^6(\mathbb{R}^d)$ and every $j \in \{1,\dots,5\}$, $\tilde{\mathcal{R}}_{f,2,j} = \tilde{\mathcal{R}}_{-f,2,j}$. □

□

4.3 Growth control

Lemma 4.2. *Let $p>0, a \in (0,1]$, $s \geqslant 1$, $\rho \in [1,2]$ and, $\psi(y)=y^p$ and $\phi(y)=y^a$. We suppose that the sequence $(U_n)_{n\in\mathbb{N}^*}$ satisfies $M_{\rho\vee(2p\rho/s)}(U)$ (see (52)). Then, for every $n \in \mathbb{N}$ and every $f \in \mathcal{D}(A)_0$,*

$$\mathbb{E}[|f(\overline{X}_{\Gamma_{n+1}})-f(\overline{X}_{\Gamma_n}+\gamma_{n+1}b(\overline{X}_{\Gamma_n})+\gamma_{n+1}^2Ab(\overline{X}_{\Gamma_n}))|^\rho|\overline{X}_{\Gamma_n}] \tag{73}$$
$$\leqslant C_f\gamma_{n+1}^{\rho/2}\operatorname{Tr}[\sigma\sigma^*(\overline{X}_{\Gamma_n})]^{\rho/2}+C_f\gamma_{n+1}^{\rho}|D\sigma|^\rho\operatorname{Tr}[\sigma\sigma^*]^{\rho/2}+C_f\gamma_{n+1}^{\rho3/2}\operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*(\overline{X}_{\Gamma_n})]^{\rho/2}.$$

with $\mathcal{D}(A)_0=\mathcal{C}^2_K(\mathbb{R}^d)$. *In other words, we have* $\mathcal{GC}_Q(\mathcal{D}(A)_0,g_\sigma,\rho,\epsilon_\mathcal{I})$ *(see (11)) with* $g_\sigma=\operatorname{Tr}[\sigma\sigma^*]^{\rho/2}+|D\sigma|^\rho\operatorname{Tr}[\sigma\sigma^*]^{\rho/2}+\operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*(\overline{X}_{\Gamma_n})]^{\rho/2}$ *and* $\epsilon_\mathcal{I}(\gamma)=\gamma^{\rho/2}$ *for every* $\gamma\in\mathbb{R}_+$.

Moreover, if (56) and $\mathfrak{B}(\phi)$ *(see (58)) hold and*

$$\mathcal{SW}_{pol}(p,a,s,\rho)\qquad ap\rho/s\leqslant p+a-1. \tag{74}$$

Then, for every $n\in\mathbb{N}$, *we have*

$$\mathbb{E}[|V^{p/s}(\overline{X}_{\Gamma_{n+1}})-V^{p/s}(\overline{X}_{\Gamma_n})|^\rho|\overline{X}_{\Gamma_n}]\leqslant C\gamma_{n+1}^{\rho/2}V^{p+a-1}(\overline{X}_{\Gamma_n}). \tag{75}$$

In other words, we have $\mathcal{GC}_Q(V^{p/s},V^{p+a-1},\rho,\epsilon_\mathcal{I})$ *(see (11)) with and* $\epsilon_\mathcal{I}(\gamma)=\gamma^{\rho/2}$ *for every* $\gamma\in\mathbb{R}_+$.

Proof. We begin by noticing that

$$|\overline{X}_{\Gamma_{n+1}}-(\overline{X}_{\Gamma_n}+\gamma_{n+1}b(\overline{X}_{\Gamma_n})+\gamma_{n+1}^2Ab(\overline{X}_{\Gamma_n}))|$$
$$\leqslant C\gamma_{n+1}^{1/2}\operatorname{Tr}[\sigma\sigma^*(\overline{X}_{\Gamma_n})]^{1/2}|U_{n+1}|+C\gamma_{n+1}|D\sigma|\operatorname{Tr}[\sigma\sigma^*]^{1/2}|\mathcal{W}_{n+1}|+\gamma_{n+1}^{3/2}\operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*(\overline{X}_{\Gamma_n})]^{1/2}|U_{n+1}|.$$

Let $f\in\mathcal{D}(A)$. Then f is Lipschitz and the previous inequality gives (77).

We focus now on the proof of (75). We first notice that $\mathfrak{B}(\phi)$ (see (58))implies that for any $n\in\mathbb{N}$,

$$|\overline{X}_{\Gamma_{n+1}}-\overline{X}_{\Gamma_n}|\leqslant C\gamma_{n+1}^{1/2}\sqrt{\phi\circ V(\overline{X}_{\Gamma_n})}(1+|U_{n+1}|+|\mathcal{W}_{n+1}||)$$

Case $2p\leqslant s$**.** We notice that $V^{p/s}$ is α-Hölder for any $\alpha\in[2p/s,1]$ (see Lemma 3. in [21]) and then $V^{p/s}$ is $2p/s$-Hölder. We deduce that

$$\mathbb{E}[|V^{p/s}(\overline{X}_{\Gamma_{n+1}})-V^{p/s}(\overline{X}_{\Gamma_n})|^\rho|\overline{X}_{\Gamma_n}]\leqslant C[V^{p/s}]_{2p/s}^\rho\gamma_{n+1}^{\rho/2}V^{a\rho/2}(\overline{X}_{\Gamma_n}).$$

In order to obtain (75), it remains to use $ap\rho/s\leqslant a+p-1$.

Case $2p\geqslant s$**.** Using the following inequality

$$\forall u,v\in\mathbb{R}_+,\forall\alpha\geqslant 1,\qquad |u^\alpha-v^\alpha|\leqslant\alpha 2^{\alpha-1}(v^{\alpha-1}|u-v|+|u-v|^\alpha), \tag{76}$$

with $\alpha=2p/s$, and since $\sqrt{V}$ is Lipschitz, we have

$$\begin{aligned}\left|V^{p/s}(\overline{X}_{\Gamma_{n+1}})-V^{p/s}(\overline{X}_{\Gamma_n})\right|\leqslant& 2^{2p/s}p/s(V^{p/s-1/2}(\overline{X}_{\Gamma_n})|\sqrt{V}(\overline{X}_{\Gamma_{n+1}})-\sqrt{V}(\overline{X}_{\Gamma_n})|\\&+|\sqrt{V}(\overline{X}_{\Gamma_{n+1}})-\sqrt{V}(\overline{X}_{\Gamma_n})|^{2p/s})\\ \leqslant& 2^{2p/s}p/s([\sqrt{V}]_1V^{p/s-1/2}(\overline{X}_{\Gamma_n})|\overline{X}_{\Gamma_{n+1}}-\overline{X}_{\Gamma_n}|\\&+[\sqrt{V}]_1^{2p/s}|\overline{X}_{\Gamma_{n+1}}-\overline{X}_{\Gamma_n}|^{2p/s}).\end{aligned}$$

In order to obtain (75), it remains to use the assumptions $\mathfrak{B}(\phi)$ (see (58)) and then $ap\rho/s \leqslant p+a-1$.

□

Lemma 4.3. *Let $\rho \in [1,2]$ and, $\psi(y) = y^p$ and $\phi(y) = y^a$. We suppose that the sequence $(U_n)_{n\in\mathbb{N}^*}$ satisfies $M_\rho(U)$ (see (52)). Then, for every $n \in \mathbb{N}$, we have: for every $f \in F = \{f \in \mathcal{C}^2(\mathbb{R}^d;\mathbb{R}), D^q f \in \mathcal{C}_b(\mathbb{R}^d;\mathbb{R}), \forall q \in \{1,2\}\}$.*

$$\begin{aligned}\mathbb{E}[|f(\overline{X}_{\Gamma_{n+1}})-&f(\overline{X}_{\Gamma_n}) - \sqrt{\gamma_{n+1}}(Df(\overline{X}_{\Gamma_n});\sigma(\overline{X}_{\Gamma_n})U_{n+1})|^\rho|\overline{X}_{\Gamma_n}] \\ \leqslant &C_f\gamma_{n+1}^\rho \operatorname{Tr}[\sigma\sigma^*(\overline{X}_{\Gamma_n})]^\rho + C_f\gamma_{n+1}^\rho|b(X_n)| + C_f\gamma_{n+1}^\rho|D\sigma(\overline{X}_{\Gamma_n})|^\rho \operatorname{Tr}[\sigma\sigma^*(\overline{X}_{\Gamma_n})]^{\rho/2} \\ &+ C_f\gamma_{n+1}^{\rho 3/2}\operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*(\overline{X}_{\Gamma_n})]^{\rho/2} + C_f\gamma_{n+1}^{2\rho}|Ab(\overline{X}_{\Gamma_n})|^\rho.\end{aligned} \tag{77}$$

In particular for $q \in \{1,2\}$, assume that $\mathbb{P}-a.s.$, $\lim_{n\to+\infty}\nu_n^\gamma(|\sigma^* Df|^2) = \nu(|\sigma^* Df|^2)$ for every $f \in F$ satisfying $Af \in \mathcal{C}_b(\mathbb{R}^d;\mathbb{R})$ when $q=2$ and that $\operatorname{Tr}[\sigma\sigma^*] = o_{|x|\to+\infty}(W)$ with $\sup_{n\in\mathbb{N}^*}\nu_n^\gamma(W) < +\infty$.

Then $\mathcal{GC}_{Q,q}(F,g,\rho,\epsilon_{\mathcal{X}},\epsilon_{\mathcal{GC}},\mathfrak{V})$ (see (26)) is satisfied with $g = \operatorname{Tr}[\sigma\sigma^*]^\rho + |b|^\rho + |D\sigma|^\rho \operatorname{Tr}[\sigma\sigma^*]^{\rho/2} + \operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*]^{\rho/2} + |Ab|^\rho$, $\epsilon_{\mathcal{X}}(\gamma) = \gamma$ and $\epsilon_{\mathcal{GC}}(\gamma) = \gamma^\rho$ for every $\gamma \in \mathbb{R}_+$ and $\mathfrak{V}f = |\sigma^* Df|^2$ for every $f \in \mathcal{C}^1(\mathbb{R}^d;\mathbb{R})$.

Proof. The first step consists in writing

$$\begin{aligned}f(\overline{X}_{\Gamma_{n+1}}) - f(\overline{X}_{\Gamma_n}) =&f(\overline{X}_{\Gamma_n} + \sqrt{\gamma_{n+1}}\sigma(\overline{X}_{\Gamma_n})U_{n+1}) - f(\overline{X}_{\Gamma_n}) \\ &+ f(\overline{X}_{\Gamma_{n+1}}) - f(\overline{X}_{\Gamma_n} + \sqrt{\gamma_{n+1}}\sigma(\overline{X}_{\Gamma_n})U_{n+1}).\end{aligned} \tag{78}$$

We study the first term of the *r.h.s.* of the above equation. Using Taylor expansion at order two and the fact that $Df \in \mathcal{C}_b(\mathbb{R}^d)$ yields

$$\begin{aligned}\big|f(\overline{X}_{\Gamma_n} + \sqrt{\gamma_{n+1}}\sigma(\overline{X}_{\Gamma_n})U_{n+1}) - f(\overline{X}_{\Gamma_n}) - \sqrt{\gamma_{n+1}}(Df(\overline{X}_{\Gamma_n});\sigma(\overline{X}_{\Gamma_n})U_{n+1})\big| \\ \leqslant \frac{1}{2}\|D^2 f\|_\infty|\sqrt{\gamma_{n+1}}\sigma(\overline{X}_{\Gamma_n})U_{n+1}|^2.\end{aligned}$$

Now we study the second term of the *r.h.s.* of (78). From Taylor expansion at order one

$$\begin{aligned}|f(\overline{X}_{\Gamma_{n+1}}) - f(\overline{X}_{\Gamma_n} + \sqrt{\gamma_{n+1}}\sigma(\overline{X}_{\Gamma_n})U_{n+1})| \leqslant &\|Df\|_\infty\Big|\gamma_{n+1}\Big(b(\overline{X}_{\Gamma_n}) + (D\sigma(\overline{X}_{\Gamma_n});\sigma(\overline{X}_{\Gamma_n})\mathcal{W}_{n+1}^*)\Big) \\ &+ \gamma_{n+1}^{3/2}\tilde{\sigma}(\overline{X}_{\Gamma_n})U_{n+1} + \gamma_{n+1}^2 Ab(\overline{X}_{\Gamma_n})\Big|.\end{aligned}$$

Gathering both terms of (78), raising to the power ρ and taking conditional expectancy thus yields (77). To obtain $\mathcal{GC}_{Q,q}(F,g,\rho,\epsilon_{\mathcal{X}},\epsilon_{\mathcal{GC}},\mathfrak{V})$ (see (26)), we observe that Af is bounded when $q=2$ and it remains to show that (27) holds with $\mathcal{X}_{f,n} = \sqrt{\gamma_{n+1}}(Df(\overline{X}_{\Gamma_n});\sigma(\overline{X}_{\Gamma_n})U_{n+1})$, $n \in \mathbb{N}$. This is already done in the seminal paper [10] (see Proposition 2.) and we invite the reader to refer to this result. □

4.4 Proof of Theorem 4.1

Proof of Theorem 4.1 point A.

This result follows from Theorem 2.1 and Theorem 2.2. The proof consists in showing that the assumptions from those theorems are satisfied.

Step 1. Mean reverting recursive control First, we show that $\mathcal{RC}_{Q,V}(\psi_p,\phi,p\tilde{\alpha},p\beta)$ and $\mathcal{RC}_{Q,V}(\psi_1,\phi,\tilde{\alpha},\beta)$ (see (6)) is satisfied for every $\tilde{\alpha}\in(0,\alpha)$.

Since (56), $\mathfrak{B}(\phi)$ (see (58)) and $\mathcal{R}_p(\alpha,\beta,\phi,V)$ (see (59)) hold, it follows from Proposition 4.1 that $\mathcal{RC}_{Q,V}(\psi_p,\phi,p\tilde{\alpha},p\beta)$ (see (6)) is satisfied for every $\tilde{\alpha}\in(0,\alpha)$ since $\liminf_{y\to+\infty}\phi(y)>\beta/\tilde{\alpha}$. Moreover let us notice that for every $p\leqslant 1$ then $\mathcal{R}_p(\alpha,\beta,\phi,V)$ (see (59)) is similar to $\mathcal{R}_1(\alpha,\beta,\phi,V)$ and then $\mathcal{RC}_{Q,V}(\psi_1,\phi,\tilde{\alpha},\beta)$ (see (6)) is satisfied for every $\tilde{\alpha}\in(0,\alpha)$

Step 2. Step weight assumption Now, we show that $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p\vee 1+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (12)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{p\vee 1+a-1})$ (see (13)) hold.

First we noticel that from Step1. the assumption $\mathcal{RC}_{Q,V}(\psi_{p\vee 1},\phi,(p\vee 1)\tilde{\alpha},(p\vee 1)\beta)$ (see (6)) is satisfied for every $\tilde{\alpha}\in(0,\alpha)$. Then, using $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(\rho,\epsilon_{\mathcal{I}})$ (see (20)) with Lemma 2.2 gives $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p\vee 1+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (12)). Similarly, $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{p\vee 1+a-1})$ (see (13) follows from $\mathcal{SW}_{\mathcal{II},\gamma,\eta}$ (see (21)) and Lemma 2.2.

Step 3. Growth control assumption Now, we prove $\mathcal{GC}_Q(F,V^{p\vee 1+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (11)) for $F=\mathcal{D}(A)_0$ and $F=\{V^{p/s}\}$.

This is a consequence of Lemma 4.2. We recall that ρ‘ $\in[1,2]$. Consequently $M_{\rho\vee(2p\rho/s)}(U)$ (see (52)) holds. Now, we notice that from $\mathfrak{B}(\phi)$ (see (58)), we have $\mathrm{Tr}[\sigma\sigma^*]^{\rho/2}+|D\sigma|^{\rho}\,\mathrm{Tr}[\sigma\sigma^*]^{\rho/2}+\mathrm{Tr}[\tilde{\sigma}\tilde{\sigma}^*]^{\rho/2}\leqslant CV^{\rho a/2}$ with $a\rho/2\leqslant p+a-1$ since $\mathcal{SW}_{pol}(p,a,s,\rho)$ (see (74)) holds. Then Lemma 4.2 implies that for $F=\mathcal{D}(A)_0$ and $F=\{V^{p/s}\}$, then $\mathcal{GC}_Q(F,V^{p\vee 1+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (11)) holds

Step 4. Conclusion

i. The first part of Theorem 4.1 (see (63)) is a consequence of Theorem 2.1. Let us observe that assumptions from Theorem 2.1 indeed hold.

On the one hand, we observe that from Step 2. and Step 3. the assumptions $\mathcal{GC}_Q(V^{p/s},V^{p\vee 1+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (11)), $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p\vee 1+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (12)) and $\mathcal{SW}_{\mathcal{II},\gamma,\eta}(V^{p\vee 1+a-1})$ (see (13)) hold which are the hypothesis from Theorem 2.1 point **A.** with $g=V^{p\vee 1+a-1}$.

On the other hand, form Step 1. the assumption$\mathcal{RC}_{Q,V}(\psi_p,\phi,p\tilde{\alpha},p\beta)$ (see (6)) is satisfied for every $\tilde{\alpha}\in(0,\alpha)$. Moreover, since L_V (see (5)) holds and that $p/s+a-1>0$, then the hypothesis from Theorem 2.1 point **B.** are satisfied.

We thus conclude from Theorem 2.1 that $(\nu_n^\eta)_{n\in\mathbb{N}^*}$ is $\mathbb{P}-a.s.$ tight and (63) holds which concludes the proof of the first part of Theorem 4.1 point **A.**.

ii. Let us now prove the second part of Theorem 4.1 (see (64)) which is a consequence of Theorem 2.2.

On the one hand,we observe that from Step 2. and Step 3. the assumptions $\mathcal{GC}_Q(\mathcal{D}(A)_0,V^{p\vee 1+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (11)) and $\mathcal{SW}_{\mathcal{I},\gamma,\eta}(V^{p\vee 1+a-1},\rho,\epsilon_{\mathcal{I}})$ (see (12)) hold which are the hypothesis from Theorem 2.2 point **A.** with $g=V^{p\vee 1+a-1}$.

On the other hand, since b, σ, $|D\sigma|\operatorname{Tr}[\sigma\sigma^*]^{1/2}$, $\tilde{\sigma}$ and Ab have sublinear growth and that $g_\sigma \leqslant CV^{p/s+a-1}$, with $g_\sigma = \operatorname{Tr}[\sigma\sigma^*]+|D\sigma|\operatorname{Tr}[\sigma\sigma^*]^{1/2}+\operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*]^{1/2}$, so that $\mathbb{P}$-a.s. $\sup_{n\in\mathbb{N}^*}\nu_n^\eta(g_\sigma)<+\infty$, it follows from Proposition 4.2 that $\mathcal{E}(\widetilde{A},A,\mathcal{D}(A)_0)$ (see (8)) is satisfied. Then, the hypothesis from Theorem 2.2 point **B.** hold and (64) follows from (18).

Proof of Theorem 4.1 point B.

First we notice that using Theorem 4.1 point **A.**, then for every $f\in F_q$, $|\sigma^*Df|^2\in\mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d)$ and $\mathfrak{M}_q f\in\mathcal{C}_{\tilde{V}_{\psi_p,\phi,s}}(\mathbb{R}^d)$,

$$\mathbb{P}-a.s.\quad \lim_{n\to\infty}\nu_n^\gamma(|\sigma^*Df|^2)=\nu(|\sigma^*Df|^2)\qquad\text{and}\qquad \lim_{n\to\infty}\nu_n^{\tilde{\eta}_q}(\mathfrak{M}_q f)=\nu(\mathfrak{M}_q f).$$

Now, we notice that using Proposition 4.2, point **B.** and point **C.**, gives $\mathcal{E}_q(F_q,\tilde{A},A,\mathfrak{M}_q,\tilde{\eta}_q)$ (see (22)).

Moreover, Lemma 4.3 gives $\mathcal{GC}_{Q,q}(F_q,g,\rho,\epsilon_{\mathcal{X}},\epsilon_{\mathcal{GC}},\mathfrak{V})$ (see (26)) with $g=\operatorname{Tr}[\sigma\sigma^*]^\rho+|b|^\rho+|D\sigma|^\rho\operatorname{Tr}[\sigma\sigma^*]^{\rho/2}+\operatorname{Tr}[\tilde{\sigma}\tilde{\sigma}^*]^{\rho/2}+|Ab|^\rho$, $\epsilon_{\mathcal{X}}(\gamma)=\gamma$ and $\epsilon_{\mathcal{GC}}(\gamma)=\gamma^\rho$ for every $\gamma\in\mathbb{R}_+$, every $\rho\in[1,2]$, and with $\mathfrak{V}f=|\sigma^*Df|^2$. Since $\mathfrak{B}(\phi)$ (see (58)) holds, then $g\leqslant CV^{\rho a/2}$ and it follows that $\mathcal{GC}_{Q,q}(F_q,V^{p\vee 1+a-1},\tilde{\rho}_q,\epsilon_{\mathcal{X}},\epsilon_{\mathcal{GC}},\mathfrak{V})$ (see (26)) is satisfied.

Observing that $\mathcal{SW}_{\mathcal{GC},\gamma}(\tilde{\rho}_q,\gamma,\gamma)$ (see (29)) holds, the proof of Theorem 4.1 point **B.** is thus a direct consequence of Theorem 3.2 and Theorem 3.3.

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