HÖRMANDER PROPERTIES OF DISCRETE-TIME MARKOV PROCESSES

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Abstract. We present an abstract framework to establish smoothing properties for a class of inhomogeneous discrete-time Markov processes. These properties, in turn, serve as a basis to demonstrate the existence of density functions for our processes or more precisely, for regularized versions of them. We also use them to show the convergence in total variation to the solution of a stochastic differential equation as the time step between two observations of the discrete-time Markov processes tends to zero. The distinctive feature of our methodology lies in the exploration of smoothing properties under some local weak Hörmander-type conditions satisfied by the discrete-time Markov processes. Moreover, these Hörmander properties are consistent with the standard local weak Hörmander conditions satisfied by the coefficients of the stochastic differential equations that arise as total variation limits of the discrete-time Markov processes.

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Contents

1. Introduction

1.1. **Context.** In this article, we work on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and for $\delta\in(0,1]$ and $d,N\in\mathbb{N}^*$, we consider a sequence of independent random variables $Z^{\delta}_t\in\mathbb{R}^N$, $t\in\pi^{\delta,*}$ (we use the notations $\pi^{\delta}:=\delta\mathbb{N}$ and $\pi^{\delta,*}:=\delta\mathbb{N}^*$), which are assumed to be centered with an identity covariance matrix and Lebesgue lower bounded distribution (see (2.8) for the definition). Then, our focus lies on the $\mathbb{R}^d$-valued discrete-time Markov process $(X^{\delta}_t)_{t\in\pi^{\delta}}$ defined as follows:

$$X^{\delta}_{t+\delta}=\psi(X^{\delta}_t,t,\delta^{\frac{1}{2}}Z^{\delta}_{t+\delta},\delta),\quad t\in\pi^{\delta},\quad X^{\delta}_0\in\mathbb{R}^d, \tag{1.1}$$

where $\psi:(x,t,z,y)\mapsto\psi(x,t,z,y)\in\mathcal{C}^{\infty}(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}^N\times[0,1];\mathbb{R}^d)$. Our primary challenge is to demonstrate that, under suitable properties on ψ, we can construct a process $(\bar{X}^{\delta}_t)_{t\in\pi^{\delta}}$ that is arbitrarily close to $(X^{\delta}_t)_{t\in\pi^{\delta}}$ in total variation distance (for any fixed $t\in\pi^{\delta}$). Additionally, this process satisfies the smoothing/regularization property: For every $\alpha,\beta\in\mathbb{N}^d$, there exists $C:\mathbb{R}^d\times\pi^{\delta,*}\to\mathbb{R}_+$ (which does not depend on δ) such that for every $T\in\pi^{\delta,*}$, $x\in\mathbb{R}^d$ and every $f\in\mathcal{C}^{\infty}(\mathbb{R}^d;\mathbb{R})$, bounded,

$$|\partial^{\alpha}_x\mathbb{E}[\partial^{\beta}f(\bar{X}^{\delta}_T)|X^{\delta}_0=x]|\leqslant C(x,T)\|f\|_{\infty}. \tag{1.2}$$

A refined version of this result is exposed in Theorem 2.1. Relying on those regularization properties, we can infer that $\bar{X}^{\delta}_t$, $t\in\pi^{\delta}$, admits a smooth density (see Corollary 2.1). A main application of those results is provided in Theorem 2.2, where we identify a total variation limit (along with explicit rate of convergence) for X^{δ}_t, $t\in\pi^{\delta}$, as δ tends to zero. This weak limit random variable is given by the solution, at time t, of the Stochastic Differential Equation (SDE),

$$X_t=X^{\delta}_0+\int_0^t V_0(X_s,s)\mathrm{d}s+\sum_{i=1}^{N}\int_0^t V_i(X_s,s)\mathrm{d}W^i_s, \tag{1.3}$$

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where $((W^i_t)_{t\geqslant 0}, i \in \{1,\dots,N\})$ are N independent $\mathbb{R}$-valued standard Brownian motions and $V_0 := \partial_y\psi(.,.,0,0) + \frac{1}{2}\sum_{i=1}^N \partial^2_{z^i}\psi(.,.,0,0)$, $V_i = \partial_{z^i}\psi(.,.,0,0)$, $i \in \{1,\dots,N\}$.
More particularly, we show that, for $\epsilon > 0$, for $T \in \pi^\delta$, $T \geqslant 2\delta$, if $X_0 = X^\delta_0 = x \in \mathbb{R}^d$,

$$
\begin{aligned}
d_{TV}(\mathrm{Law}(X_T), \mathrm{Law}(X^\delta_T)) =& \frac{1}{2} \sup_{f:\mathbb{R}^d\to[-1,1], f \text{ measurable}} |\mathbb{E}[f(X_T) - f(X^\delta_T)|X_0 = X^\delta_0 = x]| \\
\leqslant& \delta^{\frac{1}{2}-\epsilon} \frac{1+|x|^{\mathfrak{c}}_{\mathbb{R}^d}}{(\boldsymbol{\mathcal{V}}_L(x)T)^\eta} C\exp(CT),
\end{aligned}
\tag{1.4}
$$

where c, C, η are positive constants and $\boldsymbol{\mathcal{V}}_L(x) \in (0,1]$ under a local weak Hörmander-type property (of order L, see (2.5) for details and definition of $\boldsymbol{\mathcal{V}}_L$) at initial point x. Moreover, the rate $\delta^{\frac{1}{2}-\epsilon}$ can be replaced by $\delta^{1-\epsilon}$ if the third order moments of Z^δ_t, $t \in \pi^{\delta,*}$, are supposed to be zero. In addition ϵ can be set to 0 when the Hörmander property is uniform. Consequently, X_t admits a density which can be approximated (uniformly on compact sets) by the one of $\bar{X}^\delta_t$. Similar estimates also hold for the derivatives of the density. Those results are derived under polynomial type upper bounds on the derivatives of ψ in conjunction with the aforementioned local weak Hörmander-type property.

Processes such as $(X^\delta_t)_{t\in\pi^\delta}$ commonly arise in weak approximation problems, although the perspective differs from that adopted in the present work. The problem is to consider a process $(X_t)_{t\geqslant 0}$ that is a solution to a given SDE similar to (1.3). Subsequently, the aim is to build the weak approximation process $(X^\delta_t)_{t\in\pi^\delta}$ and then compute an approximation for $\mathbb{E}[f(X_t)]$ by means of $\mathbb{E}[f(X^\delta_t)]$. Two interconnected questions naturally arise. First, what is the rate of convergence of the approximation as δ tends to zero? Second, for which class of functions f does this rate hold? Among others, this paper addresses those questions by providing an upper bound for the total variation distance (that is when f is bounded and measurable) with rate $\delta^{\frac{1}{2}-\epsilon}$. It is worth noting that this rate could be improved to $\delta^{1-\epsilon}$ or even $\delta^{m-\epsilon}$, $m \in \mathbb{N}$, regarding some conditions on Z^δ_t, $t \in \pi^{\delta,*}$ and ψ. Considering smooth f bounded with bounded derivatives up to some given order, it is well established that the weak convergence of the Euler scheme ($\psi(x,t,z,y) = V_0(x,t)y + \sum_{i=1}^N V_i(x,t)z^i$) occurs with rate δ (see [32]), but various higher order methods (see *e.g.* [31], [23], [1]) propose better rates (that we refer to as weak smooth rates). This naturally raises the following question. Do these weak smooth rates still apply to convergence in total variation? In the case of the Euler scheme with Gaussian increments, the convergence in total variation with order δ is established in [8] in a homogeneous and uniform weak Hörmander setting. For higher order methods, a solution combining the use of existing results concerning weak smooth rates and regularization properties similar to (1.2) is provided in [7]. In this article, it is shown that for $(X^\delta_t)_{t\in\pi^\delta}$ defined as in (1.1), the total variation rate aligns with the weak smooth rate under the restriction that ψ has smooth derivatives and satisfies a uniform elliptic property (*i.e.* uniform Hörmander property of order 0): For every $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, $\mathrm{span}(V_i, i \in \{1,\dots,N\})(x,t) = \mathbb{R}^d$.

The results [8] and [7] offer first insights for establishing convergence in total variation under Hörmander-type conditions for processes satisfying (1.1). The complexity of our approach relies both on the abstract definition (1.1) and in the weak Hörmander properties at any order L considered in a local setting. To provide clarity on our intentions, let us give some more details. To begin, we give an alternative formulation of (1.3) by employing the Stratonovich integral:

$$
X_t = X^\delta_0 + \int_0^t \bar{V}_0(X_s,s)\mathrm{d}s + \sum_{i=1}^N \int_0^t V_i(X_s,s)\circ \mathrm{d}W^i_s,
\tag{1.5}
$$

with $\bar{V}_0 = V_0 - \frac{1}{2}\sum_{i=1}^N \nabla_x V_i V_i$. In this article, $\bar{V}_0, V_i, i \in \{1,\dots,N\})$ and their derivatives are supposed to have polynomial growth in the space variable except for the order one derivatives in space which are simply bounded so that the existence of an *a.s.* unique solution to (1.5) is guaranteed. The infinitesimal generator of the Markov process $(X_t)_{t\geqslant 0}$ is expressed as $A = \bar{V}_0 + \frac{1}{2}\sum_{i=1}^N (V_i)^2$ where for a test function f, we use the abuse of notation $\bar{V}_0 f = \langle \bar{V}_0, \nabla f\rangle_{\mathbb{R}^d}$ and similarly for $V_i f$. As demonstrated in the seminal work [18], the hypoellipticity of $A+\partial_t$ and then the existence of a smooth density for X_t is closely related to the dimension of some Lie algebras generated with the vector fields $\bar{V}_0, V_i, i \in \{1,\dots,N\}$. This type of properties are referred to as Hörmander conditions, which we now introduce.

We consider, for fixed $t \geqslant 0$, the vector fields on $\mathbb{R}^d$ given by, $x \mapsto \bar{V}_0(x,t)$ and $x \mapsto V_i(x,t)$, $i \in \{1,\dots,N\}$. Subsequently, we introduce the extended vector fields on $\mathbb{R}^d \times \mathbb{R}_+$ denoted by $\bar{V}_{*,0} : (x,t) \mapsto (\bar{V}_0(x,t), 1)$ and $V_{*,i} : (x,t) \mapsto (V_i(x,t), 0)$, $i \in \{1,\dots,N\}$. In particular, the following relationship on the

Lie bracket holds: For V, W, two vector fields in $\{\bar{V}_0, V_1, \ldots, V_N\}$ and $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, $j \in \{1, \ldots, d+1\}$, the j-th components of $[V_*, W_*]$ satisfies

$$\begin{aligned}[V_*, W_*]^j(x,t) =&(\nabla_x WV(x,t) - \nabla_x VW(x,t))^j + \partial_t W_*^j V_*^{d+1}(x,t) - \partial_t V_*^j W_*^{d+1}(x,t)\\ =&[V,W]^j(x,t) + \partial_t W_*^j V_*^{d+1}(x,t) - \partial_t V_*^j W_*^{d+1}(x,t).\end{aligned}$$

It is worth noting that $x \mapsto [V,W](x,t)$ is a vector field on $\mathbb{R}^d$ and we use convention $[V,W]^{d+1} = 0$. We are now in a position to present the Hörmander properties which mainly consist in assuming that the vector fields generated by the Lie brackets are full in $\mathbb{R}^d$. Various versions of Hörmander properties appear in the literature serving to prove hypoellipticity. We try to give a brief overview. Let us introduce

$$\begin{aligned}\mathbf{V}_*^{(0)} =&\{V_{*,i}, i \in \{1, \ldots, N\}\}.\\ \mathbf{V}_*^{(n+1)} =&\mathbf{V}_*^{(n)} \cup \{[\bar{V}_{*,0}, V], [V_{*,i}, V], i \in \{1, \ldots, N\}, V \in \mathbf{V}_*^{(n)}\}, \quad n \in \mathbb{N}.\end{aligned}$$

Similarly, we define $\mathbf{V}^{(n)}$, $n \in \mathbb{N}$, in the same way but with $\bar{V}_{*,0}$ (respectively $V_{*,1}, \ldots, V_{*,N}$) replaced by $\bar{V}_0$ (resp. $V_1, \ldots, V_N$). The weak local Hörmander assumption (at initial point $(X_0 = x, 0)$) in an inhomogeneous setting (*i.e.* when $V_0, \ldots, V_N$ depend on time), which is the one we use in this paper, consists in assuming that

$$\text{span}(\cup_{n=0}^{\infty} \mathbf{V}_*^{(n)}(x,0)) = \mathbb{R}^{d+1}.$$

In the homogeneous setting (*i.e.* $V_0, V_1, \ldots, V_N$ do not depend on the time component), it consists in assuming that: $\text{span}(\cup_{n=0}^{\infty} \mathbf{V}^{(n)})(x,0) = \mathbb{R}^d$ (see *e.g.* [21]). Obviously, if the coefficients $V_0, V_1, \ldots, V_N$ do not depend on the time component, this last condition is equivalent to assume that $\text{span}(\cup_{n=0}^{\infty} \mathbf{V}_{*,n} \cup (0, \ldots, 0, 1))(x,0) = \mathbb{R}^{d+1}$.

Notice that, when $\text{span}(\mathbf{V}^{(0)}) = \mathbb{R}^d$, we are in the elliptic setting. The hypothesis is referred to as "local" Hörmander because $\mathbf{V}_*^{(n)}$ is considered at the initial point $(X_0 = x, 0)$. In the case where, uniformly in $(u,t) \in \mathbb{R}^d \times \mathbb{R}_+$, we have $\text{span}(\cup_{n=0}^{\infty} \mathbf{V}_*^{(n)})(u,t) = \mathbb{R}^{d+1}$, we refer to it as "uniform" Hörmander property. The term "weak" Hörmander pertains to the definition of $\mathbf{V}_*^{(n)}$ (or $\mathbf{V}^{(n)}$) which differs from the "strong" Hörmander property where $\bar{V}_{*,0}$ is replaced by 0 in the computation of $\mathbf{V}_*^{(n)}$. The investigation of Hörmander properties in an inhomogeneous setting is, for example, conducted to prove existence of smooth density in [12] or [13] for the weak uniform setting, in [11] for the strong local setting or in [19] or [28] for the weak local setting. For the homogeneous case, refer *e.g.* to [21], [25], [6] or [27] for applications of local weak Hörmander properties. We finally point out that, by Remark 2.1. in [33], concerning the uniform Hörmander setting for SDE with inhomogeneous coefficient, hypoellipticity may not hold if only $\text{span}(\cup_{n=0}^{\infty} \mathbf{V}^{(n)}) = \mathbb{R}^d$.

The results presented in this paper offer, among others, the opportunity to extend the abstract framework from [7] so that, it can be applied to the total variation approximation of inhomogeneous SDEs having polynomial bounds on their coefficients and their derivatives and satisfying the usual weak local Hörmander property. In terms of the function ψ, it simply consists in supposing a weak local Hörmander-type property (see (2.5)) and assuming polynomial growth properties on the derivatives of ψ (see (2.2) and (2.3)). In the homogeneous case, those assumptions are similar to the ones made in [21] concerning the coefficients of (1.5). We also highlight that the regularization properties established in this current paper (see Theorem 2.1), enable us to demonstrate that the total variation rate of convergence in the local weak hypoelliptic setting, mimicks the weak smooth rate. (see Remark 2.2). Total variation convergence with high rates of convergence can thus be obtained for the methods presented *e.g.* in [31], [23] or [1].

Similar results have previously been explored but only restricted to the case where $(Z_t^\delta)_{t \in \pi^{\delta,*}}$ is made of standard Gaussian variables and for some specific ψ (see *e.g.* [8] when ψ is the Euler scheme of a homogeneous SDE satisfying weak uniform Hörmander property). In particular standard Malliavin calculus can be applied to derive convergence in total variation. It is worth mentioning that analogous results are also investigated under a different (and weaker) condition from the Hörmander one, called the UFG condition, but we do not discuss this type of hypothesis in this paper (see *e.g.* [20] for an order two rate scheme still in the homogeneous setting). In [8], the methodology differs from ours in the sense that the estimates are obtained relying on the proximity (in the L^p-sense for Sobolev norms built with Malliavin derivatives) between a well chosen coupling of the scheme $(X_t^\delta)_{t \in \pi^\delta}$ and the limit $(X_t)_{t \geqslant 0}$ which satisfies standard regularization results under suitable properties (see *e.g.* [21]). More particularly, a continuous time version of $(X_t^\delta)_{t \in \pi^\delta}$ which satisfies a similar SDE as (1.3) (but with frozen coefficients) can be built. In this Gaussian context, the Malliavin calculus techniques are well known and used by the authors to bound the Sobolev norms. Conversely, our approach is self-contained and regularization properties for $(\bar{X}^{\delta}_t)_{t\in\pi^{\delta}}$ are derived without using the ones satisfied by $(X_t)_{t\geqslant 0}$. Our techniques draw inspiration from Malliavin calculus, while being tailored to a discrete framework and extending beyond the Gaussian case, as the law of $(Z^{\delta}_t)_{t\in\pi^{\delta,*}}$ may be arbitrary. Due to the liberty granted to the choice of ψ and to the law of $(Z^{\delta}_t)_{t\in\pi^{\delta,*}}$, our result may be seen as an invariance principle. In particular, the law of X_t depends on ψ only through its first order derivative in y and first and second order derivatives in z evaluated at some points $(x,t,0,0)$, with $x\in\mathbb{R}^d, t\geqslant 0$. Hence a similar limit is reached for a large class of function ψ and random variables $(Z^{\delta}_t)_{t\in\pi^{\delta,*}}$.

1.2. **Organization of the paper.** Section 2 presents the main results of this paper, namely Theorem 2.1, Corollary 2.1, and Theorem 2.2. These results focus on the regularization properties of discrete-time Markov processes of the form (1.1) and on the invariance principle they satisfy. Section 3 develops a Malliavin-inspired discrete differential calculus, which constitutes the main analytical tool used to prove the results stated in Section 2. In particular, in Theorem 3.4, we establish some crucial regularization properties for a well-chosen modification of Q^{δ}. Finally, Section 4 is devoted to establishing estimates on Malliavin-Sobolev norms and moments of the Malliavin covariance matrix. These estimatesare the main ingredient of proof of Theorem 3.4.

1.3. **Notations.** We denote by $\mathcal{M}(\mathbb{R}^d)$ (respectively $\mathcal{M}_b(\mathbb{R}^d)$), the set of measurable (resp. measurable and bounded) functions defined on $\mathbb{R}^d$. For $q\in\mathbb{N}$, $\mathcal{C}^q_b(\mathbb{R}^d)$ (resp. $\mathcal{C}^q_K(\mathbb{R}^d)$, $\mathcal{C}^q_{pol}(\mathbb{R}^d)$), $q\in\mathbb{N}\cup\{+\infty\}$, is the set of functions belonging to $\mathcal{C}^q(\mathbb{R}^d)$ such that all the derivatives (of order 0 to q) are bounded (resp. have compact support, have polynomial growth). We will also denote $\mathcal{M}(\mathbb{R}^d;\mathbb{R})$ for the set of measurable functions defined on $\mathbb{R}^d$ and taking values in $\mathbb{R}$ (and similarly for other sets of functions defined above). When dealing with functions defined and taking values on Hilbert spaces, we introduce some notations: Let $\mathcal{H},\mathcal{H}'$ be two Hilbert spaces. For $f:\mathcal{H}\to\mathcal{H}'$ and $u\in\mathcal{H}$, the directional derivative $\mathscr{D}_u f$ of f along u is given by (when it exists) $\mathscr{D}_u f(x):=\lim_{\epsilon\to 0}\frac{f(x+\epsilon u)-f(x)}{\epsilon}$ for every $x\in\mathcal{H}$. When f is Fréchet differentiable, we recall that its derivative $u\mapsto\mathscr{D}_u f(x)$ is a linear application from $\mathcal{H}$ to $\mathcal{H}'$ that we simply denote $\mathscr{D} f(x)$. When $\mathcal{H}'=\mathbb{R}$, $\mathscr{D} f(x)$ is uniquely defined by Riesz theorem and for every $u\in\mathcal{H}$, $\mathscr{D}_u f(x)=\langle\mathscr{D} f(x),u\rangle_{\mathcal{H}}$. For $f\in\mathcal{M}_b(\mathbb{R}^d;\mathbb{R}^{d'})$, we introduce the supremum norm $\|f\|_{\infty}=\sup_{x\in\mathbb{R}^d}|f(x)|_{\mathbb{R}^{d'}}$ with $|.|_{\mathbb{R}^{d'}}$ the standard Euclidean norm on $\mathbb{R}^{d'}$. When f takes values in $\mathbb{R}^{d'\times d'}$, we denote $\|f\|_{\mathbb{R}^{d'}}=\sup_{\xi\in\mathbb{R}^{d'};|\xi|_{\mathbb{R}^{d'}}=1}|f\xi|_{\mathbb{R}^{d'}}$. For a multi-index $\alpha=(\alpha^1,\cdots,\alpha^d)\in\mathbb{N}^d$ we denote $|\alpha|=\alpha^1+...+\alpha^d$, $\|\alpha\|=d$ and if $f\in\mathcal{C}^{|\alpha|}(\mathbb{R}^d)$, we define $\partial^{\alpha}f=\partial^{\alpha}_x f(x)=\partial^{\alpha^1}_{x^1}\ldots\partial^{\alpha^d}_{x^d}f(x)$. Also, for $\beta\in\mathbb{N}^{d'}$, we define $(\alpha,\beta)=(\alpha^1,\cdots,\alpha^d,\beta^1,\ldots,\beta^{d'})$. In addition, we also denote $\nabla_x f=(\partial_{x^j}f_i)_{(i,j)\in\{1,\ldots,d'\}\times\{1,\ldots,d\}}$ for the Jacobian matrix of f and $\nabla^2_x f=((\partial_{x^j}\partial_{x^l}f^i)_{(l,j)\in\{1,\ldots,d\}\times\{1,\ldots,d\}})_{i\in\{1,\ldots,d'\}}$ for the Hessian matrix of f. In particular, for $v\in\mathbb{R}^d$, $v^{\mathrm{T}}\nabla^2_x f\in\mathbb{R}^{d'\times d}$ and $(v^{\mathrm{T}}\nabla^2_x f)^{i,j}=\sum_{l=1}^d\partial_{x^j}\partial_{x^l}f^i v^l$. We include the multi-index $\alpha=(0,...,0)$ and in this case $\partial^{\alpha}f=f$.
In addition, in our estimates, C stands for a constant which can change from line to line, and given some parameter $\vartheta=(\vartheta_1,\ldots,\vartheta_l)\in\mathbb{R}^l$, $l\in\mathbb{N}$, we sometimes emphasize the dependence of a constant C *w.r.t.* ϑ by denoting $C(\vartheta)$ (or by stating that the constant depends on ϑ) implying also that the constant can possibly tend to infinity if ϑ_i tends to infinity for some $i\in\{1,\ldots,l\}$. When we state that a constant depends on the moments of the random variables $Z^{\delta}_t, t\in\pi^{\delta,*}, t\leqslant T$, we precisely mean that it depends on the moments up to a given order that itself does not depend on δ, T or X^{δ}_0. Also, $\mathbf{1}_{a,b}$ stands for the Kronecker symbol, meaning $\mathbf{1}_{a,b}=1$ if $a=b$ and is zero otherwise.
Finally, for a discrete-time process $(Y_t)_{t\in\pi^{\delta}}$, we denote by $\mathcal{F}^Y_t:=\sigma(Y_w, w\in\pi^{\delta}, w\leqslant t)$ the sigma algebra generated by Y until time t. Also, for a random variable F, we adopt the short notation $\mathbb{E}_x[F]=\mathbb{E}[F|X_0=X^{\delta}_0=x]$.

2. Main results

In this section, we present our main result about the regularization properties of $(X^{\delta}_t)_{t\in\pi^{\delta}}$. Once the regularization results are established (Theorem 2.1), we infer the existence of a total variation limit for X^{δ}_t, for fixed $t\in\pi^{\delta}$, in terms of a solution to a specific SDE (Theorem 2.2).

2.1. A Class of Markov Semigroups.

Definition of the semigroups. For $\delta\in(0,1]$ and $N\in\mathbb{N}^*$, we consider a sequence of independent random variables $Z^{\delta}_t\in\mathbb{R}^N$, $t\in\pi^{\delta,*}$, and we assume that Z^{δ}_t, are centered with $\mathbb{E}[Z^{\delta,i}_t Z^{\delta,j}_t]=\mathbf{1}_{i,j}$ for every $i,j\in\mathcal{N}:=\{1,\ldots,N\}$ and every $t\in\pi^{\delta,*}$. We construct the $\mathbb{R}^d$-valued Markov process $(X^\delta_t)_{t\in\pi^\delta}$ in the following way:

$$X^\delta_{t+\delta}=\psi(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta},\delta),\quad t\in\pi^\delta,\quad X^\delta_0=\mathrm{x}^\delta_0\in\mathbb{R}^d, \tag{2.1}$$

where

$$\psi\in\mathcal{C}^\infty(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}^N\times[0,1];\mathbb{R}^d)\quad\text{and}\quad\forall(x,t)\in\mathbb{R}^d\times\pi^\delta,\psi(x,t,0,0)=x.$$

Let us now define the discrete-time semigroup associated to $(X^\delta_t)_{t\in\pi^\delta}$. For every measurable function f from $\mathbb{R}^d$ to $\mathbb{R}$, and every $x\in\mathbb{R}^d$,

$$\forall t\in\pi^\delta,\qquad Q^\delta_t f(x)=\int_{\mathbb{R}^d}f(y)Q^\delta_t(x,\mathrm{d}y):=\mathbb{E}[f(X^\delta_t)|X^\delta_0=x].$$

We will obtain regularization properties for modifications of this discrete semigroup. Our approach relies on some hypothesis on ψ and Z^δ we now present.

Hypothesis on ψ. Polynomial growth and Hörmander property. We first consider a polynomial growth assumption concerning the derivatives of ψ: We introduce some non-decreasing sequences $(\mathbf{K}_q)_{q\in\mathbb{N}^*}$ and $(\mathbf{p}_q)_{q\in\mathbb{N}^*}$, $\mathbf{K}_q\geqslant1,\mathbf{p}_q\in\mathbb{N}$. For $r\in\mathbb{N}^*$,

$\mathbf{A}^\delta_1(r)$. We assume that for every $(x,t,z,y)\in\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}^N\times[0,1]$,

$$\sum_{|\alpha^x|+|\alpha^t|=0}^{r}\ \sum_{|\alpha^z|+|\alpha^y|=1}^{r-|\alpha^x|-|\alpha^t|}|\partial_x^{\alpha^x}\partial_t^{\alpha^t}\partial_z^{\alpha^z}\partial_y^{\alpha^y}\psi|_{\mathbb{R}^d}(x,t,z,y)\leqslant\mathbf{K}_r(1+|x|_{\mathbb{R}^d}^{\mathbf{p}_r}+\delta^{-\frac{\mathbf{p}_r}{2}}|z|_{\mathbb{R}^N}^{\mathbf{p}_r}), \tag{2.2}$$

$$\{\sum_{l=1}^d|\partial_{x^l}\partial_y\psi|_{\mathbb{R}^d}+\sum_{i\in\mathcal{N}}|\partial_{x^l}\partial_{z^i}\psi|_{\mathbb{R}^d}+\sum_{i,j\in\mathcal{N}}|\partial_{x^l}\partial_{z^i}\partial_{z^j}\psi|_{\mathbb{R}^d}\}(x,t,z,y)\leqslant\mathbf{K}_3(1+\delta^{-\frac{\mathbf{p}_3}{2}}|z|_{\mathbb{R}^N}^{\mathbf{p}_3}). \tag{2.3}$$

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We denote $\mathbf{A}^\delta_1(+\infty)$ when $\mathbf{A}^\delta_1(r)$ is satisfied for every $r\in\mathbb{N}^*$. We note that we obtain exactly the same results if we add $\mathbf{K}_3\,\delta^{-1}|y|$ in the *r.h.s.* of (2.3), or if we add $\mathbf{K}_r\,\delta^{-1}|y|$ in the *r.h.s.* of (2.2). This is due to the fact that the function ψ is only used for $y=\delta$ (or $y=C\delta$, $C\leqslant1$) so the assumptions above are then satisfied replacing $\mathbf{K}_3$ (respectively $\mathbf{K}_r$) by $2\,\mathbf{K}_3$ (respectively $2\,\mathbf{K}_r$). Also, we do not give the explicit dependence of the *r.h.s* of (2.2) or (2.3) *w.r.t.* the variable t because in our results, t is taken in a compact interval of the form $[0,T]$.

At this point, let us observe that we can compare this assumption with the one in [21] where the authors directly study the existence of the density of the solution of (1.3) by means of standard Malliavin calculus but when coefficients do not depend on time. Taking ψ linear in its third and fourth variable, and homogeneous, *i.e.* $\psi:(x,t,z,y)\mapsto x+V_0(x)y+\sum_{i\in\mathcal{N}}V_i(x)z^i$ then, exactly $\mathbf{A}^\delta_1(+\infty)$ is the regularity assumption made on $V_0,\ldots,V_N$ in [21] (combined with a weak local Hörmander property) to derive similar estimates as (2.1) in Corollary 2.1.

The second hypothesis we need on ψ is local weak Hörmander property on some vector fields we now introduce. We denote the Lie bracket of two $\mathcal{C}^1$ vector fields in $\mathbb{R}^d$, $[\cdot,\cdot]:(\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d))^2\to\mathcal{C}^0(\mathbb{R}^d,\mathbb{R}^d)$, $f_1,f_2\mapsto[f_1,f_2]:=\nabla_xf_2f_1-\nabla_xf_1f_2$.
We denote $V_0=\partial_y\psi(\cdot,\cdot,0,0)+\frac{1}{2}\sum_{i\in\mathcal{N}}\partial^2_{z^i}\psi(\cdot,\cdot,0,0)$, $V_i=\partial_{z^i}\psi(\cdot,\cdot,0,0)$, $i\in\mathcal{N}$, $\bar{V}_0=V_0-\frac{1}{2}\sum_{i\in\mathcal{N}}\nabla_xV_iV_i$.
For a multi-index $\alpha\in\mathcal{N}^{\|\alpha\|}$ and $V:\mathbb{R}^d\times\mathbb{R}_+\to\mathbb{R}^d$, we define also $V^{[\alpha]}$ using the recurrence relation $V^{[(\alpha,0)]}=[\bar{V}_0,V^{[\alpha]}]+\partial_tV^{[\alpha]}+\frac{1}{2}\sum_{i\in\mathcal{N}}[V_i,[V_i,V^{[\alpha]}]]$ and $V^{[(\alpha,j)]}:=[V_j,V^{[\alpha]}]$ if $j\in\mathcal{N}$ with the convention $V^{[\emptyset]}=V$. We are now in a position to introduce our Hörmander hypothesis on ψ: For $L\in\mathbb{N}$, the order of our Hörmander condition, let us define for every $(x,t)\in\mathbb{R}^d\times\mathbb{R}_+$,

$$\boldsymbol{\mathcal{V}}_L(x,t):=1\wedge\inf_{b\in\mathbb{R}^d,|b|_{\mathbb{R}^d}=1}\sum_{i\in\mathcal{N}}\sum_{l=0}^L\sum_{\alpha\in\{0,\ldots,N\}^l}\langle V_i^{[\alpha]}(x,t),b\rangle^2_{\mathbb{R}^d}. \tag{2.4}$$

We introduce, for $x\in\mathbb{R}^d$:

$\mathbf{A}_2(x,L)$. A local weak Hörmander property of order $L\in\mathbb{N}$,

$$\boldsymbol{\mathcal{V}}_L(x,0)>0. \tag{2.5}$$

Especially, this hypothesis is used at initial point for $x = X_0^\delta$. We will sometimes consider a uniform weak Hörmander property of order L,

$$\boldsymbol{\mathcal{V}}_L^\infty := \inf_{t\in\mathbb{R}_+} \inf_{u\in\mathbb{R}^d} \boldsymbol{\mathcal{V}}_L(u,t) > 0. \tag{2.6}$$

In this case, we denote $\mathbf{A}_2^\infty(L)$ instead of $\mathbf{A}_2(x,L)$. Also, we denote $\boldsymbol{\mathcal{V}}_L(x) := \boldsymbol{\mathcal{V}}_L(x,0)$.

It is worth noticing that, with the notations introduced in the Introduction, (2.5) is satisfied for some $L \in \mathbb{N}$ if and only if $\mathrm{span}(\cup_{n=0}^\infty \mathbf{V}_*^{(n)})(x,0) = \mathbb{R}^d$, which is why, we refer to it as local weak Hörmander property. A similar observation holds for (2.6) in the uniform setting. The case $L = 0$ corresponds to the elliptic case.

Hypothesis on Z^δ. Lebesgue lower bounded distributions. A first assumption concerns the finiteness of the moment of Z^δ: For $p \geqslant 0$,

$\mathbf{A}_3^\delta(p)$.

$$\mathbf{M}_p(Z^\delta) := 1 \vee \sup_{t\in\pi^{\delta,*}} \mathbb{E}[|Z_t^\delta|_{\mathbb{R}^N}^p] < \infty. \tag{2.7}$$

We denote $\mathbf{A}_3^\delta(+\infty)$ the assumption such that $\mathbf{A}_3^\delta(p)$ is satisfied for every $p \geqslant 0$.

A second assumption is made on the distribution of Z^δ. We suppose that the distribution of Z^δ is Lebesgue lower bounded:

$\mathbf{A}_4^\delta$. There exists $z_* = (z_{*,t})_{t\in\pi^{\delta,*}}$ taking its values in $\mathbb{R}^N$ and $\varepsilon_*, r_* > 0$ such that for every Borel set $A \subset \mathbb{R}^N$ and every $t \in \pi^{\delta,*}$,

$$\mathbb{P}(Z_t^\delta \in A) \geqslant \varepsilon_* \lambda_{\mathrm{Leb}}(A \cap B_{r_*}(z_{*,t})), \tag{2.8}$$

where λ_{Leb} is the Lebesgue measure on $\mathbb{R}^N$.

Let us comment on the assumption $\mathbf{A}_4^\delta$. First, notice that (2.8) holds if and only if there exists some non-negative measures μ_t^δ with total mass $\mu_t^\delta(\mathbb{R}^N) < 1$ and a lower semi-continuous function $\varphi \geqslant 0$ such that $\mathbb{P}(Z_t^\delta \in dz) = \mu_t^\delta(dz) + \varphi(z - z_{*,t})dz$ for every $t \in \pi^{\delta,*}$. We also point out that the random variables $(Z_t^\delta)_{t\in\pi^{\delta,*}}$ are not assumed to be identically distributed. However, the fact that $r_* > 0$ and $\varepsilon_* > 0$ are the same for all k represents a mild substitute of this property. In order to construct φ, we introduce the following function: For $v > 0$, set $\varphi_v : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\varphi_v(z) = \mathbf{1}_{|z|_{\mathbb{R}^N} \leqslant v} + \exp\Big(1 - \frac{v^2}{v^2 - (|z|_{\mathbb{R}^N} - v)^2}\Big) \mathbf{1}_{v<|z|_{\mathbb{R}^N}<2v}. \tag{2.9}$$

Then $\varphi_v \in \mathcal{C}_b^\infty(\mathbb{R}^N;\mathbb{R})$, $0 \leqslant \varphi_v \leqslant 1$ and we have the following crucial property: For every $p, q \in \mathbb{N}$, every $z \in \mathbb{R}^N$

$$|\sum_{\substack{\alpha^z\in\mathbb{N}^N \\ |\alpha^z|\in\{1,\ldots,q+1\}}} |\partial_z^{\alpha^z} \ln \varphi_v(z)|^2|^{\frac{p}{2}} \varphi_v(z) \leqslant \frac{C(q,p)N^{\frac{pq}{4}}}{v^{pq}}, \tag{2.10}$$

with the convention $\ln \varphi_v(z) = 0$ for $|z| \geqslant 2v$. As an immediate consequence of (2.8), for every non negative function $f : \mathbb{R}^N \to \mathbb{R}_+$ and $t \in \pi^\delta$, $t > 0$,

$$\mathbb{E}[f(Z_t^\delta)] \geqslant \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(\ z - z_{*,t}\ ) f(z) \mathrm{d}z.$$

We denote

$$m_* = \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(z) dz = \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(z - z_{*,t}) \mathrm{d}z \in (0,1].$$

We consider a sequence of independent random variables $\chi_t^\delta \in \{0,1\}$, $U_t^\delta, V_t^\delta \in \mathbb{R}^N$, $t \in \pi^{\delta,*}$ (also independent from W the Brownian motion running (1.3), with laws given by

$$\begin{aligned}
\mathbb{P}(\chi_t^\delta = 1) =& m_*, \qquad \mathbb{P}(\chi_t^\delta = 0) = 1 - m_*, \\
\mathbb{P}(\delta^{-\frac{1}{2}} U_t^\delta \in \mathrm{d}z) =& \frac{\varepsilon_*}{m_*} \varphi_{r_*/2}(z - z_{*,t}) \mathrm{d}z, \\
\mathbb{P}(\delta^{-\frac{1}{2}} V_t^\delta \in \mathrm{d}z) =& \frac{1}{1 - m_*} (\mathbb{P}(Z_t^\delta \in \mathrm{d}z) - \varphi_{\frac{r_*}{2}}(z - z_{*,t}) \mathrm{d}z),
\end{aligned}$$

where $\varphi_{\frac{r_*}{2}}$ satisfies (2.10) with $v=\frac{r_*}{2}$. Notice that $\mathbb{P}(V_t^\delta\in \mathrm{d}z)\geqslant 0$ and a direct computation shows that

$$\mathbb{P}(\chi_t^\delta U_t^\delta+(1-\chi_t^\delta)V_t^\delta\in dz)=\mathbb{P}(\delta^{\frac{1}{2}}Z_t^\delta\in \mathrm{d}z).$$

This is the splitting procedure for Z_t^δ. From now on, we will work with this representation of the law of Z_t^δ. Consequently, we always use the decomposition

$$\delta^{\frac{1}{2}}Z_t^\delta=\chi_t^\delta U_t^\delta+(1-\chi_t^\delta)V_t^\delta.$$

Consequently, throughout this paper, we work on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ where $\mathcal{F}$ is generated by $(\chi_t^\delta,U_t^\delta,V_t^\delta)_{t\in\pi^{\delta,*}}$ and W. The above splitting procedure has already been widely used in the literature and is usually referred to as the Nummelin splitting. In [26] and [22], it is used in order to prove convergence to equilibrium of Markov processes. In [9], [10] and [35], it is used to study the Central Limit Theorem. Also, in [24], the above splitting method (with $\mathbf{1}_{B_{r_*}(z_{*,t})}$ instead of $\varphi_{r_*/2}(z-z_{*,t})$) is used in a framework which is similar to the one in this paper. Finally in [7], it is used to prove regularization properties of Markov semigroup under the uniform ellipticity property: $\inf_{(x,t)\in\mathbb{R}^d\times\pi^\delta}\boldsymbol{\mathcal{V}}_0(x,t)>0$.

We introduce a final purely technical assumption ensuring that the time step δ is small enough. For $\delta\in(0,1]$, when (2.3) holds, we first define

$$\begin{aligned}\boldsymbol{\eta}_1(\delta):=&\delta^{-d\frac{44}{89}}(\mathbf{1}_{L=0}+\mathbf{1}_{L>0}\frac{1}{2^5(1\vee T^3)^{\frac{1}{2}}})^{d13^{(L-1)}}\quad\text{and}\\ \boldsymbol{\eta}_2(\delta):=&\min(\delta^{-\frac{1}{2}}\,\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}},\frac{1}{2}|\delta^{\frac{1}{2}}8\,\mathbf{K}_3\,|^{-\frac{1}{\mathbf{p}_3+1}}),\end{aligned}\tag{2.11}$$

with $\mathbf{p}$ given in (2.3). For $T\in\pi^{\delta,*}$, $x\in\mathbb{R}^d$, we introduce the following assumption:

$\mathbf{A}_5^\delta(x,T)$. Assume that (2.3) and $\mathbf{A}_2(x,L)$ (see (2.5)) hold and that $\delta\in(0,(2^{\mathbf{p}+1}8\,\mathbf{K}_3)^{-4}]$ is small enough so that

$$\begin{aligned}\boldsymbol{\eta}_1(\delta)>&2\max((\frac{d}{2})^{\frac{d}{2}},(\frac{40(L+1)N^{\frac{L(L+1)}{2}}}{T\,\boldsymbol{\mathcal{V}}_L(x)m_*})^{d13^L},\\ &\mathbf{1}_{L=0}+\mathbf{1}_{L>0}(\frac{10}{m_*})^d(4^{\frac{13}{12}}(2^8(1\vee T))^{143}N^{\frac{L(L-1)}{2}})^{d13^{L-1}})\quad\text{and}\\ \boldsymbol{\eta}_2(\delta)>&1.\end{aligned}\tag{2.12}$$

Similarly as the assumption $\mathbf{A}_2(x,L)$, this hypothesis is used at initial point for $x=X_0^\delta$.

Considering the lower bound of $\boldsymbol{\eta}_1(\delta)$ in (2.12), it becomes apparent that while it remains independent of δ, it may take excessively large values. This dependence could potentially be decreased with modifications to the proof structure, but at the expense of possibly higher upper bounds on the semigroup's derivatives. In this paper, we tailor our proof to minimize the reliance of $C(x,T)$ in (1.2) with respect to $\boldsymbol{\mathcal{V}}_L(x)^{-1}$ and T^{-1}. Specifically, our proofs are designed so that the constant η appearing in Theorem 2.1, Corollary 2.1, and Theorem 2.2 are as small as possible. Explicit values for η are given in the proof of those results.

2.2. **An alternative regularization property.** In this section we provide the regularization property for a modified version of X^δ. We consider a d-dimensional standard (centered with covariance identity) Gaussian random variable G which is independent of $(Z_t^\delta)_{t\in\pi^{\delta,*}}$, and for $\theta>0$,

$$Q_T^{\delta,\theta}f(x)=\int_{\mathbb{R}^d}f(y)Q_T^{\delta,\theta}(x,\mathrm{d}y):=\mathbb{E}[f(X_T^\delta+\delta^\theta G)|X_0^\delta=x],\quad T\in\pi^\delta.\tag{2.13}$$

It can be seen as a regularization by convolution of the semigroup Q^δ. From a practical viewpoint, the modified version $X_T^\delta+\delta^\theta G$ is easily computable and then well adapted to simulation based approaches such as Monte Carlo methods.

Theorem 2.1. *Let $T\in\pi^{\delta,*}$, $L\in\mathbb{N}$ and $f\in\mathcal{M}(\mathbb{R}^d;\mathbb{R})$ satisfying the polynomial growth assumption: there exists $\mathbf{K}_f\geqslant 0$ and $\mathbf{p}_f\in\mathbb{N}$ such that for every $x\in\mathbb{R}^d$,*

$$|f(x)|\leqslant\mathbf{K}_f(1+|x|_{\mathbb{R}^d}^{\mathbf{p}_f}).$$

Then we have the following properties:

A. *Let $x \in \mathbb{R}^d$, $q \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{N}^d$ such that $|\alpha| + |\beta| \leqslant q$. Assume, that $\mathbf{A}_1^\delta(\max(q+3, 2L+5))$, $\mathbf{A}_2(x, L)$, $\mathbf{A}_3^\delta(+\infty)$, $\mathbf{A}_4^\delta$ and $\mathbf{A}_5^\delta(x, T)$ hold. Then, if f is $|\beta|$-times differentiable and $\partial^\beta f$ has polynomial growth,*

$$|\partial^\alpha Q_T^{\delta,\theta} \partial^\beta f(x)| \leqslant \mathbf{K}_f \, \frac{(1+|x|_{\mathbb{R}^d}^{\mathfrak{c}}) C \exp(CT)}{(\boldsymbol{\mathcal{V}}_L(x)T)^\eta}, \tag{2.14}$$

where $\eta \geqslant 0$ depends on d, L, q and θ and $\mathfrak{c}, C \geqslant 0$ depend on d, N, L, q, $\mathbf{K}_{\max(q+3,2L+5)}$, $\mathbf{p}_{\max(q+3,2L+5)}, \mathbf{p}_f, \frac{1}{m_*}, \frac{1}{r_*}, \theta$ and on the moment of Z^δ.

B. *Assume that $\mathbf{A}_1^\delta(2L+5)$, $\mathbf{A}_2(x, L)$, $\mathbf{A}_3^\delta(+\infty)$, $\mathbf{A}_4^\delta$ and $\mathbf{A}_5^\delta(x, T)$ hold. Then,*

$$|Q_T^\delta f(x) - Q_T^{\delta,\theta} f(x)| \leqslant \delta^\theta \, \mathbf{K}_f \, \frac{(1+|x|_{\mathbb{R}^d}^{\mathfrak{c}}) C \exp(CT)}{(\boldsymbol{\mathcal{V}}_L(x)T)^\eta}, \tag{2.15}$$

where $\eta \geqslant 0$ depends on d, L and θ and $\mathfrak{c}, C \geqslant 0$ depend on $d, N, L, q, \mathbf{K}_{2L+5}$, $\mathbf{p}_{2L+5}, \mathbf{p}_f$, $\frac{1}{m_*}, \frac{1}{r_*}, \theta$ and on the moment of Z^δ.

Remark 2.1. *We point out that, in the case where $\mathbf{p}_f = \mathbf{p}_r = 0$ for every $r \in \mathbb{N}^*$, then $\mathfrak{c} = 0$ in (2.14) and (2.15). This remark remains valid in Corollary 2.1 (see (2.16)) and Theorem 2.2 (see (2.18) but not (2.19)) stated later in this Section. Assuming further that $\mathbf{A}_2^\infty(L)$ holds, the upper bounds established in Theorem 2.1 thus become uniform w.r.t. x.*

A consequence of Theorem 2.1 concerns the existence of a bounded density with bounded derivatives for $X_T^\delta + \delta^\theta G$. The proof of this result is given in Section 3.3. Notice that an explicit value is given for η. This type of result is usually referred to as hypoellipticity property of the operator $Q^{\delta,\theta}$.

Corollary 2.1. *Let $T \in \pi^{\delta,*}$, $x \in \mathbb{R}^d$ and $L \in \mathbb{N}$. Let $q \in \mathbb{N}$, let $\alpha, \beta \in \mathbb{N}^d$ such that $|\alpha| + |\beta| \leqslant q$. Assume that $\mathbf{A}_1^\delta(\max(q+d+4, 2L+5))$, $\mathbf{A}_2(x, L)$, $\mathbf{A}_3^\delta(+\infty)$, $\mathbf{A}_4^\delta$ and $\mathbf{A}_5^\delta(x, T)$ hold.*

Then, for every $y \in \mathbb{R}^d$, $Q_T^{\delta,\theta}(x, dy) = q_T^{\delta,\theta}(x, y) dy$ and $q_T^{\delta,\theta} \in \mathcal{C}^q(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies, for every $p > 0$,

$$|\partial_x^\alpha \partial_y^\beta q_T^{\delta,\theta}(x, y)| \leqslant \frac{(1+|x|_{\mathbb{R}^d}^{\mathfrak{c}}) C \exp(CT)}{(\boldsymbol{\mathcal{V}}_L(x)T)^\eta (1+|y|_{\mathbb{R}^d}^p)}, \tag{2.16}$$

where $\eta \geqslant 0$ depends on d, L, q and θ and $\mathfrak{c}, C \geqslant 0$ depends on d, N, L, q, $\mathbf{K}_{\max(q+d+3,2L+5)}$, $\mathbf{p}_{\max(q+d+3,2L+5)}$, $\frac{1}{m_*}, \frac{1}{r_*}, \theta, p$ and on the moment of Z^δ.

Moreover, if $\mathbf{p}_2 = 0$ (see hypothesis $\mathbf{A}_1^\delta$) and there exists $z^\infty \geqslant 1$ such that a.s. $\sup_{t \in \pi^{\delta,*}} |Z_t^\delta|_{\mathbb{R}^N} \leqslant z^\infty$, then,

$$|\partial_x^\alpha \partial_y^\beta q_T^{\delta,\theta}(x, y)| \leqslant \frac{C \exp(CT)}{(\boldsymbol{\mathcal{V}}_L(x)T)^\eta} \exp\Big(-\frac{|y-x|_{\mathbb{R}^d}^2}{c(T \vee \delta^{2\theta})}\Big), \tag{2.17}$$

where η is the same as in (2.16), $c \geqslant 0$ depends on d, $\mathbf{K}_1$ and z^∞, and $C \geqslant 0$ depends on d, N, L, q, $\mathbf{K}_{\max(q+d+3,2L+5)}$, $\mathbf{p}_3$, $\frac{1}{m_*}, \frac{1}{r_*}, \theta$ and z^∞.

2.3. **An invariance principle.** Let us consider $(X_t)_{t \geqslant 0}$ the $\mathbb{R}^d$-valued Itô process solution to the SDE (1.3). In the following results, we show that, for a fixed $T > 0$, X_T^δ converges in total variation to X_T. Notably, our result is stronger than the convergence in total variation since we consider measurable test functions with polynomial growth. Moreover, we establish the existence of the density of $X_T$that can be approximated by the one of $X_T^\delta + \delta^\theta G$. In an ideal situation, we would like to approximate the density of X_T using the one of X_T^δ. However, due to the absence of regularization properties for the random variable X_T^δ, we cannot offer any assurance regarding the existence of its density. Actually, since the random variables $(Z_t^\delta)_{t \in \pi^{\delta,*}}$ do not necessarily have a density, we can easily build an example such that X_T^δ does not have a density, for instance by considering $X_T^\delta = \sum_{t \in \pi^{\delta,*}; t \leqslant T} Z_t^\delta$. In contrast, since $X_T^\delta + \delta^\theta G$ satisfies the regularization property, we can guarantee the existence of its density together with an upper bound on this density.

Exploiting Theorem 2.1 and Corollary 2.1, we can deduce the convergence of the law of X_T^δ towards the one of X_T as δ tends to zero. We are, among others, interested in obtaining an upper bound for

$$|\mathbb{E}_x[f(X_T) - f(X_T^\delta)]|,$$

which writes $C(x)\delta^m \sup_{u \in \mathbb{R}^d} |f(u)|$ when $f \in \mathcal{M}_b(\mathbb{R}^d)$ (and similarly when f has polynomial growth). One main technical point is that the upper bound does not depend on the derivatives of

f.

This result may be seen as an invariance principle under two aspects. First, the law of the limit X_T only depends on derivatives (of order one and two) of ψ evaluated at some points $(x,t,0,0)$ with $(x,t)\in\mathbb{R}^d\times\mathbb{R}_+$. As a consequence, if we replace ψ by any function $\tilde{\psi}$ giving the same evaluations of those derivatives, the limit of X^{δ}_T remains X_T. Another aspect is that the law of $(Z_t)_{t\in\pi^{\delta,*}}$ is not specified explicitly and can be chosen in a large set of probability measures. In particular, in the following result, we show that only $\mathbf{A}^{\delta}_3(+\infty)$ (see (2.7)) and $\mathbf{A}^{\delta}_4$ (see (2.8)) are assumed concerning the law of $(Z_t)_{t\in\pi^{\delta,*}}$.

Theorem 2.2. *Let $T\in\pi^{\delta}$, with $T\geqslant 2\delta$, $L\in\mathbb{N}$ and $m>0$. We have the following properties:*

A. *Let $f\in\mathcal{M}(\mathbb{R}^d;\mathbb{R})$ satisfying: there exists $\mathbf{K}_f\geqslant 0$ and $\mathbf{p}_f\in\mathbb{N}$ such that for every $x\in\mathbb{R}^d$,*

$$|f(x)|\leqslant \mathbf{K}_f(1+|x|^{\mathbf{p}_f}_{\mathbb{R}^d}).$$

Let $x\in\mathbb{R}^d$. Assume that $\mathbf{A}^{\delta}_1(\max(6,2L+5))$, $\mathbf{A}_2(x,L)$, $\mathbf{A}^{\delta}_3(+\infty)$, $\mathbf{A}^{\delta}_4$ and $\mathbf{A}^{\delta}_5(x,T)$ hold. Then, for every $\epsilon>0$,

$$|\mathbb{E}_x[f(X_T)-f(X^{\delta}_T)]|\leqslant\delta^{\frac{1}{2}-\epsilon}\,\mathbf{K}_f\,\frac{1+|x|^{c}_{\mathbb{R}^d}}{(\boldsymbol{\mathcal{V}}_L(x)T)^{\eta}}C\exp(CT),\tag{2.18}$$

where $\eta\geqslant 0$ depends on d,L and $\frac{1}{\epsilon}$ and $c,C\geqslant 0$ depend on d,N,L, $\sup_{r\in\mathbb{N}^*}\mathbf{K}_r$, $\sup_{r\in\mathbb{N}^*}\mathbf{p}_r$, $\mathbf{p}_f,\frac{1}{m_*}$, $\frac{1}{r_*}$, $\frac{1}{\epsilon}$.

B. *Assume that $\mathbf{A}^{\delta}_1(+\infty)$ holds and that the hypothesis from **A.** are satisfied. Then, X_T starting at point x has a density $y\in\mathbb{R}^d\mapsto p_T(x,y)$ with $p_T\in\mathcal{C}^{\infty}(\mathbb{R}^d\times\mathbb{R}^d)$.*
Moreover, for every $\theta\geqslant\frac{3}{2}$, $q\in\mathbb{N}$, $\alpha,\beta\in\mathbb{N}^d$ with $|\alpha|+|\beta|\leqslant q$, $p\geqslant 0$, $\epsilon>0$ and every $y\in\mathbb{R}^d$,

$$|\partial^{\alpha}_x\partial^{\beta}_y p_T(x,y)-\partial^{\alpha}_x\partial^{\beta}_y q^{\delta,\theta}_T(x,y)|\leqslant\delta^{\frac{1}{2}-\epsilon}\frac{(1+|x|^{c}_{\mathbb{R}^d})C\exp(CT)}{(\boldsymbol{\mathcal{V}}_L(x)T)^{\eta}(1+|y|^{p}_{\mathbb{R}^d})},\tag{2.19}$$

where $\eta\geqslant 0$ depends on d,L,q,θ and $\frac{1}{\epsilon}$ and $c,C\geqslant 0$ depend on d,N,L,q,$\sup_{r\in\mathbb{N}^*}\mathbf{K}_r$, $\sup_{r\in\mathbb{N}^*}\mathbf{p}_r$, $\mathbf{p}_f$, $\frac{1}{m_*}$, $\frac{1}{r_*}$, $\theta,p,\frac{1}{\epsilon}$ and on the moment of Z^{δ}.

Remark 2.2. *(1) Let us recall that for μ and ν two probability measures on the Borel σ-algebra of $\mathbb{R}^d$, the total variation distance between μ and ν is given by*

$$d_{TV}(\mu,\nu)=\sup_{f\in\mathcal{M}(\mathbb{R}^d;\mathbb{R}),\|f\|_\infty\leqslant 1}\frac{1}{2}|\mu(f)-\nu(f)|=\sup_{f\in\mathcal{C}^{\infty}_K(\mathbb{R}^d;\mathbb{R}),\|f\|_\infty\leqslant 1}\frac{1}{2}|\mu(f)-\nu(f)|,$$

where $\mu(f)=\int_{\mathbb{R}^d}f(x)\mu(dx)$ and similarly for $\nu(f)$. The last equality above is a direct consequence of the Lusin's Theorem.

*In particular, (2.18) provides a bound on the total variation distance between the law of X_T starting from $x\in\mathbb{R}^d$ (denoted $P_T(x,.)$) and the one of X^{δ}_T also starting from x (denoted $Q_T(x,.)$). Under the hypothesis from **A.** in Theorem 2.2, then*

$$d_{TV}(P_T(x,.),Q_T(x,.))\leqslant\delta^{\frac{1}{2}-\epsilon}\frac{1+|x|^{c}_{\mathbb{R}^d}}{(\boldsymbol{\mathcal{V}}_L(x)T)^{\eta}}C\exp(CT).\tag{2.20}$$

*(2) If we suppose in addition that $\theta\geqslant 2$ and that for every $t\in\pi^{\delta,*}$, $i\in\mathcal{N}$, $\mathbb{E}[(Z^i_t)^3]=0$ and we replace $\mathbf{A}^{\delta}_1(\max(6,2L+5))$ by $\mathbf{A}^{\delta}_1(\max(7,2L+5))$ in **A.**, then Theorem 2.2 (and also (2.20)) holds with $\delta^{\frac{1}{2}-\epsilon}$ replaced by $\delta^{1-\epsilon}$ and $(\mathbf{K}_{\max(6,2L+5)},\mathbf{p}_{\max(6,2L+5)})$ replaced by $(\mathbf{K}_{\max(7,2L+5)},\mathbf{p}_{\max(7,2L+5)})$ in the r.h.s. of (2.18) and (2.19).*

(3) More generally, let us suppose that, in addition to hypothesis from Theorem 2.2, the assumption $\mathbf{A}^{\delta}_1(+\infty)$ hold and, given $m>0$, $\theta\geqslant m+1$ and there exists $q(m)\in\mathbb{N}$ such that: For every $f\in\mathcal{C}^{\infty}_{pol}(\mathbb{R}^d;\mathbb{R})$ such that for every $\alpha\in\mathbb{N}^d$ and every $x\in\mathbb{R}^d$,

$$|\partial^{\alpha}f(x)|\leqslant\mathbf{K}_{f,\alpha}(1+|x|^{p(\alpha)}),$$

with $\mathbf{K}_{f,\alpha}\geqslant 1$ and $p(\alpha)\geqslant 0$, then, for every $t\in\pi^{\delta}$,

$$|\mathbb{E}[f(X^{\delta}_{t+\delta})-f(X_{t+\delta})|X_t=X^{\delta}_t=x]|\leqslant\delta^{m+1}\sum_{|\alpha|\leqslant q(m)}\mathbf{K}_{f,\alpha}\,C(1+|x|^{p}),\tag{2.21}$$

where C and p do not depend on $\mathbf{K}_{f,\alpha}$ or δ. Then, Theorem 2.2 holds with $\delta^{\frac{1}{2}-\epsilon}$ replaced by $\delta^{m-\epsilon}$ and $(\mathbf{K}_{\max(6,2L+5)},\mathbf{p}_{\max(6,2L+5)})$ replaced by $(\sup_{r\in\mathbb{N}^*}\mathbf{K}_r,\sup_{r\in\mathbb{N}^*}\mathbf{p}_r)$ in the r.h.s. of (2.18) and (2.19) (and also (2.20)). In this case η,c and C may depend on m.

When assuming simply that for every $t \in \pi^{\delta,*}$, $i \in \mathcal{N}$, $\mathbb{E}[(Z_t^i)^3] = 0$, we have automatically that (2.21) holds with $m = 1$, which leads to the previous remark.

(4) *By a straightforward application of Corollary 2.1 and Theorem 2.2, under the hypothesis from Theorem 2.2 point **B.**, we derive easily the following estimate of the density of X_T: Let $\alpha, \beta \in \mathbb{N}^d$ and let $p > 0$. Then, for every $y \in \mathbb{R}^d$,*

$$|\partial_x^\alpha \partial_y^\beta p_T(x,y)| \leqslant \mathbf{K}_f \, \frac{(1+|x|_{\mathbb{R}^d}^c) C \exp(CT)}{(\boldsymbol{\mathcal{V}}_L(x)T)^\eta (1+|y|_{\mathbb{R}^d}^p)}.$$

(5) *When uniform weak Hörmander property holds, that is $\mathbf{A}_2^\infty(L)$ (see (2.5)), then $\delta^{\frac{1}{2}-\epsilon}$ can be replaced by $\delta^{\frac{1}{2}}$ in (2.18) or (2.20) (but not in (2.19)). When we assume (2.21) holds, similar conclusions hold but with $\delta^{\frac{1}{2}-\epsilon}$ (respectively $\delta^{\frac{1}{2}}$) replaced by $\delta^{m-\epsilon}$ (resp. δ^m).*

Example 2.1. (1) *Let us consider $X = (X^1, X^2)$, the solution of the 2-dimensional system of $\mathbb{R}$ valued SDE, starting at point $x = (x^1, x^2) \in \mathbb{R}^2$ and given by*

$$\begin{aligned} dX_t^1 =& b(X_t^1, t)dt + \sigma(X_t^1, t) dW_t \\ dX_t^2 =& X_t^1 dt, \end{aligned}$$

where $(W_t)_{t\geqslant 0}$ is a one dimensional standard Brownian motion, b and σ are smooth with bounded derivatives of order one and polynomial bounds for higher orders. In the setting from (1.3), we have $V_0 : (u,t) \mapsto (b(u^1,t), u^1)$ and $V_1 : (u,t) \mapsto (\sigma(u^1,t), 0)$. In this example local ellipticity holds for X^1 as long as $\sigma(x^1,t) \neq 0$. However ellipticity does not hold for X since $dim(span((\sigma,0)))(x,0) \leqslant 1$. Nevertheless, let us compute the Lie brackets. In particular

$$[V_0, V_1] : (u,t) \mapsto (\partial_{u^1}\sigma(u^1,t) b(u^1,t) - \partial_{u^1} b(u^1,t)\sigma(u^1,t), -\sigma(x^1,t)),$$

and, for $\sigma(x^1,0) \neq 0$, $span((\sigma,0),(\partial_{x^1}\sigma b - \partial_{x^1} b\sigma + \partial_t \sigma, -\sigma)(x,0) = \mathbb{R}^2$ so that local weak Hörmander condition holds. Now, let us consider the Euler scheme of X, given by $(X_0^{\delta,1}, X_0^{\delta,2}) = x$ and for $t \in \pi^\delta$,

$$\begin{aligned} X_{t+\delta}^{\delta,1} =& X_t^{\delta,1} + b(X_t^{\delta,1},t)\delta + \sigma(X_t^{\delta,1},t)\sqrt{\delta} Z_{t+\delta}^\delta \\ X_{t+\delta}^{\delta,2} =& X_t^{\delta,2} + X_t^{\delta,1}\delta, \end{aligned}$$

where $Z_t^\delta \in \mathbb{R}$, $t \in \pi^{\delta,*}$, are centered with variance one and Lebesgue lower bounded distribution and moment of order three equal to zero. With notations introduced in (2.4), for $\sigma(x^1,0) \neq 0$,

$$\begin{aligned} \boldsymbol{\mathcal{V}}_1(x) =& 1 \wedge \inf_{b\in\mathbb{R}^d, |b|_{\mathbb{R}^2}=1} \langle V_1(x,0), b\rangle_{\mathbb{R}^2}^2 + \langle [V_0 - \frac{1}{2}\nabla_x V_1 V_1, V_1](x,0) + \partial_t V_1(x,0), b\rangle_{\mathbb{R}^2}^2 \\ =& 1 \wedge \inf_{b\in\mathbb{R}^2, |b|_{\mathbb{R}^2}=1} \langle (\sigma,0), b\rangle_{\mathbb{R}^2}^2 + \langle (\partial_{x^1}\sigma b - \partial_{x^1} b\sigma + \frac{1}{2}\sigma^2 \partial_{x^1}^2 \sigma + \partial_t\sigma, -\sigma), b\rangle_{\mathbb{R}^2}^2 (x^1, 0) \\ >& 0, \end{aligned}$$

*and for every $f \in \mathcal{M}(\mathbb{R}^d;\mathbb{R})$ satisfying hypothesis from Theorem 2.2, **A.**, we have, for $T \in \pi^\delta$, $T \geqslant 2\delta$, $\epsilon \in (0,1]$,*

$$|\mathbb{E}_x[f(X_T) - f(X_T^\delta)]| \leqslant \delta^{1-\epsilon} \, \mathbf{K}_f \, \frac{1+|x|_{\mathbb{R}^d}^c}{(\boldsymbol{\mathcal{V}}_1(x)T)^\eta} C\exp(CT),$$

where η, C, c can explode if ϵ tends to zero.

(2) *In a similar but simpler way, we can give an extension of the central limit theorem in total variation distance, including the iterated time integrals of the Brownian motion.*
We consider $Z_n \in \mathbb{R}$, $n \in \mathbb{N}^*$, which are centered with variance one and Lebesgue lower bounded distribution and we define $S_l^{(0)} := n^{-\frac{1}{2}} \sum_{k=1}^l Z_k$, $n, l \in \mathbb{N}$, and for $h \in \mathbb{N}^*$, $S_l^{(h)} := n^{-1}\sum_{k=1}^l S_k^{(h-1)}$.
Then, for every $h \in \mathbb{N}$, $(S_n^{(0)}, \ldots, S_n^{(h)})_{n\in\mathbb{N}}$, satisfies the uniform Hörmander property with $L = h$ and converges in total variation distance, as n tends to infinity, toward the random variable $(W_1, \int_0^1 W_s ds, \ldots, \int_0^1 \ldots \int_0^{s_2} W_{s_1} ds_1 \ldots ds_h)$ where $(W_t)_{t\geqslant 0}$ is a one dimensional standard Brownian motion.

3. A Malliavin-inspired approach to prove smoothing properties

In this Section, we prove Theorem 2.1, Corollary 2.1 and Theorem 2.2 (see Section 3.5). This will be done by establishing Theorem 3.4 which is, in essence, close to the result of Theorem 2.1 but obtained by considering a modification of Q^δ built by using a localization argument. This modification is not tractable for simulation purpose as, for instance, it cannot be computed by simulation (except for trivial cases) but is a key ingredient for the proofs of the main results.
To prove Theorem 3.4, our strategy is to derive regularization properties for this modification of Q^δ by establishing some integration by parts formulas (Theorem 3.1, (3.12)) and then by bounding the Malliavin weights appearing in those formulas (Theorem 3.1, (3.13)). These bounds on Malliavin weights are derived by bounding the Sobolev norms depending on the Malliavin derivatives (Theorem 3.2) and by bounding the moments of the inverse Malliavin covariance matrix (Theorem 3.3).

3.1. **A generic discrete-time Malliavin calculus.** In this section, we present the discrete Malliavin calculus tailored to our framework and derive integration by parts formulas and estimates on the Malliavin weights.

We recall that we work on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ where $\mathcal{F}$ is generated by $(\chi^\delta_t,U^\delta_t,V^\delta_t)_{t\in\pi^{\delta,*}}$. Since we are interested in random variables of the form (2.1), where the laws of random variables Z^δ are arbitrary (and thus not only Gaussian) the standard Malliavin calculus is not adapted anymore. Therefore, we remain inspired by Malliavin calculus but we whether develop a discrete-time differential calculus which happens to be well suited to our framework as soon as Z^δ involves a regular part *i.e.* is Lebesgue lower bounded. In this section, we always assume that $\mathbf{A}^\delta_4$ (see (2.8)) holds true. For $\mathcal{T}\subset\pi^{\delta,*}$, we denote $|\mathcal{T}|=\mathrm{Card}(\mathcal{T})$.

In the following, we will denote $\chi^\delta=(\chi^\delta_t)_{t\in\pi^{\delta,*}}$, $U^\delta=(U^\delta_t)_{t\in\pi^{\delta,*}}$ and $V^\delta=(V^\delta_t)_{t\in\pi^{\delta,*}}$. Given a finite-dimensional Hilbert space $(\mathcal{H},\langle .,.\rangle_{\mathcal{H}})$ with dimension $d_{\mathcal{H}}$, equipped with an orthonormal base $\mathcal{B}_{\mathcal{H}}:=(\mathtt{h}_n)_{n\in\{1,\ldots,d_{\mathcal{H}}\}}$, we will consider the class of random variables:

$$\begin{aligned}\mathcal{S}^\delta(\mathcal{H})=\{&F=f(\chi^\delta_t,U^\delta_t,V^\delta_t,t\in\mathcal{T}),\mathcal{T}\subset\pi^{\delta,*},|\mathcal{T}|<+\infty\\&\forall\chi\in\{0,1\}^{\mathcal{T}},(u,v)\mapsto f(\chi,u,v)\in\mathcal{C}^\infty_b(\mathbb{R}^{\mathcal{N}\times\mathcal{T}}\times\mathbb{R}^{\mathcal{N}\times\mathcal{T}};\mathcal{H})\}.\end{aligned}$$

When $\mathcal{H}=\mathbb{R}$, we simply denote $\mathcal{S}^\delta$. Our applications will be limited to cases where $\mathcal{H}=\mathbb{R}^{l_1\times\ldots\times l_m}$ with $l_1\times\ldots\times l_m, m\in\mathbb{N}^*$. However, we consider $\mathcal{H}$ as an abstract finite-dimensional Hilbert space as it does not introduce additional difficulties.
We now construct a differential calculus based on the laws of the random variables U^δ which mimics the Malliavin calculus, following the ideas from [5], [2], [3] or [7]. We begin by introducing the basic element of our differential calculus.
Derivative operator and Malliavin covariance matrix.
For $F\in\mathcal{S}^\delta(\mathcal{H})$, we define the Malliavin derivatives $\mathbf{D}\,F:=(\mathbf{D}_{(t,i)}\,F)_{(t,i)\in\pi^{\delta,*}\times\mathcal{N}}\in\mathcal{S}^\delta(\mathcal{H})^{\pi^{\delta,*}\times\mathcal{N}}$ by

$$\mathbf{D}_{(t,i)}\,F:=\chi^\delta_t\partial_{u^i_t}f(\chi^\delta,U^\delta,V^\delta),\quad(t,i)\in\pi^{\delta,*}\times\mathcal{N}.$$

We recall that there always exists a finite $\mathcal{T}\subset\pi^{\delta,*}$ such that $\mathbf{D}_{(t,i)}\,F=0$ for $t\notin\mathcal{T}$. In particular $\mathbf{D}\,F$ is finite dimensional and we use the notation $\mathbf{D}\,F:=(\mathbf{D}_{(t,i)}\,F)_{(t,i)\in\mathcal{T}\times\mathcal{N}}\in\mathcal{S}^\delta(\mathcal{H})^{\mathcal{T}\times\mathcal{N}}$. We also extend the derivative outside of the time grid $\pi^{\delta,*}$. For $s\in(t-\delta,t]$, with $t\in\pi^{\delta,*}$ we define

$$\mathbf{D}_{(s,i)}\,F:=\mathbf{D}_{(t,i)}\,F,$$

and $\mathbf{D}_{(0,i)}=0$. The higher order derivatives are defined by iterating $\mathbf{D}$. In particular, for $m\in\mathbb{N}$, we introduce $\mathbf{D}^m$. Let $\alpha=(\alpha^1,\ldots,\alpha^m)\in(\pi^{\delta,*}\times\mathcal{N})^m$, $m\in\mathbb{N}$. In the sequel, we will use the notation $\|\alpha\|=m$. We define

$$\mathbf{D}_\alpha\,F=\mathbf{D}_{\alpha^1}\cdots\mathbf{D}_{\alpha^m}\,F$$

when $m>0$ and $\mathbf{D}_\alpha\,F=\mathbf{D}_\emptyset\,F=F$ if $m=0$. In particular, we will denote

$$\mathbf{D}^m\,F=(\mathbf{D}_\alpha\,F)_{\alpha\in(\pi^{\delta,*}\times\mathcal{N})^m}=(\mathbf{D}_\alpha\,F)_{\alpha\in(\mathcal{T}\times\mathcal{N})^m}$$

The Malliavin covariance matrix of $F\in\mathcal{S}^\delta(\mathcal{H})$, is the matrix defined for every $\mathtt{h},\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}$ by

$$\begin{aligned}\boldsymbol{\sigma}_F[\mathtt{h},\mathtt{h}']&=\delta\langle\mathbf{D}\langle F,\mathtt{h}\rangle_{\mathcal{H}},\mathbf{D}\langle F,\mathtt{h}'\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\pi^{\delta,*}\times\mathcal{N}}}\\&:=\delta\sum_{t\in\pi^{\delta,*}}\sum_{l\in\mathcal{N}}\mathbf{D}_{(t,l)}\langle F,\mathtt{h}\rangle_{\mathcal{H}}\,\mathbf{D}_{(t,l)}\langle F,\mathtt{h}'\rangle_{\mathcal{H}}.\end{aligned}\tag{3.1}$$

Notice that this quantity is well defined as the series is actually a finite sum. Moreover, we have the common integral representation formula,

$$\boldsymbol{\sigma}_F[\mathtt{h},\mathtt{h}'] = \int_0^{+\infty} \sum_{l\in\mathcal{N}} \mathbf{D}_{(t,l)} \langle F,\mathtt{h}\rangle_{\mathcal{H}} \, \mathbf{D}_{(t,l)} \langle F,\mathtt{h}'\rangle_{\mathcal{H}} \mathrm{d}t.$$

We remark that $\boldsymbol{\sigma}_F$ can be seen as a linear operator on $\mathcal{H}$ such that for every $\ell \in \mathcal{H}$, $\boldsymbol{\sigma}_F\,\ell := \sum_{\mathtt{h},\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}} \boldsymbol{\sigma}_F[\mathtt{h},\mathtt{h}']\langle \ell,\mathtt{h}'\rangle_{\mathcal{H}}\mathtt{h}$, that is the standard matrix product.We then introduce the inverse Malliavin covariance matrix. In particular, when the determinant $\det \boldsymbol{\sigma}_F$ of $\boldsymbol{\sigma}_F$ is not zero, we define $\boldsymbol{\gamma}_F$, the inverse of the Malliavin covariance matrix $\boldsymbol{\sigma}_F$ satisfying, for every $\ell \in \mathcal{H}$, $\boldsymbol{\sigma}_F\,\boldsymbol{\gamma}_F\,\ell = \boldsymbol{\gamma}_F\,\boldsymbol{\sigma}_F\,\ell = \ell$.
Divergence and Ornstein Uhlenbeck operators. Let $G = (G_t)_{t\in\pi^{\delta,*}}$ with $G_t \in \mathcal{S}^\delta(\mathcal{H}^{\mathcal{N}})$.The divergence operator $\boldsymbol{\delta}$ is given by

$$\boldsymbol{\delta}(G) = \delta \sum_{t\in\pi^{\delta,*}} \sum_{i\in\mathcal{N}} G_t^i \, \mathbf{D}^\delta_{(t,i)} \, \Gamma^\delta_t + \mathbf{D}^\delta_{(t,i)} \, G^i_t \in \mathcal{S}^\delta(\mathcal{H}),$$

with, for $t \in \pi^{\delta,*}$,

$$\Gamma_t = \ln \varphi_{r_*/2}(\delta^{-\frac{1}{2}} U^\delta_t - z_{*,t}) \in \mathcal{S}^\delta(\mathbb{R}).$$

with $\varphi_{r_*/2}$ defined in (2.9). In particular, for $i \in \mathcal{N}$,

$$\mathbf{D}_{(t,i)} \, \Gamma_t = \delta^{-\frac{1}{2}} \chi^\delta_t \partial_{z^i} \ln \varphi_{r_*/2}(\delta^{-\frac{1}{2}} U^\delta_t - z_{*,t}) \in \mathcal{S}^\delta(\mathbb{R}).$$

Finally, we define the Ornstein Uhlenbeck operator, for $F \in \mathcal{S}^\delta(\mathcal{H})$,

$$\mathbf{L}\, F = -\,\boldsymbol{\delta}(\mathbf{D}\, F) = -\delta \sum_{t\in\pi^{\delta,*}} \sum_{i\in\mathcal{N}} \mathbf{D}_{(t,i)} \, \mathbf{D}_{(t,i)} \, F + \mathbf{D}_{(t,i)} \, F \, \mathbf{D}_{(t,i)} \, \Gamma_t \in \mathcal{S}^\delta(\mathcal{H}).$$

Remark 3.1. *The basic random variables in our calculus are* $Z^\delta_t, t \in \pi^{\delta,*}$, *so we precise the way in which the differential operators act on them. Since* $\delta^{\frac{1}{2}} Z^\delta_t = \chi^\delta_t U^\delta_t + (1-\chi^\delta_t) V^\delta_t$, *it follows that for* $w, t \in \pi^{\delta,*}$, $i, j \in \mathcal{N}$,

$$\delta^{\frac{1}{2}} \, \mathbf{D}_{(t,i)} \, Z^{\delta,j}_w = \chi^\delta_w \mathbf{1}_{w,t} \mathbf{1}_{i,j}, \tag{3.2}$$

and

$$\mathbf{L}\, Z^{\delta,i}_t = \chi^\delta_t \partial_{z^i} \ln \varphi_{r_*/2}(\delta^{-\frac{1}{2}} U^\delta_t - z_{*,t}). \tag{3.3}$$

We observe that in our framework, the duality formula reads as follows: For each $F \in \mathcal{S}^\delta(\mathcal{H})$ and $G = (G_t)_{t\in\pi^{\delta,*}}$ with $G_t \in \mathcal{S}^\delta(\mathcal{H})^{\mathcal{N}}$

$$\mathbb{E}[\langle F, \boldsymbol{\delta}(G)\rangle_{\mathcal{H}}] = \mathbb{E}[\langle \mathbf{D}\, F, G\rangle_{\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}}}] := \delta \sum_{t\in\pi^{\delta,*}} \sum_{i\in\mathcal{N}} \mathbb{E}[\langle \mathbf{D}_{(t,i)} \, F, G^i_t\rangle_{\mathcal{H}}]. \tag{3.4}$$

This leads to the following relationship: For each $F, G \in \mathcal{S}^\delta(\mathcal{H})$,

$$\begin{aligned} \mathbb{E}[\langle F, \mathbf{L}\, G\rangle_{\mathcal{H}}] = \mathbb{E}[\langle G, \mathbf{L}\, F\rangle_{\mathcal{H}}] =& \delta \mathbb{E}[\langle \mathbf{D}\, F, \mathbf{D}\, G\rangle_{\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}}}] \\ :=& \delta \sum_{t\in\pi^{\delta,*}} \sum_{i\in\mathcal{N}} \mathbb{E}[\langle \mathbf{D}_{(t,i)} \, F, \mathbf{D}_{(t,i)} \, G\rangle_{\mathcal{H}}]. \end{aligned} \tag{3.5}$$

We prove directly (3.5). The duality relationship (3.4) is proven by similar arguments. This follows immediately using the independence structure and standard integration by parts on $\mathbb{R}^N$: Indeed, if $f, g \in \mathcal{C}^2(\mathbb{R}^N;\mathbb{R})$ and $t \in \pi^{\delta,*}$, then

$$\begin{aligned} \sum_{i\in\mathcal{N}} \mathbb{E}[\partial_{u^i} f(U^\delta_t)\partial_{u^i} g(U^\delta_t)] &= \frac{\varepsilon_*}{m_*} \sum_{i=1}^N \int_{\mathbb{R}^N} \partial_{u^i} f(u) \partial_{u^i} g(u) \delta^{-\frac{N}{2}} \varphi_{r_*/2}(\delta^{-\frac{1}{2}} u - z_{*,t}) du \\ &= -\frac{\varepsilon_*}{m_*} \sum_{i\in\mathcal{N}} \int_{\mathbb{R}^N} f(u)(\partial^2_{u^i} g(u) + \partial_{u^i} g(u) \frac{\partial_{u^i} \varphi_{r_*/2}(\delta^{-\frac{1}{2}} u - z_{*,t})}{\varphi_{r_*/2}(\delta^{-\frac{1}{2}} u - z_{*,t})}) \delta^{-\frac{N}{2}} \varphi_{r_*/2}(\delta^{-\frac{1}{2}} u - z_{*,t}) du \\ &= -\,\mathbb{E}[f(U^\delta_t) \sum_{i\in\mathcal{N}} \partial^2_{u^i} g(U^\delta_t) + \partial_{u^i} g(U^\delta_t) \delta^{-\frac{1}{2}} \partial_{z^i} \ln \varphi_{r_*/2}(\delta^{-\frac{1}{2}} U^\delta_t - z_{*,t})]. \end{aligned}$$

Now, we consider $F, G \in \mathcal{S}^\delta(\mathcal{H})$, so that $F = f(\chi^\delta_{t_i}, U^\delta_{t_i}, V^\delta_{t_i}, i \in \{1,\dots,n\})$ and $G = g(\chi^\delta_{t_i}, U^\delta_{t_i}, V^\delta_{t_i}, i \in \{1,\dots,n\})$. Moreover, we introduce the functions $f_l := \langle f, \mathtt{h}_l\rangle_{\mathcal{H}}, g_l := \langle g, \mathtt{h}_l\rangle_{\mathcal{H}}$, $l \in \{1,\dots,d_{\mathcal{H}}\}$, which belong to $\mathcal{C}^\infty((\mathbb{R}^N)^n;\mathbb{R})$. It follows from the calculus above that

$$\mathbb{E}[\langle \mathbf{D}\, F, \mathbf{D}\, G\rangle_{\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}}}] = \sum_{l=1}^{d_{\mathcal{H}}} \sum_{t\in\pi^{\delta,*}} \sum_{i\in\mathcal{N}} \mathbb{E}[\chi^\delta_t \partial_{u^i_t} f_l(\chi^\delta, U^\delta, V^\delta)\partial_{u^i_t} g_l(\chi^\delta, U^\delta, V^\delta)]$$

$$= -\sum_{l=1}^{d_{\mathcal{H}}} \mathbb{E}[f_l(\chi^\delta, U^\delta, V^\delta) \sum_{t\in\pi^{\delta,*}} \chi^\delta_t$$
$$\times \sum_{i\in\mathcal{N}} \partial^2_{u^i_t} g_l(\chi^\delta, U^\delta, V^\delta) + \partial_{u^i_t} g_l(\chi^\delta, U^\delta, V^\delta)\delta^{-\frac{1}{2}}\partial_{z^i}\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U^\delta_t - z_{*,t})]$$
$$= -\,\mathbb{E}[\langle F, \sum_{t\in\pi^{\delta,*}} \sum_{i\in\mathcal{N}} \mathbf{D}_{(t,i)}\, \mathbf{D}^\delta_{(t,i)}\, G + \mathbf{D}_{(t,i)}\, G\, \mathbf{D}_{(t,i)}\,\Gamma_t\rangle_{\mathcal{H}}] = \delta^{-1}\mathbb{E}[\langle F, \mathbf{L}\, G\rangle_{\mathcal{H}}],$$

which is exactly (3.5). We have the following standard chain rule: Let $\phi \in \mathcal{C}^\infty(\mathcal{H};\mathcal{H}')$ with $\mathcal{H}'$ a finite-dimensional Hilbert space and $F \in \mathcal{S}^\delta(\mathcal{H})$. Then

$$\mathbf{D}\,\phi(F) = \mathscr{D}_{\mathbf{D}\,F}\,\phi(F) \in \mathcal{S}^\delta(\mathcal{H}')^{\pi^{\delta,*}\times\mathcal{N}}. \tag{3.6}$$

More particularly, when $\mathcal{H}' = \mathbb{R}$ we have

$$\mathbf{D}\,\phi(F) = \langle \mathscr{D}\,\phi(F), \mathbf{D}\,F\rangle_{\mathcal{H}} \in \mathcal{S}^\delta(\mathbb{R})^{\pi^{\delta,*}\times\mathcal{N}}. \tag{3.7}$$

Moreover, it follows from (3.6) and the duality relation (or direct computation), that

$$\mathbf{L}\,\phi(F) = \langle \mathscr{D}\,\phi(F), \mathbf{L}\,F\rangle_{\mathcal{H}} + \delta \sum_{\mathtt{h},\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}} \mathscr{D}_{\mathtt{h}'}\,\mathscr{D}_{\mathtt{h}}\,\phi(F)\langle \mathbf{D}\langle F,\mathtt{h}\rangle_{\mathcal{H}}, \mathbf{D}\langle F,\mathtt{h}'\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\pi^{\delta,*}\times\mathcal{N}}}. \tag{3.8}$$

We then build our Sobolev-Malliavin norms that, among others, will be involved in estimates of the Malliavin weights of the integration by parts formulas derived in Theorem 3.1
Sobolev-Malliavin norm. For $F \in \mathcal{S}^\delta(\mathcal{H})$, $q \in \mathbb{N}$, we begin by introducing the Malliavin-Sobolev norms:

$$|F|^2_{\mathcal{H},1,q} = \sum_{\substack{\alpha\in(\pi^{\delta,*}\times\mathcal{N})^j \\ j\in\{1,\dots,q\}}} \delta^j |\,\mathbf{D}_\alpha\,F|^2_{\mathcal{H}}, \qquad |F|^2_{\mathcal{H},q} = |F|^2_{\mathcal{H}} + |F|^2_{\mathcal{H},1,q} \tag{3.9}$$

and for $p \geqslant 1$

$$\|F\|_{\mathcal{H},1,q,p} = \mathbb{E}[|F|^p_{\mathcal{H},1,q}]^{\frac{1}{p}} \qquad \|F\|_{\mathcal{H},q,p} = \mathbb{E}[|F|^p_{\mathcal{H}}]^{\frac{1}{p}} + \|F\|_{\mathcal{H},1,q,p}. \tag{3.10}$$

Owing to the duality relationship (3.4), the operator $\mathbf{D}$, initially defined on $\mathcal{S}^\delta(\mathcal{H})$ is closable from $\mathrm{L}_p(\Omega;\mathcal{H})$ into $\mathrm{L}_p(\Omega;\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}})$ for $p \geqslant 1$. More specifically, for $(F_n)_{n\in\mathbb{N}}$ taking values in $\mathcal{S}^\delta(\mathcal{H})$ which converges to 0 in $\mathrm{L}_p(\Omega;\mathcal{H})$ and such that $(\mathbf{D}\,F_n)_{n\in\mathbb{N}}$ converges to ζ in $\mathrm{L}_p(\Omega;\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}})$, it follows from (3.6) and (3.4) that, for every $G \in \mathcal{S}^\delta(\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}})$

$$\mathbb{E}[\langle \mathbf{D}\,F_n, G\rangle_{\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}}}] = \mathbb{E}[\langle F_n, \boldsymbol{\delta}(G)\rangle_{\mathcal{H}}].$$

Since G is bounded, the *l.h.s.* above tends to $\mathbb{E}[\langle \zeta, G\rangle_{\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}}}]$ and since $\boldsymbol{\delta}(G) \in \mathcal{S}^\delta$ is also bounded, the *r.h.s.* tends to 0 so $\zeta = 0, a.s.$, proving the closability. For higher orders, the result is similar.

In particular, we introduce $\mathbb{D}^{q,p}(\mathcal{H})$ the completion of $\mathcal{S}^\delta(\mathcal{H})$ by the norm $\|\cdot\|_{\mathcal{H},q,p}$ and extend continuously the definition of the norms (3.9) (resp. (3.10)) from $F \in \mathcal{S}^\delta(\mathcal{H})$ to $F \in \mathbb{D}^{q,1}(\mathcal{H})$ (resp. $\mathbb{D}^{q,p}(\mathcal{H})$). Then, for $F \in \mathbb{D}^{q,p}(\mathcal{H})$, we define $\mathbf{D}^q\,F$ as the limit in $\mathrm{L}_p(\Omega;\mathcal{H}^{(\pi^{\delta,*}\times\mathcal{N})^{\otimes q}})$ of $\mathbf{D}^q\,F_n$ where $(F_n)_{n\in\mathbb{N}}$ is a $\mathcal{S}^\delta(\mathcal{H})$-valued sequence converging to F in the norm $\|\cdot\|_{\mathcal{H},q,p}$. Similarly, we define $\boldsymbol{\delta}(G)$ for every G that belongs to

$$\mathrm{Dom}(\boldsymbol{\delta}) := \{u \in \mathrm{L}_2(\Omega;\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}}), \exists C \geqslant 0, \forall F \in \mathbb{D}^{1,2}(\mathcal{H}), |\mathbb{E}[\langle \mathbf{D}\,F, u\rangle_{\mathcal{H}^{\pi^{\delta,*}\times\mathcal{N}}}]| \leqslant C\|F\|_2\},$$

which satisfies (3.4) for every $F \in \mathbb{D}^{1,2}(\mathcal{H})$.

We also note that, under assumptions $\mathbf{A}^\delta_1(q)$ (see (2.2) and (2.3)), and $\mathbf{A}^\delta_3(+\infty)$ (see (2.7)), then $X^\delta_t \in \mathbb{D}^{q,p}(\mathbb{R}^d)$ for every $p \geqslant 1$ and we have similar results for the tangent process introduced in (3.18).

Integration by parts formula. Below, we define the Malliavin weights that appear in our integration by parts formulas. Let $F \in \mathbb{D}^{2,p}(\mathcal{H})$, $G \in \mathbb{D}^{1,p}(\mathbb{R})$ and $\mathtt{h} \in \mathcal{B}_{\mathcal{H}}$. We define

$$\mathbf{H}(F,G)[\mathtt{h}] := -\langle G\,\boldsymbol{\gamma}_F\,\mathbf{L}\,F,\mathtt{h}\rangle_{\mathcal{H}} - \delta \sum_{\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}} \langle \mathbf{D}(G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}']),\mathbf{D}\langle F,\mathtt{h}'\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\pi^{\delta,*}}\times\mathcal{N}}.$$

Considering higher order integration by parts formulas, for $F \in \mathbb{D}^{q+1,p}(\mathcal{H})$, $G \in \mathbb{D}^{q,p}(\mathbb{R})$ and $\mathbf{h} = (\mathtt{h}^1,\ldots,\mathtt{h}^q) \in (\mathcal{B}_{\mathcal{H}})^q$ we define $\mathbf{H}(F,G)[\mathbf{h}]$ by the recurrence

$$\mathbf{H}(F,G)[\mathbf{h}] := \mathbf{H}(F,\mathbf{H}(F,G)[\mathtt{h}^1,\ldots,\mathtt{h}^{q-1}])[\mathtt{h}^q]. \tag{3.11}$$

Exploiting the relationships (3.5) and (3.7), we deduce the following result which is a integration by parts formula together with an estimate of the Sobolev norms of the weights.

Theorem 3.1. *Let $q \in \mathbb{N}^*$, $\phi \in \mathcal{C}^\infty_{pol}(\mathcal{H};\mathbb{R})$. Let F and G be such that $F \in \mathbb{D}^{q+1,p}(\mathcal{H})$, $G \in \mathbb{D}^{q,p}(\mathbb{R})$ and $\mathbb{E}[|\det \boldsymbol{\gamma}_F|^p \mathbf{1}_{|G|>0}] < +\infty$, for every $p \geqslant 1$. Then, for every $\mathbf{h} = (\mathtt{h}^1,\ldots,\mathtt{h}^q) \in (\mathcal{B}_{\mathcal{H}})^q$,*

$$\mathbb{E}[\mathscr{D}_{\mathbf{h}}\,\phi(F)G] = \mathbb{E}[\phi(F)\,\mathbf{H}(F,G)[\mathbf{h}]], \tag{3.12}$$

with $\mathbf{H}(F,G)[\mathbf{h}]$ defined in (3.11). Moreover, for every $m \in \mathbb{N}$, if $F \in \mathbb{D}^{q+m+1,p}(\mathcal{H})$ and $G \in \mathbb{D}^{q+m,p}(\mathbb{R})$ for every $p \geqslant 1$, then

$$|\,\mathbf{H}(F,G)[\mathbf{h}]|_{\mathbb{R},,m} \leqslant C(d_{\mathcal{H}},q,m)\mathbf{c}(d_{\mathcal{H}},q,m,F,G), \tag{3.13}$$

with

$$\mathbf{c}(d_{\mathcal{H}},q,m,F,G) = (1\vee\det\boldsymbol{\gamma}_F)^{q(m+q+1)}(1+|F|^{2d_{\mathcal{H}}q(q+m+2)}_{\mathcal{H},1,q+m+1} + |\,\mathbf{L}\,F|^{2q}_{\mathcal{H},q+m-1})|G|_{\mathbb{R},m+q}.$$

In order to prove Theorem 3.1, we will combine the previous identities with the following result. The reader can find the detailed proof of this result in [2], Theorem 3.4. (see also [5]).

Proposition 3.1. *Let $m,q \in \mathbb{N}$, and $\mathbf{h} = (\mathtt{h}^1,\ldots,\mathtt{h}^l) \in (\mathcal{B}_{\mathcal{H}})^l$ with $l \leqslant q$. Let F and G be such that $F \in \mathbb{D}^{q+m+1,p}(\mathcal{H})$, $G \in \mathbb{D}^{q+m,p}(\mathbb{R})$ and $\mathbb{E}[|\det\boldsymbol{\gamma}_F|^p\mathbf{1}_{|G|_{\mathcal{H},q}>0}] < +\infty$, for every $p \geqslant 1$. Then*

$$|\,\mathbf{H}(F,G)[\mathbf{h}]|_{\mathbb{R},m} \leqslant C(d_{\mathcal{H}},q,m)\mathbf{c}(d_{\mathcal{H}},q,m,F,G),$$

with $\mathbf{c}(d_{\mathcal{H}},q,m,F,G)$ defined in (3.13).

Proof of Theorem 3.1. We prove the result for $m = 1$. Then, a recurrence yields (3.12). We also restrict ourselves to the case where $F \in \mathcal{S}^\delta(\mathcal{H})$ and $G \in \mathcal{S}^\delta$ and extend to our result by density. Using the chain rule (3.7), we have for every $\mathtt{h} \in \mathcal{B}_{\mathcal{H}}$,

$$\begin{aligned}\langle \mathbf{D}\,\phi(F),\mathbf{D}\langle F,\mathtt{h}\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\pi^{\delta,*}}\times\mathcal{N}} &= \sum_{\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}} \langle \mathscr{D}\,\phi(F),\mathtt{h}'\rangle_{\mathcal{H}}\langle\mathbf{D}\langle F,\mathtt{h}'\rangle_{\mathcal{H}},\mathbf{D}\langle F,\mathtt{h}\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\pi^{\delta,*}}\times\mathcal{N}}\\ &=\delta^{-1}\sum_{\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}} \mathscr{D}_{\mathtt{h}'}\,\phi(F)\,\boldsymbol{\sigma}_F[\mathtt{h},\mathtt{h}'] = \langle\boldsymbol{\sigma}_F\,\mathtt{h},\mathscr{D}\,\phi(F)\rangle_{\mathcal{H}}.\end{aligned}$$

Using (3.8) with $F = (\langle F,\mathtt{h}'\rangle_{\mathcal{H}},\phi(F))$, $\mathcal{H} = \mathbb{R}^2$ and $\phi : (x,y) \mapsto xy$, (3.5) with $F = \phi(F)\langle F,\mathtt{h}'\rangle_{\mathcal{H}}$ (respectively $F = G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}']\langle F,\mathtt{h}'\rangle_{\mathcal{H}}$), $G = G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}']$ (resp. $G = \phi(F)$) and $\mathcal{H} = \mathbb{R}$ (resp. $\mathcal{H} = \mathbb{R}$) and finally (3.8) with $F = (\langle F,\mathtt{h}'\rangle_{\mathcal{H}}, G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}'])$, $\mathcal{H} = \mathbb{R}^2$ and $\phi : (x,y) \mapsto xy$, it follows that

$$\begin{aligned}\mathbb{E}[\mathscr{D}_{\mathtt{h}}\,\phi(F)G] &= \delta\sum_{\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}}\mathbb{E}[G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}']\langle\mathbf{D}\,\phi(F),\mathbf{D}\langle F,\mathtt{h}'\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\pi^{\delta,*}}\times\mathcal{N}}]\\ &=\frac{1}{2}\sum_{\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}}\mathbb{E}[G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}'](\mathbf{L}(\phi(F)\langle F,\mathtt{h}'\rangle_{\mathcal{H}}) - \phi(F)\,\mathbf{L}\langle F,\mathtt{h}'\rangle_{\mathcal{H}} - \langle F,\mathtt{h}'\rangle_{\mathcal{H}}\,\mathbf{L}\,\phi(F))]\\ &=\frac{1}{2}\sum_{\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}}\mathbb{E}[\phi(F)\langle F,\mathtt{h}'\rangle_{\mathcal{H}}\,\mathbf{L}(G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}']) - \phi(F)G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}']\,\mathbf{L}\langle F,\mathtt{h}'\rangle_{\mathcal{H}}\\ &\quad - \phi(F)\,\mathbf{L}(G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}']\langle F,\mathtt{h}'\rangle_{\mathcal{H}})]\\ &= -\sum_{\mathtt{h}'\in\mathcal{B}_{\mathcal{H}}}\mathbb{E}[\phi(F)(G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}']\,\mathbf{L}\langle F,\mathtt{h}'\rangle_{\mathcal{H}} + \delta\langle\mathbf{D}(G\,\boldsymbol{\gamma}_F[\mathtt{h},\mathtt{h}']),\mathbf{D}\langle F,\mathtt{h}'\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\pi^{\delta,*}}\times\mathcal{N}})],\end{aligned}$$

which is exactly (3.12) for $q = 1$. Iterating this formula, we obtain (3.12).

In order to obtain 3.13, we simply apply Proposition 3.1. □

In the sequel, we are interested in obtaining an estimate of the weights $\mathbf{H}$ which appear in the integration by parts formulas (3.12) when G is replaced by $G\Theta$ with $\Theta \in [0,1]$ the localizing random weight. We first provide a bound on the Sobolev norms of $G\Theta$.

Lemma 3.1. *Let $q \in \mathbb{N}$. Let $G \in \mathbb{D}^{q,2}(\mathcal{H})$ and $\Theta \in \mathbb{D}^{q,2}(\mathbb{R})$. Then*

$$|G\Theta|_{\mathcal{H},q} \leqslant C(q) \sum_{m=0}^{q} |G|_{\mathcal{H},m} |\Theta|_{\mathbb{R},q-m}. \tag{3.14}$$

Proof. We prove the result by recurrence. We assume, without loss of generality, that $G = g(\chi^{\delta}_{t_i}, U^{\delta}_{t_i}, V^{\delta}_{t_i}, t_i \in \mathcal{T}, i \in \{1, \ldots, n\})$ for a finite set $\mathcal{T} \subset \pi^{\delta,*}$. For $q \in \mathbb{N}$, we define $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}_{q+1} = (\mathcal{H}_q)^{\mathcal{T} \times \mathcal{N}}$. The result is true for $q = 0$. Assume it is true until some $q \in \mathbb{N}$ and let us show it still holds for $q + 1$. We have

$$|G\Theta|^2_{\mathcal{H},q+1} = |G\Theta|^2_{\mathcal{H}} + \sum_{l=0}^{q} \delta^{l+1} |\,\mathbf{D}^l(\Theta\,\mathbf{D}\,G + G\,\mathbf{D}\,\Theta)|^2_{\mathcal{H}_{l+1}},$$

with

$$\begin{aligned}|\,\mathbf{D}^l(\Theta\,\mathbf{D}\,G)|_{\mathcal{H}_{l+1}} \leqslant & \delta^{-\frac{l}{2}} |\Theta\,\mathbf{D}\,G|_{\mathcal{H}^{\mathcal{T}\times\mathcal{N}},l} \\ \leqslant & C\delta^{-\frac{l}{2}} \sum_{m=0}^{l} |\Theta|_{\mathbb{R},l-m} |\,\mathbf{D}\,G|_{\mathcal{H}_1,m} = \delta^{-\frac{l+1}{2}} C(l) \sum_{m=0}^{l} |\Theta|_{\mathbb{R},l-m} |G|_{\mathcal{H},1,m+1},\end{aligned}$$

where we have applied (3.14) with G replaced by $\mathbf{D}^{\delta}\,G$, $q = l$ and $\mathcal{H} = \mathcal{H}_1$. Similarly

$$\begin{aligned}|\,\mathbf{D}^l(G\,\mathbf{D}\,\Theta)|_{\mathcal{H}_{l+1}} =& |\sum_{|\alpha|=l} \sum_{|\beta|=1} |\,\mathbf{D}_{\alpha}(G\,\mathbf{D}_{\beta}\,\Theta)|^2_{\mathcal{H}}|^{\frac{1}{2}} = |\sum_{|\beta|=1} |\,\mathbf{D}^l(G\,\mathbf{D}_{\beta}\,\Theta)|^2_{\mathcal{H}_l}|^{\frac{1}{2}} \\ \leqslant & |\sum_{|\beta|=1} \delta^{-l} |G\,\mathbf{D}_{\beta}\,\Theta|^2_{\mathcal{H},l}|^{\frac{1}{2}} \leqslant C\delta^{-\frac{l}{2}} \sum_{m=0}^{l} |G|_{\mathcal{H},m} |\sum_{|\beta|=1} |\,\mathbf{D}\,\Theta|^2_{\mathcal{H},l-m}|^{\frac{1}{2}} \\ \leqslant & C\delta^{-\frac{l+1}{2}} \sum_{m=0}^{l} |G|_{\mathcal{H},m} |\Theta|_{\mathbb{R},l+1-m},\end{aligned}$$

and the proof is completed. □

3.2. **Sobolev Norms.** Before we state our results, we recall that $\partial_{\mathrm{X}^{\delta}_0} X^{\delta}_t$, $t \in \pi^{\delta}$, is the tangent flow and is introduced in (3.18). In a similar way, for $\alpha \in \mathbb{N}^d$, $\partial^{\alpha}_{\mathrm{X}^{\delta}_0} X^{\delta}_t$ denotes the derivatives of X^{δ}_t of order $|\alpha|$ *w.r.t.* X^{δ}_0 and is given by $\partial^{\alpha^1}_{(\mathrm{X}^{\delta}_0)^1} \ldots \partial^{\alpha^d}_{(\mathrm{X}^{\delta}_0)^d} X^{\delta}_t$. The following result provides an upper bound for the Sobolev norms appearing in the upper bound of the Malliavin weights established in Theorem 3.1.

Theorem 3.2. *Let $T \in \pi^{\delta,*}$, $\mathcal{T} = (0, T] \cap \pi^{\delta}$ and $x \in \mathbb{R}^d$. Let $q \in \mathbb{N}$, $q^{\diamond} \in \{0, 1\}$, $p \geqslant 1$ and $\alpha \in \mathbb{N}^d$ a multi-index. Assume that $\mathbf{A}^{\delta}_1(q + |\alpha| + 2)$, $\mathbf{A}^{\delta}_3(+\infty)$ and $\mathbf{A}^{\delta}_4$ hold. Then*

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}} |\partial^{\alpha}_{\mathrm{X}^{\delta}_0} X^{\delta}_t|^p_{\mathbb{R}^d,q^{\diamond},q}]^{\frac{1}{p}} \leqslant (1 + \mathbf{1}_{\mathbf{p}_{q+|\alpha|+2}>0 \cup q^{\diamond}=|\alpha|=0} |x|^C_{\mathbb{R}^d})\,\mathbf{K}^C_{q+|\alpha|+2} \exp(C(1+T)\,\mathbf{M}_C(Z^{\delta})\,\mathbf{K}^2_3), \tag{3.15}$$

with $C = C(d, N, \frac{1}{r_*}, \mathbf{p}_{\max(3,q+|\alpha|+2)}, q, p)$. Moreover, if we replace the assumption $\mathbf{A}^{\delta}_1(q + |\alpha| + 2)$, by the assumption $\mathbf{A}^{\delta}_1(q + 4)$, then

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}} |\,\mathbf{L}\,X^{\delta}_t|^p_{\mathbb{R}^d,q}]^{\frac{1}{p}} \leqslant (1 + \mathbf{1}_{\mathbf{p}_{q+4}>0} |x|^C_{\mathbb{R}^d})\,\mathbf{K}^C_{q+4} \exp(C(1+T)\,\mathbf{M}_C(Z^{\delta})\,\mathbf{K}^2_3), \tag{3.16}$$

with $C = C(d, N, \frac{1}{r_*}, \mathbf{p}_{q+4}, q, p)$.

Remark 3.2. *This result was obtained in* [7] *(see Theorem 4.2) in the case $\mathbf{p}_r = 0$ for r large enough in the assumption $\mathbf{A}^{\delta}_1(r)$ (see (2.2)).*

3.3. **Malliavin covariance matrix.** In this Section, we provide an upper bound for the localized moments of the inverse of the Malliavin covariance matrix of $(X^{\delta}_t)_{t\in\pi^{\delta}}$ defined in (3.1). In the following, we will not work under $\mathbb{P}$, but under a localized measure which we define now. The technique consists in localizing the random variables Z^{δ} and the Malliavin covariance matrix σ^{δ}_F. For the first one, we aim to control that the norm is not too high while for the latter, we aim to control that it is not too low. We first introduce a regularized version of the indicator function. For $v > 1$, we consider $\Psi_v \in \mathcal{C}^{\infty}_b(\mathbb{R}; [0, 1])$ such that $\Psi_v(x) = 1$ if $|x| \leqslant v - \frac{1}{2}$ and 0 if $|x| \geqslant v$ and that the function $z \in \mathbb{R}^N \mapsto \Psi_v(|z|_{\mathbb{R}^N})$ belongs

to $\mathcal{C}_b^\infty(\mathbb{R}^N;[0,1])$ (*e.g.* for $|x|\in(v-\frac{1}{2},v)$, $\Psi_v(x)=\exp(1-\frac{1}{1-(2|x|-2v+1)^2})$).
Given $\mathcal{T}\subset\pi^{\delta,*}$, and $\eta=(\eta_1,\eta_2)\in(1+\infty)^2$ we introduce, for ,

$$\Theta_{F,G,\eta,\mathcal{T}}=\Theta_{F,G,\eta_1}\Theta_{\eta_2,\mathcal{T}} \quad \text{with} \tag{3.17}$$
$$\Theta_{F,G,\eta_1}=\Psi_{\eta_1}(G\det\boldsymbol{\gamma}_F), \quad \text{and} \quad \Theta_{\eta_2,\mathcal{T},t}=\prod_{w\in((0,t]\cap\mathcal{T})}\Psi_{\eta_2}(|Z^\delta_w|_{\mathbb{R}^N}), \quad t\in\pi^\delta,$$

where $\Theta_{\eta_2,\mathcal{T}}=\Theta_{\eta_2,\mathcal{T},\infty}$. Moreover, we introduce the tangent flow process $(\dot{X}_t)_{t\in\pi^\delta}$ defined by $\dot{X}_0=I_{d\times d}$ and

$$\dot{X}_t:=\partial_{\mathrm{X}_0^\delta}X_t^\delta, \tag{3.18}$$

the Jacobian matrix of derivatives of X^δ *w.r.t.* the initial value X_0^δ. In the following, for $\mathcal{T}=(0,T]\cap\pi^\delta$, $T\in\pi^\delta$, we will employ the localization random variable

$$\Theta^*_T=\Theta_{X_T^\delta,\det(\dot{X}_T^\delta)^2,(\boldsymbol{\eta}_1(\delta),\boldsymbol{\eta}_2(\delta)),\mathcal{T}}, \tag{3.19}$$

where $\boldsymbol{\eta}_1(\delta)$ and $\boldsymbol{\eta}_2(\delta)$ are defined in (2.11)

Theorem 3.3. *Let $T\in\pi^{\delta,*}$, $x\in\mathbb{R}^d$ and $p\geqslant 0$. Assume that $\mathbf{A}_1^\delta(2L+5)$, $\mathbf{A}_2(x,L)$, $\mathbf{A}_3^\delta(+\infty)$, $\mathbf{A}_4^\delta$ and $\mathbf{A}_5^\delta(x,T)$ hold. Then*

$$\mathbb{E}_x[|\det\boldsymbol{\gamma}_{X_T^\delta}|^p\mathbf{1}_{\Theta^*_T>0}]\leqslant\frac{1+\mathbf{1}_{\mathbf{p}_{2L+5}>0}|x|^C_{\mathbb{R}^d}}{\boldsymbol{\mathcal{V}}_L(x)^{\frac{3}{2}13^Ldp+1}T^{13^Ldp}}\mathbf{K}^C_{2L+5}\exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4), \tag{3.20}$$

with $C=C(d,N,L,\frac{1}{m_*},\mathbf{p}_{2L+5},p)$. Moreover, if $p\geqslant 4$, for every $a>0$

$$\begin{aligned}\mathbb{P}_x(\Theta^*_T<1)\leqslant&\delta^{-1}\boldsymbol{\eta}_2(\delta)^{-a}TC(a)\,\mathbf{M}_{\lceil a\rceil}(Z^\delta)\\&+\boldsymbol{\eta}_1(\delta)^{-p}\frac{1+\mathbf{1}_{\mathbf{p}_{2L+5}>0}|x|^C_{\mathbb{R}^d}}{\boldsymbol{\mathcal{V}}_L(x)^{\frac{3}{2}13^Ldp}}\mathbf{K}^C_{2L+5}\exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4),\end{aligned} \tag{3.21}$$

with $C=C(d,N,L,\frac{1}{m_*},\mathbf{p}_{2L+5},p)$.

Remark 3.3. *We have the following observations concerning the result above.*

(1) *The terms 13^L in the r.h.s. of both (3.20) and (3.21) can be replaced by $(12+b)^L$, $b>0$, but the miscellaneous constants $C(.)$ may explode when b tends to zero or to infinity.*
(2) *When the uniform Hörmander hypothesis $\mathbf{A}_2^\infty(L)$ (see (2.5)) holds, the estimates (3.20) and (3.21) can be improved. In particular the term $\boldsymbol{\mathcal{V}}_L(x)^{-\frac{3}{2}13^Ldp+1}$ in the r.h.s. of (3.20) (resp. the term $\boldsymbol{\mathcal{V}}_L(x)^{-\frac{3}{2}13^Ldp}$ in the r.h.s. of (3.21)) can be replaced by $(\boldsymbol{\mathcal{V}}_L^\infty)^{-13^Ldp}$ (resp. by 1). In this uniform elliptic setting ($L=0$) we thus recover the results from* [7] *Proposition 4.4.*

3.4. **Regularization properties for approximations of the semigroup.** We introduce $(Q_t^{\delta,\Theta^*_T})_{t\in\pi^\delta}$ such that, for every $x\in\mathbb{R}^d$,

$$\forall T\in\pi^\delta,\quad Q_T^{\delta,\Theta^*_T}f(x):=\mathbb{E}[\Theta^*_Tf(X_T^\delta)|X_0^\delta=x]. \tag{3.22}$$

with Θ^*_T defined in (3.19). Notice that $(Q_t^{\delta,\Theta^*_T})_{t\in\pi^\delta}$, is not a semigroup. We will not be able to prove the smoothing property for Q^δ but for Q^{δ,Θ^*_T}. Our approach consists in applying the integration by part formulas derived in Theorem 3.1 and then the moments of the weights appearing in those formulas exploiting Theorem 3.2 and Theorem 3.3.

Theorem 3.4. *Let $T\in\pi^{\delta,*}$ and $f\in\mathcal{M}(\mathbb{R}^d;\mathbb{R})$ satisfying: there exists $\mathbf{K}_f\geqslant 0$ and $\mathbf{p}_f\in\mathbb{N}$ such that for every $x\in\mathbb{R}^d$,*

$$|f(x)|\leqslant\mathbf{K}_f(1+|x|^{\mathbf{p}_f}_{\mathbb{R}^d}).$$

Then we have the following properties:

A. *Let $x\in\mathbb{R}^d$, $q\in\mathbb{N}$ and $\alpha,\beta\in\mathbb{N}^d$ such that $|\alpha|+|\beta|\leqslant q$. Assume that $\mathbf{A}_1^\delta(\max(q+3,2L+5))$, $\mathbf{A}_2(x,L)$, $\mathbf{A}_3^\delta(+\infty)$, $\mathbf{A}_4^\delta$ and $\mathbf{A}_5^\delta(x,T)$ hold. Then, if f is $|\beta|$-times differentiable and $\partial^\beta f$ has polynomial, growth*

$$\begin{aligned}|\partial^\alpha Q_T^{\delta,\Theta^*_T}\partial^\beta f(x)|\leqslant&\mathbf{K}_f\,\frac{1+\mathbf{1}_{\mathbf{p}_{\max(q+3,2L+5)}+\mathbf{p}_f>0}|x|^C_{\mathbb{R}^d}}{(\boldsymbol{\mathcal{V}}_L(x)T)^{\frac{3}{2}13^Ldq(q+3)+1}}\\&\times\mathbf{K}^C_{\max(q+3,2L+5)}\exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4),\end{aligned} \tag{3.23}$$

with $C=C(d,N,L,q,\mathbf{p}_{\max(q+4,2L+5)},\mathbf{p}_f,\frac{1}{m_*},\frac{1}{r_*})\geqslant 0$.

B. *Let $h>0$. Assume that $\mathbf{A}_1^\delta(2L+5)$ $\mathbf{A}_2(x,L)$, $\mathbf{A}_3^\delta(+\infty)$, $\mathbf{A}_4^\delta$ and $\mathbf{A}_5^\delta(x,T)$ hold. Then,*

(3.24)
$$|Q_T^\delta f(x)-Q_T^{\delta,\Theta_T^*}f(x)| \leqslant \delta^h\, \mathbf{K}_f\, \frac{1+\mathbf{1}_{\mathbf{p}_{2L+5}+\mathbf{p}_f>0}|x|_{\mathbb{R}^d}^C}{\boldsymbol{\mathcal{V}}_L(x)^{\frac{3}{2}13^L d\max(2,\frac{89h}{44d})}}\, \mathbf{K}_{2L+5}^C \exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4),$$

with $C=C(d,N,L,p,\mathbf{p}_{2L+5},\mathbf{p}_f,\frac{1}{m_*},h)\geqslant 0$.

Remark 3.4. *(1) In the case of uniform Hörmander hypothesis $\mathbf{A}_2^\infty(L)$ (see (2.5)), if we consider $\delta\leqslant\delta_0$ for some δ_0 small enough, then for any $x\in\mathbb{R}^d$, $Q_T^{\delta,\Theta_T^*}f(x)$ can be replaced by the localized probability measure $\frac{1}{\mathbb{E}_x[\Theta_T^*]}\mathbb{E}_x[\Theta_T^* f(X_T^\delta)]$ and the conclusion of Theorem 3.4 still hold. In case of non uniform Hörmander property, δ_0 would depend on x so it is not uniform anymore and we can not obtain the same result.*

(2) Using our approach, we can easily show that under the uniform Hörmander hypothesis $\mathbf{A}_2^\infty(L)$ (see (2.5)), $(\boldsymbol{\mathcal{V}}_L(x)T)^{-\frac{3}{2}13^L dq(q+3)+1}$ can be replaced by $(\boldsymbol{\mathcal{V}}_L^\infty\, T)^{-13^L dq(q+3)}$ in the r.h.s. of (3.23) and $\boldsymbol{\mathcal{V}}_L(x)$ can be replaced by 1 in the r.h.s. of (3.24).

Proof. For the sake of clarity, in the proof of this result we simply denote by C the constant appearing in (3.23) (respectively (3.24)) avoiding to make appear the dependence to the parameters. Notice that we will similarly use this simplified notation as convention in all the proofs of this paper.

We prove the result for $f\in\mathcal{C}_{pol}^\infty(\mathbb{R}^d;\mathbb{R})$. The extension to our setting follows from similar arguments as in the proof of Proposition 3.2 in [16] by introducing an smooth approximating sequence for f.

The first step of our proof consists in estimating the Sobolev norm of $\partial_{\mathrm{X}_0^\delta}^\gamma\Theta_T^*$ for a multi index $\gamma\in\mathbb{N}^d$. We begin with a useful lemma establishing bounds on the derivatives of the inverse covariance matrix.

Lemma 3.2. *Let $m\in\mathbb{N}$, $\alpha\in(\mathcal{T}\times\mathcal{N})^m$ and $\gamma\in\mathbb{N}^d$ with $m+|\gamma|\geq 1$. Then*
$$\begin{aligned}|\,\mathbf{D}_\alpha\,\partial_{\mathrm{X}_0^\delta}^\gamma \det(\boldsymbol{\gamma}_{X_T^\delta})| \leqslant & C(d,N,m,|\gamma|)1\vee T^{m+|\gamma|-1}1\vee|\det\boldsymbol{\gamma}_{X_T^\delta}|^{m+|\gamma|+1}\\ &\times\delta\sum_{\alpha'\in\alpha::\mathcal{T}\times\mathcal{N}}\sum_{|\gamma'|\leqslant|\gamma|}|\,\mathbf{D}_{\alpha'}\,\partial_{\mathrm{X}_0^\delta}^{\gamma'}X_T^\delta|_{\mathbb{R}^d}^{2d(m+|\gamma|)}.\end{aligned}$$

where we denote $\alpha::\mathcal{T}\times\mathcal{N}:=\{\alpha'\in(\mathcal{T}\times\mathcal{N})^l, l\in\{1,\ldots,m+1\},\alpha'^{,j}\in\{\alpha^1,\ldots,\alpha^m\}, j=1,\ldots,l-1\}$.

Proof. We first write $|\,\mathbf{D}_\alpha\,\partial_{\mathrm{X}_0^\delta}^\gamma\det(\boldsymbol{\gamma}_{X_T^\delta})|=|\,\mathbf{D}_\alpha\,\partial_{\mathrm{X}_0^\delta}^\gamma\frac{1}{\det(\boldsymbol{\sigma}_{X_T^\delta})}|$. In particular, owing to the Faà di Bruno formula for the inverse function yields the following estimate
$$|\,\mathbf{D}_\alpha\,\partial_{\mathrm{X}_0^\delta}^\gamma\det(\boldsymbol{\gamma}_{X_T^\delta})| \leqslant C|\sum_{k=1}^{m+|\gamma|}\frac{1}{\det(\boldsymbol{\sigma}_{X_T^\delta})^{k+1}}\sum_{(\alpha',\gamma')\in\mathcal{U}_k(\alpha,\gamma)}\prod_{j=1}^k|\,\mathbf{D}_{\alpha_j'}\,\partial_{\mathrm{X}_0^\delta}^{\gamma_j'}\det(\boldsymbol{\sigma}_{X_T^\delta})|,$$
where
$$\mathcal{U}_k(\alpha,\gamma)=\{(\alpha',\gamma')\in\mathcal{U}_k^1(\alpha)\times\mathcal{U}_k^2(\gamma),\forall j\in\{1,\ldots,k\},\|\alpha_j'\|+|\gamma_j'|>0\} \tag{3.25}$$
with notation $\|\alpha\|=i$ if $\alpha\in(\mathcal{T}\times\mathcal{N})^i$ and
$$\mathcal{U}_k^1(\alpha)=\{\alpha'=(\alpha_j')_{j=1}^k,\alpha_j'\in(\mathcal{T}\times\mathcal{N})^{\|\alpha_j'\|},\alpha=\mathcal{P}(\alpha_1'^{,1},\ldots,\alpha_1'^{,\|\alpha_1'\|},\ldots,\alpha_k'^{,1},\ldots,\alpha_k'^{,\|\alpha_k'\|})\}$$
where $\mathcal{P}$ is the notation for the permutation acting on the components of $\alpha_1',\ldots,\alpha_k'$ and
$$\mathcal{U}_k^2(\alpha,\gamma)=\{\gamma'=(\gamma_1',\ldots,\gamma_k'),\gamma_j'\in\mathbb{N}^d,\sum_{j=1}^k\gamma_j'=\gamma\}.$$

Notice that we allow the cases $\alpha_j'=\emptyset$ and $\gamma_j'=0\in\mathbb{N}^d$. Remembering the definition of $\boldsymbol{\sigma}_{X_T^\delta}$ and using standard estimates, we derive also
$$|\,\mathbf{D}_{\alpha_j'}\,\partial_{\mathrm{X}_0^\delta}^{\gamma_j'}\det(\sigma_{X_T^\delta,\mathcal{T}}^\delta)|\leqslant C\delta\sum_{\tilde{\alpha}\in\alpha::\mathcal{T}\times\mathcal{N}}\sum_{|\tilde{\gamma}|\leqslant|\gamma|}|\,\mathbf{D}_{\tilde{\alpha}}\,\partial_{\mathrm{X}_0^\delta}^{\tilde{\gamma}}X_T^\delta|_{\mathbb{R}^d}^{2d}.$$

Finally, gathering all the terms together, applying the Hölder inequality and observing that $\mathrm{card}(\mathcal{U}_k(\alpha,\gamma))\leqslant C$, we obtain the announced result. □

We denote shortly $\Theta_1 = \Theta_{X_T^\delta, \det(\dot{X}_T^\delta)^2, \eta_1, \mathcal{T}} = \Psi_{\eta_1}(\det(\boldsymbol{\gamma}_{X_T^\delta}(\dot{X}_T^\delta)^2))$, where $\dot{X}_T^\delta = \partial_{\mathrm{X}_0^\delta} X_T^\delta$ is defined in (3.18). We first estimate the Sobolev norms of $\partial^\gamma_{\mathrm{X}_0^\delta}\Theta_1$ for $\gamma \in \mathbb{N}^d$. Similarly as in the proof of Lemma 3.2, using the Faà di Bruno formula, we can write (we refer to (3.25) for the definition of $\mathcal{U}_k(\alpha,\gamma)$ and to (3.9) for the definition of our Sobolev-Malliavin norms $|.|_{\mathbb{R},m}$),

$$
\begin{aligned}
|\partial^\gamma_{\mathrm{X}_0^\delta}\Theta_1|^2_{\mathbb{R},m} =& \sum_{l=0}^m \delta^l \sum_{\alpha\in(\mathcal{T}\times\mathcal{N})^l} | \sum_{k=\mathbf{1}_{l+|\gamma|>0}}^{l+|\gamma|} \partial^{(k)}\Psi_{\eta_1}(\det(\boldsymbol{\gamma}_{X_T^\delta}(\dot{X}_T^\delta)^2)) \\
& \times \sum_{(\alpha',\gamma')\in\mathcal{U}_k(\alpha,\gamma)} C(\alpha',\gamma') \prod_{j=1}^k \mathbf{D}_{\alpha'_j}\, \partial^{\gamma'_j}_{\mathrm{X}_0^\delta} \det(\boldsymbol{\gamma}_{X_T^\delta}(\dot{X}_T^\delta)^2)|^2 \\
\leqslant & C\mathbf{1}_{\Theta_1>0}\|\psi_{\eta_1}\|^2_{\infty,m+|\gamma|}(1+\sum_{l=0}^m\delta^l \sum_{\alpha\in(\mathcal{T}\times\mathcal{N})^l}\sum_{k=1}^{l+|\gamma|}\sum_{(\alpha',\gamma')\in\mathcal{U}_k(\alpha,\gamma)}\prod_{j=1}^k |\,\mathbf{D}_{\alpha'_j}\,\partial^{\gamma'_j}_{\mathrm{X}_0^\delta}\det(\boldsymbol{\gamma}_{X_T^\delta}(\dot{X}_T^\delta)^2)|^2).
\end{aligned}
$$

with convention $\sum_{(\alpha',\gamma')\in\mathcal{U}_0(\alpha,\gamma)}\prod_{j=1}^0 = 1$ and $\sum_{\alpha\in(\mathcal{T}\times\mathcal{N})^l}$ can be omitted when $l=0$. At this point, evoking again the Faà di Bruno formula combined with Lemma 3.2, we derive the following estimate (recall that $|\gamma'_j + \|\alpha'_j\| \geqslant 1$),

$$
\begin{aligned}
|\,\mathbf{D}_{\alpha'_j}\,\partial^{\gamma'_j}_{\mathrm{X}_0^\delta}\det(\boldsymbol{\gamma}_{X_T^\delta}(\dot{X}_T^\delta)^2)| \leqslant & C 1\vee T^{\|\alpha'_j\|+|\gamma'_j|-1} \\
& \times 1\vee|\det\boldsymbol{\gamma}_{X_T^\delta}|^{\|\alpha'_j\|+|\gamma'_j|+1}\delta \sum_{(\tilde{\alpha},\tilde{\gamma})\in\mathcal{I}(\alpha,\gamma)} |\,\mathbf{D}_{\tilde{\alpha}}\,\partial^{\tilde{\gamma}}_{\mathrm{X}_0^\delta}X_T^\delta|^{2d(m+|\gamma|+1)}_{\mathbb{R}^d}.
\end{aligned}
$$

where $\mathcal{I}(\alpha,\gamma) = \{(\alpha',\gamma')\in(\alpha :: \mathcal{T}\times\mathcal{N})\times\mathbb{N}^d, |\gamma'|\leqslant|\gamma|+1, \|\alpha'\|+|\gamma'|\leqslant\|\alpha\|+|\gamma|+1\}$. Moreover, since $\sum_{j=1}^k|\gamma'_j| = |\gamma|$ and $\sum_{j=1}^k\|\alpha'_j\| = \|\alpha\|$, we deduce from the Hölder inequality, that

$$
\begin{aligned}
|\partial^\gamma_{\mathrm{X}_0^\delta}\Theta_1|^2_{\mathbb{R},m} \leqslant & C\mathbf{1}_{\Theta_1>0}\|\Psi_{\eta_1}\|^2_{\infty,m+|\gamma|}(1+\sum_{l=0}^m\delta^l(1\vee|\det\boldsymbol{\gamma}_{X_T^\delta}|^{4(l+|\gamma|)})\sum_{k=1}^{l+|\gamma|}1\vee T^{2(l+|\gamma|-k)} \\
& \times\sum_{\alpha\in(\mathcal{T}\times\mathcal{N})^l}\delta T^{2k-1}\sum_{(\tilde{\alpha},\tilde{\gamma})\in\mathcal{I}(\alpha,\gamma)}|\,\mathbf{D}_{\tilde{\alpha}}\,\partial^{\tilde{\gamma}}_{\mathrm{X}_0^\delta}X_T^\delta|^{4dk(m+|\gamma|+1)}_{\mathbb{R}^d}) \\
\leqslant & C\mathbf{1}_{\Theta_1>0}\|\Psi_{\eta_1}\|^2_{\infty,m+|\gamma|}1\vee T^{2(m+|\gamma|)-1}(1\vee|\det\boldsymbol{\gamma}_{X_T^\delta}|^{4(m+|\gamma|)}) \\
& \times(1+\sum_{|\tilde{\gamma}|\leqslant|\gamma|+1}|\partial^{\tilde{\gamma}}_{\mathrm{X}_0^\delta}X_T^\delta|^{4d(m+|\gamma|)(m+|\gamma|+1)}_{\mathbb{R}^d,m+\mathbf{1}_{|\tilde{\gamma}|<|\gamma|+1}}),
\end{aligned}
$$

Therefore, using the the computations from Remark 3.1, we derive that $|\,\Psi_{\eta_2}(|Z_w^\delta|_{\mathbb{R}^N})|_{\mathbb{R},q}\leqslant C\|\Psi_{\eta_2}(|\cdot|_{\mathbb{R}^N})\|_{\infty,m}$, and, taking $(\eta_1,\eta_2) = (\boldsymbol{\eta}_1(\delta),\boldsymbol{\eta}_2(\delta))$, it follows from Lemma 3.1 that

$$
\begin{aligned}
|\partial^\gamma_{\mathrm{X}_0^\delta}\Theta_T^*|_{\mathbb{R},m} \leqslant & C\mathbf{1}_{\Theta_T^*>0}\|\Psi_{\boldsymbol{\eta}_1(\delta)}\|_{\infty,m+|\gamma|}\|\Psi_{\boldsymbol{\eta}_2(\delta)}(|\cdot|_{\mathbb{R}^N})\|_{\infty,m}1\vee T^{m+|\gamma|-\frac{1}{2}} \\
& \times 1\vee|\det\boldsymbol{\gamma}_{X_T^\delta}|^{2(m+|\gamma|)}(1+\sum_{|\tilde{\gamma}|\leqslant|\gamma|+1}|\partial^{\tilde{\gamma}}_{\mathrm{X}_0^\delta}X_T^\delta|^{2d(m+|\gamma|)(m+|\gamma|+1)}_{\mathbb{R}^d,m+\mathbf{1}_{|\tilde{\gamma}|<|\gamma|+1}}).
\end{aligned}
\tag{3.26}
$$

We now focus on establishing (3.23). We remark that,

$$
\partial^\alpha Q_T^{\delta,\Theta_T^*}\partial^\beta f(x) = \sum_{|\beta|\leqslant|\gamma|\leqslant q}\sum_{0\leqslant|\gamma'|\leqslant q-|\gamma|}\mathbb{E}_x[\partial^\gamma f(X_T^\delta)\mathscr{P}_\gamma(X_T^\delta)\partial^{\gamma'}_{\mathrm{X}_0^\delta}\Theta_T^*],
\tag{3.27}
$$

where $\mathscr{P}_\gamma(X_T^\delta)$ is a universal polynomial of $\partial^\rho_{\mathrm{X}_0^\delta}X_T^\delta, 1\leqslant|\rho|\leqslant q-|\gamma|+1,\ \gamma,\gamma'\in\mathbb{N}^d$. We do not give details about the validity of the formula above. It is actually a consequence of the dominated convergence theorem and of the Gronwall lemma to control the increments of the function $X_0^\delta\mapsto X_T^\delta$ and its derivatives. Using the integration by parts formula (3.12) and the estimate (3.13) obtained in Theorem 3.1, we derive

$$
\begin{aligned}
|\mathbb{E}_x[\partial^\gamma f(X_T^\delta)\mathscr{P}_\gamma(X_T^\delta)\partial^{\gamma'}_{\mathrm{X}_0^\delta}\Theta_T^*]| =& |\mathbb{E}_x[f(X_T^\delta)\,\mathbf{H}(X_T^\delta,\mathscr{P}_\gamma(X_T^\delta)\partial^{\gamma'}_{\mathrm{X}_0^\delta}\Theta_T^*)[\gamma]]| \\
\leqslant & \mathbf{K}_f\,\mathbb{E}_x[(1+|X_T^\delta|^{\mathbf{p}_f}_{\mathbb{R}^d})|\,\mathbf{H}(X_T^\delta,\mathscr{P}_\gamma(X_T^\delta)\partial^{\gamma'}_{\mathrm{X}_0^\delta}\Theta_T^*)[\gamma]|] \\
\leqslant & C1\vee T^{q-\frac{1}{2}}\,\mathbf{K}_f\,A_1A_2A_3A_4,
\end{aligned}
$$

where, using and (3.26) and Lemma 3.1 combined with the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
A_1 =& 1 \vee \mathbb{E}_x[|\det \boldsymbol{\gamma}_{X_T^\delta}|^{4q(q+3)} \mathbf{1}_{\Theta_T^*>0}]^{\frac{1}{4}} \\
A_2 =& 1 + \mathbb{E}_x[|X_T^\delta|_{\mathbb{R}^d,1,q+1}^{8dq(q+2)}]^{\frac{1}{4}} + \mathbb{E}_x[|\mathbf{L}\, X_T^\delta|_{\mathbb{R}^d,q-1}^{8q}]^{\frac{1}{4}} \\
A_3 =& 1 + \mathbb{E}_x[\sum_{|\tilde{\gamma}|\leqslant q-|\gamma|+1} |\partial_{X_0^\delta}^{\tilde{\gamma}} X_T^\delta|_{\mathbb{R}^d,|\gamma|+\mathbf{1}_{|\tilde{\gamma}|\leqslant q-|\gamma|}}^{8dq(q+1)}]^{\frac{1}{4}} \\
A_4 =& \mathbb{E}_x[(1+|X_T^\delta|_{\mathbb{R}^d}^{\mathbf{p}_f})^4 |\mathscr{P}_\gamma(X_T^\delta)|_{\mathbb{R},|\gamma|}^4]^{\frac{1}{4}}.
\end{aligned}
$$

In order to bound the first term, we use the estimates on the moments of inverse Malliavin covariance matrix derived Theorem 3.3 and obtain

$$
A_1 \leqslant \frac{1+\mathbf{1}_{\mathbf{p}_{2L+5}>0}|x|_{\mathbb{R}^d}^C}{(\boldsymbol{\mathcal{V}}_L(x)T)^{\frac{3}{2}13^L dq(q+3)+1}} \mathbf{K}_{2L+5}^C \exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4).
$$

Moreover, using the results from Theorem 3.2, we obtain

$$
A_2A_3A_4 \leqslant (1+\mathbf{1}_{\mathbf{p}_{q+3}+\mathbf{p}_f>0}|x|_{\mathbb{R}^d})^C\, \mathbf{K}_{q+3}^C \exp(C(T+1)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^2).
$$

We gather all the terms together and the proof of (3.23) is completed.

Now, let us prove **B.**. We have

$$
|Q_T^\delta f(x) - Q_T^{\delta,\Theta_T^*} f(x)| \leqslant \mathbb{E}_x[f(X_T^\delta)(1-\Theta_T^*)] \leqslant \mathbf{K}_f\, \mathbb{E}_x[(1+|X_T^\delta|_{\mathbb{R}^d}^{\mathbf{p}_f})^2]^{\frac{1}{2}} \mathbb{E}_x[1-\Theta_T^*]^{\frac{1}{2}}.
$$

We then obtain an upper bound for $\mathbb{E}_x[1-\Theta_T^*]$ by using (3.21). The upper bound of $\mathbb{E}_x[|X_t^\delta|^{2\,\mathbf{p}_f}]$ is obtained using Lemma 4.2. It follows that, for every $a>0$ and every $p \geqslant 4$,

$$
\begin{aligned}
|Q_T^\delta f(x) - Q_T^{\delta,\Theta_T^*} f(x)| \leqslant & (\delta^{-1}\,\boldsymbol{\eta}_2(\delta)^{-a}\,\mathbf{M}_{\lceil a\rceil}(Z^\delta) + \boldsymbol{\eta}_1(\delta)^{-p}(1+\boldsymbol{\mathcal{V}}_L(x)^{-\frac{3}{2}13^L dp}))^{\frac{1}{2}} \\
& \times \mathbf{K}_f(1+\mathbf{1}_{\mathbf{p}_{2L+5}+\mathbf{p}_f>0}|x|_{\mathbb{R}^d}^C)\,\mathbf{K}_{2L+5}^C \exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4),
\end{aligned}
$$

with C depending also on a. We fix $p=p(h)=\max(4,\frac{89h}{22d})$ so that $\boldsymbol{\eta}_1(\delta)^{-p(h)} \leqslant \delta^{2h}C(1+T^C)$. Similarly we chose $a=a(h)=2(2h+1)\max(\mathbf{p}+1,89)$ so that $\boldsymbol{\eta}_2(\delta)^{-a(h)}\delta^{-1} \leqslant \delta^{2h}C(1+T^C)$ and

$$
\begin{aligned}
|Q_T^\delta f(x) - Q_T^{\delta,\Theta_T^*} f(x)| \leqslant & \delta^h\,\mathbf{K}_f(1+\boldsymbol{\mathcal{V}}_L(x)^{-\frac{3}{4}13^L dp(h)})(1+\mathbf{1}_{\mathbf{p}_{2L+5}+\mathbf{p}_f>0}|x|_{\mathbb{R}^d}^C) \\
& \times \mathbf{K}_{2L+5}^C\, C \exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4),
\end{aligned}
$$

and the proof of (3.24) is completed.

□

3.5. **Proof of the main results.** From a practical viewpoint, an issue of this last result resides in the computation of Q^{δ,Θ_T^*}. Indeed, Θ is not simulable (at least easily) and then methods such as Monte Carlo do not seem to be applicable. A solution is provided by Theorem 2.1, where we show that the regularization properties are also satisfied by $Q^{\delta,\theta}$. In this case, Monte Carlo methods can be designed by simply simulating the sum of X_T^δ and of an independent Gaussian variable. The proof of this result exploits the one we just established in Theorem 3.4.

Proof of Theorem 2.1. Similarly as in the proof of Theorem 3.4, it is sufficient to prove our result for $f \in \mathcal{C}_{pol}^\infty(\mathbb{R}^d;\mathbb{R})$. Let us prove (2.14). Assume that $q \in \mathbb{N}^*$, the case $q=0$ being immediate. As in (3.27), we write

$$
\partial^\alpha Q_T^{\delta,\theta}\partial^\beta f(x) = \sum_{|\beta|\leqslant|\gamma|\leqslant q} \mathbb{E}_x[\partial^\gamma f(\delta^\theta G + X_T^\delta)\mathscr{P}_\gamma(X_T^\delta)],
$$

where $\mathscr{P}_\gamma(X_t^\delta)$ is a universal polynomial of $\partial_{X_0^\delta}^\rho X_t^\delta, 1 \leqslant |\rho| \leqslant q-|\gamma|+1$. We decompose

$$
\mathbb{E}_x[\partial^\gamma f(\delta^\theta G + X_T^\delta)\mathscr{P}_\gamma(X_T^\delta)] = A_1 + A_2,
$$

with $A_1 = \mathbb{E}_x[\Theta_T^*\partial^\gamma f(\delta^\theta G + X_T^\delta)\mathscr{P}_\gamma(X_T^\delta)]$ and $A_2 = \mathbb{E}_x[\partial^\gamma f(\delta^\theta G + X_T^\delta)\mathscr{P}_\gamma(X_T^\delta)(1-\Theta_T^*)]$. Applying the reasoning from the proof of Theorem 3.4 (with $\alpha=\emptyset$) we derive

$$
A_1 \leqslant \mathbf{K}_f\, \frac{1+\mathbf{1}_{\mathbf{p}_{\max(q+3,2L+5)}+\mathbf{p}_f>0}|x|_{\mathbb{R}^d}^C}{(\boldsymbol{\mathcal{V}}_L(x)T)^{\frac{3}{2}13^L dq(q+3)+1}} \mathbf{K}_{\max(q+3,2L+5)}^C \exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4).
$$

Moreover, since G follows the standard Gaussian distribution and is independent of X^δ and Θ^*_T, we have

$$A_2 = \mathbb{E}_x[\mathscr{P}_\gamma(X^\delta_T)(1-\Theta^*_T)\int_{\mathbb{R}^d}\partial^\gamma f(\delta^\theta u + X^\delta_T)(2\pi)^{-\frac{d}{2}}e^{-\frac{|u|^2}{2}}du].$$

Now, notice that $\partial^\gamma f(\delta^\theta u + X^\delta_T) = \delta^{-|\gamma|\theta}\partial^\gamma_u(f(\delta^\theta u + X^\delta_T))$, so that, using standard integration by parts, we have

$$A_2 = \delta^{-|\gamma|\theta}\mathbb{E}_x[\mathscr{P}_\gamma(X^\delta_T)(1-\Theta^*_T)\int_{\mathbb{R}^d} f(\delta^\theta u + X^\delta_T)\mathscr{H}_\gamma(u)(2\pi)^{-\frac{d}{2}}e^{-\frac{|u|^2}{2}}\mathrm{d}u],$$

where $\mathscr{H}_\gamma$ is the Hermite polynomial corresponding to the multi-index γ. Finally, using the results from Theorem 3.2 (see (3.15)), we obtain

$$|A_2| \leqslant \delta^{-|\gamma|\theta}\,\mathbf{K}_f\,\mathbb{E}_x[1-\Theta^*_T]^{\frac{1}{2}}(1+\mathbf{1}_{\mathbf{p}_{q+3}+\mathbf{p}_f>0}|x|^C_{\mathbb{R}^d})\,\mathbf{K}^C_{q+3}\exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^2_3).$$

As in the proof of Theorem 3.4, we estimate $\mathbb{E}_x[1-\Theta^*_T]$ using Theorem 3.3 (see (3.21)) with $p = p(q\theta) = \max(4, \frac{89q\theta}{22d})$ and $a = a(q\theta) = 2(2q\theta+1)\max(\mathbf{p}+1, 89)$ and deduce that

$$|\partial^\alpha Q^{\delta,\theta}_T\partial^\beta f(x)| \leqslant \mathbf{K}_f\,\frac{1+\mathbf{1}_{\mathbf{p}_{\max(q+3,2L+5)}+\mathbf{p}_f>0}|x|^C_{\mathbb{R}^d}}{(\boldsymbol{\mathcal{V}}_L(x)T)^{\max(\frac{3}{2}13^L d\frac{89q\theta}{44d},\frac{3}{2}13^L dq(q+3)+1)}}\,\mathbf{K}^C_{\max(q+3,2L+5)}\exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^4_3),$$

and the proof of (2.14) is completed. Remark that with our approach, under the uniform Hörmander hypothesis $\mathbf{A}^\infty_2(L)$ (see (2.5)), we can show that $(\boldsymbol{\mathcal{V}}_L(x)T)^{\max(\frac{3}{2}13^L d\frac{89q\theta}{44d},\frac{3}{2}13^L dq(q+3)+1)}$ can be replaced by $(\boldsymbol{\mathcal{V}}^\infty_L\,T)^{-13^L dq(q+3)}$ in the *r.h.s.* above.

Let us prove (2.15). Since f has polynomial growth, it follows that

$$\begin{aligned}|Q^\delta_T f(x) - Q^{\delta,\theta}_T f(x)| \leqslant{}& |\mathbb{E}_x[\Theta^*_T(f(X^\delta_T) - f(X^\delta_T+\delta^\theta G))]| \\ &+ \mathbf{K}_f\,C(1+\mathbb{E}_x[|X^\delta_T|^{2\mathbf{p}_f}_{\mathbb{R}^d}]^{\frac{1}{2}} + \delta^{\theta\,\mathbf{p}_f}\mathbb{E}[|G|^{2\mathbf{p}_f}_{\mathbb{R}^d}]^{\frac{1}{2}})\mathbb{E}_x[1-\Theta^*_T]^{\frac{1}{2}} \\ \leqslant{}& \delta^\theta\sum_{j=1}^d\int_0^1|\mathbb{E}_x[\Theta^*_T\partial^{(j)}f(X^\delta_T+\lambda\delta^\theta G)G^j]|\mathrm{d}\lambda \\ &+ \mathbf{K}_f\,C(1+|x|^{\mathbf{p}_f}_{\mathbb{R}^d})\exp(CT\,\mathbf{K}^2_3\,\mathbf{M}_C(Z^\delta))\mathbb{E}_x[1-\Theta^*_T]^{\frac{1}{2}}.\end{aligned}$$

Using Theorem 3.4 (see (3.23) with $q=1$) and the same estimate of $\mathbb{E}_x[1-\Theta^*_T]$ as in the proof of (2.14) with $p = p(\theta) = \max(4, \frac{89\theta}{22d})$ and $a = a(\theta) = 2(2\theta+1)\max(\mathbf{p}_3+1, 89)$, we obtain

$$|Q^\delta_T f(x) - Q^{\delta,\theta}_T f(x)| \leqslant \delta^\theta\,\mathbf{K}_f\,\frac{1+\mathbf{1}_{\mathbf{p}_{2L+5}+\mathbf{p}_f>0}|x|^C_{\mathbb{R}^d}}{(\boldsymbol{\mathcal{V}}_L(x)T)^{\max(\frac{3}{2}13^L d\frac{89\theta}{44d},13^L 6d+1)}}\,\mathbf{K}^C_{2L+5}\exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^4_3).$$

Notice that under the uniform Hörmander hypothesis $\mathbf{A}^\infty_2(L)$, $(\boldsymbol{\mathcal{V}}_L(x)T)^{\max(\frac{3}{2}13^L d\frac{89\theta}{44d},13^L 6d+1)}$ can be replaced by $(\boldsymbol{\mathcal{V}}^\infty_L\,T)^{13^L 4d}$ in the *r.h.s.* above. □

We now show the existence as well as upper bounds for the density of X^δ_T. This result is mainly a consequence of Theorem 2.1. It is noteworthy that we also propose a Gaussian type bound when relying in a simplified framework. It is derived combining a representation formula for the density, Theorem 2.1 and the Azuma-Hoeffding inequality.

Proof of Corollary 2.1 . We first prove the existence together with a representation formula of $q^{\delta,\theta}_T$. Let $f \in \mathcal{C}^\infty_b(\mathbb{R}^d;\mathbb{R})$. Let us define $g:\mathbb{R}^d\to\mathbb{R}$ such that for every $u\in\mathbb{R}^d$,

$$g(u) := \int_0^{u^d}\dots\int_0^{u^1} f(y)\mathrm{d}y^1\dots\mathrm{d}y^d.$$

Then $g \in \mathcal{C}^\infty_{pol}(\mathbb{R}^d;\mathbb{R})$ and for $\gamma_0 = (1,\dots,1)\in\mathbb{N}^d$, we have $\partial^{\gamma_0}g = f$. In order to state our representation formula, we introduce, we introduce the function $\mathbf{\Gamma} : (u,v)\in(\mathbb{R}^d)^2 \mapsto \prod_{i=1}^d(\mathbf{1}_{0<v^i<u^i} - \mathbf{1}_{u^i<v^i<0})$ which plays a fundamental role in the representation formula that will be derived.

In particular, applying Theorem 3.1 with the test function g, it follows from the Fubini identity that, with similar notations as in the proof of Theorem 2.1,

$$\begin{aligned}\partial^\alpha Q_T^{\delta,\theta}\partial^\beta f(x) =&\partial^\alpha Q_T^{\delta,\theta}\partial^{(\beta,\gamma_0)}g(x) = \sum_{0\leqslant|\gamma|\leqslant q+d}\mathbb{E}_x[\partial^\gamma g(\delta^\theta G+X_T^\delta)\mathscr{P}_\gamma(X_T^\delta)(\Theta_T^*+1-\Theta_T^*)]\\ =&\sum_{0\leqslant|\gamma|\leqslant q+d}\mathbb{E}_x[g(\delta^\theta G+X_t^\delta)\,\mathbf{H}(X_t^\delta,\Theta_T^*\mathscr{P}_\gamma(X_t^\delta))[\gamma]]\\ &+\mathbb{E}_x[g(\delta^\theta G+X_t^\delta)\delta^{-|\gamma|\theta}\mathscr{P}_\gamma(X_T^\delta)(1-\Theta_T^*)\mathscr{H}_\gamma(G))]\\ =&\int_{\mathbb{R}^d}f(y)\mathbb{E}_x[H(\alpha,\beta)\mathbf{\Gamma}(\delta^\theta G+X_t^\delta,y)]\mathrm{d}y, \qquad (3.28)\end{aligned}$$

where

$$H(\alpha,\beta)=\sum_{0\leqslant|\gamma|\leqslant q+d}\mathbf{H}(X_t^\delta,\Theta_T^*\mathscr{P}_\gamma(X_t^\delta))[\gamma]+\delta^{-|\gamma|\theta}\mathscr{P}_\gamma(X_T^\delta)(1-\Theta_T^*)\mathscr{H}_\gamma(G).$$

Notice that using the Hölder inequality, it follows from $\mathbf{A}_1^\delta(\max(q+d+3,2L+5))$ and $\mathbf{A}_3^\delta(+\infty)$ that $\delta^\theta G^i+X_t^{\delta,i}\in \mathrm{L}_p(\Omega,\mathbb{P})$ for any $p\geq 1$ (see Lemma 4.2), and that $H(\alpha,\beta)\mathbf{\Gamma}(\delta^\theta G+X_t^\delta,\cdot)\in \mathrm{L}_1(\Omega\times\mathbb{R}^d,\mathbb{P}\otimes\mathrm{d}y)$ (which makes the use of Fubini identity valid in the previous computation). Moreover, since $\mathbf{A}_1^\delta(\max(q+d+4,2L+5))$ also holds, a similar result is true in the case $|\alpha|+|\beta|=q+1$. Hence, using the same approach as in [33], Lemma 3.1 and [30] Lemma 3.1, it follows that $\delta^\theta G+X_T^\delta$ has a smooth density $q_T^{\delta,\theta}(X_0^\delta,\cdot)$ with $q_T^{\delta,\theta}\in\mathcal{C}^q(\mathbb{R}^d\times\mathbb{R}^d;\mathbb{R})$. Therefore, owing to (3.28) when f has compact support and to the dominated convergence theorem, we have the following representation formula for $q_T^{\delta,\theta}$ and its derivatives:

$$\partial_x^\alpha\partial_y^\beta q_T^{\delta,\theta}(x,y)=(-1)^{|\beta|}\mathbb{E}_x[\mathbf{\Gamma}(\delta^\theta G+X_t^\delta,y)H(\alpha,\beta)].$$

The estimate (2.16) then follows from the Cauchy Schwarz inequality, Lemma 4.2 combined with Markov inequality and a similar approach as in the proof of the previous results to bound the moments of $H(\alpha,\beta)$. In particular

$$|\partial_x^\alpha\partial_y^\beta q_T^{\delta,\theta}(x,y)|\leqslant\frac{(1+\mathbf{1}_{\mathbf{p}_{\max(q+d+3,2L+5)}>0}|x|_{\mathbb{R}^d}^C)C\exp(CT)}{(\boldsymbol{\mathcal{V}}_L(x)T)^\eta(1+|y|_{\mathbb{R}^d}^p)},$$

where $\eta=13^L3d\max(\frac{89(d+q)\theta}{44d},(d+q)^2+3(d+q)+1)$. Now let us prove (2.17). The reasonning made before still apply if we replace $\mathbf{\Gamma}$ by $\mathbf{\Gamma}_x(u,v)\in(\mathbb{R}^d)^2\mapsto\prod_{i=1}^d(\mathbf{1}_{x^i<v^i<u^i}-\mathbf{1}_{u^i<v^i<x^i}$. We first notice that $|\mathbf{\Gamma}_x(u,v)|\leq\prod_{i=1}^d\mathbf{1}_{|u^i-x^i|\geq|v^i-x^i|}\leq\mathbf{1}_{|u-x|_{\mathbb{R}^d}\geq|v-x|_{\mathbb{R}^d}}$. Using Taylor expansions of ψ and recalling that $\psi(u,t,0,0)=x$ for every $(u,t)\in\mathbb{R}^d\times\mathbb{R}_+$, and then exponential estimate on the tail probability of G and the Azuma-Hoeffding inequality yields, for $|y-x|_{\mathbb{R}^d}\geqslant 6T\,\mathbf{K}_2(1+|z^\infty|^2)$,

$$\begin{aligned}\mathbb{E}_x[\mathbf{\Gamma}_x(X_T^\delta+\delta^\theta G,y)]\leqslant&\,\mathbb{P}_x(|y-x|_{\mathbb{R}^d}-\delta^\theta|G|_{\mathbb{R}^d}|\leqslant|X_T^\delta-x|_{\mathbb{R}^d})\\ \leqslant&\,\mathbb{P}_x(|y-x|_{\mathbb{R}^d}-\delta^\theta|G|_{\mathbb{R}^d}\leqslant 3T\,\mathbf{K}_2(1+|z^\infty|^2)+\delta^{\frac12}|\sum_{t\in\pi^\delta,t<T}\sum_{i=1}^N Z_{t+\delta}^{\delta,i}\partial_{z^i}\psi(X_t^\delta,t,0,0)|_{\mathbb{R}^d})\\ \leqslant&\,\mathbb{P}_x(\frac12|y-x|_{\mathbb{R}^d}-3T\,\mathbf{K}_2(1+|z^\infty|^2)\leqslant\delta^{\frac12}|\sum_{t\in\pi^\delta,t<T}\sum_{i\in\mathcal{N}}Z_{t+\delta}^{\delta,i}\partial_{z^i}\psi(X_t^\delta,t,0,0)|_{\mathbb{R}^d})\\ &+\mathbb{P}(|G|_{\mathbb{R}^d}\geqslant\frac12\delta^{-\theta}|y-x|_{\mathbb{R}^d})\\ \leqslant&\,C\exp(CT-\frac{|y-x|_{\mathbb{R}^d}^2}{cT})+C\exp(-\frac{|y-x|_{\mathbb{R}^d}^2}{c\delta^{2\theta}}),\end{aligned}$$

Finally, we observe that when $|y-x|_{\mathbb{R}^d}<6T\,\mathbf{K}_2(1+|z^\infty|^2)$, the estimate still holds by remarking that

$$\exp(-\frac{|y-x|_{\mathbb{R}^d}^2}{c(T\vee\delta^{2\theta})})\geqslant\exp(-\frac{(6T\,\mathbf{K}_2(1+|z^\infty|^2))^2}{T}),$$

so that

$$\mathbb{E}_x[\mathbf{\Gamma}_x(X_T^\delta+\delta^\theta G,y)]\leqslant\exp(-\frac{|y-x|_{\mathbb{R}^d}^2}{c(T\vee\delta^{2\theta})})\exp((6\,\mathbf{K}_2(1+|z^\infty|^2))^2T).$$

Using the Cauchy-Schwarz inequality combined with the representation formula concludes the proof. □

We end this section with the proof of the invariance principle presented in Theorem 2.2. Our strategy is to decompose the error using the Lindeberg approach and semigroup properties. Our focus is then on the short time estimate *i.e.* the error made on simply one time step of size δ. Then, we replace Q^{δ} by $Q^{\delta,\theta}$. Applying Taylor expansion techniques leads to a representation of the error involving some slight modifications of the process $X^{\delta,\theta}$ satisfying also regularization properties. Exploiting them leads to the expected result. A similar strategy can be designed to prove higher order convergence.

Proof of Theorem 2.2. For $u \in \mathbb{R}^d$, $s,t \in \pi^{\delta}$, $s \leqslant t$, we define, when the expactations are well defined, $Q^{\delta}_{s,t}f(u) := \mathbb{E}[f(X^{\delta}_t)|X^{\delta}_s = u]$, $Q^{\delta,\theta}_{s,t}f(u) := \mathbb{E}[f(X^{\delta}_t + \delta^{\theta}G)|X^{\delta}_s = u]$,$P_{s,t}f(u) := \mathbb{E}[f(X_t)|X_s = u] = \mathbb{E}[f(X_t(s,u))]$ ($X_t(s,u)$, being the solution of (1.5) at time t and starting from u at time s), $\Delta f(u) := Q^{\delta}_{t,t+\delta}f(u) - P_{t,t+\delta}f(u)$ and $\Delta^{\theta}f(u) := Q^{\delta,\theta}_{t,t+\delta}f(u) - P_{t,t+\delta}f(u)$. We observe that the results from Theorem 3.4 remains true replacing $(Q^{\delta,\theta}_t)_{t\geqslant 0}$ by $(Q^{\delta,\theta}_{s,t})_{t\geqslant s}$ for any $s \in \pi^{\delta}$. For sake of clarity, we assume that P satisfies the same regularization property (2.14) as $Q^{\delta,\theta}$. Similar ideas as in [7] can be used to conclude under the actual hypothesis of Theorem 2.2.

We prove the result for $f \in \mathcal{C}^{\infty}_{pol}(\mathbb{R}^d)$. The extension to f simply measurable with polynomial growth follows from the Lusin's theorem. We provide the main key points avoiding heavy calculus which can be dealt with using similar arguments as the one we already developed to derive Theorem 2.1. Using the semigroup property satisfied by Q^{δ} and P, we have

$$\begin{aligned} Q^{\delta}_T f(x) - P_T f(x) &= \sum_{t\in\pi^{\delta},t<T} Q^{\delta}_{0,t}\Delta_{t,t+\delta}P_{t+\delta,T}f(x) \\ &= \sum_{t\in\pi^{\delta},t<T} Q^{\delta,\theta}_{0,t}\Delta^{\theta}_{t,t+\delta}P_{t+\delta,T}f(x) + (Q^{\delta}_{0,t}(Q^{\delta}_{t,t+\delta} - Q^{\delta,\theta}_{t,t+\delta}) + (Q^{\delta}_{0,t} - Q^{\delta,\theta}_{0,t})Q^{\delta,\theta}_{t,t+\delta})P_{t+\delta,T}f(x). \end{aligned}$$

The last two terms can be bounded using (2.15) from Theorem 2.1. and we focus only on the estimate of the first term. As a result of the Taylor expansions, $\Delta^{\theta}_{t,t+\delta}f(x)$ can be written as a finite sum of term with form

$$\mathbb{E}[\int_0^1 \partial^{\alpha}f(Y^{\lambda}_{t+\delta}(x))B(x,t,\delta,\lambda)\mathrm{d}\lambda], \quad \alpha \in \mathbb{N}^d, |\alpha| \leqslant 3,$$

where $Y^{\lambda}_{t+\delta}(x)$ takes values in $\{x, x+\lambda(\psi(x,t,\delta^{\frac{1}{2}}Z^{\delta}_{t+\delta},\delta) - x + \delta^{\theta}G), X_{t+\lambda\delta}(t,x)\}$ and, for any $p \geqslant 1$,

$$\sup_{t\in\pi^{\delta},t<T} \mathbb{E}_x[|B(X^{\delta,\theta}_t,t,\delta,\lambda)|^p_{\mathbb{R},|\alpha|}]^{\frac{1}{p}} \leqslant \delta^{\frac{3}{2}}(1+|x|^c_{\mathbb{R}^d})C\exp(CT), \tag{3.29}$$

It follows that $Q^{\delta,\theta}_{0,t}\Delta^{\theta}_{t,t+\delta}P_{t+\delta,T}f(x)$ is a finite sum of terms with form

$$\mathbb{E}_x[\int_0^1 \partial^{\alpha}P_{t+\delta,T}f(Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t))B(X^{\delta,\theta}_t,t,\delta,\lambda)\mathrm{d}\lambda].$$

At this point, we observe that a similar approach as the one developed in this paper ensures that the results from Theorem 2.1 remains true taking $T = t$ and replacing $X^{\delta,\theta}_t$ by $Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t)$. It relies on the fact that our Malliavin derivatives of $Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t) - X^{\delta,\theta}_t$ can be bounded by a term of order δ. Moreover, $P_{t+\delta,T}f$ has polynomial growth. It follows that for $t \geqslant \frac{1}{3}T\delta^{\varepsilon}$, ε small enough, exploiting the integration by part from Theorem 3.1 (with $F = Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t)$ and $\phi = P_{t+\delta,T}f$) in a similar way as in the proof of Theorem 2.1 and using (3.29) yields

$$\mathbb{E}_x[\int_0^1 \partial^{\alpha}P_{t+\delta,T}f(Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t))B(X^{\delta,\theta}_t,t,\delta,\lambda)\mathrm{d}\lambda] \leqslant \delta^{\frac{3}{2}-\epsilon}\mathbf{K}_f\,\frac{1+|x|^c_{\mathbb{R}^d}}{|\boldsymbol{\mathcal{V}}_L(x)T|^{\eta}}C\exp(CT).$$

Now let $t < \frac{1}{3}T\delta^{\varepsilon}$ so that $T - t - \delta > T(1-\frac{1}{3}\delta^{\varepsilon}) - \delta \geqslant \frac{2}{3}T - \delta \geqslant \frac{1}{3}T$. We write $Q^{\delta,\theta}_{0,t}\Delta^{\theta}_{t,t+\delta}P_{t+\delta,T}f(x)$ as a finite sum of term with form

$$\mathbb{E}_x[\int_0^1 \partial^{\alpha}(\phi_{\boldsymbol{\mathcal{V}}_L(x)}P_{t+\delta,T}f)(Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t))B(X^{\delta,\theta}_t,t,\delta,\lambda)\mathrm{d}\lambda] + Q^{\delta,\theta}_{0,t}\Delta^{\theta}_{t,t+\delta}((1-\phi_{\boldsymbol{\mathcal{V}}_L(x)})P_{t+\delta,T}f)(x),$$

where $\phi_{\boldsymbol{\mathcal{V}}_L(x)}$ is a smooth localizing function satisfying, for every $u \in \mathbb{R}^d$,

$$\mathbf{1}_{|\boldsymbol{\mathcal{V}}_L(u)-\boldsymbol{\mathcal{V}}_L(x)|\leqslant\frac{\boldsymbol{\mathcal{V}}_L(x)}{4}} \leqslant \phi_{\boldsymbol{\mathcal{V}}_L(x)}(u) \leqslant \mathbf{1}_{|\boldsymbol{\mathcal{V}}_L(u)-\boldsymbol{\mathcal{V}}_L(x)|\leqslant\frac{\boldsymbol{\mathcal{V}}_L(x)}{2}},$$

and having derivatives uniformly bounded by a polynomial of $\boldsymbol{\mathcal{V}}_L(x)^{-1}$. Since $T - t - \delta > \frac{1}{3}T$, applying (2.14) for $\phi_{\boldsymbol{\mathcal{V}}_L(x)}P_{t+\delta,T}f$ enables to bound the first term of the *r.h.s.* above. To bound the second term, we remark that, since f has polynomial growth then so has $P_{t+\delta,T}f$ and we can show that $\mathbf{K}_{P_{t+\delta,T}f} \leqslant C\exp(CT)\,\mathbf{K}_f$ where C doest not depend on f, T or δ. Hence

$$(1-\phi_{\boldsymbol{\mathcal{V}}_L(x)}(u))P_{t+\delta,T}f(u) \leqslant \ \mathbf{K}_f(1+|u|^c_{\mathbb{R}^d})\mathbf{1}_{|\boldsymbol{\mathcal{V}}_L(u)-\boldsymbol{\mathcal{V}}_L(x)|>\frac{\boldsymbol{\mathcal{V}}_L(x)}{4}}C\exp(CT).$$

Moreover, using that $t<\frac{1}{3}T\delta^{\varepsilon}$, we can combine Taylor expansion of $\boldsymbol{\mathcal{V}}_L$, Markov and Doob (see (4.17)) inequalities with Theorem 3.2, to derive

$$Q^{\delta,\theta}_{0,t}\Delta^{\theta}_{t,t+\delta}(\mathbf{1}_{|\boldsymbol{\mathcal{V}}_L(.)-\boldsymbol{\mathcal{V}}_L(x)|>\frac{\boldsymbol{\mathcal{V}}_L(x)}{4}})(x) \leqslant \delta^{\frac{3}{2}-\epsilon}\frac{1+|x|^c_{\mathbb{R}^d}}{|\boldsymbol{\mathcal{V}}_L(x)T|^{\eta}}C\exp(CT).$$

The bound on the second term thus follows from the Cauchy-Schwarz inequality and the proof of (2.18) is completed. If $\mathbf{A}^{\infty}_2(L)$ is assumed, the localization procedure with the function $\phi_{\boldsymbol{\mathcal{V}}_L(x)}$ is not necessary anymore and the achieved convergence rate $\delta^{\frac{1}{2}-\epsilon}$ in (2.18) can be replaced by $\delta^{\frac{1}{2}}$.

Approximation (2.19) follows from an application of Theorem 2.6 in [4]. Notice that this application is also a reason why the convergence happens with rate $\delta^{\frac{1}{2}-\epsilon}$ instead of $\delta^{\frac{1}{2}}$ even in the uniform Hörmander setting $\mathbf{A}^{\infty}_2(L)$. □

4. Estimates on Sobolev norms and on the Malliavin covariance matrix.

In this section, we aim to prove Theorem 3.2 and Theorem 3.3 presented in Section 3.

4.1. **Proof of Theorem 3.2.** We begin by introducing for every $(x,t,z,y)\in\mathbb{R}^d\times\pi^{\delta}\times\mathbb{R}^N\times[0,1]$ and $(i,j)\in\mathcal{N}$,

$$A^i_1(x,t)=\partial_{z^i}\psi(x,t,0,0),\quad A^{i,j}_2(x,t,z)=\int_0^1(1-\lambda)\partial_{z^i}\partial_{z^j}\psi(x,t,\lambda z,0)\mathrm{d}\lambda \tag{4.1}$$

$$A_3(x,t,z,y)=\int_0^1\partial_y\psi(x,t,z,\lambda y)\mathrm{d}\lambda.$$

and remark that, applying the Taylor expansion at order one *w.r.t.* to the fourth variable and then at order two *w.r.t.* the thierd variables yields

$$\psi(x,t,z,y)=x+\sum_{i\in\mathcal{N}}z^iA^i_1(x,t)+\sum_{i,j\in\mathcal{N}}z^iz^jA^{i,j}_2(x,t,z)+yA_3(x,t,z,y). \tag{4.2}$$

We will also denote $A_1:=(A^i_1)_{i\in\mathcal{N}}$ and $A_2:=(A^{i,j}_2)_{i,j\in\mathcal{N}^2}$. Before we treat the Sobolev norms of X^{δ} and $\mathbf{L}\,X^{\delta}$ we establish some preliminary results. The first one gives an estimate of the Sobolev norms of $\mathbf{L}\,Z^{\delta}$.

Lemma 4.1. *Let $t\in\pi^{\delta}$, $t>0$. We have the following properties.*

A. *For every $i\in\mathcal{N}$, we have*

$$\mathbb{E}[\mathbf{L}\,Z^{\delta,i}_t]=0. \tag{4.3}$$

B. *Assume that (2.10) holds for $v=\frac{r_*}{2}$. Then, for every $q\in\mathbb{N}$ and $p\geqslant 1$,*

$$\|\,\mathbf{L}\,Z^{\delta}_t\|_{\mathbb{R}^N,q,p}\leqslant\frac{C(N,p,q)m_*^{\frac{1}{p}}}{r_*^{q+1}}. \tag{4.4}$$

Proof. We prove **A.**. Using the duality relation (3.5) with $\mathcal{H}=\mathbb{R}$, we obtain immediatly $\mathbb{E}[\mathbf{L}\,Z^{\delta,i}_t]=\sum_{(w,j)\in\in\mathcal{T}\times\mathcal{N}}\mathbb{E}[\mathbf{D}_{(w,j)}\,1\,\mathbf{D}_{(w,j)}\,Z^{\delta,i}_t]=0$. In order to prove **B.** we recall (see (3.3)) that

$$\mathbf{L}\,Z^{\delta}_t=\chi^{\delta}_t\nabla_z\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U^{\delta}_t-z_{*,t}).$$

For $\alpha=(\alpha^1,\ldots,\alpha^q)$ with $\alpha^j=(t_j,i_j)$, $t_j\in\pi^{\delta},t_j>0$, $i_j\in\mathcal{N}$,

$$\mathbf{D}_{\alpha}\,\mathbf{L}\,Z^{\delta,i}_t=\delta^{-\frac{|\alpha|}{2}}\chi^{\delta}_t\partial^{\alpha^u_i}_u\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U^{\delta}_t-z_{*,t})\mathbf{1}_{\cap^q_{j=1}\{t=t_j\}},$$

with $\alpha^u_i:=((\alpha^u_i)^j)_{j\in\mathcal{N}}$, $(\alpha^u_i)^j=\mathbf{1}_{i=j}+\sum_{l=1}^q\mathbf{1}_{i_l=j}$. In particular,

$$\sum_{\substack{\alpha\in(\pi^{\delta,*}\times\mathcal{N})^j\\ j\leqslant q}}\delta^j|\,\mathbf{D}_{\alpha}\,\mathbf{L}\,Z^{\delta}_t|^2_{\mathbb{R}^N}=\chi^{\delta}_t\sum_{\substack{\alpha^u\in\mathbb{N}^N\\ |\alpha^u|\in\{1,\ldots,q+1\}}}|\partial^{\alpha^u}_u\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U^{\delta}_t-z_{*,t})|^2.$$

Since the function $\varphi_{r_*/2}$ is constant on $B_{r_*/2}(0)$ and on $\mathbb{R}^d \setminus \bar{B}_{r_*}(0)$, using (2.10), we obtain

$$\begin{aligned}
&\mathbb{E}[|\sum_{\substack{\alpha\in(\pi^{\delta,*}\times\mathcal{N})^j\\ j\leqslant q}} \delta^j |\,\mathbf{D}_\alpha\,\mathbf{L}\,Z^\delta_t|^2_{\mathbb{R}^N}|^{\frac{p}{2}}]\\
&=\frac{\varepsilon_*\mathbb{E}\left[|\chi^\delta_t|^p\right]}{m_*}\int_{\mathbb{R}^N}|\sum_{\substack{\alpha^u\in\mathbb{N}^N\\ |\alpha^u|\in\{1,\dots,q+1\}}}|\partial^{\alpha^u}_u \ln\varphi_{\frac{r_*}{2}}(\delta^{-\frac12}u-z_{*,t})|^2|^{\frac{p}{2}}\delta^{\frac{N}{2}}\varphi_{\frac{r_*}{2}}(\delta^{-\frac12}u-z_{*,t})du\\
&=\varepsilon_*\int_{r_*/2\leqslant|u|\leqslant r_*}|\sum_{\substack{\alpha^u\in\mathbb{N}^N\\ |\alpha^u|\in\{1,\dots,q+1\}}}|\partial^{\alpha^u}_u \ln\varphi_{\frac{r_*}{2}}(u)|^2|^{\frac{p}{2}}\varphi_{\frac{r_*}{2}}(u)du\\
&\leqslant\frac{C\delta^{\frac{p}{2}}\varepsilon_*|\pi^{\frac12}r_*|^N}{r_*^{p(q+1)}}.
\end{aligned}$$

In order to derive (4.4), we observe that $m_* \geqslant \varepsilon_*\lambda_{\text{Leb}}(B(0,\frac{r_*}{2}))$ so that $\varepsilon_*|\pi^{\frac12}\frac{r_*}{2}|^N \leqslant Cm_*$. □

Now, we establish a bound on the moments of $(X^\delta_t)_{t\in\pi^\delta}$.

Lemma 4.2. *Let $T>0$, $\mathcal{T}=[0,T]\cap\pi^\delta$, $x\in\mathbb{R}^d$ and $p\geqslant 1$. Assume that $\mathbf{A}^\delta_1(2)$ and $\mathbf{A}^\delta_3((\mathbf{p}+1)(p\vee 2))$ hold. Then*

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}}|X^\delta_t|^p_{\mathbb{R}^d}]^{\frac1p}\leqslant(1+|x|_{\mathbb{R}^d})\exp(C(p)T\,\mathbf{M}_{(\mathbf{p}_3+1)(p\vee2)}(Z^\delta)^{\frac1p}\,\mathbf{K}_3^{\frac{2}{p}\vee1}). \tag{4.5}$$

Proof. In a first step, we show that, for every $(x,t,z,y)\in\mathbb{R}^d\times\pi^\delta\times\mathbb{R}^N\times[0,1]$,

$$\{|\partial_y\psi|_{\mathbb{R}^d}+\sum_{i\in\mathcal{N}}|\partial_{z^i}\psi|_{\mathbb{R}^d}+\sum_{i,j\in\mathcal{N}}|\partial_{z^i}\partial_{z^j}\psi|_{\mathbb{R}^d}\}(x,t,z,y)\leqslant\mathbf{C}(x,z,\delta). \tag{4.6}$$

with $\mathbf{C}(x,z,\delta)=\mathbf{K}_2(1+\delta^{-\frac{\mathbf{p}_2}{2}}|z|^{\mathbf{p}_2}_{\mathbb{R}^N})+\mathbf{K}_3\,|x|_{\mathbb{R}^d}(1+\delta^{-\frac{\mathbf{p}_3}{2}}|z|^{\mathbf{p}_3}_{\mathbb{R}^N})$. Remark that the result remains valid if we replace $\partial_{z^i}\partial_{z^j}\psi$ by 0 in the *l.h.s.* above and replace $\mathbf{K}_2$ and $\mathbf{p}_2$ by $\mathbf{K}_1$ and $\mathbf{p}_1$ in the definition of $\mathbf{C}$. The proof of this last extension is left to the reader. To prove (4.6), we simply combined the following two observation. First from the Tylor expansion at order one, we write

$$\partial_y\psi(x,z,t,y)=\partial_y\psi(0,z,t,y)+\sum_{l=1}^d x^l\int_0^1\partial_{x^l}\partial_y\psi(\lambda x,z,t,y)\mathrm{d}\lambda,$$

with similar formulas for the derivatives *w.r.t.* z. In addition, it follows from assumption $\mathbf{A}^\delta_1(2)$, (2.2) that

$$\{|\partial_y\psi|_{\mathbb{R}^d}+\sum_{i\in\mathcal{N}}|\partial_{z^i}\psi|_{\mathbb{R}^d}+\sum_{i,j\in\mathcal{N}}|\partial_{z^i}\partial_{z^j}\psi|_{\mathbb{R}^d}\}(0,t,z,y)\leqslant\mathbf{K}_2(1+\delta^{-\frac{\mathbf{p}_2}{2}}|z|^{\mathbf{p}_2}_{\mathbb{R}^N}).$$

Gathering those terms and using $\mathbf{A}^\delta_1(2)$, (see 2.3) yields (4.6). Moreover, owing again to the Taylor expansion at order two applied to the function $\mathbf{m}_p : x\in\mathbb{R}^d\mapsto|x|^p_{\mathbb{R}^d}$, we get

$$\begin{aligned}
|X^\delta_t|^p_{\mathbb{R}^d}-|X^\delta_{t-\delta}|^p_{\mathbb{R}^d}=&\sum_{i=1}^d(X^{\delta,i}_t-X^{\delta,i}_{t-\delta})\partial_{x_i}\mathbf{m}_p(X^\delta_{t-\delta})\\
&+\frac12\sum_{i,j=1}^d(X^\delta_t-X^\delta_{t-\delta})_i(X^\delta_t-X^\delta_{t-\delta})_j\int_0^1(1-\lambda)\partial_{x_i,x_j}\mathbf{m}_p(X^\delta_{t-\delta}+\lambda(X^\delta_t-X^\delta_{t-\delta}))\mathrm{d}\lambda,
\end{aligned}$$

To estimate the first term of the *r.h.s.* above, we recall that Z^δ_t is centered and consider the decompositon (4.2) together with (4.6) and obtain

$$|\mathbb{E}[\sum_{i=1}^d(X^{\delta,i}_t-X^{\delta,i}_{t-\delta})\partial_{x_i}\mathbf{m}_p(X^\delta_{t-\delta})|X^\delta_{t-\delta}]|\leqslant C(p)|X^\delta_{t-\delta}|^{p-1}_{\mathbb{R}^d}\mathbb{E}[\mathbf{C}(X^\delta_{t-\delta},Z^\delta_t,\delta)(1+|Z^\delta_t|^2_{\mathbb{R}^N})|X^\delta_{t-\delta}].$$

To bound the second term, we apply similar arguments (but using the estimate $\mathbf{C}$ that depends on $\mathbf{K}_1$ and $\mathbf{p}_1$ but does not involved second order derivatives in (4.6)). Since $\mathbf{K}_3\geqslant\mathbf{K}_2$ and $\mathbf{p}_3\geqslant\mathbf{p}_2$, we finally obtain for $p\geqslant 2$,

$$|\mathbb{E}_x[|X^\delta_t|^p_{\mathbb{R}^d}]-\mathbb{E}_x[|X^\delta_{t-\delta}|^p_{\mathbb{R}^d}]|\leqslant\delta C(p)\,\mathbf{M}_{p(\mathbf{p}_3+1)}(Z^\delta)\,\mathbf{K}_3^p\,\mathbb{E}_x[1+|X^\delta_{t-\delta}|^p_{\mathbb{R}^d}],$$

and (4.5) follows from the Gronwall lemma. For $p \in [1,2)$, it simply remains to use the Cauchy-Schwarz inequality.

$\square$

In order to obtain estimates of the Sobolev norms which appear in Theorem 3.2, we derive some estimates for a generic class of processes which involves the Malliavin derivatives of $\partial^{\alpha}_{X^{\delta}_0} X^{\delta}$ and $\mathbf{L}\, X^{\delta}_t$. We first recall that for $t \in \pi^{\delta}$, $X^{\delta}_{t+\delta} = \psi(X^{\delta}_t, t, \delta^{\frac{1}{2}} Z^{\delta}_{t+\delta}, \delta)$ with ψ satisfying the decomposition (4.1). We then introduce the $\mathbb{R}^{d\times d}$-valued process $(B_t)_{t\in\pi^{\delta}}$ such that for every $t \in \pi^{\delta}$,

$$B_t = \delta^{\frac{1}{2}} \sum_{i\in\mathcal{N}} Z^{\delta,i}_{t+\delta} \nabla_x A^i_1(X^{\delta}_t, t) + \delta \sum_{i,j\in\mathcal{N}} Z^{\delta,i}_{t+\delta} Z^{\delta,j}_{t+\delta} \nabla_x A^{i,j}_2(X^{\delta}_t, t, \delta^{\frac{1}{2}} Z^{\delta}_{t+\delta}) + \delta \nabla_x A_3(X^{\delta}_t, t, \delta^{\frac{1}{2}} Z^{\delta}_{t+\delta}, \delta).$$

We now consider a Hilbert space $\mathcal{H}$ and introduce some $\mathcal{H}^d$-valued processes $(B^{1,i}_t)_{t\in\pi^{\delta}}$, $(B^{2,i}_t)_{t\in\pi^{\delta}}$, which are both adapted to the filtration $(\sigma(Z^{\delta}_{\delta}, \ldots, Z^{\delta}_t))_{t\in\pi^{\delta}}$ and $(B^3_t)_{t\in\pi^{\delta}}$ which is adapted to the filtration $(\sigma(Z^{\delta}_{\delta}, \ldots, Z^{\delta}_{t+\delta}))_{t\in\pi^{\delta}}$ and for every $h \in \mathcal{H}$, $\langle B^{l,i}, h\rangle_{\mathcal{H}}$, $l = 1, 2$, and $\langle B^3, h\rangle_{\mathcal{H}}$, all belong to $(\mathcal{S}^{\delta})^d$. In this proof, we will consider a $\mathcal{H}^d$-valued generic process $(Y_t)_{t\in\pi^{\delta}}$ which satisfies,for every $t \in \pi^{\delta}$,

$$Y_{t+\delta} = Y_t + B_t Y_t + \delta^{\frac{1}{2}} \sum_{i\in\mathcal{N}} Z^{\delta,i}_{t+\delta} B^{1,i}_t + \delta^{\frac{1}{2}} \sum_{i\in\mathcal{N}} \mathbf{L}\, Z^{\delta,i}_{t+\delta} B^{2,i}_t + B^3_t. \tag{4.7}$$

Moreover for $q \in \mathbb{N}$ and $p \geqslant 1$, $\mathcal{T} = (0,T] \cap \pi^{\delta}$ with $T > 0$, we denote

$$\mathscr{C}_{\mathcal{H}^d,q,p}(B^1, B^2, B^3) = 1 + \sup_{t\in\mathcal{T}} (\|B^{1,\cdot}_{t-\delta}\|_{(\mathcal{H}^d)^{\mathcal{N}},q,p} + \|B^{2,\cdot}_{t-\delta}\|_{(\mathcal{H}^d)^{\mathcal{N}},q,p} + \|\sum_{\substack{w\in\pi^{\delta}\\ w<t}} B^3_w\|_{\mathcal{H}^d,q,p}).$$

where for $(B(i,l))_{(i,l)\in\mathcal{N}\times\{1,\ldots,d\}}$ taking values in $\mathcal{H}$, $|B|_{(\mathcal{H}^d)^{\mathcal{N}}} = |\sum_{i\in\mathcal{N}} \sum_{l=1}^{d} |B(i,l)|^2_{\mathcal{H}}|^{\frac{1}{2}}$. We denote $\mathscr{C}^x_{\mathcal{H}^d,q,p}(B^1, B^2, B^3)$ when the expectations in the norm above are conditional to $X^{\delta}_0 = X_0 = x$. Before we estimate the Sobolev norms, we recall the Burkholder inequality for Hilbert space. We consider a separable Hilbert space $\mathcal{H}$, we denote $|.|_{\mathcal{H}}$ the norm of $\mathcal{H}$ and, for a random variable $F \in \mathcal{H}$, we denote $\|F\|_{\mathcal{H},p} = \mathbb{E}[|F|^p_{\mathcal{H}}]^{\frac{1}{p}}$. Moreover we consider a martingale $M_n \in \mathcal{H}$, $n \in \mathbb{N}$ and we recall the Burkholder inequality in this framework: For each $p \geqslant 2$ there exists a constant $\mathscr{B}_p \geqslant 1$ such that

$$\forall n \in \mathbb{N}, \quad \|\sup_{k\in\{0,\ldots,n\}} M_k\|_{\mathcal{H},p} \leqslant \mathscr{B}_p\, \mathbb{E}[(\sum_{k=1}^{n} |M_k - M_{k-1}|^2_{\mathcal{H}})^{\frac{p}{2}}]^{\frac{1}{p}}. \tag{4.8}$$

As an immediate consequence

$$\|\sup_{k\in\{0,\ldots,n\}} M_k\|_{\mathcal{H},p} \leqslant \mathscr{B}_p\, |\sum_{k=1}^{n} \|M_k - M_{k-1}\|^2_{\mathcal{H},p}|^{\frac{1}{2}}. \tag{4.9}$$

This first result gives an estimate of the Sobolev norms of $(X^{\delta}_t)_{t\in\mathcal{T}}$, and $(Y_t)_{t\in\mathcal{T}}$ *w.r.t.* the quantities above.

Proposition 4.1. *Let $T > 0$, $\mathcal{T} = (0,T] \cap \pi^{\delta}$ and $x \in \mathbb{R}^d$. Let $q \in \mathbb{N}$ and $p \geqslant 1$. Assume that $\mathbf{A}^{\delta}_1(q+2)$ (see (2.2) and (2.3)), $\mathbf{A}^{\delta}_3(+\infty)$ (see (2.7)) and $\mathbf{A}^{\delta}_4$ (see (2.8)) hold. Then, when $q \geqslant 1$,*

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}} |X^{\delta}_t|^p_{\mathbb{R}^d,1,q}]^{\frac{1}{p}} \leqslant (1 + \mathbf{1}_{\mathbf{p}_{q+2}>0} |x|^C_{\mathbb{R}^d})\, \mathbf{K}^C_{q+2} \exp(C(1+T)\, \mathbf{M}_C(Z^{\delta})\, \mathbf{K}^2_3). \tag{4.10}$$

with $C = C(d, \mathbf{p}_{q+2}, q, p)$. Moreover, for $(Y_t)_{t\in\pi^{\delta}}$ satisfying (4.7), if we assume that $\mathbf{A}^{\delta}_1(q+2)$ holds, then

$$\begin{aligned}\mathbb{E}_x[\sup_{t\in\mathcal{T}} |Y_t|^p_{\mathcal{H}^d,q}]^{\frac{1}{p}} \leqslant & (\mathbb{E}_x[|Y_0|^{2^q p}_{\mathcal{H}^d,q}]^{\frac{1}{2^q p}} + \mathscr{C}^x_{\mathcal{H}^d,q,2^q p}(B^1, B^2, B^3)) \\ & \times (1 + \mathbf{1}_{\mathbf{p}_{q+3}>0} |x|^C_{\mathbb{R}^d})\, \mathbf{K}^C_{q+3} \exp(C(1+T)\, \mathbf{M}_C(Z^{\delta})\, \mathbf{K}^2_3).\end{aligned} \tag{4.11}$$

with $C = C(d, N, \frac{1}{r_*}, \mathbf{p}_{q+3}, q, p)$.

Proof. In this proof, our strategy is to derive estimates adapted to use the Gronwall lemma, first for the generic process Y (**Step 1.**) then for X and its Malliavin derivatives (**Step 2.**) and then for Y and its Malliaivin derivative (**Step 3.**). **Step 1.** We begin by proving (4.11) in the specific case $q = 0$. We aim to use the Gronwall lemma so we study the moments of the terms which appear in the *r.h.s.* of (4.7). We consider $i, j \in \mathcal{N}$. Notice that for every $t \in \pi^\delta$, $\mathbb{E}[\mathbf{L}\, Z^{\delta,i}_{t+\delta}] = 0$ (see (4.3)) and $B^{2,i}_t$ is $\mathcal{F}^{Z^\delta}_t$-measurable. It follows from (4.9) (with $\mathcal{H}$ replaced by $\mathcal{H}^d$) and (4.4) that

$$\begin{aligned}\mathbb{E}_x[\sup_{t\in\mathcal{T}} |\delta^{\frac12} \sum_{i\in\mathcal{N}} \sum_{\substack{w\in\pi^\delta\\ w<t}} \mathbf{L}\, Z^{\delta,i}_{w+\delta} B^{2,i}_w|^p_{\mathcal{H}^d}]^{\frac2p} \leqslant & \mathscr{B}_p^2\, \delta \sum_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[| \sum_{i\in\mathcal{N}} \mathbf{L}\, Z^{\delta,i}_{t+\delta} B^{2,i}_t|^p_{\mathcal{H}^d}]^{\frac2p} \\ \leqslant & C\delta \sum_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[| \sum_{i\in\mathcal{N}} |B^{2,i}_t|^2_{\mathcal{H}^d}|^{\frac p2}]^{\frac2p} \leqslant CT \sup_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[|B^{2,\cdot}_t|^p_{(\mathcal{H}^d)^{\mathcal{N}}}]^{\frac2p}.\end{aligned}$$

In the same way,

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}} |\delta^{\frac12} \sum_{i\in\mathcal{N}} \sum_{\substack{w\in\mathcal{T}\\ w<t}} Z^{\delta,i}_{w+\delta} B^{1,i}_w|^p_{\mathcal{H}^d}]^{\frac2p} \leqslant C\, \mathbf{M}_C(Z^\delta) T \sup_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[|B^{1,\cdot}_t|^p_{(\mathcal{H}^d)^{\mathcal{N}}}]^{\frac2p}.$$

We now study the term $B_t Y_t$. Applying similar arguments as previously and remembering that Y_w is $\mathcal{F}^{Z^\delta}_w$-measurable, it follows from $\mathbf{A}^\delta_1$ (see (2.3)) together with (4.9) (with $\mathcal{H}$ replaced by $\mathcal{H}^d$), that

$$\begin{aligned}\mathbb{E}_x[\sup_{t\in\mathcal{T}} |\delta^{\frac12} \sum_{i\in\mathcal{N}} \sum_{\substack{w\in\pi^\delta\\ w<t}} Z^{\delta,i}_{w+\delta} \nabla_x A^i_1(X^\delta_w, w) Y_w|^p_{\mathcal{H}^d}]^{\frac2p} \leqslant & \mathscr{B}_p^2\, \delta \sum_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[| \sum_{i\in\mathcal{N}} Z^{\delta,i}_{t+\delta} \nabla_x A^i_1(X^\delta_t, t) Y_t|^p_{\mathcal{H}^d}]^{\frac2p} \\ \leqslant & C\, \mathbf{M}_C(Z^\delta)\, \mathbf{K}_3^2\, \delta \sum_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[|Y_t|^p_{\mathcal{H}^d}]^{\frac2p}.\end{aligned}$$

Applying $\mathbf{A}^\delta_1$ (see (2.3)) with the triangle inequality also gives

$$\begin{aligned}&\mathbb{E}_x[\sup_{t\in\mathcal{T}} |\delta \sum_{i,j\in\mathcal{N}} \sum_{\substack{w\in\pi^\delta\\ w<t}} Z^{\delta,i}_{w+\delta} Z^{\delta,j}_{w+\delta} \nabla_x A^{i,j}_2(X^\delta_w, w, \delta^{\frac12} Z^\delta_{w+\delta}) Y_w|^p_{\mathcal{H}^d}]^{\frac1p} \\ &\leqslant \delta \sum_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[\sum_{i,j\in\mathcal{N}} |Z^{\delta,i}_{t+\delta} Z^{\delta,j}_{t+\delta} \nabla_x A^{i,j}_2(X^\delta_t, t, \delta^{\frac12} Z^\delta_{t+\delta}) Y_t|^p_{\mathcal{H}^d}]^{\frac1p} \leqslant C\, \mathbf{M}_C(Z^\delta)\, \mathbf{K}_3\, \delta \sum_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[|Y_t|^p_{\mathcal{H}^d}]^{\frac1p}.\end{aligned}$$

Moreover, a similar estimate holds for the terms involving A_3. We gather all the terms and using the triangle and Cauchy-Schwarz inequalities provides the following estimate

$$\mathbb{E}_x[\sup_{\substack{t\in\pi^\delta\\ t\leqslant T}} |Y_t|^p_{\mathcal{H}^d}]^{\frac1p} \leqslant \mathbb{E}_x[|Y_0|^p_{\mathcal{H}^d}]^{\frac1p} + C\, \mathbf{M}_C(Z^\delta)^{\frac1p} (1+T^{\frac12}) (\mathscr{C}^x_{\mathcal{H}^d,0,p}(B^1,B^2,B^3) + \mathbf{K}_3 (\delta \sum_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[|Y_t|^p_{\mathcal{H}^d}]^{\frac2p})^{\frac12}).$$

Hence, using the Gronwall lemma yields (4.11) when $q=0$.
Step 2. Let us prove (4.10). For $q\in\mathbb{N}$, we define $\mathcal{R}_0 = \mathbb{R}$ and $\mathcal{R}_{q+1} = (\mathcal{R}_q)^{\mathcal{T}\times\mathcal{N}}$ and we have

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}} |X^\delta_t|^p_{\mathbb{R}^d,1,q}]^{\frac1p} = \mathbb{E}_x[\sup_{t\in\mathcal{T}} \sum_{q^\diamond=1}^{q} |\, \mathbf{D}^{q^\diamond} X^\delta_t|^p_{\mathcal{R}^d_{q^\diamond}}]^{\frac1p}.$$

First, we focus on the case $q=1$. We remark that for every $t\in\pi^\delta$, $w\in\mathcal{T}$, and every $i\in\mathcal{N}$,

$$\delta^{\frac12}\, \mathbf{D}_{(w,i)}\, X^\delta_{t+\delta} = (I_{d\times d} + B_t) \delta^{\frac12}\, \mathbf{D}_{(w,i)}\, X^\delta_t + (B^3_{1,t})_{w,i},$$

with, for $(w,i)\in\mathcal{T}\times\mathcal{N}$,

$$\begin{aligned}(B^3_{1,t})_{w,i} = & \chi^\delta_{t+\delta} \mathbf{1}_{w=t+\delta} (\delta^{\frac12} A^i_1(X^\delta_t, t) + \delta \sum_{j\in\mathcal{N}} Z^{\delta,j}_{t+\delta} (1+\mathbf{1}_{i=j}) A^{i,j}_2(X^\delta_t, t, \delta^{\frac12} Z^\delta_{t+\delta}) \\ & + \delta^{\frac32} \sum_{j,l\in\mathcal{N}} Z^{\delta,j}_{t+\delta} Z^{\delta,l}_{t+\delta} \partial_{z^i} A^{j,l}_2(X^\delta_t, t, \delta^{\frac12} Z^\delta_{t+\delta}) + \delta^{\frac32} \partial_{z^i} A_3(X^\delta_t, t, \delta^{\frac12} Z^\delta_{t+\delta}, \delta)).\end{aligned}$$

In particular, $\delta^{\frac12}\, \mathbf{D}\, X^\delta_t = (\delta^{\frac12}\, \mathbf{D}_{(w,i)}\, X^\delta_t)_{(w,i)\in\mathcal{T}\times\mathcal{N}}$ is a $\mathcal{R}^d_1$-valued random variable and, for $t\in\pi^\delta$, we have $\delta^{\frac12}\, \mathbf{D}\, X^\delta_{t+\delta} = (I_{d\times d} + B_t)\delta^{\frac12}\, \mathbf{D}^\delta\, X^\delta_t + B^3_{1,t}$. Then, (4.10) for $q=1$, follows from Lemma 4.2 (see (4.5)) and (4.11) with $q=0$ and $Y = \delta^{\frac12}\, \mathbf{D}^\delta\, X^\delta$, $\mathcal{H} = \mathcal{R}_1$, and B^3 thus defined since the assumption $\mathbf{A}^\delta_1(3)$ (see (2.2)) implies that

$$\mathscr{C}^x_{\mathcal{R}^d_1,0,p}(0,0,B^3_{1,.}) = 1+\sup_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[|\sum_{\substack{w\in\pi^\delta\\ w<t}}\sum_{i\in\mathcal{N}} |(B^3_{1,w})_{w+\delta,i}|^2_{\mathbb{R}^d}|^{\frac{p}{2}}]^{\frac{1}{p}}$$
$$\leqslant 1+\mathbb{E}_x[|\sum_{\substack{t\in\pi^\delta\\ t<T}} |(B^3_{1,t})_{t+\delta,.}|^2_{(\mathbb{R}^d)^\mathcal{N}}|^{\frac{p}{2}}]^{\frac{1}{p}} \leqslant 1+T^{\frac12}\delta^{-\frac12}\sup_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}_x[|(B^3_{1,t})_{t+\delta,.}|^p_{(\mathbb{R}^d)^\mathcal{N}}]^{\frac1p}$$
$$\leqslant 1+CT^{\frac12}\,\mathbf{K}_3\,\mathbf{M}_C(Z^\delta)(1+\mathbb{E}_x[\sup_{t\in\mathcal{T}}|X^\delta_{t-\delta}|^{p\,\mathbf{p}_3}_{\mathbb{R}^d}]^{\frac1p}).$$

Now let us focus on the case $q\in\mathbb{N}$, $q\geqslant 2$. Similarly as in the case $q=1$, $\delta^{\frac{q}{2}}\,\mathbf{D}^q\,X^\delta_t$ is a $\mathcal{R}^d_q$-valued random variable and, for $t\in\pi^\delta$, we have

$$\delta^{\frac{q}{2}}\,\mathbf{D}^q\,X^\delta_{t+\delta} = (I_{d\times d}+B_t)\delta^{\frac{q}{2}}\,\mathbf{D}^q\,X^\delta_t+\delta^{\frac12}\sum_{i\in\mathcal{N}}Z^{\delta,i}_{t+\delta}B^{1,i}_{q,t}+B^3_{q,t},$$

with, $B^{1,i}_{1,.}=0$, $B^3_{1,.}$ defined in the beginning of **Step 2**, and for $q\geqslant 2$,

$$B^{1,i}_{q,t} = \delta^{\frac{q}{2}}(\mathbf{D}\,X^\delta_t)^{\mathrm{T}}\,\nabla^2_x\,A^i_1(X^\delta_t,t)\,\mathbf{D}^{q-1}\,X^\delta_t+\delta^{\frac12}\,\mathbf{D}^\delta\,B^{1,i}_{q-1,t}$$
$$B^3_{q,t} = \delta^{\frac{q-1}{2}}(B^{3,1}_t+B^{3,2}_t)\,\mathbf{D}^{q-1}\,X^\delta_t+\delta^{\frac12}\,\mathbf{D}\,B^3_{q-1,t}+\delta\sum_{i\in\mathcal{N}}B^{1,i}_{q-1,t}\,\mathbf{D}\,Z^{\delta,i}_{t+\delta},$$

with, for $(w,v)\in\mathcal{T}\times\mathcal{N}$,

$$B^{3,1}_t = \delta\sum_{i,j\in\mathcal{N}}Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}(\delta^{\frac12}\,\mathbf{D}\,X^\delta_t)^{\mathrm{T}}\,\nabla^2_x\,A^{i,j}_2(X^\delta_t,t,\delta^{\frac12}\,Z^\delta_{t+\delta}) \tag{4.12}$$
$$+\,\delta(\delta^{\frac12}\,\mathbf{D}\,X^\delta_t)^{\mathrm{T}}\,\nabla^2_x\,A_3(X^\delta_t,t,\delta^{\frac12}\,Z^\delta_{t+\delta},\delta)$$
$$(B^{3,2}_t)_{w,v} = \chi^\delta_{t+\delta}\mathbf{1}_{w=t+\delta}(\delta^{\frac12}\nabla_x A^v_1(X^\delta_t,t)+\delta\sum_{j\in\mathcal{N}}Z^{\delta,j}_{t+\delta}(1+\mathbf{1}_{v=j})\nabla_x A^{v,j}_2(X^\delta_t,t,\delta^{\frac12}\,Z^\delta_{t+\delta})$$
$$+\,\delta^{\frac32}\sum_{i,j\in\mathcal{N}}Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}\partial_{z^v}\nabla_x A^{i,j}_2(X^\delta_t,t,\delta^{\frac12}\,Z^\delta_{t+\delta})+\delta^{\frac32}\,\partial_{z^i}\nabla_x A_3(X^\delta_t,t,\delta^{\frac12}\,Z^\delta_{t+\delta},\delta)).$$

First, we remark that, since $B^1_{1,.}=0$, it follows from Lemma 3.1 and (4.10) that, since $\mathbf{A}^\delta_1(q+1)$ (see (2.2)) holds, then

$$\mathscr{C}^x_{\mathcal{R}^d_q,0,p}(B^1_{q,.},0,0)\leqslant\mathscr{C}^x_{\mathcal{R}^d_q,0,p}(\delta^{\frac{q}{2}}(\mathbf{D}\,X^\delta)^{\mathrm{T}}\,\nabla^2_x\,A_1(X^\delta,.)\,\mathbf{D}^{q-1}\,X^\delta,0,0)+\mathscr{C}^x_{\mathcal{R}^d_{q-1},1,p}(B^1_{q-1,.},0,0)$$
$$\leqslant\sum_{q'=1}^{q-1}\mathscr{C}^x_{\mathcal{R}^d_{q-q'+1},q'-1,p}(\delta^{\frac{q-q'+1}{2}}(\mathbf{D}\,X^\delta)^{\mathrm{T}}\,\nabla^2_x\,A_1(X^\delta,.)\,\mathbf{D}^{q-q'}\,X^\delta,0,0)$$
$$\leqslant C\,\mathbf{K}_{q+1}\,\mathbb{E}_x[\sup_{t\in\mathcal{T}}(1+|X^\delta_t|^{p(q)}_{\mathbb{R}^d,1,q-1})(1+|X^\delta_t|^{p\,\mathbf{p}_{q+1}}_{\mathbb{R}^d})]^{\frac1p}.$$

Moreover

$$\mathscr{C}^x_{\mathcal{R}^d_q,0,p}(0,0,B^3_{q,.})\leqslant\mathscr{C}^x_{\mathcal{R}^d_q,0,p}(0,0,\delta^{\frac{q-1}{2}}(B^{3,1}+B^{3,2})\,\mathbf{D}^{q-1}\,X^\delta)$$
$$+\,\mathscr{C}^x_{\mathcal{R}^d_{q-1},1,p}(0,0,B^3_{q-1,.})+\mathscr{C}^x_{\mathcal{R}^d_q,0,p}(0,0,\delta\sum_{i\in\mathcal{N}}B^{1,i}_{q-1,.}\,\mathbf{D}\,Z^{\delta,i}_{.+\delta})$$
$$\leqslant\sum_{q'=1}^{q-1}\mathscr{C}^x_{\mathcal{R}^d_{q-q'+1},q'-1,p}(0,0,\delta^{\frac{q-q'}{2}}(B^{3,1}+B^{3,2})\,\mathbf{D}^{q-q'}\,X^\delta)$$
$$+\sum_{q'=1}^{q-1}\mathscr{C}^x_{\mathcal{R}^d_{q-q'+1},q'-1,p}(0,0,\delta\sum_{i\in\mathcal{N}}B^{1,i}_{q-q',.}\,\mathbf{D}\,Z^{\delta,i}_{.+\delta})+\mathscr{C}^x_{\mathcal{R}^d_1,q-1,p}(0,0,B^3_{1,.}).$$

Since $\mathbf{A}^\delta_1(q+2)$ holds, using a similar approach as for the case $q=1$ yields

$$\mathscr{C}^x_{\mathcal{R}^d_1,q-1,p}(0,0,B^3_{1,.}) = 1+\sup_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}_x[|\sum_{\substack{w\in\pi^\delta\\ w<t}}\sum_{i\in\mathcal{N}}\sum_{q'=0}^{q-1}\delta^{q'}|\,\mathbf{D}^{q'}(B^3_{1,w})_{w+\delta,i}|^2_{\mathcal{R}_{q'}}|^{\frac{p}{2}}]^{\frac1p}$$
$$\leqslant 1+CT^{\frac12}\delta^{-\frac12}\sup_{\substack{t\in\pi^\delta\\ t<T}}\sup_{q'\in\{0,\ldots,q-1\}}\mathbb{E}_x[|\delta^{\frac{q'}{2}}\,\mathbf{D}^{\delta,q'}(B^3_{1,t})_{t+\delta,.}|^p_{(\mathcal{R}^d_{q'})^\mathcal{N}}]^{\frac1p}$$
$$\leqslant 1+CT^{\frac12}\,\mathbf{K}_{q+2}\,\mathbf{M}_C(Z^\delta)^{\frac1p}\,\mathbb{E}_x[\sup_{t\in\mathcal{T}}|1+|X^\delta_{t-\delta}|^{q-1}_{\mathbb{R}^d,1,q-1}|^p|1+|X^\delta_{t-\delta}|^{\mathbf{p}_{q+2}}_{\mathbb{R}^d}|^p]^{\frac1p}.$$

Moreover, for $q' \in \{1, \dots, q-1\}$, we deduce from the Cauchy-Schwartz inequality that

$$\begin{aligned}
&\mathscr{C}^x_{\mathcal{R}^d_{q-q'+1}, q'-1, p}(0, 0, \delta^{\frac{q-1}{2}} (B^{3,1} + B^{3,2}) \mathbf{D}^{q-q'} X^\delta) \\
&\quad = 1 + \sup_{\substack{t \in \pi^\delta \\ t < T}} \mathbb{E}_x[| \sum_{\substack{w \in \pi^\delta \\ w < t}} \delta^{\frac{q-1}{2}} (B^{3,1}_w + B^{3,2}_w) \mathbf{D}^{q-q'} X^\delta_w|^p_{\mathcal{R}^d_{q-q'+1}, q'-1}]^{\frac{1}{p}} \\
&\quad \leqslant 1 + \sum_{\substack{t \in \pi^\delta \\ t < T}} \mathbb{E}_x[|\delta^{\frac{q-1}{2}} B^{3,1}_t \mathbf{D}^{q-q'} X^\delta_t|^p_{\mathcal{R}^d_{q-q'+1}, q'-1}]^{\frac{1}{p}} \\
&\quad\quad + |\delta \sum_{\substack{t \in \pi^\delta \\ t < T}} \sum_{i \in \mathcal{N}} \mathbb{E}_x[|\delta^{\frac{q-2}{2}} (B^{3,2}_t)_{t+\delta, i} \mathbf{D}^{q-q'} X^\delta_t|^p_{\mathcal{R}^d_{q-q'}, q'-1}]^{\frac{2}{p}}|^{\frac{1}{2}},
\end{aligned}$$

with, since $\mathbf{A}^\delta_1(q+2)$ (see (2.2)) holds,

$$\begin{aligned}
&\mathbb{E}_x[|\delta^{\frac{q-1}{2}} B^{3,1}_t \mathbf{D}^{q-q'} X^\delta_t|^p_{\mathcal{R}^d_{q-q'+1}, q'-1}]^{\frac{1}{p}} + \mathbb{E}_x[|\delta^{\frac{q-2}{2}} (B^{3,2}_t)_{t,i} \mathbf{D}^{q-q'} X^\delta_t|^p_{\mathcal{R}^d_{q-q'}, q'-1}]^{\frac{1}{p}} \\
&\quad \leqslant \delta C \, \mathbf{M}_C(Z^\delta)^{\frac{1}{p}} \, \mathbf{K}_{q+2} \, \mathbb{E}_x[\sup_{t \in \mathcal{T}} (1 + |X^\delta_t|^{p(q)}_{\mathbb{R}^d, 1, q-1})(1 + |X^\delta_t|^{p \, \mathbf{p}_{q+2}}_{\mathbb{R}^d})]^{\frac{1}{p}}.
\end{aligned}$$

Finally, for $q' \in \{1, \dots, q-1\}$, recalling (3.2) and owing to $\mathbf{A}^\delta_1(q)$ (see (2.2)) yields by similar techniques

$$\begin{aligned}
\mathscr{C}^x_{\mathcal{R}^d_{q-q'+1}, q'-1, p}(0, 0, \delta \sum_{i \in \mathcal{N}} B^{1,i}_{q-q', .} \, \mathbf{D} \, Z^{\delta, i}_{.+\delta}) &\leqslant 1 + \mathbb{E}_x[| \sum_{\substack{w \in \pi^\delta \\ w < t}} \sum_{i \in \mathcal{N}} \delta^2 |B^{1,i}_{q-q', w} \, \mathbf{D}_{(w+\delta, i)} \, Z^{\delta, i}_{w+\delta}|^2_{\mathcal{R}^d_{q-q'}, q'-1}|^{\frac{p}{2}}]^{\frac{1}{p}} \\
&\leqslant 1 + C T^{\frac{1}{2}} \, \mathbf{K}_q \, \mathbb{E}_x[\sup_{t \in \mathcal{T}} (1 + |X^\delta_t|^{p(q-1)}_{\mathbb{R}^d, 1, q-1})(1 + |X^\delta_t|^{p \, \mathbf{p}_q}_{\mathbb{R}^d})]^{\frac{1}{p}}.
\end{aligned}$$

More specifically, we have shown that

$$\mathscr{C}^x_{\mathcal{R}^d_q, 0, p}(0, 0, B^3_{q,.}) \leqslant C(1+T) \, \mathbf{M}_C(Z^\delta)^{\frac{1}{p}} \, \mathbf{K}_{q+2} \, \mathbb{E}_x[\sup_{t \in \mathcal{T}} (1 + |X^\delta_t|^{pq}_{\mathbb{R}^d, 1, q-1})(1 + |X^\delta_t|^{p \, \mathbf{p}_{q+2}}_{\mathbb{R}^d})]^{\frac{1}{p}}.$$

Applying (4.11) with $q = 0$, yields, for $q \geqslant 2$,

$$\mathbb{E}_x[\sup_{t \in \mathcal{T}} |X^\delta_t|^p_{\mathbb{R}^d, 1, q}]^{\frac{1}{p}} \leqslant \mathbf{K}_{q+2} \exp(C(T+1) \, \mathbf{M}_C(Z^\delta)^{\frac{2}{p}} \, \mathbf{K}^2_3) \mathbb{E}_x[\sup_{t \in \mathcal{T}} (1 + |X^\delta_t|^{pq}_{\mathbb{R}^d, 1, q-1})(1 + |X^\delta_t|^{p \, \mathbf{p}_{q+2}}_{\mathbb{R}^d})]^{\frac{1}{p}}.$$

Using a recursive approach combined with (4.11) with $q = 0$, gives (4.10).

Step 3. In this last step, we prove (4.11). For $q \in \mathbb{N}$, we define $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}_{q+1} = (\mathcal{H}_q)^{\mathcal{T} \times \mathcal{N}}$. For Y satisfying (4.7),we have (remember that $\mathbf{D}^q \, Y_t$, $t \in \pi^\delta$, belongs to $\mathcal{H}^d_q$), for every $t \in \pi^\delta$

$$\delta^{\frac{q}{2}} \, \mathbf{D}^q \, Y_{t+\delta} = \delta^{\frac{q}{2}} \, \mathbf{D}^q \, Y_t + B_t \delta^{\frac{q}{2}} \, \mathbf{D}^q \, Y_t + \delta^{\frac{1}{2}} \sum_{i=1}^N Z^{\delta, i}_{t+\delta} B^{q,i}_{1,t} + \delta^{\frac{1}{2}} \sum_{i=1}^N \mathbf{L} \, Z^{\delta, i}_{t+\delta} B^{2,i}_{q,t} + B^3_{q,t},$$

with $B^{1,i}_{q,t} = \delta^{\frac{1}{2}} \, \mathbf{D} \, B^{1,i}_{q-1,t} + \delta^{\frac{q}{2}} (\mathbf{D} \, X^\delta_t)^{\mathrm{T}} \, \nabla^2_x \, A^i_1(X^\delta_t, t) \, \mathbf{D}^{q-1} \, Y_t$, $B^{2,i}_{q,t} = \delta^{\frac{1}{2}} \, \mathbf{D} \, B^{2,i}_{q-1,t}$ and

$$\begin{aligned}
B^3_{q,t} = &\delta^{\frac{1}{2}} \, \mathbf{D} \, B^3_{q-1,t} + \delta^{\frac{q}{2}} \sum_{i=1}^N \nabla_x A^i_1(X^\delta_t, t) \, \mathbf{D}(\delta^{\frac{1}{2}} Z^{\delta, i}_{t+\delta}) \, \mathbf{D}^{q-1} \, Y_t \\
&+ \delta^{\frac{q}{2}} \frac{1}{2} \sum_{i,j \in \mathcal{N}} \mathbf{D}^\delta (\delta Z^{\delta, i}_{t+\delta} Z^{\delta, j}_{t+\delta} \nabla_x A^{i,j}_2(X^\delta_t, t, \delta^{\frac{1}{2}} Z^\delta_{t+\delta}, \delta)) \, \mathbf{D}^{q-1} \, Y_t \\
&+ \delta^{\frac{q}{2}} \, \mathbf{D}(\delta \nabla_x A_3(X^\delta_t, t, \delta^{\frac{1}{2}} Z^\delta_{t+\delta}, \delta)) \, \mathbf{D}^{q-1} \, Y_t + \delta^{\frac{1}{2}} \sum_{i \in \mathcal{N}} B^{1,i}_{q-1,t} \, \mathbf{D}(\delta^{\frac{1}{2}} Z^{\delta, i}_{t+\delta}) + B^{2,i}_{q-1,t} \, \mathbf{D} \, \mathbf{L}(\delta^{\frac{1}{2}} Z^{\delta, i}_{t+\delta}).
\end{aligned}$$

Now, we aim to obtain an upper bound on $\mathscr{C}^x_{\mathcal{H}^d_q, 0, p}(B^1_{q,.}, B^3_{q,.}, B^3_{q,.})$.

We study $\mathscr{C}^x_{\mathcal{H}^d_q, 0, p}(B^1_{q,.}, B^3_{q,.}, B^3_{q,.})$. We first remark that, using (3.14) and $\mathbf{A}^\delta_1(q+2)$,

$$\begin{aligned}
&\mathscr{C}^x_{\mathcal{H}^d_q, 0, p}(B^1_{q,.}, 0, 0) \leqslant \mathscr{C}^x_{\mathcal{H}^d_{q-1}, 1, p}(B^1_{q-1,.}, 0, 0) + \mathscr{C}^x_{\mathcal{H}^d_q, 0, p}(\delta^{\frac{q}{2}} (\mathbf{D} \, X^\delta)^{\mathrm{T}} \, \nabla^2_x \, A_1(X^\delta, .) \, \mathbf{D}^{q-1} \, Y, 0, 0) \\
&\quad \leqslant \mathscr{C}^x_{\mathcal{H}^d, q, p}(B^1, 0, 0) + \sum_{q'=1}^q \mathscr{C}^x_{\mathcal{H}^d_{q-q'+1}, q'-1, p}(\delta^{\frac{q-q'+1}{2}} (\mathbf{D} \, X^\delta)^{\mathrm{T}} \, \nabla^2_x \, A_1(X^\delta, .) \, \mathbf{D}^{q-q'} \, Y^\delta, 0, 0) \\
&\quad \leqslant \mathscr{C}^x_{\mathcal{H}^d, q, p}(B^1, 0, 0) + C \, \mathbf{K}_{q+2} (1 + \mathbb{E}_x[\sup_{t \in \mathcal{T}} |Y_t|^{2p}_{\mathcal{H}^d, q-1}]^{\frac{1}{2p}}) \mathbb{E}_x[\sup_{t \in \mathcal{T}} (1 + |X^\delta_t|^{2pq}_{\mathbb{R}^d, 1, q})(1 + |X^\delta_t|^{3p \, \mathbf{p}_{q+2}}_{\mathbb{R}^d})]^{\frac{1}{2p}},
\end{aligned}$$

and similarly $\mathscr{C}^x_{\mathcal{H}^d_q,0,p}(0,B^2_{q,.},0) \leqslant \mathscr{C}^x_{\mathcal{H}^d,q,p}(0,B^2,0)$. Computing the Malliavin derivatives involved in $B^3_{q,.}$ (recall that $B^{3,1}$ and $B^{3,1}$ are defined in (4.12)), we also derive the following estimate

$$\begin{aligned}\mathscr{C}^x_{\mathcal{H}^d_q,0,p}(0,0,B^3_{q,.}) \leqslant& \mathscr{C}^x_{\mathcal{H}^d_{q-1},1,p}(0,0,B^3_{q-1,.}) + \mathscr{C}^x_{\mathcal{H}^d_q,0,p}(0,0,\delta^{\frac{q-1}{2}}(B^{3,1}+B^{3,2})\,\mathbf{D}^{q-1}\,Y)\\ &+ \mathscr{C}^x_{\mathcal{H}^d_q,0,p}(0,0,\delta^{\frac{1}{2}}\sum_{i\in\mathcal{N}} B^{1,i}_{q-1,.}\,\mathbf{D}(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta}) + B^{2,i}_{q-1,.}\,\mathbf{D}\,\mathbf{L}(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta}))\\ \leqslant& \mathscr{C}^x_{\mathcal{H}^d,q,p}(0,0,B^3) + \sum_{q'=1}^{q}\mathscr{C}^x_{\mathcal{H}^d_{q-q'+1},q'-1,p}(0,0,\delta^{\frac{q-q'}{2}}(B^{3,1}+B^{3,2})\,\mathbf{D}^{q-q'}\,Y)\\ &+ \sum_{q'=1}^{q}\mathscr{C}^x_{\mathcal{H}^d_{q-q'+1},q'-1,p}(0,0,\delta^{\frac{1}{2}}\sum_{i\in\mathcal{N}} B^{1,i}_{q-q',.}\,\mathbf{D}(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta}) + B^{2,i}_{q-q',.}\,\mathbf{D}\,\mathbf{L}(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta})).\end{aligned}$$

We study the first term of the estimate above. Similarly as in **Step 2.** we have, for $q' \in \{1,\ldots,q\}$,

$$\begin{aligned}\mathscr{C}^x_{\mathcal{H}^d_{q-q'+1},q'-1,p}(0,0,\delta^{\frac{q-q'}{2}}(B^{3,1}+B^{3,2})\,\mathbf{D}^{q-q'}\,Y) \leqslant 1 + \sum_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}_x[|B^{3,1}_t\delta^{\frac{q-q'}{2}}\,\mathbf{D}^{q-q'}\,Y_t|^p_{\mathcal{H}^d_{q-q'+1},q'-1}]^{\frac{1}{p}}\\ + |\delta\sum_{\substack{t\in\pi^\delta\\ t<T}}\sum_{i\in\mathcal{N}} \mathbb{E}_x[|\delta^{\frac{q-q'-1}{2}}(B^{3,2}_t)_{t+\delta,i}\,\mathbf{D}^{q-q'}\,Y_t|^p_{\mathcal{R}^d_{q-q'},q'-1}]^{\frac{2}{p}}|^{\frac{1}{2}}.\end{aligned}$$

Moreover, using (3.14) and $\mathbf{A}^\delta_1(q+3)$ (see (2.2)) holds, we have the estimates

$$\begin{aligned}\mathbb{E}_x[|B^{3,1}_t\delta^{\frac{q-q'}{2}}\,\mathbf{D}^{q-q'}\,Y_t|^p_{\mathcal{R}^d_{q-q'+1},q'-1}]^{\frac{1}{p}} \leqslant \delta C\,\mathbf{M}_C(Z^\delta)^{\frac{1}{p}}\,\mathbf{K}_{q+3}\\ \times\mathbb{E}_x[\sup_{t\in\mathcal{T}}(1+|X^\delta_t|^{2pq}_{\mathbb{R}^d,1,q})(1+|X^\delta_t|^{2p\,\mathbf{p}_{q+3}}_{\mathbb{R}^d})]^{\frac{1}{2p}}\,\mathbb{E}_x[\sup_{t\in\mathcal{T}}|Y_t|^{2p}_{\mathbb{R}^d,q-1}]^{\frac{1}{2p}},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_x[|\delta^{-\frac{1}{2}}(B^{3,2}_t)_{t,.}\delta^{\frac{q-q'}{2}}\,\mathbf{D}^{q-q'}\,Y_t|^p_{(\mathcal{H}^d_{q-q'})^{\mathcal{N}},q'-1}]^{\frac{1}{p}} \leqslant \delta C\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_{q+3}\\ \times\mathbb{E}_x[\sup_{t\in\mathcal{T}}(1+|X^\delta_t|^{2p(q-1)}_{\mathbb{R}^d,1,q-1})(1+|X^\delta_t|^{2p\,\mathbf{p}_{q+l+3}}_{\mathbb{R}^d})]^{\frac{1}{2p}}\,\mathbb{E}_x[1+\sup_{t\in\mathcal{T}}|Y_t|^{2p}_{\mathcal{H}^d,q-1}]^{\frac{1}{2p}}.\end{aligned}$$

Now we focus on the last two terms of the estimate of $\mathscr{C}_{\mathcal{H}^d_q,0,p}(0,0,B^3_{q,.})$. For $q' \in \{1,\ldots,q\}$, using the same calculus as a few lines above, we have

$$\begin{aligned}\mathscr{C}^x_{\mathcal{R}^d_{q-q'+1},q'-1,p}(0,0,\delta^{\frac{1}{2}}\sum_{i\in\mathcal{N}} B^{1,i}_{q-q',.}\,\mathbf{D}(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta})) \leqslant 1 + T^{\frac{1}{2}}\mathscr{C}^x_{\mathcal{H}^d,q-1,p}(B^1,0,0)\\ + CT^{\frac{1}{2}}\,\mathbf{K}_{q+1}\,\mathbb{E}_x[\sup_{t\in\mathcal{T}}(1+|X^\delta_t|^{2p(q-1)}_{\mathbb{R}^d,1,q-1})(1+|X^\delta_t|^{2p\,\mathbf{p}_{q+1}}_{\mathbb{R}^d})]^{\frac{1}{2p}}\,\mathbb{E}_x[1+\sup_{t\in\mathcal{T}}|Y|^{2p}_{\mathcal{H}^d,q-1}]^{\frac{1}{2p}}.\end{aligned}$$

Using (3.14) with the estimate (4.4) from Lemma 4.1 yields, for every $q' \in \{1,\ldots,q\}$,

$$\begin{aligned}&\mathscr{C}^x_{\mathcal{H}^d_{q-q'+1},q'-1,p}(0,0,\delta^{\frac{1}{2}}\sum_{i\in\mathcal{N}} B^{2,i}_{q-q',.}\,\mathbf{D}\,\mathbf{L}(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta}))\\ &\leqslant 1 + CT^{\frac{1}{2}}\sup_{t\in\mathcal{T}}\mathbb{E}_x[|B^2_{q-q',t}|^{2p}_{(\mathcal{H}^d_{q-q'})^{\mathcal{N}},q'-1}]^{\frac{1}{2p}}\,\|\,\mathbf{L}\,Z^\delta_t\|_{\mathbb{R}^N,q',2p} \leqslant 1 + CT^{\frac{1}{2}}\mathscr{C}^x_{\mathcal{H}^d,q-1,2p}(0,B^2,0).\end{aligned}$$

In particular, we have shown that

$$\begin{aligned}\mathscr{C}^x_{\mathcal{H}^d_q,0,p}(0,0,B^3_{q,.}) \leqslant& (1+T^{\frac{1}{2}})\mathscr{C}^x_{\mathcal{H}^d,q-1,p}(B^1,0,B^3) + CT^{\frac{1}{2}}\mathscr{C}^x_{\mathcal{H}^d,q-1,2p}(0,B^2,0)\\ &+ C(1+T^{\frac{1}{2}})\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_{q+3}\,\mathbb{E}_x[1+\sup_{t\in\mathcal{T}}|X^\delta_t|^{2p(q+\mathbf{p}_{q+3})}_{\mathbb{R}^d,q}]^{\frac{1}{2p}}\,\mathbb{E}_x[1+\sup_{t\in\mathcal{T}}|Y_t|^{2p}_{\mathbb{R}^d,q-1}]^{\frac{1}{2p}}.\end{aligned}$$

Applying (4.11) with $q=0$ and (4.10) concludes the proof of (4.11) □

Now, we are in a position to prove Theorem 3.2.

Proof of Theorem 3.2. We do not treat the case $(\mathbf{p}_n)_{n\in\mathbb{N}^*} \equiv 0$ which is similar but simpler because we do not use Lemma 4.2 in this case. Let us focus on the case $(\mathbf{p}_n)_{n\in\mathbb{N}^*} \not\equiv 0$. We treat the Sobolev norms of $\partial^\alpha_{X^\delta_0}X^\delta_t$. In the case $|\alpha|=1$, (3.15) is a direct consequence of Proposition 4.1, since

$$\partial^\alpha_{X^\delta_0}X^\delta_{t+\delta} = \partial^\alpha_{X^\delta_0}X^\delta_t + B_t\partial^\alpha_{X^\delta_0}X^\delta_t.$$

For $\alpha=(\alpha^1,\ldots,\alpha^d)\in\mathbb{N}^d$ with $|\alpha|\in\mathbb{N}^*$, we consider $i_0\in\{1,\ldots,d\}$ such that $\alpha^{i_0}\in\mathbb{N}^*$ and denote $\alpha^-=\{\alpha^1,\ldots,\alpha^{i_0-1},\alpha^{i_0}-1,\alpha^{i_0+1},\ldots,\alpha^d\}$. Then

$$\partial^{\alpha}_{\mathrm{X}_0^\delta}X^\delta_{t+\delta}=\partial^{\alpha}_{\mathrm{X}_0^\delta}X^\delta_t+B_t\partial^{\alpha}_{\mathrm{X}_0^\delta}X^\delta_t+\delta^{\frac12}\sum_{i\in\mathcal{N}}Z^{\delta,i}_{t+\delta}B^{1,i}_{\alpha,t}+B^3_{\alpha,t},$$

with $B^1_\alpha=B^3_\alpha=0$ if $|\alpha|=1$ and for $|\alpha|\geqslant 2$, $B^{1,i}_{\alpha,t}=(\partial_{\mathrm{X}_0^{\delta,i_0}}X^\delta_t)^{\mathrm{T}}\,\nabla^2_x A^i_1(X^\delta_t,t)\partial^{\alpha^-}_{\mathrm{X}_0^\delta}X^\delta_t+\partial_{\mathrm{X}^{\delta,i_0}}B^{1,i}_{\alpha^-,t}$, $B^3_{\alpha,t}=\tilde{B}^{i_0}_t\partial^{\alpha^-}_{\mathrm{X}_0^\delta}X^\delta_t+\partial_{\mathrm{X}_0^{\delta,i_0}}B^3_{\alpha^-,t}$, where we have introduce the quantity

$$\begin{aligned}\tilde{B}^{i_0}_t=&\delta\sum_{i,j\in\mathcal{N}}Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}(\partial_{\mathrm{X}_0^{\delta,i_0}}X^\delta_t)^{\mathrm{T}}\,\nabla^2_x A^{i,j}_2(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta})\\&+\delta(\partial_{\mathrm{X}_0^{\delta,i_0}}X^\delta_t)^{\mathrm{T}}\,\nabla^2_x A_3(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta},\delta).\end{aligned}$$

We proove our estimate using Proposition 4.1. Therefore, it is sufficient to study the quantity $\mathscr{C}_{\mathbb{R}^d,q,p}(B^1_\alpha,B^2_\alpha,B^3_\alpha)$ for $p\geqslant 1$. In particular, since $\mathbf{A}^\delta_1(q+3)$ (see (2.2)) holds, for every $p\geqslant 1$, and every $i\in\mathcal{N}$, and every multi-index $\beta\in\mathbb{N}^d$, using a recursive approach and that $B^1_{\alpha'}=0$ if $|\alpha'|=1$, we obtain (the expectation in the norms below are computed conditionalyl to $X^\delta_0=x$)

$$\begin{aligned}\|B^1_{\alpha,t}\|_{(\mathbb{R}^d)^{\mathcal{N}},q,p}\leqslant&\|\partial_{x_0^{\delta,i_0}}B^{1,i}_{\alpha^-,t}\|_{\mathbb{R}^d,q,p}+C\,\mathbf{K}_{q+3}\sup_{t\in\mathcal{T}}\sum_{\substack{q'\in\{0,1\},\alpha'\in\mathbb{N}^d\\1-q'\leqslant|\alpha'|<|\alpha|}}\mathbb{E}_x[(1+|\partial^{\alpha'}_{x_0^\delta}X^\delta_t|^{p(q+2)}_{\mathbb{R}^d,q',q})(1+|X^\delta_t|^{p\,\mathbf{p}_{q+3}}_{\mathbb{R}^d})]^{\frac1p}\\\leqslant&C\,\mathbf{K}_{q+|\alpha|+1}\sup_{t\in\mathcal{T}}\sum_{\substack{q'\in\{0,1\},\alpha'\in\mathbb{N}^d\\1-q'\leqslant|\alpha'|<|\alpha|}}\mathbb{E}_x[(1+|\partial^{\alpha'}_{x_0^\delta}X^\delta_t|^{p(q+|\alpha|)}_{\mathbb{R}^d,q',q})(1+|X^\delta_t|^{p\,\mathbf{p}_{q+|\alpha|+1}}_{\mathbb{R}^d})]^{\frac1p}.\end{aligned}$$

In the same way

$$\begin{aligned}\|\sum_{\substack{w\in\pi^\delta\\w<t}}B^3_{\alpha,w}\|_{\mathbb{R}^d,q,p}\leqslant&C(1+T)\,\mathbf{K}_{q+|\alpha|+2}\,\mathbf{M}_C(Z^\delta)\\&\times\sup_{t\in\mathcal{T}}\sum_{\substack{q'\in\{0,1\},\alpha'\in\mathbb{N}^d\\1-q'\leqslant|\alpha'|<|\alpha|}}\mathbb{E}_x[(1+|\partial^{\alpha'}_{x_0^\delta}X^\delta_t|^{p(q+|\alpha|)}_{\mathbb{R}^d,q',q})(1+|X^\delta_t|^{p\,\mathbf{p}_{q+|\alpha|+2}}_{\mathbb{R}^d})]^{\frac1p}.\end{aligned}$$

Then (3.15) follows from Proposition 4.1 combined with a recursive approach. We now study the Sobolev norms of $\mathbf{L}\,X^\delta_t$. We have

$$\mathbf{L}\,X^\delta_{t+\delta}=\mathbf{L}\,X^\delta_t+B_t\,\mathbf{L}\,X^\delta_t+\delta^{\frac12}\sum_{i\in\mathcal{N}}Z^{\delta,i}_{t+\delta}B^{1,i}_t+\delta^{\frac12}\sum_{i\in\mathcal{N}}\mathbf{L}\,Z^{\delta,i}_{t+\delta}B^{2,i}_t+B^3_t,$$

with $B^{1,i}_t=\mathrm{Tr}(\boldsymbol{\sigma}_{X^\delta_t}\,\nabla^2_x A^i_1(X^\delta_t,t))$, $B^{2,i}_t=A^i_1(X^\delta_t,t)$ and

$$\begin{aligned}B^3_t=&\delta\sum_{i,j\in\mathcal{N}}(Z^{\delta,i}_{t+\delta}\,\mathbf{L}\,Z^{\delta,j}_{t+\delta}+Z^{\delta,j}_{t+\delta}\,\mathbf{L}\,Z^{\delta,i}_{t+\delta}+\chi^\delta_{t+\delta}\mathbf{1}_{i,j})A^{i,j}_2(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta})\\&+Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}(\mathrm{Tr}(\boldsymbol{\sigma}_{X^\delta_t}\,\nabla^2_x A^{i,j}_2(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta}))+\delta^{\frac12}\sum_{l\in\mathcal{N}}\partial_{z^l}A^{i,j}_2(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta})\,\mathbf{L}\,Z^{\delta,l}_{t+\delta}\\&+\chi^\delta_{t+\delta}\delta\sum_{l\in\mathcal{N}}\partial^2_{z^l}A^{i,j}_2(X^\delta_t,t+\delta,\delta^{\frac12}Z^\delta_{t+\delta}))+\chi^\delta_{t+\delta}\delta\sum_{l\in\mathcal{N}}\partial^2_{z^l}A_3(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta},\delta)\\&+\mathrm{Tr}(\boldsymbol{\sigma}_{X^\delta_t}\,\nabla^2_x A_3(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta},\delta)+\delta^{\frac12}\sum_{l\in\mathcal{N}}\partial_{z^l}A_3(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta},\delta)\,\mathbf{L}\,Z^{\delta,l}_{t+\delta}.\end{aligned}$$

Moreover, for every $p\geqslant 1$, and every $i\in\mathcal{N}$, using $\mathbf{A}^\delta_1(q+4)$ (see (2.2)),

$$\begin{aligned}\|B^{1,i}_t\|_{(\mathbb{R}^d)^{\mathcal{N}},q,p}\leqslant&C\,\mathbf{K}_{q+3}\sup_{t\in\mathcal{T}}\mathbb{E}_x[(1+|X^\delta_t|^{p(q+2)}_{\mathbb{R}^d,1,q+1})(1+|X^\delta_t|^{p\,\mathbf{p}_{q+3}}_{\mathbb{R}^d})]^{\frac1p},\\\|B^{2,i}_t\|_{(\mathbb{R}^d)^{\mathcal{N}},q,p}\leqslant&C\,\mathbf{K}_{q+1}\sup_{t\in\mathcal{T}}\mathbb{E}_x[(1+|X^\delta_t|^{pq}_{\mathbb{R}^d,1,q})(1+|X^\delta_t|^{p\,\mathbf{p}_{q+1}}_{\mathbb{R}^d})]^{\frac1p},\end{aligned}$$

and

$$\| \sum_{\substack{w\in\pi^{\delta}\\ w<t}} B^3_w\|_{\mathbb{R}^d,,q,p} \leqslant C(1+T)\,\mathbf{K}_{q+4}\,\mathbf{M}_C(Z^{\delta})(1+\sup_{t\in\mathcal{T}}\|\,\mathbf{L}\,Z^{\delta}_t\|_{\mathbb{R}^N,q,2p})$$
$$\times \sup_{t\in\mathcal{T}}\mathbb{E}_x[(1+|X^{\delta}_t|^{p(q+2)}_{\mathbb{R}^d,1,q+1})(1+|X^{\delta}_t|^{p\,\mathbf{p}_{q+4}}_{\mathbb{R}^d})]^{\frac{1}{p}}.$$

We finally use (4.4) from Lemma 4.1 and Proposition 4.1 to complete the proof of (3.16).

□

4.2. **Proof of Theorem 3.3.**

4.2.1. *Preliminaries.* Before we focus on the proof of Theorem 3.3, we provide a representation formula for the Malliavin derivatives using the variation of constant formula and some technical results we will employ in our proof.

Representations formula. Let $w,t\in\pi^{\delta,*}, i\in\mathcal{N}$. Then $\mathbf{D}_{(w,i)}\,X^{\delta}_t(x)=0$ for every $w>t$ and for $w\leqslant t$,

$$\mathbf{D}_{(w,i)}\,X^{\delta}_t=\chi^{\delta}_t\partial_{z^i}\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\delta)\mathbf{1}_{w=t}+\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\delta)\,\mathbf{D}_{(w,i)}\,X^{\delta}_{t-\delta}(x).$$

We consider the tanget flow process $(\dot{X}_t)_{t\in\pi^{\delta}}$ defined by $\dot{X}_0=I_{d\times d}$ and

$$\dot{X}_t:=\partial_{\mathrm{X}^{\delta}_0}X^{\delta}_t=\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\delta)\dot{X}_{t-\delta}.$$

We now define the inverse tangent flow. To prove the invertibility, we consider the Hilbert space $(\mathbb{R}^{d\times d},\langle,\rangle_{\mathbb{R}^{d\times d}})$, with the Frobenius scalar product defined by $\langle M,M'\rangle_{\mathbb{R}^{d\times d}}:=\mathrm{Trace}(M'M^{\mathrm{T}})=\sum_{i=1}^d(M'M^{\mathrm{T}})_{i,i}$, $M,M'\in\mathbb{R}^{d\times d}$. Notice that for $M\in\mathbb{R}^{d\times d}$, $\|M\|_{\mathbb{R}^d}\leqslant|M|_{\mathbb{R}^{d\times d}}\leqslant d^{\frac12}\|M\|_{\mathbb{R}^d}$. Also, for $k\in\mathbb{N}^*$, $|M^k|_{\mathbb{R}^{d\times d}}\leqslant\|M\|_{\mathbb{R}^d}|M^{k-1}|_{\mathbb{R}^{d\times d}}\leqslant|M|^k_{\mathbb{R}^{d\times d}}$ (with $M^0=I_{d\times d}$ and $M^l=MM^{l-1}$, $l\in\{1,\ldots,k\}$).

Now, since $\nabla_x\psi(x,t,0,0)=I_{d\times d}$ for every $(x,t)\in\mathbb{R}^d\times\pi^{\delta}$, it follows from the Taylor expansion of $\nabla_x\psi$, that

$$\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\delta)=I_{d\times d}+\delta^{\frac12}\sum_{l\in\mathcal{N}}Z^{\delta,l}_t\int_0^1\partial_{z^l}\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\lambda\delta^{\frac12}Z^{\delta}_t,0)\mathrm{d}\lambda.$$
$$+\,\delta\int_0^1\partial_y\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\lambda\delta)\mathrm{d}\lambda,$$

and using the assumption $\mathbf{A}_1$ (see (2.3)) yields

$$|I_{d\times d}-\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\delta)|_{\mathbb{R}^{d\times d}}\leqslant\delta^{\frac12}4\,\mathbf{K}_3\max(|Z^{\delta}_t|^{\mathbf{p}_3+1}_{\mathbb{R}^N},1). \tag{4.13}$$

In particular, if we assume that

$$\delta^{\frac12}\eta_2^{\mathbf{p}_3+1}8\,\mathbf{K}_3<1, \tag{4.14}$$

we remark that, on the set $\{|Z^{\delta}_t|_{\mathbb{R}^N}\leqslant\eta_2\}$, we have

$$\begin{aligned}|\det\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\delta)|^{\frac{2}{d}}\geqslant&\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}|\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\delta)\xi|_{\mathbb{R}^d}\\ \geqslant&1-\|I_{d\times d}-\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\delta)\|_{\mathbb{R}^d}\\ \geqslant&1-\delta^{\frac12}2\,\mathbf{K}_3(1+\eta_2^{\mathbf{p}_3+1})\geqslant\frac12.\end{aligned}$$

The matrix $\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,\delta^{\frac12}Z^{\delta}_t,\delta)$ is thus invertible on the set $\{|Z^{\delta}_t|_{\mathbb{R}^N}\leqslant\eta_2\}$. We are now in a position to introduce the inverse tangent flow, namely $(\mathring{X}_t)_{t\in\pi^{\delta}}$ satisfying $\mathring{X}_0=I_{d\times d}$ and which is well defined for every $t\in\pi^{\delta,*}$ as soon as we are on the set $\{\Theta_{\eta_2,\pi^{\delta,*},t}>0\}$ and (4.14) holds. In this case

$$\mathring{X}_t:=\dot{X}_t^{-1}=\mathring{X}_{t-\delta}\nabla_x\psi(X^{\delta}_{t-\delta},t-\delta,Z^{\delta}_t,\delta)^{-1}. \tag{4.15}$$

In particular we introduce $\mathring{X}_{\eta_2,t}:=\mathring{X}_t\mathbf{1}_{\Theta_{\eta_2,\pi^{\delta,*},t}>0}$ which is well defined for every $t\in\pi^{\delta}$.

We conclude this introduction observing that we have the so-called variation of constant formula. Under assumption (4.14) holds, on the set $\{\Theta_{\eta_2,\pi^{\delta,*},t}>0\}$, for every $(w,i)\in\pi^{\delta,*}\cap(0,t]\times\mathcal{N}$, we have

$$\mathbf{D}_{(w,i)}\,X^{\delta}_t=\chi^{\delta}_w\dot{X}_t\mathring{X}_w\partial_{z^i}\psi(X^{\delta}_{w-\delta},w-\delta,\delta^{\frac12}Z^{\delta}_w,\delta). \tag{4.16}$$

Before we give the proof Theorem 3.3, we start with some preliminary results which are crucial in the study of the determinant of the inverse of the Malliavin covariance matrix.

Preliminary results. Two standard results will be used in our approach, namely the Burkholder inequality (see (4.8)) and an exponential martigale inequality, we recall thereafter. First, let us introduce some notations. Given a $\mathbb{R}$-valued process $(Y_t)_{t\in\pi^\delta}$ progressively measurable *w.r.t.* a filtration $(\mathcal{F}_t^Y)_{t\in\pi^\delta}$, we denote $\mathscr{M}(Y)_t = \delta^{-\frac{1}{2}}(Y_{t+\delta} - \mathbb{E}[Y_{t+\delta}|\mathcal{F}_t^Y])$, $\mathscr{K}(Y)_t = \delta^{-1}\mathbb{E}[Y_{t+\delta} - Y_t|\mathcal{F}_t^Y]$.
Let $(M_t)_{t\in\pi^\delta}$ be a $\mathbb{R}$-valued local square integrable $(\mathcal{F}_t)_{t\in\pi^\delta}$-martingale. We denote $[M]_t = |M_0|^2 + \delta\sum_{w\in\pi^\delta, w<t} |\mathscr{M} M_w|^2$ and $\langle M\rangle_t = \mathbb{E}[|M_0|^2] + \delta\sum_{w\in\pi^\delta, w<t}\mathbb{E}[|\mathscr{M} M_w|^2|\mathcal{F}_w^M]$. Then (see [14] Corollary 3.4 or [15]), we have the following extension of the Freedman inequality [17]: For $a, b > 0$ and $t \in \pi^\delta$,

$$\mathbb{P}(\sup_{\substack{w\in\pi^\delta\\ w\leqslant t}} |M_w| \geqslant a, [M]_t + \langle M\rangle_t < b) \leqslant 2\exp(-\frac{a^2}{2b}). \tag{4.17}$$

Now, let us give some additional intermediate results which are proved in the Appendix A. The first one is a technical result that is used to bound the probability that the determinant of a random matrix Σ is under some threshold by studying $\mathbb{P}(\xi^{\mathrm{T}}\Sigma\xi \leqslant \epsilon)$ for $\xi \in \mathbb{R}^d$.

Lemma 4.3. *Let Σ be a $\mathbb{R}^{d\times d}$-valued random variable and $\epsilon \in (0, \frac{2^{\frac{1}{2}}}{d^{\frac{1}{2}}})$. Then*

$$\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1} \xi^T\Sigma\xi \leqslant \frac{1}{2}\epsilon) \leqslant C(d)\epsilon^{-2d} \sup_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1} \mathbb{P}(\xi^T\Sigma\xi \leqslant \epsilon) + \mathbb{P}(\|\Sigma\|_{\mathbb{R}^d} > \frac{1}{3\epsilon}). \tag{4.18}$$

The second result provides an estimate of the moments of the inverse tangent flow.

Lemma 4.4. *Let $T > 0$, $\mathcal{T} = (0, T] \cap \pi^\delta$, $x \in \mathbb{R}^d$, let $p \geqslant 2$ and let $\eta_2 > 1$. Assume that (2.3) from $\mathbf{A}_1$, $\mathbf{A}_3^\delta(p(q^\delta_{\eta_2} \vee (2\,\mathbf{p}_3 + 2)))$ with $q^\delta_{\eta_2} := 1 + \lceil -\frac{\ln(\delta)}{2\ln(\eta_2)}\rceil$ (see (2.7)) and that (4.14) hold. Then,*

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}} \|\mathring{X}_t\|^p_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t}>0}]^{\frac{1}{p}} \leqslant C(d)\exp(C(p)T\,\mathbf{M}_{p(q^\delta_{\eta_2}\vee(2\,\mathbf{p}_3+2))}(Z^\delta)^{\frac{2}{p}}\,\mathbf{K}_3^4). \tag{4.19}$$

The next result is a discrete-time Lie expansion satisfied by our process X^δ together with a control of the remainder appearing. Before we state the result, we introduce a set of polynomially bounded function (together with their derivatives of order $q \in \mathbb{N}$) that is $\mathcal{C}^q_{\text{pol}}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^d) := \{f \in \mathcal{C}^q(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^d), \exists\,\mathbf{K}_{f,q} \geqslant 1, \mathbf{p}_{f,q} \in \mathbb{N}, \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \sum_{|\alpha|\leqslant q} |\partial_x^\alpha f(x,t)|_{\mathbb{R}^d} \leqslant \mathbf{K}_{f,q}(1 + |x|^{\mathbf{p}_{f,q}}_{\mathbb{R}^d})\}$

Lemma 4.5 (Discrete-time Lie expansion)**.** *Let $V \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{R}_+)$ and let $\eta_2 > 1$. Assume that $\psi \in \mathcal{C}^3(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^N \times [0,1])$. Then, for every $t \in \pi^{\delta,*}$, we have the Lie expansion*

$$\begin{aligned}\mathring{X}_{\eta_2,t}V(X_t^\delta, t) =&\mathring{X}_{\eta_2,t-\delta}V(X^\delta_{t-\delta}, t-\delta) + \delta^{\frac{1}{2}}\sum_{i\in\mathcal{N}} Z_t^{\delta,i}\mathring{X}_{\eta_2,t-\delta}V^{[i]}(X^\delta_{t-\delta}, t-\delta)\\ &+ \delta\mathring{X}_{\eta_2,t-\delta}V^{[0]}(X^\delta_{t-\delta}, t-\delta)) + \mathring{X}_{\eta_2,t-\delta}\mathscr{R}V(X^\delta_{t-\delta}, t-\delta, Z_t^\delta).\end{aligned}$$

Moreover, let $\alpha^x \in \mathbb{N}^d$ and let us introduce the $\mathbb{R}^d$-valued functions defined for every $(x,t,z) \in \mathbb{R}^d \times \pi^{\delta,*} \times \in \mathbb{R}^N$ by

$$\begin{aligned}\tilde{\mathscr{R}}V(x, t-\delta, z) =&\mathscr{R}V(x, t-\delta, z) - \mathbb{E}[\mathscr{R}V(x, t-\delta, Z_t^\delta)]\\ \bar{\mathscr{R}}V(x, t-\delta) =&\mathbb{E}[\mathscr{R}V(x, t-\delta, Z_t^\delta)].\end{aligned}$$

Assume that $\mathbf{A}_1^\delta(|\alpha^x| + 4)$ (see (2.2) and (2.3)) and $\mathbf{A}_3^\delta(+\infty)$ (see (2.7)) hold. Assume also that $V \in \mathcal{C}^{|\alpha^x|+3}_{pol}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R}^d)$ and that (4.14) holds. Then, for every $(x,t,z) \in \mathbb{R}^d \times \pi^{\delta,*} \times \mathbb{R}^N$,

$$|\tilde{\mathscr{R}}V(x, t-\delta, z)|_{\mathbb{R}^d} \leqslant \delta C\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_4^C\,\mathbf{K}_{V,3}^C(1 + |x|^C_{\mathbb{R}^d} + |z|^C_{\mathbb{R}^N}), \tag{4.20}$$

with $C = C(\mathbf{p}_4, \mathbf{p}_{V,3}, q^\delta_{\eta_2})$. Moreover

$$|\partial_x^{\alpha^x}\bar{\mathscr{R}}V(x, t-\delta)|_{\mathbb{R}^d} \leqslant \delta^{\frac{3}{2}}C\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^C_{|\alpha^x|+4}\,\mathbf{K}^C_{V,|\alpha^x|+3}(1 + |x|^C_{\mathbb{R}^d}), \tag{4.21}$$

with $C = C(\mathbf{p}_{|\alpha^x|+4}, \mathbf{p}_{V,|\alpha^x|+3}, q^\delta_{\eta_2}, |\alpha^x|)$.

We point out that, Lemma 4.4 and Lemma 4.5 will be used in the specific situation where $\eta_2 = \boldsymbol{\eta}_2(\delta)$ under assumption $\mathbf{A}_5^\delta(x,T)$ hold. In this case $\delta \leqslant (2^{\mathbf{p}_3+1}8\,\mathbf{K}_3)^{-4}$ so that $\boldsymbol{\eta}_2(\delta) \geqslant \min(\delta^{-\frac{1}{188}}, \delta^{-\frac{1}{4(1+\mathbf{p}_3)}})$ and then $q^\delta_{\boldsymbol{\eta}_2(\delta)} \leqslant C(\mathbf{p}_3)$.
The last result is a Norris Lemma adapted to discrete-time processes. In the continuous case, this lemma can be found in [25], Lemma 2.3.2. Before giving this result, we introduce some notations. Let $q > 0$

and $\mathcal{T} \subset \pi^{\delta,*}$. Given a $\mathbb{R}$-valued process $(Y_t)_{t\in\pi^\delta}$ progressively measurable *w.r.t.* a filtration $(\mathcal{F}^Y_t)_{t\in\pi^\delta}$, we denote,

$$\begin{aligned}\mathscr{N}_{Y,\mathcal{T}}(q) :=&1 + \sup_{t\in\mathcal{T}} \mathbb{E}[|Y_{t-\delta}|^q] + \mathbb{E}[\sup_{t\in\mathcal{T}} |\mathscr{K}(Y)_{t-\delta}|^q] + \mathbb{E}[\sup_{t\in\mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]] \\ &+ \mathbb{E}[\sup_{t\in\mathcal{T}} |\mathscr{K}(\mathscr{K}(Y))_{t-\delta}|^q] + \mathbb{E}[\sup_{t\in\mathcal{T}} \mathbb{E}[|\mathscr{M}(\mathscr{K}(Y))_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]].\end{aligned} \tag{4.22}$$

Lemma 4.6 (Discrete-time Norris Lemma)**.** *Let $T \geqslant \delta$, $\mathcal{T} = (0,T] \cap \pi^\delta$. Let $(Y_t)_{t\in\pi^\delta}$ be a $\mathbb{R}$-valued random process progressively measurable with respect to a filtration $(\mathcal{F}^Y_t)_{t\in\pi^\delta}$, let $r \in (0, \frac{1}{12})$ and let $p > 0$. Let us introduce $q(r,p) = \max(4, \frac{44p}{1-12r})$ and assume that*

$$\mathscr{N}_{Y,\mathcal{T}}(q(r,p)) < +\infty.$$

Then, for every $\epsilon \in [(2^{10}(1\vee T^3)\delta)^{\frac{44}{91-36r}}, (2^8(1\vee T))^{-\frac{11}{1-12r}}]$,

$$\begin{aligned}&\mathbb{P}(\delta\sum_{t\in\mathcal{T}}|Y_t|^2 < \epsilon, \delta\sum_{t\in\mathcal{T}}\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}] + |\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \epsilon^r) \\ &\qquad\leqslant \epsilon^p(1\vee T^{2q(r,p)})2^{5q(r,p)+3}\mathscr{N}_{Y,\mathcal{T}}(q(r,p)) + 12\exp(-\frac{\epsilon^{-\frac{1-12r}{22}}}{2^{11}(1\vee T^2)}).\end{aligned} \tag{4.23}$$

4.2.2. *Proof of Theorem 3.3.*

Proof. The strategy of this proof consists in showing that, for every $p \geqslant 0$,

$$\mathbb{E}_x[|\det\tilde{\boldsymbol{\gamma}}_{X^\delta_T}|^p\mathbf{1}_{\Theta^*_T>0}] \leqslant \frac{1+\mathbf{1}_{\mathbf{p}_{2L+5}>0}|x|^C_{\mathbb{R}^d}}{\boldsymbol{\mathcal{V}}_L(x)^{\frac{3}{2}13^L d(p+4)}T^{13^L dp}}\mathbf{K}^C_{2L+5}\exp(C(1+T)\mathbf{M}_C(Z^\delta)\mathbf{K}^4_3), \tag{4.24}$$

where $\tilde{\boldsymbol{\gamma}}_{X^\delta_T} = \dot{X}^{\mathrm{T}}_T\boldsymbol{\gamma}_{X^\delta_T}\dot{X}_T$ and (3.20) follows from the Hölder inequality together with Lemma 4.4. To show this estimate, we first establish a sufficient condition (see (4.25) and (4.26)) to derive (4.24) (**Step 1.**). Then we prove that (4.25) (resp. (4.26)) holds in **Step 2** (resp. **Step 3**). **Step 4** is dedicated to the proof of (3.21). We begin with some notations. For every $i \in \mathcal{N}$, we introduce the $\mathbb{R}^d$-valued process $(\mathscr{V}_{i,t})_{t\in\mathcal{T}}$ defined for every $t \in \mathcal{T}$ by $\mathscr{V}_{i,t} = \mathring{X}_{t-\delta}\nabla_x\psi^{-1}\partial_{z^i}\psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^\delta_t, \delta)$. Notice that, we have also $\mathscr{V}_{i,t} = \mathring{X}_t\partial_{z^i}\psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^\delta_t, \delta)$. We now introduce the notation $v^2 = vv^{\mathrm{T}} \in \mathbb{R}^{d\times d}$ for a vector $v \in \mathbb{R}^d$. Using the variation of constant formula (4.16), denoting $\tilde{\boldsymbol{\sigma}}_{X^\delta_T} = \delta\sum_{(t,i)\in\mathcal{T}\times\mathcal{N}}\chi^\delta_t(\mathscr{V}_{i,t})^2$, on the set $\{\Theta_{\eta_2,\mathcal{T},t} > 0\}$, we have

$$\begin{aligned}\boldsymbol{\sigma}_{X^\delta_T} =&\delta\sum_{(t,i)\in\mathcal{T}\times\mathcal{N}}(\mathbf{D}_{(t,i)}X^\delta_T)^2 = \delta\sum_{(t,i)\in\mathcal{T}\times\mathcal{N}}\chi^\delta_t(\dot{X}_T\mathring{X}_t\partial_{z^i}\psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^\delta_t, \delta))^2 \\ =&\delta\sum_{(t,i)\in\mathcal{T}\times\mathcal{N}}\chi^\delta_t(\dot{X}_T\mathscr{V}_{i,t})^2 = \dot{X}_T\tilde{\boldsymbol{\sigma}}_{X^\delta_T}\dot{X}^{\mathrm{T}}_T.\end{aligned}$$

Step 1. We first show that the proof of (4.24), boils down to prove that there exists $\bar{\boldsymbol{\epsilon}} \in (\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}}, \frac{2^{\frac{1}{2}}}{d^{\frac{1}{2}}}]$ and $\boldsymbol{\kappa} \geqslant 1$ (which do not depend on δ and will be made explicit in the sequel) such that, for every $\epsilon \in (\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}}, \bar{\boldsymbol{\epsilon}})$,

$$\sup_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\mathbb{P}_x(\xi^{\mathrm{T}}\tilde{\boldsymbol{\sigma}}_{X^\delta_T}\xi \leqslant 2\epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \leqslant \boldsymbol{\kappa}\epsilon^{d(p+4)}, \tag{4.25}$$

and

$$\mathbb{P}_x(\|\tilde{\boldsymbol{\sigma}}_{X^\delta_T}\|_{\mathbb{R}^d} > \frac{1}{6\epsilon}, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \leqslant \boldsymbol{\kappa}\epsilon^{d(p+2)}. \tag{4.26}$$

In this case

$$\mathbb{E}_x[|\det\tilde{\boldsymbol{\gamma}}_{X^\delta_T}|^p\mathbf{1}_{\Theta^*_T>0}] \leqslant C\boldsymbol{\kappa} + \lceil\bar{\boldsymbol{\epsilon}}^{-d}\rceil^p.$$

The result of **Step 1** is mainly a consequence of Lemma 4.3. We begin by noticing that

$$\mathbb{P}_x(|\det\tilde{\boldsymbol{\gamma}}_{X^\delta_T}|\mathbf{1}_{\Theta^*_T>0} \geqslant \epsilon^{-d}) = \mathbb{P}_x(|\det\tilde{\boldsymbol{\sigma}}_{X^\delta_T}| \leqslant \epsilon^d, \Theta^*_T > 0).$$

Since $|\det\tilde{\boldsymbol{\sigma}}^\delta_{X^\delta_T,\mathcal{T}}| > \boldsymbol{\eta}_1(\delta)^{-1}$ on $\{\Theta_{X^\delta_T,\det(\dot{X}^\delta_T)^2,\boldsymbol{\eta}_1(\delta),\mathcal{T}} > 0\}$, the quantity above is equal to zero as soon as $\epsilon^d \leqslant \boldsymbol{\eta}_1(\delta)^{-1}$ and for every $\epsilon^d > \boldsymbol{\eta}_1(\delta)^{-1}$,

$$\begin{aligned}\mathbb{P}_x(|\det\tilde{\boldsymbol{\sigma}}_{X^\delta_T}| \leqslant \epsilon^d, \Theta^*_T > 0) \leqslant&\mathbb{P}_x(|\det\tilde{\boldsymbol{\sigma}}_{X^\delta_T}| \leqslant \epsilon^d, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \\ \leqslant&\mathbb{P}_x(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^{\mathrm{T}}\tilde{\boldsymbol{\sigma}}_{X^\delta_T}\xi \leqslant \epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0).\end{aligned}$$

Applying Lemma 4.3 (with (4.25) and (4.26)), for every $\epsilon \in (\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}}, \bar{\boldsymbol{\epsilon}})$,

$$\mathbb{P}_x(|\det \tilde{\boldsymbol{\sigma}}_{X_T^\delta}| \leqslant \epsilon^d, \Theta_T^* > 0) \leqslant C\boldsymbol{\kappa}\epsilon^{d(p+2)}.$$

Therefore

$$\begin{aligned}\mathbb{E}_x[|\det \tilde{\boldsymbol{\gamma}}_{X_T^\delta}|^p \mathbf{1}_{\Theta_T^*>0}] \leqslant& C\boldsymbol{\kappa} \sum_{k=\lceil \bar{\boldsymbol{\epsilon}}^{-d}\rceil}^{\lceil \boldsymbol{\eta}_1(\delta)\rceil -1} \frac{(k+1)^p}{k^{p+2}} + \lceil \bar{\boldsymbol{\epsilon}}^{-d}\rceil^p \\ \leqslant& C\boldsymbol{\kappa} \sum_{k=1}^{+\infty} \frac{(k+1)^p}{k^{p+2}} + \lceil \bar{\boldsymbol{\epsilon}}^{-d}\rceil^p \leqslant C\boldsymbol{\kappa} 2^p \frac{\pi^2}{6} + \lceil \bar{\boldsymbol{\epsilon}}^{-d}\rceil^p,\end{aligned}$$

and the proof of **Step 1** is completed.

Step 2. In this part, we focus on te proof of (4.25). More particularly, we demonstrate that, if we fix,

$$\bar{\boldsymbol{\epsilon}} \in [\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}}, \min(\frac{2^{\frac{1}{2}}}{d^{\frac{1}{2}}}, (\frac{T\,\boldsymbol{\mathcal{V}}_L(x) m_*}{40(L+1)N^{\frac{L(L+1)}{2}}})^{13^L}, \mathbf{1}_{L=0} + \mathbf{1}_{L>0}|m_* \frac{(2^8(1\vee T))^{-143}}{10N^{\frac{L(L-1)}{2}}}|^{13^{L-1}})),$$

then, for every $\epsilon \in [\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}}, \bar{\boldsymbol{\epsilon}})$,

$$\begin{aligned}\sup_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1} \mathbb{P}_x(\xi^{\mathrm{T}} \tilde{\boldsymbol{\sigma}}_{X_t^\delta}\xi \leqslant 2\epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \leqslant& \epsilon^{d(p+4)}\,\boldsymbol{\mathcal{V}}_L(x)^{-\frac{3}{2}13^L d(p+4)} \\ &\times (1 + \mathbf{1}_{\mathbf{p}_{2L+5}>0}|x|^C_{\mathbb{R}^d})\,\mathbf{K}^C_{2L+5} \exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^4_3),\end{aligned} \tag{4.27}$$

Step 2.1. For every $l \in \{0,\dots,L\}$ and $\xi \in \mathbb{R}^d$, we introduce the $\mathbb{R}_+$-valued process $(\mathscr{W}^\xi_{l,t})_{t\in\mathcal{T}}$ defined for every $t \in \mathcal{T}$ by $\mathscr{W}^\xi_{l,t} = \sum_{\alpha\in\mathcal{N}^l} \sum_{i\in\mathcal{N}} \langle \xi, \mathring{X}_{t-\delta} V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta)\rangle^2_{\mathbb{R}^d}$. Let us denote $N_l = (\frac{10}{m_*})^{13^{-l}} 4^{\frac{1-13^{-l}}{1-13^{-1}}} \prod_{j=1}^l N^{j13^{j-l}}$. Then, for every $\xi \in \mathbb{R}^d$ with $|\xi| = 1$ and every $\epsilon \in [\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}}, 1)$,

$$\begin{aligned}\mathbb{P}_x(\delta &\sum_{(t,i)\in\mathcal{T}\times\mathcal{N}} \chi^\delta_t \langle \xi, \mathscr{V}_{i,t}\rangle^2_{\mathbb{R}^d} \leqslant 2\epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \\ \leqslant& \epsilon^{d(p+4)}(1 + \mathbf{1}_{\mathbf{p}_3>0}|x|^C_{\mathbb{R}^d})\,\mathbf{K}^C_3 \exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^4_3) \\ &+ \sum_{l=0}^{L-1} \mathbb{P}_x(\delta\sum_{t\in\mathcal{T}} \mathscr{W}^\xi_{l,t} \leqslant N_l \epsilon^{13^{-l}}, \delta\sum_{t\in\mathcal{T}} \mathscr{W}^\xi_{l+1,t} > N_{l+1}\epsilon^{13^{-l-1}}, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \\ &+ \mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\sum_{l=0}^{L} \mathscr{W}^\xi_{l,t} \leqslant (L+1)\frac{10}{m_*} N^{\frac{L(L+1)}{2}} \epsilon^{13^{-L}}, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0).\end{aligned} \tag{4.28}$$

First, we notice that, using the standard inequality $a^2 \geqslant \frac{1}{2}b^2 - (a-b)^2$ with $a = \langle \xi, \mathscr{V}_{i,t}\rangle_{\mathbb{R}^d}$ and $b = \langle \xi, \mathring{X}_{t-\delta} V_i(X^\delta_{t-\delta}, t-\delta)\rangle_{\mathbb{R}^d}$ for $(t,i) \in \mathcal{T}\times\mathcal{N}$ and considering separately the cases $(a-b)^2 \leqslant 2\epsilon$ and $(a-b)^2 > 2\epsilon$, yields

$$\begin{aligned}\mathbb{P}_x(\delta \sum_{(t,i)\in\mathcal{T}\times\mathcal{N}} \chi^\delta_t \langle \xi, \mathscr{V}_{i,t}\rangle^2_{\mathbb{R}^d} \leqslant 2\epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \leqslant& \mathbb{P}_x(\delta\sum_{t\in\mathcal{T}} \chi^\delta_t\,\mathscr{W}^\xi_{0,t} \leqslant 8\epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \\ &+ \mathbb{P}_x(\delta \sum_{(t,i)\in\mathcal{T}\times\mathcal{N}} \langle \xi, \mathscr{V}_{i,t} - \mathring{X}_{t-\delta} V_i(X^\delta_{t-\delta}, t-\delta)\rangle^2_{\mathbb{R}^d} > 2\epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0),\end{aligned}$$

with, by decomposing again into two estimates,

$$\begin{aligned}\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}} \chi^\delta_t\,\mathscr{W}^\xi_{0,t} \leqslant 8\epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \leqslant& \mathbb{P}_x(\delta|\sum_{t\in\mathcal{T}} (\chi^\delta_t - m_*)\,\mathscr{W}^\xi_{0,t}| > 2\epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0) \\ &+ \mathbb{P}_x(\delta\sum_{t\in\mathcal{T}} \mathscr{W}^\xi_{0,t} \leqslant \frac{10}{m_*}\epsilon, \Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}} > 0).\end{aligned}$$

We focus on the estimate of the second term of the *r.h.s.* above. Our strategy is to handle this term by combining Lemma 4.6 (which accounts for the appearance of the exponent 13^{-1} corresponding to the choice of the parameter $r \in (0, 12^{-1})$ in the application of the lemma) for a process related to $(\mathscr{W}^\xi_{l,t})_{t\in\pi^\delta}$ and assumption $\mathbf{A}^\delta_5(x,T)$. Recalling that $N_{0,r} = \frac{1}{m_*}$, we derive

$$\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{0,t}\leqslant\frac{10}{m_*}\epsilon,\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\leqslant\mathbb{P}_x(\bigcap_{l=0}^{L}\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{l,t}\leqslant N_l\epsilon^{13^{-l}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)$$
$$+\sum_{l=0}^{L-1}\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{l,t}\leqslant N_l\epsilon^{13^{-l}},\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{l+1,t}>N_{l+1}\epsilon^{13^{-l-1}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0).$$

Moreover, since $\sup_{l\in\{0,\dots,L\}}N_l\leqslant N_0N^{\frac{L(L+1)}{2}}=\frac{10}{m_*}N^{\frac{L(L+1)}{2}}$,

$$\mathbb{P}_x(\bigcap_{l=0}^{L}\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{l,t}\leqslant N_{l,r}\epsilon^{13^{-l}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\leqslant\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\sum_{l=0}^{L}\mathscr{W}^{\xi}_{l,t}\leqslant(L+1)\frac{10}{m_*}N^{\frac{L(L+1)}{2}}\epsilon^{13^{-L}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0),$$

which is the expected bound on $\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{0,t}\leqslant\frac{10}{m_*}\epsilon,\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)$. A bound on those terms will be established in the next step. We now focus on the study of $\mathbb{P}_x(\delta|\sum_{t\in\mathcal{T}}(\chi^{\delta}_t-m_*)\mathscr{W}^{\xi}_{0,t}|>2\epsilon,\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)$. Our idea is to employ the exponential martingale inequality so, keeping in mind that χ^{δ}_t follows a Bernoulli distribution with mean m_*, we write

$$\begin{aligned}&\mathbb{P}_x(\delta|\sum_{t\in\mathcal{T}}(\chi^{\delta}_t-m_*)\mathscr{W}^{\xi}_{0,t}|>2\epsilon,\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\&\leqslant\mathbb{P}_x(\delta|\sum_{t\in\mathcal{T}}(\chi^{\delta}_t-m_*)\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T},t-\delta}>0}\mathscr{W}^{\xi}_{0,t}|>2\epsilon,\\&\qquad\delta^2\sum_{t\in\mathcal{T}}(m_*(1-m_*)+(\chi^{\delta}_t-m_*)^2)\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T},t-\delta}>0}|\mathscr{W}^{\xi}_{0,t}|^2\leqslant2\epsilon^{2+\frac{1}{22}})\\&\quad+\mathbb{P}_x(\delta^2|\sum_{t\in\mathcal{T}}\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T},t-\delta}>0}|\mathscr{W}^{\xi}_{0,t}|^2>2\epsilon^{2+\frac{1}{22}}).\end{aligned}$$

Notice that the choice of the exponent $\frac{1}{22}$ is specific to our approach in order to ensure that we can estimate the quantity above with the expected bounds when $\epsilon\geqslant\boldsymbol{\eta}_1(\delta)$ as described thereafter. Using (4.17), with $M_t=\sum_{w\in\pi^{\delta,*},\,w\leqslant t}\delta(\chi^{\delta}_t-m_*)\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T},t-\delta}>0}\mathscr{W}^{\xi}_{0,t}$, the first term of the *r.h.s.* of the inequality above is bounded by $2\exp(-\epsilon^{-\frac{1}{22}})$, itself being bounded by $C\epsilon^{d(p+4)}$. In order to treat the second term, we remark that, $\mathscr{W}^{\xi}_{0,t}=\sum_{i\in\mathcal{N}}\langle\xi,\mathring{X}_{t-\delta}V_i(X^{\delta}_{t-\delta})\rangle^2_{\mathbb{R}^d}$ and using the Markov inequality, for every $a>0$,

$$\begin{aligned}&\mathbb{P}_x(\delta^2\sum_{t\in\mathcal{T}}|\sum_{i\in\mathcal{N}}\langle\xi,\mathring{X}_{t-\delta}V_i(X^{\delta}_{t-\delta},t-\delta)\rangle^2_{\mathbb{R}^d}|^2\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T},t-\delta}>0}>2\epsilon^{\frac{45}{22}})\\&\leqslant\delta^a\epsilon^{-a\frac{45}{22}}\mathbf{K}_1^{4a}T^a\mathbb{E}_x[\sup_{t\in\mathcal{T}}\|\mathring{X}_{t-\delta}\|^{4a}_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T},t-\delta}>0}(1+\sup_{t\in\mathcal{T}}|X_{t-\delta}|^{\mathbf{p}_1}_{\mathbb{R}^d})^{4a}].\end{aligned}$$

In particular we chose $a=\frac{d(p+4)\ln(\boldsymbol{\eta}_1(\delta))}{-\frac{1}{2}\ln(\boldsymbol{\eta}_1(\delta))-d\ln(\delta)}$ (remember that $\delta\leqslant\boldsymbol{\eta}_1(\delta)^{-\frac{89}{44d}}<\boldsymbol{\eta}_1(\delta)^{-\frac{1}{2d}}$ so that $a\in(0,d(p+4)]$) and apply Lemma 4.4 (see (4.19)) and Lemma 4.2 to conclude this estimate.

Now, we study $\mathbb{P}_x(\delta\sum_{(t,i)\in\mathcal{T}\times\mathcal{N}}\langle\xi,\mathscr{V}_{i,t}-\mathring{X}_{t-\delta}V_i(X^{\delta}_{t-\delta},t-\delta)\rangle^2_{\mathbb{R}^d}>2\epsilon,\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)$. Recall that, on the set $\{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0\}$, we have $|Z^{\delta,j}_t|\leqslant\boldsymbol{\eta}_2(\delta)$. Moreover, for $\eta_2>1$, let $\mathcal{D}_{\eta_2}=\{z\in\mathbb{R}^N,|z|_{\mathbb{R}^N}\leqslant\delta^{\frac{1}{2}}\eta_2\}$. We fix $(x,t,z,y)\in\mathbb{R}^d\times\mathcal{T}\times\mathcal{D}_{\eta_2}\times(0,1]$. Using the Taylor expansion thus yields

$$\begin{aligned}|\nabla_x\psi^{-1}\partial_{z^i}\psi(x,t-\delta,z,y)-V_i(x,t-\delta)|_{\mathbb{R}^d}\leqslant&\delta^{\frac{1}{2}}\eta_2\sum_{j\in\mathbb{N}}|\partial_{z^j}(\nabla_x\psi^{-1}\partial_{z^i}\psi)(x,t-\delta,z,y)|_{\mathbb{R}^d}\\&+\delta|\partial_y(\nabla_x\psi^{-1}\partial_{z^i}\psi)(x,t-\delta,z,y)|_{\mathbb{R}^d},\end{aligned}$$

with $\partial_{z^j}(\nabla_x\psi^{-1}\partial_{z^i}\psi)=\nabla_x\psi^{-1}\partial_{z^j}\nabla_x\psi\nabla_x\psi^{-1}\partial_{z^i}\psi+\nabla_x\psi^{-1}\partial_{z^j,z^i}\psi$. Moreover, a similar formula for $\partial_y(\nabla_x\psi^{-1}\partial_{z^i}\psi)$. The study of this last term is similar to the one of $\partial_{z^j}(\nabla_x\psi^{-1}\partial_{z^i}\psi)$ and is left to the reader. Owing to the estimate

$$\sum_{i,j\in\mathcal{N}}|\partial_{z^j}(\nabla_x\psi^{-1}\partial_{z^i}\psi)|_{\mathbb{R}^d}\leqslant\|\nabla_x\psi^{-1}\|^2_{\mathbb{R}^d}\sum_{i,j\in\mathcal{N}}\|\partial_{z^j}\nabla_x\psi\|_{\mathbb{R}^d}|\partial_{z^i}\psi|_{\mathbb{R}^d}+|\partial_{z^j,z^i}\psi|_{\mathbb{R}^d},$$

we derive our estimate by showing that the function $\|\nabla_x\psi^{-1}\|_{\mathbb{R}^d}$ is bounded on $\mathbb{R}^d\times\mathcal{T}\times\mathcal{D}_{\eta_2}\times(0,1]$. We consider the following decomposition

$$\nabla_x\psi^{-1}(x,t-\delta,z,\delta)=I_{d\times d}-(\nabla_x\psi(x,t-\delta,z,\delta)-I_{d\times d})\nabla_x\psi^{-1}(x,t-\delta,z,\delta).$$

Now, assumption $\mathbf{A}_1$ (see (2.3)) implies that (4.13) holds. It follows that, under the assumption (4.14), for every $(x,t,z)\in\mathbb{R}^d\times\mathcal{T}\times\mathcal{D}_{\eta_2}$, $\|\nabla_x\psi(x,t-\delta,z,\delta)-I_{d\times d}\|_{\mathbb{R}^d}\leqslant\frac{1}{2}$ and then $\|\nabla_x\psi^{-1}\|_{\mathbb{R}^d}\leqslant 2$. The exepected estimate thus follows by observing that

$$\sum_{j\in\mathcal{N}}\|\partial_{z^j}\nabla_x\psi\|_{\mathbb{R}^d}=\sum_{j\in\mathcal{N}}|\sum_{l=1}^{d}|\partial_{z^j}\partial_{x^l}\psi|^2_{\mathbb{R}^d}|^{\frac{1}{2}}\leqslant\sum_{j\in\mathcal{N}}\sum_{l=1}^{d}|\partial_{z^j}\partial_{x^l}\psi|_{\mathbb{R}^d}.$$

Using similar estimates for the term $\partial_y(\nabla_x\psi^{-1}\partial_{z^i}\psi)$ together with $\mathbf{A}_1(3)$ (see (2.2)) and since (4.14) holds for $\eta_2=\boldsymbol{\eta}_2(\delta)$, we obtain, for every $a\geqslant\frac{1}{2}$, (with C that may also depend on a in the following)

$$\begin{aligned}&\mathbb{P}_x(\delta\sum_{(t,i)\in\mathcal{T}\times\mathcal{N}}\langle\xi,\mathscr{V}_{i,t}-\mathring{X}_{t-\delta}V_i(X^\delta_{t-\delta},t-\delta)\rangle^2_{\mathbb{R}^d}>2\epsilon,\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\ &\leqslant C\delta^a\,\boldsymbol{\eta}_2(\delta)^{2a}\epsilon^{-a}\,\mathbf{K}_3^{4a}\,T^a(\mathbb{E}_x[\sup_{t\in\mathcal{T}}\|\mathring{X}_{t-\delta}\|^{2a}_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T},t-\delta}>0}(1+\sup_{t\in\mathcal{T}}|X_{t-\delta}|^{4a\,\mathbf{p}_3}_{\mathbb{R}^d})]\\ &\quad+C\delta^a\,\boldsymbol{\eta}_2(\delta)^{2a}\epsilon^{-a}\mathbb{E}_x[\sup_{t\in\mathcal{T}}\|\mathring{X}_{t-\delta}\|^{2a}_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T},t-\delta}>0}|\delta\sum_{t\in\mathcal{T}}|Z^\delta_t|^{4\,\mathbf{p}_3}_{\mathbb{R}^N}|^a].\end{aligned}$$

Moreover, the Hölder inequality (since $2a\geqslant 1$) yields

$$\mathbb{E}[|\delta\sum_{t\in\mathcal{T}}|Z^\delta_t|^{4\,\mathbf{p}_3}_{\mathbb{R}^N}|^{2a}]\leqslant T^{2a-1}\mathbb{E}[\delta\sum_{t\in\mathcal{T}}|Z^\delta_t|^{8a\,\mathbf{p}_3}_{\mathbb{R}^N}|]\leqslant T^{2a}\,\mathbf{M}_{8a\,\mathbf{p}_3}(Z^\delta).$$

We chose $a=\max(\frac{1}{2},\lceil\frac{d(p+4)\ln(\boldsymbol{\eta}_1(\delta))}{-\ln(\boldsymbol{\eta}_1(\delta))-d\ln(\delta)-2d\ln(\boldsymbol{\eta}_2(\delta))}\rceil)$ (remember that $\delta\leqslant\boldsymbol{\eta}_2(\delta)^{-2}\,\boldsymbol{\eta}_1(\delta)^{-\frac{2}{d}}$ so that $a\leqslant d(p+4)$) and conclude using Cauchy-Schwarz inequality, Lemma 4.4 (see (4.19)) and Lemma 4.2. Gathering all the upper bounds together, we obtain (4.28).

Step 2.2. Let us show that, for every $\epsilon\in(0,(\frac{T\,\boldsymbol{\mathcal{V}}_L(x)m_*}{40(L+1)N^{\frac{L(L+1)}{2}}})^{13^L}]$,

$$\begin{aligned}&\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\sum_{l=0}^{L}\mathscr{W}^\xi_{l,t}\leqslant(L+1)\frac{10}{m_*}N^{\frac{L(L+1)}{2}}\epsilon^{13^{-L}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\ &\qquad\leqslant\epsilon^{d(p+4)}\,\boldsymbol{\mathcal{V}}_L(x)^{-\frac{3}{2}d(p+4)13^L}(1+\mathbf{1}_{\mathbf{p}_{4+2L}>0}|x|^C_{\mathbb{R}^d})\,\mathbf{K}^C_{2L+4}\exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4).\end{aligned}$$

It is worth noting that, in case of uniform Hörmander properties, we have a similar result but with $\boldsymbol{\mathcal{V}}_L(x)$ replaced by 1 in the r.h.s. above.

Now let us focus on the proof of **Step 2.2**. Let us denote $\epsilon_L=(L+1)10m_*^{-1}N^{\frac{L(L+1)}{2}}\epsilon^{13^{-L}}$. Let $\mathcal{T}_0:=\{\delta,\dots,\lceil\frac{4\epsilon_L}{\delta\,\boldsymbol{\mathcal{V}}_L(x)}\rceil\delta\}$. Since $\epsilon\leqslant(\frac{T\,\boldsymbol{\mathcal{V}}_L(x)m_*}{40(L+1)N^{\frac{L(L+1)}{2}}})^{13^L}$, then $\mathcal{T}_0\subset\mathcal{T}$. Therefore,

$$\begin{aligned}&\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\sum_{l=0}^{L}\mathscr{W}^\xi_{l,t}\leqslant(L+1)N_L\epsilon^{13^{-L}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\leqslant\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}_0}\sum_{l=0}^{L}\mathscr{W}^\xi_{l,t}\leqslant\epsilon_L,\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\ &\leqslant\mathbb{P}_x(\frac{1}{2}\delta|\mathcal{T}_0|\sum_{|\alpha|\leqslant L}\sum_{i\in\mathcal{N}}\langle\xi,V_i^{[\alpha]}(x,0)\rangle^2_{\mathbb{R}^d}-\epsilon_L\\ &\qquad\leqslant\delta\sum_{t\in\mathcal{T}_0}\sum_{|\alpha|\leqslant L}\sum_{i\in\mathcal{N}}|\langle\xi,\mathring{X}_{t-\delta}V_i^{[\alpha]}(X^\delta_{t-\delta},t-\delta)-V_i^{[\alpha]}(x,0)\rangle_{\mathbb{R}^d}|^2,\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\ &\leqslant\mathbb{P}_x(\sup_{t\in\mathcal{T}_0}\sum_{|\alpha|\leqslant L}\sum_{i\in\mathcal{N}}|\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}V_i^{[\alpha]}(X^\delta_{t-\delta},t-\delta)-V_i^{[\alpha]}(x,0)\rangle_{\mathbb{R}^d}|^2\geqslant\frac{\boldsymbol{\mathcal{V}}_L(x)}{4})\\ &\leqslant\mathbb{P}_x(\sup_{t\in\mathcal{T}_0}\sum_{|\alpha|\leqslant L}\sum_{i\in\mathcal{N}}|M_{\alpha,i,t-\delta}|^2\geqslant\frac{\boldsymbol{\mathcal{V}}_L(x)}{8}-\sup_{t\in\mathcal{T}_0}\sum_{|\alpha|\leqslant L}\sum_{i\in\mathcal{N}}|B_{\alpha,i,t-\delta}|^2)\\ &\leqslant\sum_{|\alpha|\leqslant L}\sum_{i\in\mathcal{N}}\mathbb{P}_x(\sup_{t\in\mathcal{T}_0}|M_{\alpha,i,t-\delta}|^2\geqslant\frac{\boldsymbol{\mathcal{V}}_L(x)}{8N\binom{N+L}{N}}-\sup_{t\in\mathcal{T}_0}|B_{\alpha,i,t-\delta}|^2),\end{aligned}$$

with for every $t\in\mathcal{T}$, and using notations introduced in (4.17),

$$M_{\alpha,i,t}=\delta^{\frac{1}{2}}\sum_{w\in\mathcal{T};0<w\leqslant t}\mathscr{M}(Y_{\alpha,i})_{w-\delta},\qquad B_{\alpha,i,t}=\delta\sum_{w\in\mathcal{T};0<w\leqslant t}\mathscr{K}(Y_{\alpha,i})_{w-\delta},$$

where $Y_{\alpha,i,0}=0$ and for every $t\in\mathcal{T}$,

$$Y_{\alpha,i,t}=\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t}V_i^{[\alpha]}(X_t^\delta,t)-V_i^{[\alpha]}(x,0)\rangle_{\mathbb{R}^d}.$$

Now we decompose our estimate in the following way

$$\begin{aligned}\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\sum_{l=0}^{L}\mathscr{W}^{\xi}_{l,t}\leqslant(L+1)N_L\epsilon^{13^{-L}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\ \leqslant\sum_{|\alpha|\leqslant L}\sum_{i\in\mathcal{N}}\mathbb{P}_x(\sup_{t\in\mathcal{T}_0}|B_{\alpha,i,t-\delta}|^2>\frac{\boldsymbol{\mathcal{V}}_L(x)}{16N\binom{N+L}{N}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\ +\mathbb{P}_x(\sup_{t\in\mathcal{T}_0}|M_{\alpha,i,t-\delta}|^2\geqslant\frac{\boldsymbol{\mathcal{V}}_L(x)}{16N\binom{N+L}{N}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0).\end{aligned}$$

We study the first term of the *r.h.s.* above. Using the Markov inequality, for every $a>0$, we have

$$\begin{aligned}\mathbb{P}_x(\sup_{t\in\mathcal{T}_0}|B_{\alpha,i,t-\delta}|^2>\frac{\boldsymbol{\mathcal{V}}_L(x)}{16N\binom{N+L}{N}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\leqslant&\mathbb{P}_x(|\delta\sum_{t\in\mathcal{T}_0}|\mathscr{K}(Y_{\alpha,i})_{t-\delta}||^2>\frac{\boldsymbol{\mathcal{V}}_L(x)}{16N\binom{N+L}{N}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\ \leqslant&C\delta^a|\mathcal{T}_0|^a\sup_{t\in\mathcal{T}_0}\mathbb{E}_x[|\mathscr{K}(Y_{\alpha,i})_{t-\delta}|^a]|\frac{N\binom{N+L}{N}}{\boldsymbol{\mathcal{V}}_L(x)}|^{\frac{a}{2}}.\end{aligned}$$

At this point, we chose $a=d(p+4)13^L$ so that $\delta^a|\mathcal{T}_0|^a\leqslant C\,\boldsymbol{\mathcal{V}}_L(x)^{-a}\epsilon^{d(p+4)}$. In addition, as a consequence of Lemma 4.5 with $V=V_i^{[\alpha]}$ (recall that owing to $\mathbf{A}_5^\delta(x,T)$ we have $\underline{q}^\delta_{\boldsymbol{\eta}_2(\delta)}\leqslant C$)). Using the Cauchy-Schwarz inequality and remarking that $\mathbf{K}_{V_i^{[\alpha]},3}\leqslant C(|\alpha|)\,\mathbf{K}_{4+2|\alpha|}^{C(|\alpha|)}$ and $\mathbf{p}_{V_i^{[\alpha]},3}\leqslant C(|\alpha|)\,\mathbf{p}_{4+2|\alpha|}$, we derive

$$\mathbb{E}_x[|\mathscr{K}(Y_{\alpha,i})_{t-\delta}|^a]\leqslant C\,\mathbf{K}_{4+2|\alpha|}^C\,\mathbf{M}_C(Z^\delta)\mathbb{E}_x[\|\mathring{X}_{t-\delta}\|_{\mathbb{R}^d}^{2a}\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0}]^{\frac{1}{2}}(1+\mathbb{E}_x[|X_{t-\delta}^\delta|_{\mathbb{R}^d}^C]^{\frac{1}{2}}),$$

To complete our estimate, we bound the *r.h.s.* above using Lemma 4.4 (see (4.19)) and Lemma 4.2 (when $4a<2$ we also use the Hölder inequality to conclude). We consider now the temrminvolving $M_{\alpha,i,t-\delta}$ and first observe that we have the following inequality

$$\begin{aligned}\mathbb{P}_x(\sup_{t\in\mathcal{T}_0}|M_{\alpha,i,t-\delta}|^2\geqslant\frac{\boldsymbol{\mathcal{V}}_L(x)}{16N\binom{N+L}{N}})\leqslant\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}_0}\mathbb{E}[|\mathscr{M}(M_{\alpha,i})_{t-\delta}|^2|\mathcal{F}^{X^\delta}_{t-\delta}]+|\mathscr{M}(M_{\alpha,i})_{t-\delta}|^2\geqslant\epsilon^{\frac{13^{-L}}{3}})\\ +\mathbb{P}_x(\sup_{t\in\mathcal{T}_0}|M_{\alpha,i,t}|^2\geqslant\frac{\boldsymbol{\mathcal{V}}_L(x)}{16N\binom{N+L}{N}},\delta\sum_{t\in\mathcal{T}_0}\mathbb{E}_x[|\mathscr{M}(M_{\alpha,i})_{t-\delta}|^2|\mathcal{F}^{X^\delta}_{t-\delta}]+|\mathscr{M}(M_{\alpha,i})_{t-\delta}|^2<\epsilon^{\frac{13^{-L}}{3}}).\end{aligned}$$

Using the Doob exponential inequality (4.17), the first term is bounded by $2\exp(-\frac{\boldsymbol{\mathcal{V}}_L(x)}{32\epsilon^{\frac{13^{-L}}{3}}N\binom{N+L}{N}})$. In order to bound the second term we take $a\geqslant1$ and using again the Markov and Hölder inequalities and that $\mathscr{M}(Y_{\alpha,i})=\mathscr{M}(M_{\alpha,i})$, yields

$$\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}_0}\mathbb{E}[|\mathscr{M}(M_{\alpha,i})_{t-\delta}|^2|\mathcal{F}^{X^\delta}_{t-\delta}]+|\mathscr{M}(M_{\alpha,i})_{t-\delta}|^2)\geqslant\epsilon^{\frac{13^{-L}}{3}})\leqslant\delta^a|\mathcal{T}_0|^a\epsilon^{-a\frac{13^{-L}}{3}}2^{a+1}\sup_{t\in\mathcal{T}_0}\mathbb{E}_x[|\mathscr{M}(Y_{\alpha,i})_{t-\delta}|^{2a}].$$

At this point, we chose $a=2^{-1}3d(p+4)13^L$ so that $\delta^a|\mathcal{T}_0|^a\epsilon^{-a\frac{13^{-L}}{3}}\leqslant C\,\boldsymbol{\mathcal{V}}_L(x)^{-a}\epsilon^{d(p+4)}$. In order to bound the *r.h.s.* above we use Lemma 4.5 and the Cauchy-Schwarz inequality to derive

$$\mathbb{E}_x[|\mathscr{M}(Y_{\alpha,i})_{t-\delta}|^{2a}]\leqslant C\,\mathbf{K}_{4+2|\alpha|}^C\,\mathbf{M}_C(Z^\delta)^2\mathbb{E}_x[\|\mathring{X}_{t-\delta}\|_{\mathbb{R}^d}^{4a}\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0}]^{\frac{1}{2}}(1+\mathbb{E}_x[|X_{t-\delta}^\delta|_{\mathbb{R}^d}^C]^{\frac{1}{2}}),$$

and then use Lemma 4.4 (see (4.19)) and Lemma 4.2. Gathering all the estimates and using standard inequalities between exponantial and polynomial functions concludes the proof of **Step 2.2**.

Step 2.3. Consider the case $L\in\mathbb{N}^*$. Let $l\in\{0,\ldots,L-1\}$. Let us show that for every

$$\epsilon\in[\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}},|\frac{(2^8(1\vee T))^{-143}}{N_l}|^{13^l}],$$

then

$$\begin{aligned}\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{l,t}\leqslant N_{l,r}\epsilon^{13^{-l}},\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{l+1,t}>N_{l+1}\epsilon^{13^{-l-1}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\ \leqslant\epsilon^{d(p+4)}(1+\mathbf{1}_{\mathbf{p}_{2l+7}>0}|x|_{\mathbb{R}^d}^C)\,\mathbf{K}_{2l+7}^C\exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_3^4).\end{aligned}$$

First, for $\alpha\in\mathcal{N}^l$ and $i\in\mathcal{N}$, we introduce the $\mathbb{R}$-valued process $(\tilde{Y}_{\alpha,i,t})_{t\in\pi^\delta}$ such that $\tilde{Y}_{\alpha,i,0}=0$ and $\tilde{Y}_{\alpha,i,t}=\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}V_i^{[\alpha]}(X^\delta_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}$, $t\in\pi^{\delta,*}$. In particular, on the set $\{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0\}$, we have

$\mathscr{W}^{\xi}_{l,t}=\sum_{\alpha\in\mathcal{N}^l}\sum_{i\in\mathcal{N}}|\tilde{Y}_{\alpha,i,t}|^2$, $t\in\pi^{\delta}$. Furthermore, it follows from Lemma 4.5 with $V=V_i^{[\alpha]}$, that, for $t\in\pi^{\delta,*}$,

$$\begin{aligned}\tilde{Y}_{\alpha,i,t+\delta}-\tilde{Y}_{\alpha,i,t}=&\,\delta^{\frac{1}{2}}\sum_{j\in\mathcal{N}}Z_t^{\delta,j}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}V_i^{[(\alpha,j)]}(X^{\delta}_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}\\&+\delta\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}V_i^{[(\alpha,0)]}(X^{\delta}_{t-\delta},t-\delta))\rangle_{\mathbb{R}^d}+\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\mathscr{R}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta,Z^{\delta}_t)\rangle_{\mathbb{R}^d}\\=&\,\delta^{\frac{1}{2}}\sum_{j\in\mathcal{N}}Z_t^{\delta,j}\tilde{Y}_{(\alpha,j),i,t}+\delta\tilde{Y}_{(\alpha,0),i,t}+\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\mathscr{R}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta,Z^{\delta}_t)\rangle_{\mathbb{R}^d},\end{aligned}$$

and

$$\begin{aligned}\mathscr{W}^{\xi}_{l+1,t}=&\sum_{\alpha\in\mathcal{N}^l}\sum_{i\in\mathcal{N}}\mathbb{E}[|\mathscr{M}(\tilde{Y}_{\alpha,i})_t-\delta^{-\frac{1}{2}}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\tilde{\mathscr{R}}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta,Z^{\delta}_t)\rangle_{\mathbb{R}^d}|^2|\mathcal{F}^{\tilde{Y}_{\alpha,i}}_{t-\delta}]\\&+|\mathscr{K}(\tilde{Y}_{\alpha,i})_t-\delta^{-1}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\bar{\mathscr{R}}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{l,t}\leqslant N_l\epsilon^{13^{-l}},\delta\sum_{t\in\mathcal{T}}\mathscr{W}^{\xi}_{l+1,t}>N_{l+1}\epsilon^{13^{-l-1}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\&\leqslant\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}^-}\sum_{\alpha\in\mathcal{N}^l}\sum_{i\in\mathcal{N}}|\tilde{Y}_{\alpha,i,t}|^2\leqslant N_l\epsilon^{13^{-l}},\\&\quad\delta\sum_{t\in\mathcal{T}}\sum_{\alpha\in\mathcal{N}^l}\sum_{i\in\mathcal{N}}\mathbb{E}_x[|\mathscr{M}(\tilde{Y}_{\alpha,i})_t-\delta^{-\frac{1}{2}}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\tilde{\mathscr{R}}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta,Z^{\delta}_t)\rangle_{\mathbb{R}^d}|^2|\mathcal{F}^{\tilde{Y}_{\alpha,i}}_{t-\delta}]\\&\quad+|\mathscr{K}(\tilde{Y}_{\alpha,i})_t-\delta^{-1}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\bar{\mathscr{R}}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}|^2>N_{l+1}\epsilon^{13^{-l-1}})\\&\leqslant\sum_{\alpha\in\mathcal{N}^l}\sum_{i\in\mathcal{N}}\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}^-}|\tilde{Y}_{\alpha,i,t}|^2\leqslant N_l\epsilon^{13^{-l}},\delta\sum_{t\in\mathcal{T}}\mathbb{E}_x[|\mathscr{M}(\tilde{Y}_{\alpha,i})_t|^2]+|\mathscr{K}(\tilde{Y}_{\alpha,i})_t|^2>\frac{1}{4}N^{-l-1}N_{l+1}\epsilon^{13^{-l-1}})\\&\quad+\sum_{\alpha\in\mathcal{N}^l}\sum_{i\in\mathcal{N}}\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\mathbb{E}_x[|\delta^{-\frac{1}{2}}\langle\xi,\mathring{X}_{\eta_2,t-\delta}\tilde{\mathscr{R}}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta,Z^{\delta}_t)\rangle_{\mathbb{R}^d}|^2]\\&\quad+|\delta^{-1}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\bar{\mathscr{R}}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}|^2>\frac{1}{4}N^{-l-1}N_{l+1}\epsilon^{13^{-l-1}}),\end{aligned}$$

where $\mathcal{T}^-=\mathcal{T}\setminus\{\sup\{t,t\in\mathcal{T}\}\}$. We bound the the first term of the *r.h.s.* above. Since $N_{l+1}=4N^{l+1}N_l^{\frac{1}{13}}$ and, since $N_l\in[1,(\frac{10}{m_*})^{13^{-l}}4^{\frac{13}{12}}N^{\frac{l(l+1)}{2}}]$, we have $N_l\epsilon^{13^{-l}}\in[|2^{10}(1\vee T^3)\delta|^{\frac{44}{91-\frac{36}{13}}},(2^8(1\vee T))^{-143}]$ and the expected bound is obtained by applying Lemma 4.6 with $Y=\tilde{Y}_{\alpha,i}$, $\mathcal{T}=\mathcal{T}^-$, $r=13^{-1}$, $\epsilon=N_l\epsilon^{13^{-l}}$, and $p=13^ld(p+4)$. To obtain our estimate we use standard inequalities on polynomial and exponential functions and we are left to show that $\mathscr{N}_{\tilde{Y}_{\alpha,i},\mathcal{T}^-}(q(d,l,p))$ (this quantity being defined in (4.22)) with $q(d,l,p)=13^{l+1}44d(p+4))$, is bounded. We notice that $\mathscr{K}(\mathscr{K}(\tilde{Y}_{\alpha,i}))_0=\langle\xi,V_i^{[\alpha]}(x,0)\rangle_{\mathbb{R}^d}$, $\mathscr{M}(\mathscr{K}(\tilde{Y}_{\alpha,i}))_0=0$ and that, for $t\in\pi^{\delta,*}$, as a consequence of Lemma 4.5,

$$\begin{aligned}\mathscr{K}(\mathscr{K}(\tilde{Y}_{\alpha,i}))_t=&\,\tilde{Y}_{(\alpha,0,0),i,t}+\delta^{-1}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\bar{\mathscr{R}}V_i^{[(\alpha,0)]}(X^{\delta}_{t-\delta},t-\delta)\\&+\delta^{-1}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}(\bar{\mathscr{R}}V_i^{[\alpha]})^{[0]}(X^{\delta}_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}+\delta^{-2}\langle\xi,\mathring{X}_{\eta_2,(\delta)t-\delta}\bar{\mathscr{R}}\bar{\mathscr{R}}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d},\end{aligned}$$

and

$$\begin{aligned}\mathscr{M}(\mathscr{K}(\tilde{Y}_{\alpha,i}))_t=&\sum_{j\in\mathcal{N}}Z_t^{\delta,j}\tilde{Y}_{(\alpha,0,j),i,t}+\delta^{-\frac{1}{2}}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\tilde{\mathscr{R}}V_i^{[(\alpha,0)]}(X^{\delta}_{t-\delta},t-\delta,Z^{\delta}_t)\\&+\delta^{-1}Z_t^{\delta,j}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}(\bar{\mathscr{R}}V_i^{[\alpha]})^{[j]}(X^{\delta}_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}\\&+\delta^{-\frac{3}{2}}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\tilde{\mathscr{R}}\bar{\mathscr{R}}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta,Z^{\delta}_t)\rangle_{\mathbb{R}^d}.\end{aligned}$$

Applying (4.21) and (4.20), we obtain

$$\mathscr{N}_{\tilde{Y}_{\alpha,i},\mathcal{T}^-}(q(d,l,p))\leqslant C\,\mathbf{M}_C(Z^{\delta})\,\mathbf{K}^C_{2l+7}\,\mathbb{E}_x[\sup_{t\in\mathcal{T}}\|\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\|^{2q(d,l,p)}_{\mathbb{R}^d}]^{\frac{1}{2}}(1+\mathbb{E}_x[\sup_{t\in\mathcal{T}}|X^{\delta}_{t-\delta}|^C_{\mathbb{R}^d}]^{\frac{1}{2}}).$$

Using the Markov and Cauchy-Schwarz inequalities together with Lemma 4.5 gives also, for every $a>0$,

$$\begin{aligned}\mathbb{P}_x(\delta\sum_{t\in\mathcal{T}}\mathbb{E}_x[|\delta^{-\frac{1}{2}}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\tilde{\mathscr{R}}V_i^{[\alpha]}(X^\delta_{t-\delta},t-\delta,Z^\delta_t)\rangle_{\mathbb{R}^d}|^2]\\ +|\delta^{-1}\langle\xi,\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\bar{\mathscr{R}}V_i^{[\alpha]}(X^\delta_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}|^2>\frac{1}{4}N^{-l-1}N_{l+1}\epsilon^{13^{-l-1}})\\ \leqslant\delta^{\frac{a}{2}}\epsilon^{-a13^{-l-1}}T^a\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^C_{2l+4}\,\mathbb{E}_x[\sup_{t\in\mathcal{T}}\|\mathring{X}_{\boldsymbol{\eta}_2(\delta),t-\delta}\|^{2a}_{\mathbb{R}^d}]^{\frac{1}{2}}(1+\mathbb{E}_x[\sup_{t\in\mathcal{T}}|X^\delta_{t-\delta}|^{2aC}_{\mathbb{R}^d}]^{\frac{1}{2}}).\end{aligned}$$

In particular, we chose $a=\frac{d(p+4)\ln(\boldsymbol{\eta}_1(\delta))}{-13^{-l-1}\ln(\boldsymbol{\eta}_1(\delta))-\frac{d}{2}\ln(\delta)}$ so that $\delta^{\frac{a}{2}}\epsilon^{-a13^{-l-1}}\leqslant\epsilon^{d(p+4)}$ (remember that $\delta\leqslant\boldsymbol{\eta}_1(\delta)^{-\frac{2}{d}}$ so $a\in(0,2d(p+4)]$) and then apply Lemma 4.4 (see (4.19)) and Lemma 4.2 to conclude the proof of **Step 2.3** (when $4a<2$ we also use the Hölder inequality).

Step 2.4 To conclude the proof of **Step 2** we simply gather the estimates obtained in **Step 2.1**, **2.2** and **2.3**.

Step 3. We now focus on the proof of (4.26). In particular, we show that for every $\epsilon\in\mathbb{R}^*$,

$$\mathbb{P}_x(\|\tilde{\boldsymbol{\sigma}}_{X^\delta_t}\|_{\mathbb{R}^d}>\frac{1}{6\epsilon})\leqslant\epsilon^{d(p+2)}(1+\mathbf{1}_{\mathbf{p}_3>0}|x|^C_{\mathbb{R}^d})\,\mathbf{K}^C_3\exp(C(T+1)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^4_3).\tag{4.29}$$

First, we notice that, using Cauchy-Schwarz inequality, we have

$$\|\tilde{\boldsymbol{\sigma}}_{X^\delta_t}\|_{\mathbb{R}^d}\leqslant\|\sigma^\delta_{X^\delta_t,\mathcal{T}}\|_{\mathbb{R}^d}\|\mathring{X}_T\|^2_{\mathbb{R}^d}\leqslant|X^\delta_t|^2_{\mathbb{R}^d,1,1}\|\mathring{X}_T\|^2_{\mathbb{R}^d}.$$

As a consequence of the Markov inequality and again the Cauchy-Schwarz inequality, we obtain

$$\mathbb{P}_x(\|\tilde{\boldsymbol{\sigma}}_{X^\delta_t}\|_{\mathbb{R}^d}>\frac{1}{6\epsilon},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\leqslant C\epsilon^{d(p+2)}\mathbb{E}_x[\sup_{t\in\mathcal{T}}|X^\delta_{t-\delta}|^{4d(p+2)}_{\mathbb{R}^d}]^{\frac{1}{2}}\mathbb{E}_x[\sup_{t\in\mathcal{T}}\|\mathring{X}_t\|^{4d(p+2)}_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T},t}>0}]^{\frac{1}{2}}.$$

To conclude the proof of **Step 3**, we then apply Proposition 4.1 (see (4.10)) and Lemma 4.4 and obtain (4.29).

Step 4. In order to complete the proof of Theorem 3.3, it remains to show that (3.21) holds. Similarly as in **Step 1**, we have

$$\begin{aligned}\mathbb{P}_x(|\det\tilde{\boldsymbol{\gamma}}_{X^\delta_T}|\geqslant\frac{\boldsymbol{\eta}_1(\delta)}{2},\Theta^*_T>0)\leqslant&\mathbb{P}_x(|\det\tilde{\boldsymbol{\sigma}}_{X^\delta_T}|\leqslant2\,\boldsymbol{\eta}_1(\delta)^{-1},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\\ \leqslant&\mathbb{P}_x(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^{\mathrm{T}}\tilde{\boldsymbol{\sigma}}_{X^\delta_T}\xi\leqslant2^{\frac{1}{d}}\,\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0).\end{aligned}$$

Using the result from **Step 2**, (see (4.27) with $\epsilon=2^{\frac{1}{d}}\,\boldsymbol{\eta}_1(\delta)^{-\frac{1}{d}}$), for $p\geqslant4$, we have

$$\begin{aligned}\mathbb{P}_x(|\det\tilde{\boldsymbol{\gamma}}_{X^\delta_T}|\geqslant\frac{\boldsymbol{\eta}_1(\delta)}{2},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)\leqslant\boldsymbol{\eta}_1(\delta)^{-p}(1+\boldsymbol{\mathcal{V}}_L(x)^{-\frac{3}{2}13^Ldp})\\ \times(1+\mathbf{1}_{\mathbf{p}_{2L+5}>0}|x|^C_{\mathbb{R}^d})\,\mathbf{K}^C_{2L+5}\exp(C(1+T)\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^4_3).\end{aligned}$$

To conclude the proof, we simply observe that, since $\boldsymbol{\eta}_1(\delta),\boldsymbol{\eta}_2(\delta)>1$,

$$\begin{aligned}\mathbb{P}_x(\Theta^*_T<1)\leqslant&\mathbb{P}_x(|\det\tilde{\boldsymbol{\gamma}}_{X^\delta_T}|\geqslant\boldsymbol{\eta}_1(\delta)-\frac{1}{2},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)+\sum_{t\in\mathcal{T}}\mathbb{P}(|Z^\delta_t|_{\mathbb{R}^N}\geqslant\boldsymbol{\eta}_2(\delta)-\frac{1}{2})\\ \leqslant&\mathbb{P}_x(|\det\tilde{\boldsymbol{\gamma}}_{X^\delta_T}|\geqslant\frac{\boldsymbol{\eta}_1(\delta)}{2},\Theta_{\boldsymbol{\eta}_2(\delta),\mathcal{T}}>0)+\sum_{t\in\mathcal{T}}\mathbb{P}(|Z^\delta_t|_{\mathbb{R}^N}\geqslant\frac{\boldsymbol{\eta}_2(\delta)}{2}),\end{aligned}$$

and apply the Markov inequality.

□

Appendix A. Proof of technical lemmas

A.1. **Proof of Lemma 4.3.**

Proof. First notice that, since $\epsilon\in(0,\sqrt{\frac{2}{d}})$, there exists $\{\xi_1,\ldots,\xi_{N(\epsilon)}\}$ with $\xi_i\in\mathbb{R}^d$, $N(\epsilon)\leqslant7d^32^d\epsilon^{-2d}$ (see *e.g.* or [29] Theorem 2 or [34] Theorem 1.1 for a refined constant) such that $\{\xi\in\mathbb{R}^d,|\xi|_{\mathbb{R}^d}=1\}\subset$

$\cup_{i=1}^{N(\epsilon)}\{\xi\in\mathbb{R}^d, |\xi_i-\xi|_{\mathbb{R}^d}\leqslant\frac{\epsilon^2}{2}\}$. Moreover

$$\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^{\mathrm{T}}\Sigma\xi\leqslant\frac{1}{2}\epsilon)=\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^{\mathrm{T}}\Sigma\xi\leqslant\frac{1}{2}\epsilon,\|\Sigma\|_{\mathbb{R}^d}\leqslant\frac{1}{3\epsilon})+\mathbb{P}(\|\Sigma\|_{\mathbb{R}^d}>\frac{1}{3\epsilon}).$$

In particular for every $\xi\in\mathbb{R}^d, |\xi|_{\mathbb{R}^d}=1$,

$$\begin{aligned}\xi^{\mathrm{T}}\Sigma\xi=&\xi_i^{\mathrm{T}}\Sigma\xi_i+(\xi-\xi_i)^{\mathrm{T}}(\Sigma\xi_i+\Sigma^{\mathrm{T}}\xi)\\ \geqslant&\xi_i^{\mathrm{T}}\Sigma\xi_i-2|\xi_i-\xi|_{\mathbb{R}^d}\|\Sigma\|_{\mathbb{R}^d}-|\xi_i-\xi|^2_{\mathbb{R}^d}\|\Sigma\|_{\mathbb{R}^d}.\end{aligned}$$

Therefore

$$\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^{\mathrm{T}}\Sigma\xi\leqslant\frac{1}{2}\epsilon,\|\Sigma\|_{\mathbb{R}^d}\leqslant\frac{1}{3\epsilon})\leqslant\mathbb{P}(\cup_{i=1}^{N(\epsilon)}\xi_i^{\mathrm{T}}\Sigma\xi_i\leqslant\epsilon),$$

and the proof of (4.18) is completed taking $C(d)=7d^3 2^d$. □

A.2. **Proof of Lemma 4.4.** In this proof, we are going to use the Burkholder inequality (see (4.8)) on the Hilbert space $(\mathbb{R}^{d\times d},\langle,\rangle_{\mathbb{R}^{d\times d}})$, with the scalar product defined by $\langle M,M'\rangle_{\mathbb{R}^{d\times d}}:=\mathrm{Trace}(M'M^{\mathrm{T}})=\sum_{i=1}^d(M'M^{\mathrm{T}})_{i,i}$, $M,M'\in\mathbb{R}^{d\times d}$. Recall that for $M\in\mathbb{R}^{d\times d}$, $\|M\|_{\mathbb{R}^d}\leqslant|M|_{\mathbb{R}^{d\times d}}$.

Proof. **Step 1.** First we show that

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}}|\mathring{X}_t|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t}>0}]^{\frac{1}{p}}\leqslant d+\mathbb{E}_x[\sup_{t\in\mathcal{T}}|\sum_{w\in\pi^\delta\cap(0,t]}\hat{\Upsilon}_w|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}}+\mathbb{E}_x[\sup_{t\in\mathcal{T}}|\sum_{w\in\pi^\delta\cap(0,t]}\tilde{\Upsilon}_w|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}},$$

where we have introduced $\Upsilon_t=\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t}>0}\mathring{X}_{t-\delta}(I_{d\times d}-\nabla_x\psi^{-1})(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)$, $\hat{\Upsilon}_t=\mathbb{E}_x[\Upsilon_t|\mathcal{F}^{Z^\delta}_{t-\delta}]$ and $\tilde{\Upsilon}_t=\Upsilon_t-\hat{\Upsilon}_t$, $t\in\pi^{\delta,*}$. On the set $\{\Theta_{\eta_2,\mathcal{T},t}>0\}$, owing to (4.15), we have

$$\mathring{X}_t=I_{d\times d}-\sum_{w\in\pi^\delta\cap(0,t]}\mathring{X}_{w-\delta}(I_{d\times d}-\nabla_x\psi^{-1}(X^\delta_{w-\delta},w-\delta,\delta^{\frac{1}{2}}Z^\delta_w,\delta)).$$

Now, using the triangle inequality yields

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}}|\mathring{X}_t|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T}}>0}]^{\frac{1}{p}}\leqslant\sqrt{d}+\mathbb{E}[\sup_{t\in\mathcal{T}}|\sum_{w\in\pi^\delta\cap(0,t]}\Upsilon_w|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}},$$

and, using the triangle inequality once again, the proof of **Step 1** is completed.
Step 2. Let us show that, for $t\in\mathcal{T}$,

$$|\hat{\Upsilon}_t|_{\mathbb{R}^{d\times d}}\leqslant\delta|\mathring{X}_{t-\delta}|_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}C\,\mathbf{K}_3^2\,\mathbf{M}_{q^\delta_{\eta_2}\vee(2\,\mathbf{p}_3+2)}(Z^\delta).$$

We begin by noticing that, since $\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t}>0}=\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}$ (with $\mathcal{D}_{\eta_2}=\{z\in\mathbb{R}^N,|z|_{\mathbb{R}^N}\leqslant\delta^{\frac{1}{2}}\eta_2\}$ introduced in the proof of Theorem 3.3), for every $t\in\pi^{\delta,*}$,

$$|\hat{\Upsilon}_t|_{\mathbb{R}^{d\times d}}=|\mathring{X}_{t-\delta}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}\mathbb{E}[I_{d\times d}-\nabla_x\psi^{-1}(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]|_{\mathbb{R}^{d\times d}}.$$

Now, we remark that

$$\mathbb{E}_x[\delta^{\frac{1}{2}}\sum_{l\in\mathcal{N}}Z^{\delta,l}_t\nabla_xV_l(X^\delta_{t-\delta},t-\delta)(\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}+\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\notin\mathcal{D}_{\eta_2}})|\mathcal{F}^{Z^\delta}_{t-\delta}]=0.$$

Moreover, the Markov inequality, combined with (2.3) implies that,

$$\mathbb{E}_x[\delta^{\frac{1}{2}}|\sum_{l\in\mathcal{N}}Z^{\delta,l}_t\nabla_xV_l(X^\delta_{t-\delta},t-\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\notin\mathcal{D}_{\eta_2}}|_{\mathbb{R}^{d\times d}}\;|\mathcal{F}^{Z^\delta}_{t-\delta}]\leqslant\delta\,\mathbf{K}_3\,\mathbb{E}[|Z^\delta_t|^{q^\delta_{\eta_2}}_{\mathbb{R}^N}].$$

In particular, applying the Taylor expansion to $\nabla_x\psi$, we can show that

$$|\mathbb{E}_x[(I_{d\times d}-\nabla_x\psi)(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]|_{\mathbb{R}^{d\times d}}\leqslant\delta\,\mathbf{K}_3\,C\,\mathbf{M}_{q^\delta_{\eta_2}\vee(\mathbf{p}_3+2)}(Z^\delta).$$

On the other hand, using (4.13), for every $k\in\mathbb{N},k\geqslant2$, we have

$$\mathbb{E}_x[|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^k_{\mathbb{R}^{d\times d}}\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]\leqslant\delta^{\frac{k}{2}}\eta_2^{(k-2)(\mathbf{p}+1)}4^k\,\mathbf{K}_3^k\,2\,\mathbf{M}_{2(\mathbf{p}_3+1)}(Z^\delta).$$

Since $\delta^{\frac{1}{2}}\eta_2^{\mathbf{p}+1}4\,\mathbf{K}_3<\frac{1}{2}$, the following geometric series converge and satisfies

$$\mathbb{E}_x[\sum_{k=2}^\infty|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^k_{\mathbb{R}^{d\times d}}\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]\leqslant\delta C\,\mathbf{K}_3^2\,\mathbf{M}_{2(\mathbf{p}+1)}(Z^\delta).$$

In particular, using the Neumann series to write $\nabla_x\psi^{-1}=\sum_{k=0}^{\infty}(I_{d\times d}-\nabla_x\psi)^k$, we have

$$\mathbb{E}_x[|(\nabla_x\psi^{-1}-2I_{d\times d}+\nabla_x\psi)(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|_{\mathbb{R}^{d\times d}}\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]$$
$$\leqslant\mathbb{E}_x[\sum_{k=2}^{\infty}|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^k_{\mathbb{R}^{d\times d}}\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}],$$

so that

$$|\hat{\Upsilon}_t|_{\mathbb{R}^{d\times d}}\leqslant|\mathring{X}_{t-\delta}|_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}(|\mathbb{E}_x[(I_{d\times d}-\nabla_x\psi)(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]|_{\mathbb{R}^{d\times d}}$$
$$+\mathbb{E}_x[\sum_{k=2}^{\infty}|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^k_{\mathbb{R}^{d\times d}}\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]).$$

We already obtained an estimate of the second term of the *r.h.s.* above. To bound the other term, we compute $\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)$ using the decomposition (4.2) and by standard estimates we complete the proof of **Step 2**.

Step 3. Let us show that

$$\mathbb{E}_x[|\tilde{\Upsilon}_t|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}]^{\frac{1}{p}}\leqslant\delta^{\frac{1}{2}}\mathbb{E}_x[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}}C\,\mathbf{K}_3^2\,\mathbf{M}_{p(q^\delta_{\eta_2}\vee(2\,\mathbf{p}_3+2))}(Z^\delta)^{\frac{1}{p}}).$$

First, we remark that $|\tilde{\Upsilon}_t|_{\mathbb{R}^{d\times d}}\leqslant|\Upsilon_t|_{\mathbb{R}^{d\times d}}+|\hat{\Upsilon}_t|_{\mathbb{R}^{d\times d}}$. We have already studied the second term of the *r.h.s.* in **Step 2** so we focus on the first one. Proceeding similarly as in **Step 2**, we have

$$|\Upsilon_t|_{\mathbb{R}^{d\times d}}\leqslant|\mathring{X}_{t-\delta}|_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}(|(I_{d\times d}-\nabla_x\psi)(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}|_{\mathbb{R}^{d\times d}}$$
$$+\sum_{k=2}^{\infty}|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^k_{\mathbb{R}^{d\times d}}\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}).$$

Using (4.13), it follows that

$$\mathbb{E}_x[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}]$$
$$\leqslant\delta^{\frac{p}{2}}\mathbb{E}_x[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}]\,\mathbf{K}_3^p\,4^pC\,\mathbf{M}_{p(\mathbf{p}_3+1)}(Z^\delta).$$

Moreover, since $\delta^{\frac{1}{2}}\eta_2^{\mathbf{p}_3+1}4\,\mathbf{K}_3<\frac{1}{2}$, on the space $\{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}\}$, we have (4.13) and

$$\sum_{k=2}^{\infty}|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^k_{\mathbb{R}^{d\times d}}\leqslant\delta C\,\mathbf{K}_3^2(1\vee|Z^\delta_t|^{2(\mathbf{p}_3+1)}_{\mathbb{R}^N}),$$

and

$$\mathbb{E}_x[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}|\sum_{k=2}^{\infty}|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^k_{\mathbb{R}^{d\times d}}|^p\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}]$$
$$\leqslant\delta^p\mathbb{E}[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}]C\,\mathbf{K}_3^{2p}\,\mathbf{M}_{2p(\mathbf{p}_3+1)}(Z^\delta).$$

Gathering all the terms concludes the proof of **Step 3**.

Step 4. We are now in a position to conclude the proof. First, employing the Burkholder inequality (4.8), we have for every $p\geqslant 2$,

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}}|\sum_{w\in\pi^\delta\cap(0,t]}\tilde{\Upsilon}_t|^p_{\mathbb{R}^{d\times d}}]\leqslant\mathscr{B}_p\,\mathbb{E}_x[(\sum_{t\in\mathcal{T}}|\tilde{\Upsilon}_t|^2_{\mathbb{R}^{d\times d}})^{\frac{p}{2}}]\leqslant C(\sum_{t\in\mathcal{T}}\mathbb{E}_x[|\tilde{\Upsilon}_t|^p_{\mathbb{R}^{d\times d}}]^{\frac{2}{p}})^{\frac{p}{2}}.$$

We deduce from **Step 1,2,3** that

$$\mathbb{E}_x[\sup_{t\in\mathcal{T}\cup\{0\}}|\mathring{X}_t|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t}>0}]^{\frac{1}{p}}\leqslant d+\mathbb{E}_x[\sup_{t\in\mathcal{T}}|\sum_{w\in\pi^\delta\cap(0,t]}\hat{\Upsilon}_w|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}}+C(\sum_{t\in\mathcal{T}}\mathbb{E}_x[|\tilde{\Upsilon}_t|^p_{\mathbb{R}^{d\times d}}]^{\frac{2}{p}})^{\frac{1}{2}}$$
$$\leqslant d+C\,\mathbf{K}_3^2\,\mathbf{M}_{q^\delta_{\eta_2}\vee(2\,\mathbf{p}_3+2)}(Z^\delta)\mathbb{E}_x[|\sum_{t\in\mathcal{T}}\delta|\mathring{X}_{t-\delta}|_{\mathbb{R}^{d\times d}}|^p\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}]^{\frac{1}{p}}$$
$$+C\,\mathbf{K}_3^2\,\mathbf{M}_{p(q^\delta_{\eta_2}\vee(2\,\mathbf{p}_3+2))}(Z^\delta)^{\frac{1}{p}}(\sum_{t\in\mathcal{T}}\delta\mathbb{E}_x[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}]^{\frac{2}{p}})^{\frac{1}{2}}$$
$$\leqslant d+C\,\mathbf{K}_3^2\,\mathbf{M}_{p(q^\delta_{\eta_2}\vee(2\,\mathbf{p}_3+2))}(Z^\delta)^{\frac{1}{p}}(\sum_{t\in\mathcal{T}}\delta\mathbb{E}_x[\sup_{w\in\mathcal{T}\cup\{0\},w<t}|\mathring{X}_w|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathcal{T},t-\delta}>0}]^{\frac{2}{p}})^{\frac{1}{2}}.$$

Therefore, (4.19) follows from the Gronwall lemma.

□

A.3. **Proof of Lemma 4.5.**

Proof. In this proof we first derive the expansion formula and give an explicit value for $\mathbb{R}V$ (**Step 1,2,3**) and then establish the estimates on this remainder in **Step 4**.
Step 1. Let us show that for every $t \in \pi^{\delta,*}$,

$$\begin{aligned}V(X_t^\delta, t) - V(X_{t-\delta}^\delta, t-\delta) &= \delta^{\frac{1}{2}} \sum_{l=1}^N Z_t^{\delta,i} \nabla_x V(X_{t-\delta}^\delta, t-\delta) V_l(X_{t-\delta}^\delta, t-\delta) \\ &+ \delta \nabla_x V(X_{t-\delta}^\delta, t-\delta) V_0(X_{t-\delta}^\delta, t-\delta) + \delta \partial_t V(X_{t-\delta}^\delta, t-\delta) \\ &+ \delta \frac{1}{2} \sum_{l=1}^N V_l(X_{t-\delta}^\delta, t-\delta)^{\mathrm{T}} \nabla_x^2 V(X_{t-\delta}^\delta, t-\delta) V_l(X_{t-\delta}^\delta, t-\delta) \\ &+ R^{\delta,1}(X_{t-\delta}^\delta, t-\delta, Z_t^\delta),\end{aligned}$$

with for every $(x,t,z) \in \mathbb{R}^d \times \pi^\delta \times \mathbb{R}^N$,

$$\begin{aligned}R^{\delta,1}(x,t,z) =& \frac{1}{2}\delta \sum_{i,l\in\mathcal{N}} (z^i z^l - \mathbf{1}_{i=l}) V_i(x,t)^{\mathrm{T}} \nabla_x^2 V(x,t) V_l(x,t) \\ &+ (R^{\delta,1,1}(x,t,z) + 2\delta^{\frac{1}{2}} \sum_{l=1}^N z^l V_l(x,t))^{\mathrm{T}} \nabla_x^2 V(x,t) R^{\delta,1,1}(x,t,z) \\ &+ \nabla_x V(x,t) R^{\delta,1,2}(x,t,z) + R^{\delta,1,3}(x,t,z),\end{aligned}$$

with (remembering notation (4.1), $R^{\delta,1,1}(x,t,z) = \delta A_3(x,t,\delta^{\frac{1}{2}} z, \delta) + \delta \sum_{i,j\in\mathcal{N}} z^i z^j A_2^{i,j}(x,t,\delta^{\frac{1}{2}} z)$,

$$\begin{aligned}R^{\delta,1,2}(x,t,z) =& \delta \frac{1}{2} \sum_{i,j\in\mathcal{N}} (z^i z^j - \mathbf{1}_{i=j}) \partial_{z^i} \partial_{z^j} \psi(x,t,0,0) + \delta^2 \int_0^1 (1-\lambda) \partial_y^2 \psi(x,t,\delta^{\frac{1}{2}} z, \lambda\delta) \mathrm{d}\lambda \\ &+ \delta^{\frac{3}{2}} \sum_{i\in\mathcal{N}} z^i \int_0^1 \partial_{z^i} \partial_y \psi(x,t,\lambda\delta^{\frac{1}{2}} z, 0) \mathrm{d}\lambda \\ &+ \delta^{\frac{3}{2}} \frac{1}{2} \sum_{i,j,l\in\mathcal{N}} z^i z^j z^l \int_0^1 (1-\lambda)^2 \partial_{z^i} \partial_{z^j} \partial_{z^l} \psi(x,t,\lambda\delta^{\frac{1}{2}} z, 0) \mathrm{d}\lambda,\end{aligned}$$

and

$$\begin{aligned}R^{\delta,1,3}(x,t,z) =& \delta^2 \int_0^1 \partial_t^2 V(x, t+\lambda\delta) \mathrm{d}\lambda, \\ &+ \sum_{i=1}^d \int_0^1 \partial_{x^i} \mathscr{T} V(x + \lambda R^{\delta,1,0}(x,t,z), t) \mathrm{d}\lambda R^{\delta,1,0}(x,t,z)_i \\ &+ \frac{1}{2} \sum_{i,j,k=1}^d R^{\delta,1,0}(x,t,z)_{i,j,k}^{\otimes 3} \int_0^1 (1-\lambda)^2 \partial_x^{(i,j,k)} V(x + \lambda R^{\delta,1,0}(x,t,z), t) \mathrm{d}\lambda,\end{aligned}$$

with $R^{\delta,1,0}(x,t,z) = \delta A_3(x,t,\delta^{\frac{1}{2}} z, \delta) + \delta^{\frac{1}{2}} \sum_{i\in\mathcal{N}} z^i \int_0^1 \partial_{z^i} \psi(x,t,\lambda\delta^{\frac{1}{2}} z, 0) \mathrm{d}\lambda$ and

$$\mathscr{T} V(x,t) := \delta \int_0^1 \partial_t V(x, t+\lambda\delta) \mathrm{d}\lambda = \delta \partial_t V(x,t) + \delta^2 \int_0^1 \partial_t^2 V(x, t+\lambda\delta) \mathrm{d}\lambda.$$

We begin by noticing that, using the Taylor expansion of ψ *w.r.t.* its fourth and then third variables, we have

(A.1)

$$\begin{aligned}\psi(X_{t-\delta}^\delta, t-\delta, \delta^{\frac{1}{2}} Z_t^\delta, \delta) =& X_{t-\delta}^\delta + R^{\delta,1,0}(X_{t-\delta}^\delta, t-\delta, Z_t^\delta) \\ =& X_{t-\delta}^\delta + \delta^{\frac{1}{2}} \sum_{l\in\mathcal{N}} Z_t^{\delta,l} V_l(X_{t-\delta}^\delta, t-\delta) + R^{\delta,1,1}(X_{t-\delta}^\delta, t-\delta, Z_t^\delta) \\ =& X_{t-\delta}^\delta + \delta^{\frac{1}{2}} \sum_{l\in\mathcal{N}} Z_t^{\delta,l} V_l(X_{t-\delta}^\delta, t-\delta) + \delta V_0(X_{t-\delta}^\delta, t-\delta) + R^{\delta,1,2}(X_{t-\delta}^\delta, Z_t^\delta).\end{aligned}$$

Now, using again the Taylor expansion on the function V *w.r.t.* its second variable,

$$V(X_t^\delta, t) - V(X_{t-\delta}^\delta, t-\delta) = (\mathscr{T} V + V)(X_t^\delta, t-\delta) - V(X_{t-\delta}^\delta, t-\delta).$$

The Taylor expansion on the function $\mathscr{T} V$ *w.r.t* its first variable combined with (A.1) yields

$$\mathscr{T} V(X_t^\delta, t-\delta) = \mathscr{T} V(X_{t-\delta}^\delta, t-\delta) + \sum_{i=1}^d R^{\delta,1,0}(X_t^\delta, t-\delta, Z_t^\delta)_i \int_0^1 \partial_{x^i} \mathscr{T} V(X_t^\delta + \lambda R^{\delta,1,0}(X_t^\delta, t-\delta, Z_t^\delta), t)\mathrm{d}\lambda.$$

To conclude the proof of **Step 1.** it remains to study $V(X_t^\delta, t-\delta) - V(X_{t-\delta}^\delta, t-\delta)$. To do so, we employ the third order Taylor expansion of V *w.r.t.* its first variable together with the representation formulas given in (A.1) and gathering all the temrs together completes the proof.

Step 2. Let us show that for every $t \in \pi^{\delta,*}$, on the set $\{\delta^{\frac{1}{2}} Z_t^\delta \in \mathcal{D}_{\eta_2}\}$ (with $\mathcal{D}_{\eta_2} = \{z \in \mathbb{R}^N, |z| \leqslant \delta^{\frac{1}{2}} \eta_2\}$ introduced in the proof of Theorem 3.3), we have

$$\nabla_x \psi^{-1}(X_{t-\delta}^\delta, t-\delta, \delta^{\frac{1}{2}} Z_t^\delta, \delta) = I_{d\times d} - \delta^{\frac{1}{2}} \sum_{i=1}^N Z_t^{\delta,i} \nabla_x V_l(X_{t-\delta}^\delta, t-\delta) - \delta \nabla_x V_0(X_{t-\delta}^\delta, t-\delta) + \delta \sum_{i=1}^N \nabla_x V_i(X_{t-\delta}^\delta, t-\delta)^2 + \hat{R}^{\delta,2}(X_{t-\delta}^\delta, t-\delta, Z_t^\delta),$$

with, for every $(x, t, z) \in \mathbb{R}^d \times \pi^\delta \times \mathbb{R}^N$,

$$\hat{R}^{\delta,2}(x,t,z) = \hat{R}^{\delta,2,3}(x,t,z) - \hat{R}^{\delta,2,2}(x,t,z) + \delta \sum_{i,l\in\mathcal{N}} (z^i z^l - \mathbf{1}_{i=l}) \nabla_x V_i(x,t) \nabla_x V_l(x,t) - \delta^{\frac{1}{2}} \sum_{l\in\mathcal{N}} z^l (\nabla_x V_l(x,t) R^{\delta,2,1}(x,t,z) + R^{\delta,2,1}(x,t,z) \nabla_x V_l(x,t)) - \hat{R}^{\delta,2,1}(x,t,z)^2,$$

where $\hat{R}^{\delta,2,1}(x,t,z) = \nabla_x \hat{R}^{\delta,1,1}(x,t,z)$, $\hat{R}^{\delta,2,2}(x,t,z) = \nabla_x \hat{R}^{\delta,1,2}(x,t,z)$ and $\hat{R}^{\delta,2,3}(x,t,z) = (\nabla_x \psi^{-1} - I_{d\times d} - (I_{d\times d} - \nabla_x \psi) - (I_{d\times d} - \nabla_x \psi)^2)(x, t, \delta^{\frac{1}{2}} z, \delta)$ where for a matrix $M \in \mathbb{R}^{d\times d}$, $M^2 = MM$. The proof simply boils down to notice that we have the identites

$$\begin{aligned}\nabla_x \psi(X_{t-\delta}^\delta, t-\delta, \delta^{\frac{1}{2}} Z_t^\delta, \delta) &= I_{d\times d} + \delta^{\frac{1}{2}} \sum_{l\in\mathcal{N}} Z_t^{\delta,l} \nabla_x V_l(X_{t-\delta}^\delta, t-\delta) + \hat{R}^{\delta,2,1}(X_{t-\delta}^\delta, t-\delta, Z_t^\delta),\\ &= I_{d\times d} + \delta^{\frac{1}{2}} \sum_{l\in\mathcal{N}} Z_t^{\delta,l} \nabla_x V_l(X_{t-\delta}^\delta, t-\delta) + \delta \nabla_x V_0(X_{t-\delta}^\delta, t-\delta) + \hat{R}^{\delta,2,2}(X_{t-\delta}^\delta, t-\delta, Z_t^\delta),\end{aligned}$$

and to rearrange the terms together.

Step 3. Let us show that for every $t \in \pi^{\delta,*}$, on the set $\{\Theta_{\eta_2,\mathcal{T},t} > 0\}$, we have

$$\mathring{X}_t V(X_t^\delta, t) = \mathring{X}_{t-\delta} V(X_{t-\delta}^\delta, t-\delta) + \delta^{\frac{1}{2}} \sum_{i\in\mathcal{N}} Z_t^{\delta,i} \mathring{X}_{t-\delta} V^{[i]}(X_{t-\delta}^\delta, t-\delta) + \delta \mathring{X}_{t-\delta} V^{[0]}(X_{t-\delta}^\delta, t-\delta)) + \mathring{X}_{t-\delta} R^\delta V(X_{t-\delta}^\delta, t-\delta, Z_t^\delta),$$

with, for every $(x,t,z) \in \mathbb{R}^d \times \pi^\delta \times \mathbb{R}^N$, $R^\delta V(x,t,z) = R^{\delta,1}(x,t,z) + R^{\delta,2}(x,t,z) + R^{\delta,3}(x,t,z)$, where $R^{\delta,2}(x,t,z) = \hat{R}^{\delta,2}(x,t,z) V(x,t)$ and

$$\begin{aligned}R^{\delta,3}(x,t,z) = &-\delta \sum_{i,\in\mathcal{N}} (z^i z^l - \mathbf{1}_{i=l}) \nabla_x V_i(x,t) \nabla_x V(x,t) V_l(x,t)\\ &+ (-\delta \nabla_x V_0(x,t) + \delta \sum_{l\in\mathcal{N}} \nabla_x V_l(x,t)^2 + \hat{R}^{\delta,2}(x,t,z))(\delta^{\frac{1}{2}} \sum_{i\in\mathcal{N}} z^i \nabla_x V(x,t) V_l(x,t)\\ &+ \delta \nabla_x V(x,t) V_0(x,t) + \delta \partial_t V(x,t) + \delta \frac{1}{2} \sum_{i\in\mathcal{N}} V_i(x,t)^{\mathrm{T}} \nabla_x^2 V(x,t) V_i(x,t) + R^{\delta,1}(x,t,z))\\ &- \delta^{\frac{1}{2}} \sum_{i\mathcal{N}} z^i \nabla_x V_i(x,t) (\delta \nabla_x V(x,t) V_0(x,t) + \delta \partial_t V(x,t)\\ &+ \delta \frac{1}{2} \sum_{i\in\mathcal{N}} V_i(x,t)^{\mathrm{T}} \nabla_x^2 V(x,t) V_i(x,t) + R^{\delta,1}(x,t,z)).\end{aligned}$$

First, owing to (4.15), we write the following decomposition

$$\mathring{X}_t V(X^\delta_t,t) - \mathring{X}_{t-\delta} V(X^\delta_{t-\delta},t-\delta) = \mathring{X}_{t-\delta}\nabla_x\psi^{-1}(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)\left(V(X^\delta_t,t)-V(X^\delta_{t-\delta},t-\delta)\right)$$
$$+\mathring{X}_{t-\delta}\left(\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)^{-1}-I_{d\times d}\right)V(X^\delta_{t-\delta},t-\delta).$$

Then, using **Step 1** and **Step 2**, a direct computation yields

$$\begin{aligned}\nabla_x\psi^{-1}&(X^\delta_{t-\delta},t-\delta,Z^\delta_t,\delta)(V(X^\delta_t,t)-V(X^\delta_{t-\delta},t-\delta))\\ =&\delta^{\frac{1}{2}}\sum_{l\in\mathcal{N}}Z^{\delta,l}_t\nabla_x V(X^\delta_{t-\delta},t-\delta)V_l(X^\delta_{t-\delta},t-\delta)\\ &+\delta\nabla_x V(X^\delta_{t-\delta},t-\delta)V_0(X^\delta_{t-\delta},t-\delta)+\delta\partial_t V(X^\delta_{t-\delta},t-\delta)\\ &+\delta\frac{1}{2}\sum_{l\in\mathcal{N}}V_l(X^\delta_{t-\delta},t-\delta)^{\mathrm{T}}\nabla^2_x V(X^\delta_{t-\delta},t-\delta)V_l(X^\delta_{t-\delta},t-\delta)\\ &-\delta\sum_{l\mathcal{N}}\nabla_x V_l(X^\delta_{t-\delta},t-\delta)\nabla_x V(X^\delta_{t-\delta},t-\delta)V_l(X^\delta_{t-\delta},t-\delta)\\ &+R^{\delta,3}(X^\delta_{t-\delta},t-\delta,Z^\delta_t)+R^{\delta,1}(X^\delta_{t-\delta},t-\delta,Z^\delta_t).\end{aligned}$$

The study of the other term was done in **Step 2** and the proof of **Step 3** is completed.

Step 4. Let us prove (4.21) and (4.20) begining with the proof of (4.21). For $i\in\{1,2,3\}$, $t\in\pi^{\delta,*}$, we introduce the functions defined for every $x\in\mathbb{R}^d$ by $\bar{R}^i_t(x)=\mathbb{E}[R^{\delta,i}(x,t-\delta,Z^\delta_t)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}]$ and for $i\in\{1,2\},j\in\{1,2,3\}$, $\bar{R}^{i,j}_t(x)=\mathbb{E}[R^{\delta,i,j}(x,t-\delta,Z^\delta_t)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}]$ (with the notation $R^{\delta,2,j}=\hat{R}^{\delta,2,j}V$). In particular, since $\{\Theta_{\eta_2,\mathcal{T},t}>0\}=\{\Theta_{\eta_2,\mathcal{T},t-\delta}>0\}\cap\{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}\}$, then $\mathscr{R}V(x,t-\delta)=\sum_{i=1}^5\bar{R}^i_t(x)=\mathbb{E}[R^\delta(x,t-\delta,Z^\delta_t)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}]+\bar{R}^4_t(x)+\bar{R}^5_t(x)$ with

$$\begin{aligned}\bar{R}^4_t(x)=&-\delta^{\frac{1}{2}}\sum_{i=1}^N V^{[i]}(x,t-\delta)\mathbb{E}[Z^{\delta,i}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\notin\mathcal{D}_{\eta_2}}],\\ \bar{R}^5_t(x)=&-\delta V^{[0]}(x,t-\delta)\mathbb{P}(\delta^{\frac{1}{2}}Z^\delta_t\notin\mathcal{D}_{\eta_2}).\end{aligned}$$

We first study $\partial^{\alpha^x}_x\bar{R}^1_t$ for $\alpha^x\in\mathbb{N}^d$. We first remark that $\bar{R}^1_t=\bar{R}^{1,3}_t+\nabla_x V(\cdot,t-\delta)\bar{R}^{1,2}_t+\bar{R}^{1,4}_t$, with, for every $x\in\mathbb{R}^d$ and $t\in\pi^{\delta,*}$,

$$\begin{aligned}\bar{R}^{1,4}_t(x)=&\mathbb{E}[(2\delta^{\frac{1}{2}}\sum_{l=1}^N Z^{\delta,l}_t V_l(x,t-\delta)+R^{\delta,1,1}(x,t-\delta,Z^\delta_t))^{\mathrm{T}}\\ &\times\nabla^2_x V(x,t-\delta)R^{\delta,1,1}(x,t-\delta,Z^\delta_t)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}]\end{aligned}$$

We begin with the study of $\bar{R}^{1,2}_t(x)$. We observe that, for every $t\in\pi^{\delta,*}$,

$$\begin{aligned}&\sum_{i,l\in\mathcal{N}}(Z^{\delta,i}_t Z^{\delta,l}_t-\mathbf{1}_{i=l})\partial_{z^i}\partial_{z^l}\psi(x,t-\delta,0,0)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}\\ &=\sum_{i,l\in\mathcal{N}}(Z^{\delta,i}_t Z^{\delta,l}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}-\mathbb{E}[Z^{\delta,i}_t Z^{\delta,l}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}]+\mathbb{E}[Z^{\delta,i}_t Z^{\delta,l}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathcal{D}_{\eta_2}}])\partial_{z^i}\partial_{z^l}\psi(x,t,0,0)\end{aligned}$$

with $|\sum_{i,l\in\mathcal{N}}\mathbb{E}[Z^{\delta,i}_t Z^{\delta,l}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\notin\mathcal{D}_{\eta_2}}]|\leqslant\eta_2^{-a}\mathbb{E}[|Z^\delta_t|^{2+a}_{\mathbb{R}^N}]$, for every $a>0$. In particular we take $a=\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil$. Combining this estimate with standard calculus together with hypothesis $\mathbf{A}_1(|\alpha^x|+3)$ (see (2.2)) and $\mathbf{A}^\delta_3(+\infty)$ (see (2.7)), we obtain, for every $x\in\mathbb{R}^d$,

$$|\partial^{\alpha^x}_x\bar{R}^{1,2}_t(x)|_{\mathbb{R}^d}\leqslant\delta^{\frac{3}{2}}C\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_{|\alpha^x|+3}(1+|x|^C_{\mathbb{R}^d}).$$

By similar arguments, it follows from $\mathbf{A}_1(|\alpha^x|+2)$ (see (2.2)), that

$$|\partial^{\alpha^x}_x\bar{R}^{1,3}_t(x)|_{\mathbb{R}^d}+|\partial^{\alpha^x}_x\bar{R}^{1,4}_t(x)|_{\mathbb{R}^d}\leqslant\delta^{\frac{3}{2}}C\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^C_{|\alpha^x|+2}\,\mathbf{K}_{V,|\alpha^x|+3}(1+|x|^C_{\mathbb{R}}).$$

We conclude that, under the assumptions $\mathbf{A}_1(|\alpha^x|+3)$ and $\mathbf{A}^\delta_3(+\infty)$, then, for every $x\in\mathbb{R}^d$,

$$|\partial^{\alpha^x}_x\bar{R}^1_t(x)|_{\mathbb{R}^d}\leqslant\delta^{\frac{3}{2}}C\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}^C_{|\alpha^x|+3}\,\mathbf{K}_{V,|\alpha^x|+3}(1+|x|^C_{\mathbb{R}^d}).$$

Now, we focus on the study of $\partial_x^{\alpha^x}\bar{R}_t^2$. We first observe that $\bar{R}_t^2=(\bar{R}_t^{2,3}-\bar{R}_t^{2,2}-\bar{R}_t^{2,4})V(\cdot,t-\delta)$, where we have introduced the function $\bar{R}_t^{2,4}$ defined for every $x\in\mathbb{R}^d$ by

$$\begin{aligned}\bar{R}_t^{2,4}(x)=&\mathbb{E}[\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\in\mathcal{D}_{\eta_2}}(\hat{R}^{\delta,2,1}(x,t-\delta,Z_t^\delta)^2\\&+\delta^{\frac{1}{2}}\sum_{l\in\mathcal{N}}Z_t^{\delta,l}(\nabla_xV_l(x,t-\delta)\hat{R}^{\delta,2,1}(x,t-\delta,Z_t^\delta)+\hat{R}^{\delta,2,1}(x,t-\delta,Z_t^\delta)\nabla_xV_l(x,t-\delta)))],\end{aligned}$$

We begin with the study of $\bar{R}_t^{2,2}$. Remembering that $\bar{R}_t^{2,2}=\nabla_x\bar{R}_t^{1,2}V(\cdot,t-\delta)$ and using similar arguments as in the study of $\bar{R}_t^{1,2}$, under the assumptions $\mathbf{A}_1(|\alpha^x|+4)$ (see (2.2)) and $\mathbf{A}_3^\delta(+\infty)$ (see (2.7)), then, for every $x\in\mathbb{R}^d$,

$$|\partial_x^{\alpha^x}\bar{R}_t^{2,2}(x)|_{\mathbb{R}^d}\leqslant\delta^{\frac{3}{2}}C\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_{|\alpha^x|+4}\,\mathbf{K}_{V,|\alpha^x|}(1+|x|_{\mathbb{R}}^C).$$

We then bound the derivatives of $\bar{R}_t^{2,3}$. We first remark that $\hat{R}^{\delta,2,3}(x,t,z)=\sum_{k=3}^\infty(I_{d\times d}-\nabla_x\psi(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta))^k$, where for a matrix $M\in\mathbb{R}^{d\times d}$, $M^{k+1}=MM^k$, $k\in\mathbb{N}$. If $|\alpha^x|=1$, it follows from standard calculus that

$$\begin{aligned}\partial_x^{\alpha^x}\bar{R}_t^{2,3}(x)=&\mathbb{E}[\sum_{k=3}^\infty(I_{d\times d}-\nabla_x\psi(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta))^k]\partial_x^{\alpha^x}V(x,t-\delta)\\&-\mathbb{E}[\sum_{k=3}^\infty\sum_{l=1}^k((I_{d\times d}-\nabla_x\psi)^{l-1}\partial_x^{\alpha^x}\nabla_x\psi(I_{d\times d}-\nabla_x\psi)^{k-l})(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta)]V(x,t-\delta).\end{aligned}$$

When $|\alpha^x|>1$, we iterate the above formula. Combining this decomposition with the following estimate

$$\begin{aligned}|\partial_x^{\alpha^x}\nabla_x\psi(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta)|_{\mathbb{R}^{d\times d}}=&|\delta^{\frac{1}{2}}\sum_{l\in\mathcal{N}}Z_t^{\delta,l}\partial_x^{\alpha^x}\nabla_xV_l(x,t-\delta)+\partial_x^{\alpha^x}\hat{R}^{\delta,2,1}(x,t-\delta,Z_t^\delta)|_{\mathbb{R}^{d\times d}}\\\leqslant&2\delta^{\frac{1}{2}}\,\mathbf{K}_{|\alpha^x|+3}(1+|x|_{\mathbb{R}^d}^{\mathbf{p}_{|\alpha^x|+3}}+|Z_t^\delta|_{\mathbb{R}^N}^{\mathbf{p}_{|\alpha^x|+3}})(1+|Z_t^\delta|_{\mathbb{R}^N}^2),\end{aligned}$$

we deduce that

$$\begin{aligned}|\partial_x^{\alpha^x}\bar{R}_t^{2,3}(x)|_{\mathbb{R}^d}\leqslant&C\,\mathbf{K}_{|\alpha^x|+3}^C\,\mathbf{K}_{V,|\alpha^x|}\,\mathbb{E}[(1+|x|_{\mathbb{R}^d}^C+|Z_t^\delta|_{\mathbb{R}^N}^C)\\&\times\sum_{k=0}^\infty\delta^{\frac{\max(3-k,0)}{2}}(k+1)^{|\alpha^x|}|I_{d\times d}-\nabla_x\psi|_{\mathbb{R}^{d\times d}}^k(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\in\mathcal{D}_{\eta_2}}].\end{aligned}$$

Using $\mathbf{A}_1^\delta$ (see (2.3)), we have (4.13). Moreover, when $k\geqslant3$, we use $|Z_t^\delta|_{\mathbb{R}^N}^k\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\in\mathcal{D}_{\eta_2}}\leqslant|Z_t^\delta|_{\mathbb{R}^N}^3\eta_2^{k-3}$ and we obtain (where the constant C in the first line below is the same as in the previous estimate)

$$\begin{aligned}&\mathbb{E}[\delta^{\frac{\max(3-k,0)}{2}}(k+1)^{|\alpha^x|}(1+|x|_{\mathbb{R}^d}^C+|Z_t^\delta|_{\mathbb{R}^N}^C)|I_{d\times d}-\nabla_x\psi|_{\mathbb{R}^{d\times d}}^k(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\in\mathcal{D}_{\eta_2}}]\\&\leqslant\delta^{\frac{\max(k,3)}{2}}\eta_2^{\max(k-3,0)(\mathbf{p}+1)}(k+1)^{|\alpha^x|}4^k\,\mathbf{K}_3^k\,\mathbf{M}_C(Z_t^\delta)(1+|x|_{\mathbb{R}^d}^C).\end{aligned}$$

Since $\delta^{\frac{1}{2}}\eta_2^{\mathbf{p}_3+1}4\,\mathbf{K}_3<\frac{1}{2}$ (see (4.14)), we also obtain the estimate

$$|\partial_x^{\alpha^x}\bar{R}_t^{2,3}(x)|_{\mathbb{R}^d}+|\partial_x^{\alpha^x}\bar{R}_t^{2,4}(x)|_{\mathbb{R}^d}\leqslant\delta^{\frac{3}{2}}C\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_{|\alpha^x|+3}^C\,\mathbf{K}_{V,|\alpha^x|}(1+|x|_{\mathbb{R}^d}^C).$$

We conclude that, under the assumptions $\mathbf{A}_1^\delta(|\alpha^x|+4)$ and (2.3)), $\mathbf{A}_3^\delta(+\infty)$, and $\delta^{\frac{1}{2}}\eta_2^{\mathbf{p}_3+1}4\,\mathbf{K}_3<\frac{1}{2}$, then, for every $x\in\mathbb{R}^d$,

$$|\partial_x^{\alpha^x}\bar{R}_t^2(x)|_{\mathbb{R}^d}\leqslant\delta^{\frac{3}{2}}C\,\mathbf{M}_C(Z^\delta)\,\mathbf{K}_{|\alpha^x|+4}^C\,\mathbf{K}_{V,|\alpha^x|}(1+|x|_{\mathbb{R}^d}^C).$$

We now focus on the study of $\bar{R}_t^3$. We have

$$\bar{R}_t^3=\bar{R}_t^{3,1}-\bar{R}_t^{3,2}+\bar{R}_t^{3,3}-\bar{R}_t^{3,4},$$

where we have introduced

$$\begin{aligned}
\bar{R}_t^{3,1}(x) =& \delta^{\frac{1}{2}} \sum_{l\in\mathcal{N}} \mathbb{E}[Z_t^{\delta,l} \hat{R}^{\delta,2}(x,t-\delta,Z_t^\delta)\nabla_x V(x,t-\delta) V_l(x,t-\delta)\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathcal{D}_{\eta_2}}]\\
\bar{R}_t^{3,2}(x) =& \delta^{\frac{1}{2}} \sum_{l\in\mathcal{N}} \mathbb{E}[Z_t^{\delta,l} \nabla_x V_l(x,t-\delta) R^{\delta,1}(x,t-\delta,Z_t^\delta))\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathcal{D}_{\eta_2}}]\\
\bar{R}_t^{3,3}(x) =& \mathbb{E}[\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathcal{D}_{\eta_2}} \hat{R}^{\delta,2}(x,t-\delta,Z_t^\delta)\\
&\times (\delta \nabla_x V(x,t-\delta) V_0(x,t-\delta) + \delta \partial_t V(x,t-\delta)\\
&+ \delta \frac{1}{2} \sum_{l\in\mathcal{N}} V_l(x,t-\delta)^{\mathrm{T}} \nabla_x^2 V(x,t-\delta) V_l(x,t-\delta) + R^{\delta,1}(x,t-\delta,Z_t^\delta))]\\
\bar{R}_t^{3,4}(x) =& \delta^2 (\nabla_x V_0(x,t-\delta) - \sum_{l=1}^N \nabla_x V_l(x,t-\delta)^2)\\
&\times (\nabla_x V(x,t-\delta) V_0(x,t-\delta) + \partial_t V(x,t-\delta)\\
&+ \frac{1}{2} \sum_{l=1}^N V_l(x,t-\delta)^{\mathrm{T}} \nabla_x^2 V(x,t-\delta) V_l(x,t-\delta)).
\end{aligned}$$

Using a similar approach as in the study of $\bar{R}_t^1$ and $\bar{R}_t^2$, we also deduce that, under the assumptions (4.14), it follows from $\mathbf{A}_1^\delta(|\alpha^x|+4)$ (see (2.2) and (2.3)) and $\mathbf{A}_3^\delta(+\infty)$ (see (2.7)) that

$$|\partial_x^{\alpha^x} \bar{R}_t^3(x)|_{\mathbb{R}^d} \leqslant \delta^{\frac{3}{2}} C\, \mathbf{M}_C(Z^\delta)\, \mathbf{K}_{|\alpha^x|+4}^C\, \mathbf{K}_{V,|\alpha^x|+3}^C (1+|x|_{\mathbb{R}^d}^C).$$

To complete the proof, it remains to study $\bar{R}_t^4$ and $\bar{R}_t^5$. As a direct consequence of the Markov inequality, we have

$$\mathbb{E}[\sum_{i\in\mathcal{N}} Z_t^{\delta,i} \mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \notin \mathcal{D}_{\eta_2}}] \leqslant \delta^{\frac{3}{2}}\, \mathbf{M}_C(Z^\delta) \quad \text{and} \quad \mathbb{P}(\delta^{\frac{1}{2}} Z_t^\delta \notin \mathcal{D}_{\eta_2}) \leqslant \delta^{\frac{3}{2}}\, \mathbf{M}_{\lceil - \frac{C}{\rceil}}(Z^\delta).$$

Consequently

$$|\partial_x^{\alpha^x} \bar{R}_t^4(x)|_{\mathbb{R}^d} \leqslant \delta^{\frac{3}{2}} C\, \mathbf{M}_C(Z^\delta)\, \mathbf{K}_{|\alpha^x|+2}\, \mathbf{K}_{V,|\alpha^x|+1} (1+|x|_{\mathbb{R}}^{\mathbf{p}_{|\alpha^x|+2}+\mathbf{p}_{V,|\alpha^x|+1}}),$$

and

$$|\partial_x^{\alpha^x} \bar{R}_t^5(x)|_{\mathbb{R}^d} \leqslant \delta^{\frac{3}{2}} C\, \mathbf{M}_C(Z^\delta)\, \mathbf{K}_{|\alpha^x|+3}^2\, \mathbf{K}_{V,|\alpha^x|+2} (1+|x|_{\mathbb{R}}^{2\,\mathbf{p}_{|\alpha^x|+3}+\mathbf{p}_{V,|\alpha^x|+2}}).$$

and this complete the proof of (4.21). To conclude, it remains to study $\tilde{\mathscr{R}}V$. We first observe that $\tilde{\mathscr{R}}V(x,t-\delta) = (R^\delta(x,t-\delta,Z_t^\delta)\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathcal{D}_{\eta_2}} - \mathbb{E}[R^\delta(x,t-\delta,Z_t^\delta)\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathcal{D}_{\eta_2}}]) + \tilde{R}_t^4(x) + \tilde{R}_t^5(x)$, with

$$\begin{aligned}
\tilde{R}_t^4(x,z) =& -\delta^{\frac{1}{2}} \sum_{i\in\mathcal{N}} V^{[i]}(x,t-\delta)(z^i \mathbf{1}_{\delta^{\frac{1}{2}} z \notin \mathcal{D}_{\eta_2}} - \mathbb{E}[Z_t^{\delta,i} \mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \notin \mathcal{D}_{\eta_2}}]),\\
\tilde{R}_t^5(x,z) =& -\delta V^{[0]}(x,t-\delta)(\mathbf{1}_{\delta^{\frac{1}{2}} z \notin \mathcal{D}_{\eta_2}} - \mathbb{P}(\delta^{\frac{1}{2}} Z_t^\delta \notin \mathcal{D}_{\eta_2})).
\end{aligned}$$

Using $\mathbf{A}_1^\delta(2)$ (see (2.2)) and $\mathbf{A}_3^\delta(\tilde{q}_{\eta_2}^\delta)$ (see (2.7)), we obtain

$$\begin{aligned}
|\tilde{R}_t^4(x,z)|_{\mathbb{R}^d} =& |\delta^{\frac{1}{2}} \sum_{i\in\mathcal{N}} V^{[i]}(x,t-\delta)(z^i \mathbf{1}_{\delta^{\frac{1}{2}} z \notin \mathcal{D}_{\eta_2}} - \mathbb{E}[Z_t^{\delta,i} \mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \notin \mathcal{D}_{\eta_2}}])|_{\mathbb{R}^d}\\
\leqslant& \delta C\, \mathbf{K}_2\, \mathbf{K}_{V,1}\, \mathbf{M}_C(Z^\delta)(1+|x|_{\mathbb{R}^d}^{2(\mathbf{p}_2+\mathbf{p}_{V,1})} + |z|_{\mathbb{R}^N}^C),
\end{aligned}$$

and using $\mathbf{A}_1^\delta(3)$ (see (2.2)), we get $|\tilde{R}_t^5(x,z)|_{\mathbb{R}^d} \leqslant \delta C\, \mathbf{K}_3^2\, \mathbf{K}_{V,2}(1+|x|_{\mathbb{R}^d}^{2\,\mathbf{p}_3+\mathbf{p}_{V,2}})$. We treat the other terms by a similar but simpler (since it does not involves derivatives) method used to study $\bar{\mathscr{R}}$ and complete the proof of (4.20). □

A.4. **Proof of Lemma 4.6.**

Proof. The strategy of this proof is to derive localized estimates on our probability together with (small) estimates of the probability that the localizing spaces are not reached. **Step 1.** To begin we show that for every $\epsilon \in [\underline{\epsilon}_1(\delta), \bar{\epsilon}_1(\delta)]$, every $s \in (3r, \frac{1}{2})$, $u \in (0, \frac{1}{2} - s)$, every $p, v, v' > 0$, and every $q \geqslant 4$,

$$\begin{aligned}
&\mathbb{P}(\delta \sum_{t \in \mathcal{T}} |Y_t|^2 < \epsilon, \delta \sum_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] + |\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \epsilon^r, \mathcal{A}^Y_{u,q}) \\
&\leqslant \epsilon^p \mathbb{E}[|Y_0|^{\frac{p}{v}}] + \mathbb{P}(\delta |Y_0|^2 \geqslant \epsilon) \\
&\quad + \delta^{\frac{q}{4}} (\delta^{\frac{q}{4}} \epsilon^{-q(s+2u)} + \epsilon^{-q(s+u)} + \epsilon^{-q\frac{(2+v')}{4}}) 2^{5q} (1 \vee T^{2q})(1 + \sup_{t \in \mathcal{T}} \mathbb{E}[|Y_{t-\delta}|^q]) \\
&\quad + 2\exp(-\frac{\epsilon^{-4s}}{16}) + 2\exp(-2\epsilon^{-v'}) + 2\exp(-\frac{\epsilon^{2(s+u)-1}}{2^{11} T^2}) \\
&\quad + \mathbb{P}(\delta \sum_{t \in \mathcal{T}} |Y_t|^2 < \epsilon, \delta \sum_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] + |\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \epsilon^r, \mathcal{A}_1, |Y_0|^2 < \frac{\epsilon^s}{\delta |\mathcal{T}|}, \mathcal{A}^Y_{u,q}),
\end{aligned}$$

with $\underline{\epsilon}_1(\delta) = \max((16\delta T^2)^{\frac{1}{s+2u}}, (2^{10}\delta T^3)^{\frac{1}{2u+2s+2v}}), \bar{\epsilon}_1(\delta) = 2^{-\frac{1}{1-s}}$ and

$$\begin{aligned}
\mathcal{A}^Y_{u,q} =& \{\sup_{t \in \mathcal{T}} |\mathscr{K}(Y)_{t-\delta}| \leqslant \epsilon^{-u}\} \cap \{\sup_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^q | \mathcal{F}^Y_{t-\delta}] \leqslant \epsilon^{-qu}\} \\
& \cap \{\sup_{t \in \mathcal{T}} |\mathscr{K}(\mathscr{K}(Y))_{t-\delta}| \leqslant \epsilon^{-u}\} \cap \{\sup_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(\mathscr{K}(Y))_{t-\delta}|^q | \mathcal{F}^Y_{t-\delta}] \leqslant \epsilon^{-qu}\}, \\
\mathcal{A}_1 :=& \{\delta^2 \sum_{\substack{w,t \in \mathcal{T} \\ w \leqslant t}} \mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2 | \mathcal{F}^Y_{w-\delta}] < \epsilon^s\} \cap \{\delta^2 \sum_{\substack{w,t \in \mathcal{T} \\ w \leqslant t}} |\mathscr{M}(Y)_{w-\delta}|^2 < \epsilon^s\}.
\end{aligned}$$

We begin by writing, with notation introduced in (4.17), for every $t \in \mathcal{T}$,

$$\begin{aligned}
Y_t^2 =& Y_{t-\delta}^2 + \delta^{\frac{1}{2}} 2\mathscr{M}(Y)_{t-\delta} Y_{t-\delta} + \delta(2\mathscr{K}(Y)_{t-\delta} Y_{t-\delta} + |\mathscr{M}(Y)_{t-\delta}|^2) + \delta^{\frac{3}{2}} 2\mathscr{M}(Y)_{t-\delta} \mathscr{K}(Y)_{t-\delta} + \delta^2 |\mathscr{K}(Y)_{t-\delta}|^2 \\
=& Y_0^2 + \sum_{\substack{w \in \mathcal{T} \\ w \leqslant t}} \delta^{\frac{1}{2}} 2\mathscr{M}(Y)_{w-\delta} Y_{w-\delta} + \delta(2\mathscr{K}(Y)_{w-\delta} Y_{w-\delta} + |\mathscr{M}(Y)_{w-\delta}|^2) + \delta^{\frac{3}{2}} 2\mathscr{M}(Y)_{w-\delta} \mathscr{K}(Y)_{w-\delta} + \delta^2 |\mathscr{K}(Y)_{w-\delta}|^2.
\end{aligned}$$

We now introduce the following event, on which we can control $\delta \sum_{t \in \mathcal{T}} |Y_t|^2$,

$$\begin{aligned}
\mathcal{A}_2 :=& \{\delta^{\frac{3}{2}} |\sum_{\substack{w,t \in \mathcal{T} \\ w \leqslant t}} 2\mathscr{M}(Y)_{w-\delta} Y_{w-\delta}| < \frac{\epsilon^s}{8}\} \cap \{|\delta^2 \sum_{\substack{w,t \in \mathcal{T} \\ w \leqslant t}} 2\mathscr{K}(Y)_{w-\delta} Y_{w-\delta}| < \frac{\epsilon^s}{8}\} \\
& \cap \{|\delta^2 \sum_{\substack{w,t \in \mathcal{T} \\ w \leqslant t}} |\mathscr{M}(Y)_{w-\delta}|^2 - \mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2 | \mathcal{F}^Y_w]| < \frac{\epsilon^s}{8}\} \\
& \cap \{\delta^3 \sum_{\substack{w,t \in \mathcal{T} \\ w \leqslant t}} |\delta^{\frac{1}{2}} 2\mathscr{M}(Y)_{w-\delta} \mathscr{K}(Y)_{w-\delta} + \delta |\mathscr{K}(Y)_{w-\delta}|^2| < \frac{\epsilon^s}{8}\}.
\end{aligned}$$

Our idea is to distinguish the cases *w.r.t.* the event $\mathcal{A}_2$. We first estimate $\mathbb{P}(\delta \sum_{t \in \mathcal{T}} |Y_t|^2 < \epsilon, \mathcal{A}_2^c, \mathcal{A}^Y_{u,q})$. We then treat each event appearing the definition of $\mathcal{A}_2$ separately. In the sequel, for $t \in \pi^\delta$ we will denote $\mathfrak{n}_{\mathcal{T},\delta,t} = |\mathcal{T}| - t\delta^{-1}$. Concerning the first term, since $\sup_{t \in \mathcal{T}} |\mathscr{M}(Y)_{t-\delta}|^2, \sup_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] \leqslant \epsilon^{-2u}$ on $\mathcal{A}^Y_{u,q}$, for every $s \in (3r, \frac{1}{2})$, $u \in (0, \frac{1}{2} - s)$, we have

$$\begin{aligned}
&\mathbb{P}(|\delta^{\frac{3}{2}} \sum_{\substack{w,t \in \mathcal{T} \\ w \leqslant t}} 2\mathscr{M}(Y)_{w-\delta} Y_{w-\delta}| \geqslant \frac{\epsilon^s}{8}, \delta \sum_{t \in \mathcal{T}} |Y_t|^2 < \epsilon, \mathcal{A}^Y_{u,q}) \\
&\leqslant \mathbb{P}(|\delta^{\frac{3}{2}} \sum_{t \in \mathcal{T}} \mathfrak{n}_{\mathcal{T},\delta,t-\delta} \mathscr{M}(Y)_{t-\delta} Y_{t-\delta}| \geqslant \frac{\epsilon^s}{16}, \delta \sum_{t \in \mathcal{T}} |Y_{t-\delta}|^2 < 2\epsilon, \mathcal{A}^Y_{u,q}) + \mathbb{P}(\delta |Y_0|^2 \geqslant \epsilon) \\
&\leqslant \mathbb{P}(|\delta^{\frac{3}{2}} \sum_{t \in \mathcal{T}} \mathfrak{n}_{\mathcal{T},\delta,t-\delta} \mathscr{M}(Y)_{t-\delta} Y_{t-\delta}| \geqslant \frac{\epsilon^s}{16}, \\
&\qquad \delta^3 \sum_{t \in \mathcal{T}} |\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^2 (|\mathscr{M}(Y)_{t-\delta}|^2 + \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}]) |Y_{t-\delta}|^2 < 4(\delta |\mathcal{T}|)^2 \epsilon^{1-2u}) \\
&\quad + \mathbb{P}(\delta^3 \sum_{t \in \mathcal{T}} |\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^2 (|\mathscr{M}(Y)_{t-\delta}|^2 + \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}]) |Y_{t-\delta}|^2 \geqslant 4(\delta |\mathcal{T}|)^2 \epsilon^{1-2u}, \\
&\qquad \delta \sum_{t \in \mathcal{T}} |Y_{t-\delta}|^2 < 2\epsilon, \mathcal{A}^Y_{u,q}) + \mathbb{P}(\delta |Y_0|^2 \geqslant \epsilon).
\end{aligned}$$

Using the martingale exponential inequality (4.17), the first term of the *r.h.s.* above is bounded by $2\exp(-\frac{\epsilon^{2(s+u)-1}}{2^{11}(\delta|\mathcal{T}|)^2})$. We now study the second term of the *r.h.s.* above. Let us denote $H_t = |\mathscr{M}(Y)_t|^2 - \mathbb{E}[|\mathscr{M}(Y)_t|^2|\mathcal{F}_t^Y]$, $t \in \pi^\delta$. Then the aforementioned term is bounded by

$$\mathbb{P}(\delta^3 \sum_{t\in\mathcal{T}} |\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^2 H_{t-\delta}|Y_{t-\delta}|^2 \geqslant 2(\delta|\mathcal{T}|)^2\epsilon^{1-2u}, \mathcal{A}_{u,q}^Y)$$
$$+ \mathbb{P}(\delta^3 \sum_{t\in\mathcal{T}} |\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^2 |\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2|\mathcal{F}_{t-\delta}^Y]|Y_{t-\delta}|^2 \geqslant (\delta|\mathcal{T}|)^2\epsilon^{1-2u}, \delta\sum_{t\in\mathcal{T}}|Y_{t-\delta}|^2 < 2\epsilon, \mathcal{A}_{u,q}^Y).$$

Since $\mathfrak{n}_{\mathcal{T},\delta,t} \leqslant |\mathcal{T}|$ for every $t\in\mathcal{T}$ and $\sup_{t\in\mathcal{T}}\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2|\mathcal{F}_{t-\delta}^Y] \leqslant \epsilon^{-2u}$ on $\mathcal{A}_{u,q}^Y$, the second term of the *r.h.s.* above is equal to zero. We then focus to the first term of the *r.h.s.* above. Let $v' > 0$. In preparation to use the martingale exponential inequality, we consider the following estimate

$$\begin{aligned}
&\mathbb{P}(\delta^3 \sum_{t\in\mathcal{T}} |\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^2 H_{t-\delta}|Y_{t-\delta}|^2 \geqslant 2(\delta|\mathcal{T}|)^2\epsilon^{1-2u}, \mathcal{A}_{u,q}^Y)\\
\leqslant&\mathbb{P}(\delta^3 \sum_{t\in\mathcal{T}} |\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^2 H_{t-\delta}|Y_{t-\delta}|^2 \geqslant 2(\delta|\mathcal{T}|)^2\epsilon^{1-2u},\\
&\delta^6\sum_{t\in\mathcal{T}}|\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^4(|H_{t-\delta}|^2+\mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}_{t-\delta}^Y])|Y_{t-\delta}|^4 < (\delta|\mathcal{T}|)^4\epsilon^{2+v'-4u}, \mathcal{A}_{u,q}^Y)\\
&+\mathbb{P}(\delta^6\sum_{t\in\mathcal{T}}|\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^4(|H_{t-\delta}|^2+\mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}_{t-\delta}^Y])|Y_{t-\delta}|^4 \geqslant (\delta|\mathcal{T}|)^4\epsilon^{2+v'-4u}, \mathcal{A}_{u,q}^Y).
\end{aligned}$$

Using (4.17), the first term of the *r.h.s.* above is bounded by $2\exp(-2\epsilon^{-v'})$. To study the second term, we use the Markov and the Hölder inequalities and for every $q' \in [1, \frac{q}{4}]$ (more specifically, triangle inequality when $q'=1$), we obtain

$$\begin{aligned}
&\mathbb{P}(\delta^6\sum_{t\in\mathcal{T}}|\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^4(|H_{t-\delta}|^2+\mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}_{t-\delta}^Y])|Y_{t-\delta}|^4 \geqslant (\delta|\mathcal{T}|)^4\epsilon^{2+v'-4u}, \mathcal{A}_{u,q}^Y)\\
\leqslant&\mathbb{P}((\delta|\mathcal{T}|)^{q'-1}\delta\sum_{t\in\mathcal{T}}(|H_{t-\delta}|^2+\mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}_{t-\delta}^Y])^{q'}|Y_{t-\delta}|^{4q'} \geqslant \delta^{-q'}\epsilon^{q'(2+v'-4u)}, \mathcal{A}_{u,q}^Y)\\
\leqslant&\delta^{q'}\epsilon^{-q'(2+v'-4u)}(\delta|\mathcal{T}|)^{q'-1}\delta\sum_{t\in\mathcal{T}}2^{q'}\mathbb{E}[|H_{t-\delta}|^{2q'}|Y_{t-\delta}|^{4q'}\mathbf{1}_{\sup_{t\in\mathcal{T}}\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^q|\mathcal{F}_{t-\delta}^Y]\leqslant\epsilon^{-qu}}]\\
\leqslant&\delta^{q'}\epsilon^{-q'(2+v'-4u)}2^{3q'-1}(\delta|\mathcal{T}|)^{q'-1}(\delta\sum_{t\in\mathcal{T}}\mathbb{E}[|Y_{t-\delta}|^{4q'}\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^{4q'}|\mathcal{F}_t^Y]\mathbf{1}_{\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^q|\mathcal{F}_{t-\delta}^Y]\leqslant\epsilon^{-qu}}]\\
&+\delta\sum_{t\in\mathcal{T}}\mathbb{E}[|Y_{t-\delta}|^{4q'}|\mathscr{M}(Y)_{t-\delta}|^{4q'}\mathbf{1}_{\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^q|\mathcal{F}_{t-\delta}^Y]\leqslant\epsilon^{-qu}}])\\
\leqslant&\delta^{q'}\epsilon^{-(2+v')q'}2^{3q'}(\delta|\mathcal{T}|)^{q'-1}\sup_{t\in\mathcal{T}}\mathbb{E}[|Y_{t-\delta}|^{4q'}],
\end{aligned}$$

where for the last inequality, we use that, since $q' \leqslant \frac{q}{4}$, it follows from the Hölder inequality that

$$\begin{aligned}
\mathbb{E}[|Y_{t-\delta}|^{4q'}|\mathscr{M}(Y)_t|^{4q'}\mathbf{1}_{\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^q|\mathcal{F}_{t-\delta}^Y]\leqslant\epsilon^{-qu}}] \leqslant&\mathbb{E}[|Y_{t-\delta}|^{4q'}\mathbb{E}[|\mathscr{M}(Y)_t|^{4q'}|\mathcal{F}_t^Y]\mathbf{1}_{\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^{4q'}|\mathcal{F}_{t-\delta}^Y]\leqslant\epsilon^{-4q'u}}]\\
\leqslant&\epsilon^{-4q'u}\mathbb{E}[|Y_{t-\delta}|^{4q'}].
\end{aligned}$$

Gathering all the previous bounds, we complete the estimation of $\mathbb{P}(|\delta^{\frac{3}{2}}\sum_{w,t\in\mathcal{T};w\leqslant t}2\mathscr{M}(Y)_{w-\delta}Y_{w-\delta}| \geqslant \frac{\epsilon^s}{8}, \delta\sum_{t\in\mathcal{T}}|Y_t|^2 < \epsilon, \mathcal{A}_{u,q}^Y)$. Conducting a similar approach we also obtain

$$\begin{aligned}
&\mathbb{P}(|\delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{M}(Y)_{w-\delta}|^2-\mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2|\mathcal{F}_{w-\delta}^Y]| \geqslant \frac{\epsilon^s}{8}, \mathcal{A}_{u,q}^Y) = \mathbb{P}(\delta^2|\sum_{t\in\mathcal{T}}\mathfrak{n}_{\mathcal{T},\delta,t-\delta}H_{t-\delta}| \geqslant \frac{\epsilon^s}{8}, \mathcal{A}_{u,q}^Y)\\
\leqslant&\mathbb{P}(\delta^2\sum_{t\in\mathcal{T}}\mathfrak{n}_{\mathcal{T},\delta,t-\delta}H_{t-\delta} \geqslant \frac{\epsilon^s}{8}, \delta^4\sum_{t\in\mathcal{T}}|\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^2(|H_{t-\delta}|^2+\mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}_{t-\delta}^Y]) < \frac{\epsilon^{4s}}{8}, \mathcal{A}_{u,q}^Y)\\
&+\mathbb{P}(\delta^4\sum_{t\in\mathcal{T}}|\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^2(|H_{t-\delta}|^2+\mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}_{t-\delta}^Y]) \geqslant \frac{\epsilon^{4s}}{8}, \mathcal{A}_{u,q}^Y),
\end{aligned}$$

where, as a consequence of the exponential martingale inequality (4.17), the first term is bounded by $2\exp(-\frac{1}{16}\epsilon^{-4s})$. To complete the upper bound on the *l.h.s.* above, we notive that,owing to the Hölder and Markov inequalities that, for every $q' \in [1, \frac{q}{4}]$,

$$\mathbb{P}(\delta^4 \sum_{t\in\mathcal{T}} |\mathfrak{n}_{\mathcal{T},\delta,t-\delta}|^2(|H_{t-\delta}|^2+\mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]) \geqslant \frac{\epsilon^{4s}}{8}, \mathcal{A}^Y_{u,q}) \\ \leqslant \delta^{q'}\epsilon^{-4(s+u)q'}2^{6q'}(\delta|\mathcal{T}|)^{3q'-1}.$$

For the next term, we remark that, since $\sup_{t\in\mathcal{T}} |\mathscr{K}(Y)_{t-\delta}|\mathbf{1}_{\mathcal{A}^Y_{u,q}} \leqslant \epsilon^{-u}$, it follows from the Cauchy-Schwarz inequality that

$$|\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathscr{K}(Y)_{w-\delta}Y_{w-\delta}|\mathbf{1}_{\mathcal{A}^Y_{u,q}}\mathbf{1}_{|Y_0|<\epsilon^{-v}} < |\mathcal{T}|^{\frac{3}{2}}\epsilon^{-u}(\epsilon^{-2v} + \sum_{t\in\mathcal{T}}|Y_t|^2)^{\frac{1}{2}},$$

and, for $v>0$, as soon as $\epsilon \in [(32(\delta|\mathcal{T}|)^{\frac{3}{2}}\delta^{\frac{1}{2}})^{\frac{1}{u+s+v}}, 1]$,

$$\mathbb{P}(\delta\sum_{t\in\mathcal{T}}|Y_t|^2 < \epsilon, |\delta^2 \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} 2\mathscr{K}(Y)_{w-\delta}Y_{w-\delta}| \geqslant \frac{\epsilon^s}{8}, \mathcal{A}^Y_{u,q}, |Y_0| < \epsilon^{-v}) = 0.$$

Moreover, from the Markov inequality, we deduce that for every $q' \geqslant \frac{p}{v}$, $\mathbb{P}(|Y_0| \geqslant \epsilon^{-v}) \leqslant \epsilon^p\mathbb{E}[|Y_0|^{q'}]$.

Finally, for every $\epsilon \geqslant |16\delta^3|\mathcal{T}|^2|^{\frac{1}{s+2u}}$, using the Markov and Hölder inequalities and then the tower property, yields

$$\begin{aligned}
&\mathbb{P}(\delta^2 \sum_{\substack{w\in\mathcal{T}\\ w\leqslant t}} |\delta^{\frac{1}{2}}2\mathscr{M}(Y)_{w-\delta}\mathscr{K}(Y)_{w-\delta} + \delta|\mathscr{K}(Y)_{w-\delta}|^2| \geqslant \frac{\epsilon^s}{8}, \mathcal{A}^Y_{u,q})\\
&\quad\leqslant \mathbb{P}(\delta^{5/2}\sum_{\substack{w\in\mathcal{T}\\ w\leqslant t}} |2\mathscr{M}(Y)_{w-\delta}\mathscr{K}(Y)_{w-\delta}| \geqslant \frac{\epsilon^s}{8} - \delta^3|\mathcal{T}|^2\epsilon^{-2u}, \mathcal{A}^Y_{u,q})\\
&\quad\leqslant \mathbb{E}[32^q|\mathcal{T}|^{2q-2}\delta^{5q/2}\sum_{\substack{w\in\mathcal{T}\\ w\leqslant t}} |\mathscr{M}(Y)_{w-\delta}|^q\epsilon^{-q(s+u)}\mathbf{1}_{\mathcal{A}^Y_{u,q}}]\\
&\quad\leqslant 32^q|\mathcal{T}|^{2q-2}\epsilon^{-q(s+u)}\delta^{5q/2}\sum_{\substack{w\in\mathcal{T}\\ w\leqslant t}} \mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^q\mathbf{1}_{\mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^q|\mathcal{F}^Y_{w-\delta}]\leqslant\epsilon^{-qu}}]\\
&\quad\leqslant 32^q\delta^{\frac{q}{2}}\epsilon^{-q(s+2u)}(\delta|\mathcal{T}|)^{2q}.
\end{aligned}$$

In particular, taking $q' = \frac{q}{4}$, we have proved that for every $\epsilon \in [\underline{\epsilon}_1(\delta), 1]$,

$$\begin{aligned}
\mathbb{P}(\delta\sum_{t\in\mathcal{T}}|Y_t|^2 <\epsilon, \mathcal{A}_2^c, \mathcal{A}^Y_{u,q}) &\leqslant \epsilon^p\mathbb{E}[|Y_0|^{\frac{p}{v}}]) + \mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon) + \delta^{\frac{q}{4}}\epsilon^{-q(s+u)}2^{\frac{3q}{2}}(\delta|\mathcal{T}|)^{\frac{3q}{4}-1}\\
&\quad + 2^{5q}\delta^{\frac{q}{2}}\epsilon^{-q(s+2u)}(\delta|\mathcal{T}|)^{2q} + \delta^{\frac{q}{4}}\epsilon^{-\frac{(2+v')q}{4}}2^{\frac{3q}{4}}(\delta|\mathcal{T}|)^{\frac{q}{4}-1}\sup_{t\in\mathcal{T}}\mathbb{E}[|Y_{t-\delta}|^q]\\
&\quad + 2\exp(-\frac{\epsilon^{-4s}}{16}) + 2\exp(-2\epsilon^{-v'}) + 2\exp(-\frac{\epsilon^{2(s+u)-1}}{2^{12}(\delta|\mathcal{T}|)^2}).
\end{aligned}$$

At this point, we remark that

$$\begin{aligned}
\mathcal{A}_2 \subset &\{\delta\sum_{t\in\mathcal{T}}|Y_0|^2 + \delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{M}(Y)_{w-\delta}|^2 < \delta\sum_{t\in\mathcal{T}}|Y_t|^2 + \frac{\epsilon^s}{2}\}\\
&\cap\{\delta\sum_{t\in\mathcal{T}}|Y_0|^2 + \delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}\mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}] < \delta\sum_{t\in\mathcal{T}}|Y_t|^2 + \frac{\epsilon^s}{2}\}.
\end{aligned}$$

In particular, for every $\epsilon \leqslant 2^{-\frac{1}{1-s}}$, we have $\epsilon \leqslant \frac{\epsilon^s}{2}$ and

$$\{\delta\sum_{t\in\mathcal{T}}|Y_t|^2 < \epsilon\}\cap\mathcal{A}_2 \subset \{\delta\sum_{t\in\mathcal{T}}|Y_t|^2 < \epsilon\}\cap\mathcal{A}_1\cap\{|Y_0|^2 < \frac{\epsilon^s}{\delta|\mathcal{T}|}\},$$

and, remarking that $\delta|\mathcal{T}| \leqslant T$. the proof of **Step 1** is completed.

Step 2. We show that for every $\epsilon \in (0, \bar\epsilon_2(\delta)]$ and $u \in (0, \frac{s}{4} - \frac{3r}{4})$,

$$\mathbb{P}(\delta \sum_{t\in\mathcal{T}} |Y_t|^2 < \epsilon, \delta \sum_{t\in\mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] \geqslant \frac{\epsilon^r}{2}, \mathcal{A}_1, \mathcal{A}^Y_{u,q}) = 0.$$

with $\bar\epsilon_2(\delta) = \min(4\delta|\mathcal{T}|)^{\frac{1}{2u+r}}, |2^7\delta|\mathcal{T}||^{-\frac{1}{s-3r-4u}})$. First, we recall that $\mathcal{A}^Y_{u,q} \subset \{\sup_{t\in\mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] \leqslant \epsilon^{-2u}\}$ and we observe that

$$\frac{\epsilon^{r+2u}}{4}\epsilon^{-2u} + \delta(|\mathcal{T}| - \frac{\epsilon^{r+2u}}{4})\frac{\epsilon^r}{4\delta|\mathcal{T}|} < \frac{\epsilon^{r+2u}}{4}\epsilon^{-2u} + \delta|\mathcal{T}|\frac{\epsilon^r}{4\delta|\mathcal{T}|} = \frac{\epsilon^r}{2},$$

with $\epsilon^{r+2u} \leqslant 4|\mathcal{T}|$ and $\frac{\epsilon^r}{4\delta|\mathcal{T}|} \leqslant \epsilon^{-2u}$ for $\epsilon \leqslant (4\delta|\mathcal{T}|)^{\frac{1}{2u+r}}$. Therefore, on the set $\{\delta \sum_{t\in\mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] \geqslant \frac{\epsilon^r}{2}\} \cap \{\sup_{t\in\mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] \leqslant \epsilon^{-2u}\}$, the following minoration holds

$$\delta \sum_{t\in\mathcal{T}} \mathbf{1}_{\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] \geqslant \frac{\epsilon^r}{4\delta|\mathcal{T}|}} \geqslant \frac{\epsilon^{r+2u}}{4}.$$

It follows that

$$\begin{aligned}
\delta^2 \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2 | \mathcal{F}^Y_{w-\delta}] \geqslant& \delta^2 \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2 | \mathcal{F}^Y_{w-\delta}] \mathbf{1}_{\mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2 | \mathcal{F}^Y_{w-\delta}] \geqslant \frac{\epsilon^r}{4\delta|\mathcal{T}|}} \\
\geqslant& \frac{\epsilon^r}{4\delta|\mathcal{T}|}\delta^2 \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathbf{1}_{\mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2 | \mathcal{F}^Y_{w-\delta}] \geqslant \frac{\epsilon^r}{4\delta|\mathcal{T}|}} \\
\geqslant& \frac{\epsilon^r}{4\delta|\mathcal{T}|}\frac{1}{2}\frac{\epsilon^{r+2u}}{4}(\frac{\epsilon^{r+2u}}{4} + 1) \geqslant \frac{\epsilon^{3r+4u}}{2^7\delta|\mathcal{T}|}.
\end{aligned}$$

In particular, since $\epsilon \leqslant |2^7\delta|\mathcal{T}||^{-\frac{1}{s-3r-4u}}$

$$\{\delta \sum_{t\in\mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] \geqslant \frac{\epsilon^r}{2}\} \cap \{\delta^2 \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2 | \mathcal{F}^Y_{w-\delta}] < \epsilon^s\} \cap \mathcal{A}^Y_{u,q} = \emptyset.$$

Recalling that we have $\mathcal{A}_1 \subset \{\delta^2 \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2 | \mathcal{F}^Y_{w-\delta}] < \epsilon^s\}$ the proof of **Step 2** is completed.

Step 3. In this part we show that for every $\epsilon \in (\underline{\epsilon}_3(\delta), \bar\epsilon_3(\delta))$, every $h, s \in (3r, \frac{1}{2})$ with $2h < s$, $u \in (0, \min(\frac{s}{2} - h, \frac{h}{4} - \frac{3r}{4}))$,

$$\begin{aligned}
\mathbb{P}(\delta \sum_{t\in\mathcal{T}} |Y_t|^2 < \epsilon, \delta \sum_{t\in\mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_t] < \frac{\epsilon^r}{2}, \delta \sum_{t\in\mathcal{T}} |\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \frac{\epsilon^r}{2}, \mathcal{A}_1, |Y_0|^2 < \frac{\epsilon^s}{\delta|\mathcal{T}|}, \mathcal{A}^Y_{u,q}) \\
\leqslant \delta^{\frac{q}{4}}(\delta^{\frac{q}{4}}\epsilon^{-q(h+2u)} + \epsilon^{-\frac{(2+v')q}{4}})2^{5q}(1 \vee T^{2q})(1 + \sup_{t\in\mathcal{T}} \mathbb{E}[|Y_{t-\delta}|^q]) \\
+ \epsilon^{\frac{q(s-2h-2u)}{2}} 2^{3q} T^q + \mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon) \\
+ 2\exp(-\frac{\epsilon^{2(h+u)-1}}{2^9 T^2}) + 2\exp(-2\epsilon^{-v'}) + 2\exp(-\frac{\epsilon^{2h+2u}}{2^8\delta T^2}),
\end{aligned}$$

with

$$\begin{aligned}
\underline{\epsilon}_3(\delta) =& (16\delta T^2)^{\frac{1}{h+2u}}, \\
\bar\epsilon_3(\delta) =& \min((2^8\delta|\mathcal{T}|)^{-\frac{1}{h-3r-4u}}, (4\delta|\mathcal{T}|)^{-\frac{1}{s-2h-2u}}, 1).
\end{aligned}$$

In the same way as in the proof of **Step 1**, we begin by writing for every $t \in \mathcal{T}$,

$$\begin{aligned}
Y_t\mathscr{K}(Y)_t =& Y_0\mathscr{K}(Y)_0 + \sum_{\substack{w\in\mathcal{T}\\ w\leqslant t}} \delta^{\frac{1}{2}}(\mathscr{M}(Y)_{w-\delta}\mathscr{K}(Y)_{w-\delta} + \mathscr{M}(\mathscr{K}(Y))_{w-\delta}Y_{w-\delta}) \\
&+ \delta(|\mathscr{K}(Y)_{w-\delta}|^2 + \mathscr{M}(\mathscr{K}(Y))_{w-\delta}\mathscr{M}(Y)_{w-\delta}) \\
&+ \delta^{\frac{3}{2}}(\mathscr{M}(\mathscr{K}(Y))_{w-\delta}\mathscr{K}(Y)_{w-\delta} + \mathscr{M}(Y)_{w-\delta}\mathscr{K}(\mathscr{K}(Y))_{w-\delta}) + \delta^2\mathscr{K}(\mathscr{K}(Y))_{w-\delta}\mathscr{K}(Y)_{w-\delta},
\end{aligned}$$

and we define for $h \in (3r, \frac{s}{2})$

$$\mathcal{A}_3 := \{|\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \delta^{\frac{3}{2}}\mathscr{M}(\mathscr{K}(Y))_{w-\delta}Y_{w-\delta}| < \frac{\epsilon^h}{8}\} \cap \{\delta^{\frac{3}{2}}|\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathscr{M}(Y)_{w-\delta}\mathscr{K}(Y)_{w-\delta}| < \frac{\epsilon^h}{8}\}$$
$$\cap \{|\delta^2 \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathscr{M}(\mathscr{K}(Y))_{w-\delta}\mathscr{M}(Y)_{w-\delta}| < \frac{\epsilon^h}{8}\}$$
$$\cap \{\delta^2 \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} |\delta^{\frac{1}{2}}(\mathscr{M}(\mathscr{K}(Y))_{w-\delta}\mathscr{K}(Y)_{w-\delta} + \mathscr{M}(Y)_{w-\delta}\mathscr{K}(\mathscr{K}(Y))_{w-\delta}) + \delta\,\mathscr{K}(\mathscr{K}(Y))_{w-\delta}\mathscr{K}(Y)_{w-\delta}| < \frac{\epsilon^h}{8}\}.$$

We take $u \in (0, \frac{s}{2} - h)$. Using the exact same approach as in **Step 1**, the exponential martingale inequality (4.17) together with the Markov and Hölder inequalities imply that, for every $v' > 0$, we obtain the following first estimate

$$\begin{aligned}\mathbb{P}(|\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \delta^{\frac{3}{2}}\mathscr{M}(\mathscr{K}(Y))_{w-\delta}Y_{w-\delta}| \geqslant \frac{\epsilon^h}{8}, \delta\sum_{t\in\mathcal{T}}|Y_t|^2 < \epsilon, \mathcal{A}^Y_{u,q})\\ \leqslant \delta^{\frac{q}{4}}\epsilon^{-\frac{(2+v')q}{4}}2^{\frac{3q}{4}}(\delta|\mathcal{T}|)^{\frac{3q}{4}-1}\sup_{t\in\mathcal{T}}\mathbb{E}[|Y_{t-\delta}|^q]\\ + \mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon) + 2\exp(-\frac{\epsilon^{2h+2u-1}}{2^9(\delta|\mathcal{T}|)^2}) + 2\exp(-2\epsilon^{-v'}).\end{aligned}$$

Moreover, since $\mathcal{A}_3 \subset \{\delta^2\sum_{w,t\in\mathcal{T};w\leqslant t}\mathbb{E}[|\mathscr{M}(Y)_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}] + |\mathscr{M}(Y)_{w-\delta}|^2 < 2\epsilon^s\}$ and $\sup_{t\in\mathcal{T}}|\mathscr{K}(Y)_{t-\delta}| \leqslant \epsilon^{-u}$ on $\mathcal{A}^Y_{u,q}$, the inequality (4.17) yields

$$\mathbb{P}(\delta^{\frac{3}{2}}|\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathscr{M}(Y)_{w-\delta}\mathscr{K}(Y)_{w-\delta}| \geqslant \frac{\epsilon^h}{8}, \mathcal{A}_3, \mathcal{A}^Y_{u,q}) \leqslant 2\exp(-\frac{\epsilon^{2h+2u-s}}{2^8\delta|\mathcal{T}|}).$$

It remains to study thlas term steming from the definition of $\mathcal{A}_3$. Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\mathbb{P}(|\delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} \mathscr{M}(\mathscr{K}(Y))_{w-\delta}\mathscr{M}(Y)_{w-\delta}| \geqslant \frac{\epsilon^h}{8}, \delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{M}(Y)_{w-\delta}|^2 < \epsilon^s, \mathcal{A}^Y_{u,q})\\ \leqslant \mathbb{P}(\delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{M}(\mathscr{K}(Y))_{w-\delta}|^2 \geqslant \frac{\epsilon^{2h-s}}{64}, \delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{M}(Y)_{w-\delta}|^2 < \epsilon^s, \mathcal{A}^Y_{u,q}),\end{aligned}$$

In addition, by the Markov and Hölder inequalities, we derive

$$\begin{aligned}\mathbb{P}(\delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{M}(\mathscr{K}(Y))_{w-\delta}|^2 \geqslant \frac{\epsilon^{2h-s}}{64}, \mathcal{A}^Y_{u,q}) &\leqslant \epsilon^{\frac{q(s-2h)}{2}}2^{3q}\mathbb{E}[|\delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{M}(\mathscr{K}(Y))_{w-\delta}|^2|^{\frac{q}{2}}\mathbf{1}_{\mathcal{A}^Y_{u,q}}]\\ &\leqslant \epsilon^{\frac{q(s-2h)}{2}}2^{3q}\delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}\mathbb{E}[|\mathscr{M}(\mathscr{K}(Y))_{w-\delta}|^q\mathbf{1}_{\mathcal{A}^Y_{u,q}}](\delta^2|\mathcal{T}|^2)^{q/2-1}\\ &\leqslant \epsilon^{\frac{q(s-2h-2u)}{2}}2^{3q}(\delta|\mathcal{T}|)^q.\end{aligned}$$

Besides, for every $\epsilon \geqslant (16\delta^3|\mathcal{T}|^2)^{\frac{1}{h+2u}}$, we have $\delta^3 \sum_{w,t\in\mathcal{T};w\leqslant t} |\mathscr{K}(\mathscr{K}(Y))_{w-\delta}\mathscr{K}(Y)_{w-\delta}| \leqslant \frac{\epsilon^h}{16}$ on the set $\mathcal{A}^Y_{u,q}$. Using again the triangle, Markov and Hölder inequalities yields

$$\begin{aligned}
&\mathbb{P}(\delta^2 \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} |\delta^{\frac{1}{2}}(\mathscr{M}(\mathscr{K}(Y))_{w-\delta}\mathscr{K}(Y)_{w-\delta} + \mathscr{M}(Y)_{w-\delta}\mathscr{K}(\mathscr{K}(Y))_{w-\delta}) + \delta\,\mathscr{K}(\mathscr{K}(Y))_{w-\delta}\mathscr{K}(Y)_{w-\delta}| \geqslant \frac{\epsilon^h}{8}, \mathcal{A}^Y_{u,q})\\
&\qquad \leqslant \mathbb{P}(\delta^{5/2} \sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}} |\mathscr{M}(\mathscr{K}(Y))_{w-\delta}\mathscr{K}(Y)_{w-\delta} + \mathscr{M}(Y)_{w-\delta}\mathscr{K}(\mathscr{K}(Y))_{w-\delta}| \geqslant \frac{\epsilon^h}{16}, \mathcal{A}^Y_{u,q})\\
&\qquad \leqslant 2^{5q}\delta^{\frac{q}{2}}\epsilon^{-q(h+2u)}(\delta|\mathcal{T}|)^{2q}.
\end{aligned}$$

In particular, for every $\epsilon \geqslant \underline{\epsilon}_3(\delta)$, we have just shown that

$$\begin{aligned}
\mathbb{P}(\delta\sum_{t\in\mathcal{T}}|Y_t|^2 <\epsilon, \delta\sum_{t\in\mathcal{T}}\mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}] < \frac{\epsilon^r}{2}, \mathcal{A}_1, \mathcal{A}_3^c, \mathcal{A}^Y_{u,q}) \leqslant&\\
\delta^{\frac{q}{4}}(2^{5q}\delta^{\frac{q}{4}}\epsilon^{-q(h+2u)}T^{2q} + \epsilon^{-\frac{(2+v')q}{4}}2^{\frac{3q}{4}}T^{\frac{3q}{4}-1}\sup_{t\in\mathcal{T}}\mathbb{E}[|Y_{t-\delta}|^q])&\\
+\epsilon^{\frac{q(s-2h-2u)}{2}}2^{3q}T^q + \mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon)&\\
+2\exp(-\frac{\epsilon^{2h+2u-1}}{2^9(\delta|\mathcal{T}|)^2}) + 2\exp(-2\epsilon^{-v'}) + 2\exp(-\frac{\epsilon^{2h+2u-s}}{2^8\delta|\mathcal{T}|}).&
\end{aligned}$$

We notice that, similarly as in **Step 1**,

$$\mathcal{A}_3 \subset \{\delta\sum_{t\in\mathcal{T}}Y_0\,\mathscr{K}(Y)_0 + \delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{K}(Y)_{w-\delta}|^2 < \delta|\sum_{t\in\mathcal{T}}Y_t\,\mathscr{K}(Y)_t| + \frac{\epsilon^h}{2}\}.$$

It follows from the Cauchy-Schwarz inequality, applied to $\sum_{t\in\mathcal{T}}Y_t\,\mathscr{K}(Y)_t$, and the triangle inequality that, for every $\epsilon \leqslant 1 \wedge (4\delta|\mathcal{T}|)^{\frac{1}{s-2u-2h}}$,

$$\begin{aligned}
&\mathbb{P}(\delta\sum_{t\in\mathcal{T}}|Y_t|^2 <\epsilon, \delta\sum_{t\in\mathcal{T}}|\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \frac{\epsilon^r}{2}, \mathcal{A}_1, \mathcal{A}_3, |Y_0|^2 < \frac{\epsilon^s}{\delta|\mathcal{T}|}, \mathcal{A}^Y_{u,q})\\
&\qquad \leqslant \mathbb{P}(\delta\sum_{t\in\mathcal{T}}|\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \frac{\epsilon^r}{2}, \delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{K}(Y)_{w-\delta}|^2 < \frac{\epsilon^h}{2} + (\delta|\mathcal{T}|)^{\frac{1}{2}}\epsilon^{-u}(\epsilon^{\frac{s}{2}}+\epsilon^{\frac{1}{2}}), \mathcal{A}^Y_{u,q})\\
&\qquad \leqslant \mathbb{P}(\delta\sum_{t\in\mathcal{T}}|\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \frac{\epsilon^r}{2}, \delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{K}(Y)_{w-\delta}|^2 < \epsilon^h, \mathcal{A}^Y_{u,q}).
\end{aligned}$$

Similarly as in **Step 2**, we notice that, on the set $\{\delta\sum_{t\in\mathcal{T}}|\mathscr{K}(Y)_{t-\delta}|^2 > \frac{\epsilon^r}{2}\} \cap \mathcal{A}^Y_{u,q}$ then

$$\delta\sum_{t\in\mathcal{T}}\mathbf{1}_{|\mathscr{K}(Y)_t|^2 \geqslant \frac{\epsilon^r}{4\delta|\mathcal{T}|}} \geqslant \frac{\epsilon^{r+2u}}{4} \quad\text{and}\quad \delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{K}(Y)_{w-\delta}|^2 \geqslant \frac{\epsilon^{3r+4u}}{2^7\delta|\mathcal{T}|}.$$

In particular for every $\epsilon \leqslant |2^7\delta|\mathcal{T}||^{-\frac{1}{h-3r-4u}}$,

$$\{\delta\sum_{t\in\mathcal{T}}|\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \frac{\epsilon^r}{2}\} \cap \{\delta^2\sum_{\substack{w,t\in\mathcal{T}\\ w\leqslant t}}|\mathscr{K}(Y)_{w-\delta}|^2 < \epsilon^h\} = \emptyset.$$

and the proof of **Step 3** is completed.

Step 4. We now show (4.23). In the first three steps we have proved that, for every $h, s \in (3r, \frac{1}{2})$ with $2h < s$, $u \in (0, \min(\frac{1}{2}-s, \frac{s}{2}-h, \frac{h}{4}-\frac{3r}{4}))$, every $p, v, v' > 0$, every $q \geqslant 4$ and every

$\epsilon \in [\max(\underline{\epsilon}_1(\delta), \underline{\epsilon}_3(\delta), 8\delta), \min(1, \bar{\epsilon}_1(\delta), \bar{\epsilon}_2(\delta), \bar{\epsilon}_3(\delta))]$,

$$\begin{aligned}
&\mathbb{P}(\delta \sum_{t \in \mathcal{T}} |Y_t|^2 < \epsilon, \delta \sum_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] + |\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \epsilon^r, \mathcal{A}^Y_{u,q}) \\
&\leqslant \epsilon^p \mathbb{E}[|Y_0|^{\frac{p}{v}}]) + 2\mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon) + \epsilon^{\frac{q(s-2h-2u)}{2}} 2^{3q} T^q \\
&\quad + \delta^{\frac{q}{4}}(2\delta^{\frac{q}{4}} \epsilon^{-q(s+2u)} + 3\epsilon^{-q\frac{(2+v')}{4}}) 2^{5q}(1 \vee T^{2q})(1 + \sup_{t \in \mathcal{T}} \mathbb{E}[|Y_{t-\delta}|^q]) \\
&\quad + 2\exp(-\frac{\epsilon^{-4s}}{16}) + 4\exp(-2\epsilon^{-v'}) + 6\exp(-\frac{\epsilon^{2s+2u-1}}{2^{11}T^2}).
\end{aligned}$$

We first observe that, as a direct consequence of the Markov inequality, for $p > 0$,

$$\begin{aligned}
\mathbb{P}((\mathcal{A}^Y_{u,q})^c) \leqslant & \epsilon^p (\mathbb{E}[\sup_{t \in \mathcal{T}} |\mathscr{K}(\mathscr{K}(Y))_{t-\delta}|^{\frac{p}{u}}] + \mathbb{E}[\sup_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^q | \mathcal{F}^Y_{t-\delta}]^{\frac{p}{qu}}] \\
& + \mathbb{E}[\sup_{t \in \mathcal{T}} |\mathscr{K}(\mathscr{K}(Y))_{t-\delta}|^{\frac{p}{u}}] + \mathbb{E}[\sup_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(\mathscr{K}(Y))_{t-\delta}|^q | \mathcal{F}^Y_{t-\delta}]^{\frac{p}{qu}}]).
\end{aligned}$$

At this point, we assume that $q \geqslant \frac{2p}{s-2h-2u}$. Then, for every $v' > 0$ such that $\epsilon \geqslant \delta^{\frac{1}{2+v'+\frac{p}{q}}}$, we have

$$\begin{aligned}
&\mathbb{P}(\delta \sum_{t \in \mathcal{T}} |Y_t|^2 < \epsilon, \delta \sum_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] + |\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \epsilon^r, \mathcal{A}^Y_{u,q}) \\
&\leqslant \epsilon^p \mathbb{E}[|Y_0|^{\frac{p}{v}}]) + 2\mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon) \\
&\quad + \epsilon^p 2^{5q} 5(1 \vee T^{2q})(1 + \sup_{t \in \mathcal{T}} \mathbb{E}[|Y_{t-\delta}|^q]) \\
&\quad + 2\exp(-\frac{\epsilon^{-4s}}{16}) + 4\exp(-2\epsilon^{-v'}) + 6\exp(-\frac{\epsilon^{2s+2u-1}}{2^{11}T^2}).
\end{aligned}$$

Moreover, for every $q' > 0$ such that $\epsilon \geqslant \delta^{\frac{q'}{q'+2p}}$, $\mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon) \leqslant \epsilon^p \mathbb{E}[|Y_0|^{q'}]$. In particular, we take $q' = \frac{2pq}{q(1+v')+p}$ so that this inequality is satisfied when $\epsilon \geqslant \delta^{\frac{1}{2+v'+\frac{p}{q}}}$.

Now we fix $s = s(r) := \frac{5}{11} + \frac{6}{11}r$, $h = h(r) := \frac{2}{11} + \frac{9}{11}r$ and take $u < \frac{1}{22} - \frac{6}{11}r$. Notice that, snce $r \in (0, \frac{1}{12})$, we have $s(r) \in (6r, \frac{1}{2})$, $h(r) \in (3r, \frac{s(r)}{2})$. Therefore, taking $v = \frac{6}{11} - \frac{6}{11}r - u + \frac{v'}{2} + \frac{p}{2q}$, and $q \geqslant \max(4, \frac{2p}{\frac{1}{11} - \frac{12}{11}r - 2u})$, we have, for every $\epsilon \in [|2^{10}(1 \vee T^3)\delta|^{\frac{1}{2+v'+\frac{p}{q}}}, \min(|2^8 T|^{-\frac{1}{\frac{2}{11} - \frac{24}{11}r - 4u}}, 2^{-\frac{1}{\frac{6}{11} - \frac{6}{11}r}})]$,

$$\begin{aligned}
&\mathbb{P}(\delta \sum_{t \in \mathcal{T}} |Y_t|^2 < \epsilon, \delta \sum_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] + |\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \epsilon^r,) \\
&\leqslant \epsilon^p \mathbb{E}[|Y_0|^{\frac{p}{\frac{6}{11} - \frac{6}{11}r - u + \frac{v'}{2} + \frac{p}{2q}}}]) + 2\epsilon^p \mathbb{E}[|Y_0|^{\frac{2pq}{q(1+v')+p}}] \\
&\quad + \epsilon^p 2^{5q} 5(1 \vee T^{2q})(1 + \sup_{t \in \mathcal{T}} \mathbb{E}[|Y_{t-\delta}|^q]) \\
&\quad + \epsilon^p (\mathbb{E}[\sup_{t \in \mathcal{T}} |\mathscr{K}(\mathscr{K}(Y))_{t-\delta}|^{\frac{p}{u}}] + \mathbb{E}[\sup_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^q | \mathcal{F}^Y_{t-\delta}]^{\frac{p}{qu}}] \\
&\quad + \mathbb{E}[\sup_{t \in \mathcal{T}} |\mathscr{K}(\mathscr{K}(Y))_{t-\delta}|^{\frac{p}{u}}] + \mathbb{E}[\sup_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(\mathscr{K}(Y))_{t-\delta}|^q | \mathcal{F}^Y_{t-\delta}]^{\frac{p}{qu}}]) \\
&\quad + 4\exp(-2\epsilon^{-v'})) + 8\exp(-\frac{\epsilon^{-\frac{1}{11} + \frac{12}{11}r + 2u}}{2^{11}(1 \vee T^2)}).
\end{aligned}$$

Now we take $u = \frac{1}{44} - \frac{3}{11}r$ and $q = q(r,p) = \max(4, \frac{2p}{\frac{1}{11} - \frac{12}{11}r - 2u}, \frac{p}{u}) = \max(4, \frac{44p}{1-12r})$ (in particular $q(r,p) \geqslant \frac{2p}{s-2h-2u}$). It follows that, for every $\epsilon \in [|2^{10}(1 \vee T^3)\delta|^{\frac{1}{2+v'+\frac{p}{q(r,p)}}}, (2^8(1 \vee T))^{-\frac{11}{1-12r}}]$,

$$\begin{aligned}
&\mathbb{P}(\delta \sum_{t \in \mathcal{T}} |Y_t|^2 < \epsilon, \delta \sum_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] + |\mathscr{K}(Y)_{t-\delta}|^2 \geqslant \epsilon^r) \\
&\leqslant 3\epsilon^p \mathbb{E}[|Y_0|^{\frac{2pq(r,p)}{q(r,p)(1+v')+p}}] \\
&\quad + \epsilon^p 2^{5q(r,p)} 5(1 \vee T^{2q(r,p)})(1 + \sup_{t \in \mathcal{T}} \mathbb{E}[|Y_{t-\delta}|^{q(r,p)}]) \\
&\quad + \epsilon^p (2 + \mathbb{E}[\sup_{t \in \mathcal{T}} |\mathscr{K}(\mathscr{K}(Y))_{t-\delta}|^{q(r,p)}] + \mathbb{E}[\sup_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(Y)_{t-\delta}|^{q(r,p)} | \mathcal{F}^Y_{t-\delta}]] \\
&\quad + \mathbb{E}[\sup_{t \in \mathcal{T}} |\mathscr{K}(\mathscr{K}(Y))_{t-\delta}|^{q(r,p)}] + \mathbb{E}[\sup_{t \in \mathcal{T}} \mathbb{E}[|\mathscr{M}(\mathscr{K}(Y))_{t-\delta}|^{q(r,p)} | \mathcal{F}^Y_{t-\delta}]]) \\
&\quad + 4\exp(-2\epsilon^{-v'}) + 8\exp(-\frac{\epsilon^{-\frac{1}{22}+\frac{6}{11}r}}{2^{11}(1 \vee T^2)}).
\end{aligned}$$

Since $q(r,p) > p$ and $v' > 0$, $\mathbb{E}[|Y_0|^{\frac{2pq(r,p)}{q(r,p)(1+v')+p}}] \leqslant 1 + \mathbb{E}[|Y_0|^{q(r,p)}]$. We fix $v' = \frac{1}{22} - \frac{6}{11}r$ and the proof of (4.23) is completed.

□

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