HÖRMANDER PROPERTIES OF DISCRETE TIME MARKOV PROCESSES

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ABSTRACT. We present an abstract framework for establishing smoothing properties within a specific class of inhomogeneous discrete-time Markov processes. These properties, in turn, serve as a basis for demonstrating the existence of density functions for our processes or more precisely for regularized versions of them. We also use them to show the total variation convergence towards the solution of a Stochastic Differential Equation as the time step between two observations of the discrete time Markov processes tends to zero. The distinctive feature of our methodology lies in the exploration of smoothing properties under some local weak Hörmander type conditions satisfied by the discrete-time Markov processes. Our Hörmander properties are demonstrated to align with the standard local weak Hörmander properties satisfied by the coefficients of the Stochastic Differential Equations which are the total variation limits of our discrete time Markov processes.

Keywords : Discrete time Markov processes, Hörmander properties, Regularization properties, Malliavin Calculus, Invariance principle.
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1. INTRODUCTION

1.1. **Context.** For $\delta \in (0,1]$ and $d, N \in \mathbb{N}^*$, we study a sequence of independent random variables $Z_t^\delta \in \mathbb{R}^N$, $t \in \pi^{\delta,*}$ (we use the notations $\pi^\delta := \delta\mathbb{N}$ and $\pi^{\delta,*} := \delta\mathbb{N}^*$), which are supposed to be centered with covariance matrix identity and Lebesgue lower bounded distribution (see (2.8) for definition). In this paper, our focus is on the $\mathbb{R}^d$-valued discrete time Markov process $(X_t^\delta)_{t\in\pi^\delta}$ defined as follows:

$$X_{t+\delta}^\delta = \psi(X_t^\delta, t, \delta^{\frac{1}{2}} Z_{t+\delta}^\delta, \delta), \quad t \in \pi^\delta, \quad X_0^\delta = \mathrm{x}_0^\delta \in \mathbb{R}^d. \tag{1.1}$$

where $\psi : (x,t,z,y) \mapsto \psi(x,t,z,y) \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^N \times [0,1]; \mathbb{R}^d)$. Our primary challenge is to demonstrate that, under suitable properties on ψ, we can construct a process $(\overline{X}_t^\delta)_{t\in\pi^\delta}$ that is arbitrarily close to $(X_t^\delta)_{t\in\pi^\delta}$ in total variation distance (for any fixed $t \in \pi^\delta$). Additionally, this process satisfies the smoothing/regularization property: For every $\alpha, \beta \in \mathbb{N}^d$, there exists $C : \mathbb{R}^d \times \pi^{\delta,*} \to \mathbb{R}_+$ (which does not depend on δ) such that for every $T \in \pi^{\delta,*}$, $\mathrm{x} \in \mathbb{R}^d$ and every $f \in \mathcal{C}^\infty(\mathbb{R}^d;\mathbb{R})$, bounded,

$$|\partial_x^\alpha \mathbb{E}[\partial^\beta f(\overline{X}_T^\delta) | X_0^\delta = \mathrm{x}]| \leqslant C(\mathrm{x}, T)\|f\|_\infty. \tag{1.2}$$

A refined version of this result is exposed in Theorem 2.1. Relying on those regularization properties, we can infer that $\overline{X}_t^\delta$, $t \in \pi^\delta$, admits a smooth density (see Corollary 2.1). A main application of those results is provided in Theorem 2.2, where we identify a total variation limit (along with explicit rate of convergence) for X_t^δ, $t \in \pi^\delta$, as δ tends to zero. This weak limit random variable is given by the solution, at time t, of the Stochastic Differential Equation (SDE),

$$X_t = \mathrm{x}_0^\delta + \int_0^t V_0(X_s, s)\mathrm{d}s + \sum_{i=1}^N \int_0^t V_i(X_s, s)\mathrm{d}W_s^i, \tag{1.3}$$

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where $((W^i_t)_{t\geqslant 0}, i \in \{1,\dots,N\})$ are N independent $\mathbb{R}$-valued standard Brownian motions and $V_0 := \partial_y\psi(.,.,0,0) + \frac{1}{2}\sum_{i=1}^N \partial^2_{z^i}\psi(.,.,0,0)$, $V_i = \partial_{z^i}\psi(.,.,0,0)$, $i \in \{1,\dots,N\}$.
More particularly, we show that, for $\epsilon > 0$, for $T \in \pi^\delta$, $T \geqslant 2\delta$, if $X_0 = X^\delta_0 = \mathrm{x} \in \mathbb{R}^d$,

$$\begin{aligned} d_{TV}(\mathrm{Law}(X_T), \mathrm{Law}(X^\delta_T)) =& \frac{1}{2} \sup_{f:\mathbb{R}^d\to[-1,1], f \text{ measurable}} |\mathbb{E}[f(X_T) - f(X^\delta_T)]| \\ \leqslant& \delta^{\frac{1}{2}-\epsilon} \frac{1+|\mathrm{x}|^c_{\mathbb{R}^d}}{|\mathcal{V}_L(\mathrm{x})T|^\eta} C \exp(CT). \end{aligned} \tag{1.4}$$

where c, C, η are positive constants and $\mathcal{V}_L(\mathrm{x}) \in (0,1]$ under a local weak Hörmander type property (of order L, see (2.5) for details and definititon of $\mathcal{V}_L$) at initial point x. It is noteworthy that, the rate $\delta^{\frac{1}{2}}$ can be replaced by δ if the third order moment of Z^δ_t, $t \in \pi^{\delta,*}$, are supposed to be equal to zero. Consequently, X_t admits a density which can be approximated (uniformly on compact sets) by the one of $\overline{X}^\delta_t$. Similar estimates also hold for the derivatives of the density. Those results are derived under polynomial type upper bounds on the derivatives of ψ in conjunction with the aforementioned local weak Hörmander type property.

Processes such as $(X^\delta_t)_{t\in\pi^\delta}$ commonly appear in weak approximation problems where the perspective differs from the introduction of the earlier results. The problematic is to consider a process $(X_t)_{t\geqslant 0}$ solution to a given SDE similar to (1.3). Subsequently, the aim is to build the weak approximation process $(X^\delta_t)_{t\in\pi^\delta}$ and then compute an approximation for $\mathbb{E}[f(X_t)]$ by means of $\mathbb{E}[f(X^\delta_t)]$. Two interconnected questions naturally arise. First, what is the rate of convergence of the approximation as δ tends to zero? Second, for which class of functions f does this rate hold? Among others, this paper addresses those questions by providing an upper bound for the total variation distance (that is when f is bounded and measurable) with rate $\delta^{\frac{1}{2}-\epsilon}$. It's worth noting that this rate could be improved to $\delta^{1-\epsilon}$ or even $\delta^{m-\epsilon}$, $m \in \mathbb{N}$, regarding some conditions on Z^δ_t, $t \in \pi^{\delta,*}$ and ψ. Considering smooth f bounded with bounded derivatives up to some given order, it is well established that the weak convergence of the Euler scheme ($\psi(x,t,z,y) = V_0(x,t)y + \sum_{i=1}^N V_i(x,t)z^i$) occurs with rate δ (see [30]), but various higher order methods (see *e.g.* [29], [22], [1]) propose better rates (that are referred to as weak smooth rates in this paper). An intriguing question emerges: do these weak smooth rates still apply to total variation convergence? In the case of the Euler scheme with Gaussian increments, the total variation convergence with order δ is established in [8] in a homogeneous uniform weak Hörmander setting. For higher order methods, a solution combining the use of existing results concerning weak smooth rates and regularization properties similar to (1.2) is provided in [7]. In this article, it is shown that for $(X^\delta_t)_{t\in\pi^\delta}$ defined as in (1.1), the total variation rate aligns with the weak smooth rate under the restriction that ψ has smooth derivatives and satisfies a uniform elliptic property (*i.e.* uniform Hörmander property of order 0): For every $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, $\mathrm{span}(V_i, i \in \{1,\dots,N\})(x,t) = \mathbb{R}^d$.

The results [8] and [7] offer first insights for establishing total variation convergence under Hörmander type conditions for processes satisfying (1.1). The complexity of our approach relies both in the abstract definition (1.1) and in the weak Hörmander properties at any order L considered in a local setting. To provide clarity on our intentions, let's delve into specifics. To begin, we give an alternative formulation of (1.3) by employing the Stratonovich integral:

$$X_t = \mathrm{x}^\delta_0 + \int_0^t \bar{V}_0(X_s,s)\mathrm{d}s + \sum_{i=1}^N \int_0^t V_i(X_s,s)\circ \mathrm{d}W^i_s, \tag{1.5}$$

with $\bar{V}_0 = V_0 - \frac{1}{2}\sum_{i=1}^N \nabla_x V_i V_i$. In this article, $\bar{V}_0, V_i, i \in \{1,\dots,N\})$ and its derivatives are supposed to have polynomial growth in the space variable except for the order one derivatives in space which are simply bounded so that the existence of an *a.s.* unique solution to (1.5) is guaranteed. The infinitesimal generator of the Markov process $(X_t)_{t\geqslant 0}$ expresses as $A = \bar{V}_0\partial_{x_0} + \frac{1}{2}\sum_{i=1}^N (V_i\partial_{x_i})^2$. As demonstrated in the seminal work [17], the hypoellipticity of $A + \partial_t$ and then the existence of a smooth density for X_t is closely related the dimension of some Lie algebras generated with the vector fields $\bar{V}_0, V_i, i \in \{1,\dots,N\})$. This type of properties are referred to as Hörmander conditions, which we now introduce.

We consider, for fixed $t \geqslant 0$, the vector fields on $\mathbb{R}^d$ given by, $x \mapsto \bar{V}_0(x,t)$ and $x \mapsto V_i(x,t)$, $i \in \{1,\dots,N\}$. Subsequently, we introduce the extended vector fields on $\mathbb{R}^d \times \mathbb{R}_+$ denoted by $\bar{V}_{*,0} : (x,t) \mapsto (\bar{V}_0(x,t),t)$ and $V_{*,i} : (x,t) \mapsto (V_i(x,t),0)$, $i \in \{1,\dots,N\}$. In particular, the following relationship on Lie bracket holds: For V, W, two vector fields in $\{\bar{V}_0, V_1, \ldots, V_N\}$ and $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, $j \in \{1, \ldots, d+1\}$,

$$\begin{aligned}[V_*, W_*](x,t)^j =&(\nabla_x W V(x,t) - \nabla_x V W(x,t))^j + \partial_t W_*^j V_*^{d+1}(x,t) - \partial_t V_*^j W_*^{d+1}(x,t)\\ =&[V,W](x,t)^j + \partial_t W_*^j V_*^{d+1}(x,t) - \partial_t V_*^j W_*^{d+1}(x,t).\end{aligned}$$

It's worth noting that $x \mapsto [V,W](x,t)$ is a vector field on $\mathbb{R}^d$ and we use convention $[V,W]^{d+1} = 0$. We are now in a position to present the Hörmander properties which mainly consists in assuming that the vector fields generated by the Lie brackets is full in $\mathbb{R}^d$. Various versions of Hörmander properties appear in the literature serving to prove hypoellipticity. We try to give a brief overview. Let us introduce

$$\begin{aligned}\mathbf{V}_{*,0} =&\{V_{*,i}, i \in \{1, \ldots, N\}\}.\\ \mathbf{V}_{*,n+1} =&\mathbf{V}_{*,n} \cup \{[\bar{V}_{*,0}, V], [V_{*,i}, V], i \in \{1, \ldots, N\}, V \in \mathbf{V}_{*,n}\}, \quad n \in \mathbb{N}.\end{aligned}$$

Similarly, we define $\mathbf{V}_n$, $n \in \mathbb{N}$, in the same way but with $\bar{V}_{*,0}$ (respectively $V_{*,1}, \ldots, V_{*,N}$) replaced by $\bar{V}_0$ (resp. $V_1, \ldots, V_N$). The weak local Hörmander assumption (at initial point $(X_0 = \mathrm{x}, 0)$) in inhomogeneous setting (*i.e.* when $V_0, \ldots, V_N$ depend on time), which is the one we use in this paper, consists in assuming that

$$\mathrm{span}(\cup_{n=0}^{\infty} \mathbf{V}_{*,n})(\mathrm{x}, 0) = \mathbb{R}^d.$$

In the homogeneous setting (*i.e.* $V_0, V_1, \ldots, V_N$ do not depend on the time component), it consists in assuming that: $\mathrm{span}(\cup_{n=0}^{\infty} \mathbf{V}_n)(\mathrm{x}, 0) = \mathbb{R}^d$ (see *e.g.* [20]).Obviously, if coefficients $V_0, V_1, \ldots, V_N$ do not depend on the time component, this last condition is equivalent to assume that $\mathrm{span}(\cup_{n=0}^{\infty} \mathbf{V}_{*,n})(\mathrm{x}, 0) = \mathbb{R}^d$.

Notice that, when $\mathrm{span}(\mathbf{V}_{*,0}) = \mathbb{R}^d$, we are in the elliptic setting. The hypothesis is termed "local" Hörmander because $\mathbf{V}_{*,n}$ is considered at the initial point $(X_0 = x, 0)$. In the case where, for every $(y,t) \in \mathbb{R}^d \times \mathbb{R}_+$, we have $\mathrm{span}(\cup_{n=0}^{\infty} \mathbf{V}_{*,n})(y,t) = \mathbb{R}^d$, we refer to it as "uniform" Hörmander property. The term "weak" Hörmander pertains to the definition of $\mathbf{V}_{*,n}$ (or $\mathbf{V}_n$). Specifically, the "strong" Hörmander property corresponds to the case where $\bar{V}_{*,0}$ is replaced by 0 in the computation of $\mathbf{V}_{*,n}$. The investigation of Hörmander properties in inhomogeneous setting is, for example, conducted to prove existence of smooth density in [12] or [13] for the weak uniform setting, in [11] for the strong local setting or in [18] or [27] for the weak local setting. For the homogeneous case, refer *e.g.* to [20], [24], [6] or [26] for applications of local weak Hörmander properties. We finally point out that, following the observation made [31] in the uniform Hörmander setting for SDE with inhomogeneous coefficient, hypoellipticity may not hold if only $\mathrm{span}(\cup_{n=0}^{\infty} \mathbf{V}_n) = \mathbb{R}^d$.

The results presented in this paper offer, among others, the opportunity to extend the abstract framework from [7] so that, it can be applied to the total variation approximation of inhomogeneous SDE having polynomial bounds on their coefficients and their derivatives and satisfying the ususal weak local Hörmander property. In terms of the function ψ, it simply consists in supposing a weak local Hörmander type property (see (2.5)) and assuming polynomial growth properties on the derivatives of ψ (see (2.2) and (2.3)). In the homogeneous case, those assumptions are similar to the ones made in [20] concerning the coefficients of (1.5). We also highlight that the regularization properties established in this current paper (see Theorem 2.1), enables to demonstrate that the total variation rate of convergence in the local weak hypoelliptic setting, aligns with the weak smooth rate. (see Remark 2.2). Total variation convergence with high rates of convergence can thus be obtained for the methods presented *e.g.* in [29], [22] or [1].

Similar results have previously been explored but only restricted to the case where $(Z_t^{\delta})_{t \in \pi^{\delta,*}}$ is made of standard Gaussian variables and for some specific ψ (see *e.g.* [8] when ψ is the Euler scheme of a homogeneous SDE satisfying weak uniform Hörmander property). In particular standard Malliavin calculus can be applied to derive total variation convergence. It is worth mentioning that analogous results are also investigated under a different (and weaker) condition from the Hörmander one, called the UFG condition, but we do not discuss this type of hypothesis in this paper (see *e.g.* [19] for an order two rate scheme still in the homogeneous setting). In [8], the methodology differs from ours in the sense that the estimates are obtained relying on the proximity (in the L^p-sense for Sobolev norms built with Malliavin derivatives) between a well chosen coupling of the scheme $(X_t^{\delta})_{t \in \pi^{\delta}}$ and the limit $(X_t)_{t \geqslant 0}$ which satisfies standard regularization results under suitable properties (see *e.g.* [20]). More particularly, a continuous time version of $(X_t^{\delta})_{t \in \pi^{\delta}}$ which satisfies a similar SDE as (1.3) (but with freezed coefficient) can be built. In this SDE context, specific to the Euler scheme, the Malliavin calculus techniques are well known and used by the authors to bound the Sobolev norms. Conversely, our approach is self contained and regularization properties for $(\overline{X}_t^\delta)_{t\in\pi^\delta}$ are derived without using the ones satisfied by $(X_t)_{t\geqslant 0}$. Our techniques draw inspiration from Malliavin calculus but is adapted to our discrete setting and also to not only Gaussian random variables because the law of $(Z_t^\delta)_{t\in\pi^{\delta,*}}$ may be arbitrary. Due to the liberty granted to the choice of ψ and to the law of $(Z_t^\delta)_{t\in\pi^{\delta,*}}$, our result may be seen as an invariance principle. Moreover, the law of X_t depends on ψ only through his first order derivative in y and first and second order derivatives in z evaluated at some points $(x,t,0,0)$, with $x\in\mathbb{R}^d, t\geqslant 0$. Hence a similar limit is reached for a large class of function ψ and random variables $(Z_t^\delta)_{t\in\pi^{\delta,*}}$.

1.2. **Organization of the paper.** Section 2 introduces the key technical result of this paper, focusing on regularization properties of discrete time Markov process with form (1.1), namely Theorem 2.1. Additionally, the hypoellipticity result, meaning existence of smooth density for solution of (1.5) is exposed in Theorem 2.2 as well as a slightly more general version of approximation (1.4) and a density estimate result. Then, in Section 3, we delve into the development of a Malliavin inspired discrete differential calculus in order to prove the smoothing properties of Theorem 2.1. Finally, Section 4 is dedicated to prove some estimates on Malliavin weights as well as on Sobolev norms and Malliavin covariance matrix moments. These estimates collectively contribute to the recovery of the regularization properties detailed in Theorem 2.1.

1.3. **Notations.** For E and $E^\diamond$ two sets, we denote by $E^{E^\diamond}$ the set of funtions from $E^\diamond$ to E, and for $d\in\mathbb{N}^*$, we use the standard notation $E^d := E^{\{1,\dots,d\}}$. We denote by $\mathcal{M}(\mathbb{R}^d)$ (respectively $\mathcal{M}_b(\mathbb{R}^d)$), the set of measurable (resp. measurable and bounded) functions defined on $\mathbb{R}^d$. $\mathcal{C}^q(\mathbb{R}^d)$, $q\in\mathbb{N}\cup\{+\infty\}$, is the set of functions admitting derivatives up to order q and such that all those derivatives (including order 0) are continuous and $\mathcal{C}_b^q(\mathbb{R}^d)$ (resp. $\mathcal{C}_K^q(\mathbb{R}^d)$, $\mathcal{C}_{pol}^q(\mathbb{R}^d)$), $q\in\mathbb{N}\cup\{+\infty\}$, is the set of functions belonging to $\mathcal{C}^q(\mathbb{R}^d)$ such that all the derivatives (of order 0 to q) are bounded (resp. have compact support, have polynomial growth).
We will also denotes $\mathcal{M}(\mathbb{R}^d;\mathbb{R})$ for measurable function on $\mathbb{R}^d$ taking values in $\mathbb{R}$ (and similarly for other set of functions defined above).

When dealing with functions defined and taking values on Hilbert spaces, we introduce some notations: Let $\mathcal{H},\mathcal{H}^\diamond$ be two Hilbert spaces. For $f:\mathcal{H}\to\mathcal{H}^\diamond$ and $u\in\mathcal{H}$, the directional derivative $\partial_u^{\mathrm{F}} f$ of f along u is given by (when it exists) $\partial_u^{\mathrm{F}} f(x) := \lim_{\epsilon\to 0}\frac{f(x+\epsilon u)-f(x)}{\epsilon}$ for every $x\in\mathcal{H}$. When f is Frechet differentiable, we recall that $u\mapsto \partial_u^{\mathrm{F}} f(x)$ is a linear application from $\mathcal{H}$ to $\mathcal{H}^\diamond$ that we simply denote $\partial^{\mathrm{F}} f(x)$. When $\mathcal{H}^\diamond=\mathbb{R}$, we denote $\mathrm{d}^{\mathrm{F}} f(x)$ (which is uniquely defined by Riesz theorem) such that for every $u\in\mathcal{H}$, $\partial_u^{\mathrm{F}} f(x) = \langle \mathrm{d}^{\mathrm{F}} f(x), u\rangle_{\mathcal{H}}$. For $f\in\mathcal{M}_b(\mathbb{R}^d;\mathbb{R}^{d^\diamond})$, we introduce the supremum norm $\|f\|_\infty = \sup_{x\in\mathbb{R}^d}|f(x)|_{\mathbb{R}^{d^\diamond}}$ with $|.|_{\mathbb{R}^{d^\diamond}}$ the norm induced by the scalar product $\langle f, f^\diamond\rangle_{\mathbb{R}^{d^\diamond}} = \sum_{j=1}^{d^\diamond} f^j f^{\diamond,j}$. When f takes values in $\mathbb{R}^{d^\diamond\times d^\diamond}$, we denote $\|f\|_{\mathbb{R}^{d^\diamond}} = \sup_{\xi\in\mathbb{R}^{d^\diamond};|\xi|_{\mathbb{R}^{d^\diamond}}=1}|f\xi|_{\mathbb{R}^{d^\diamond}}$. For a multi-index $\alpha=(\alpha^1,\cdots,\alpha^d)\in\mathbb{N}^d$ we denote $|\alpha|=\alpha^1+...+\alpha^d$, $\|\alpha\|=d$ and if $f\in\mathcal{C}^{|\alpha|}(\mathbb{R}^d)$, we define $\partial^\alpha f = (\partial_1)^{\alpha^1}\dots(\partial_d)^{\alpha^d} f = \partial_x^\alpha f(x) = \partial_{x^1}^{\alpha^1}\dots\partial_{x^d}^{\alpha^d} f(x)$. Also, for $\beta\in\mathbb{N}^{d^\diamond}$, we define $(\alpha,\beta) = (\alpha^1,\cdots,\alpha^d,\beta^1,\dots,\beta^{d^\diamond})$. In addition, we also denote $\nabla_x f = (\partial_{x^j} f_i)_{(i,j)\in\{1,\dots,d^\diamond\}\times\{1,\dots,d\}}$ for the Jacobian matrix of f and $\mathbf{H}_x f = ((\partial_{x^j}\partial_{x^l} f^i)_{(l,j)\in\{1,\dots,d\}\times\{1,\dots,d\}})_{i\in\{1,\dots,d^\diamond\}}$ for the Hessian matrix of f. In particular, for $v\in\mathbb{R}^d$, $v^T\mathbf{H}_x f\in\mathbb{R}^{d^\diamond\times d}$ and $(v^T\mathbf{H}_x f)^{i,j} = \sum_{l=1}^d \partial_{x^j}\partial_{x^l} f^i v^l$. We include the multi-index $\alpha=(0,...,0)$ and in this case $\partial^\alpha f = f$.
In addition, unless it is stated otherwise, C stands for a universal constant which can change from line to line, and given some parameter ϑ, $C(\vartheta)$ is a constant depending on ϑ.
Also, $\mathbf{1}_{a,b}$ stands for the Kronecker symbol, meaning $\mathbf{1}_{a,b}=1$ if $a=b$ and is zero otherwise.
Finally, for a discrete time process $(Y_t)_{t\in\pi^\delta}$, we denote by $\mathcal{F}_t^Y := \sigma(Y_w, w\in\pi^\delta, w\leqslant t)$ the sigma algebra generated by Y until time t.

2. Main results

In this section, we present our main result about the regularization properties of $(X_t^\delta)_{t\in\pi^\delta}$. Once the regularization results are established (Theorem 2.1), we infer the existence of a total variation limit for X_t^δ, for fixed $t\in\pi^\delta$, in terms of a solution to a specific SDE (Theorem 2.2).

2.1. A Class of Markov Semigroups.

Definition of the semigroups. We work on a probability space $(\Omega,\mathcal{F},\mathbb{P})$. For $\delta\in(0,1]$ and $N\in\mathbb{N}^*$, we consider a sequence of independent random variables $Z_t^\delta\in\mathbb{R}^N$, $t\in\pi^{\delta,*}$, and we assume that Z_t^δ, are centered with $\mathbb{E}[Z_t^{\delta,i}Z_t^{\delta,j}] = \mathbf{1}_{i,j}$ for every $i, j \in \mathbf{N} := \{1, \ldots, N\}$ and every $t \in \pi^{\delta,*}$. We construct the $\mathbb{R}^d$-valued Markov process $(X_t^\delta)_{t\in\pi^\delta}$ in the following way:

$$X_{t+\delta}^\delta = \psi(X_t^\delta, t, \delta^{\frac{1}{2}} Z_{t+\delta}^\delta, \delta), \quad t \in \pi^\delta, \quad X_0^\delta = \mathrm{x}_0^\delta \in \mathbb{R}^d \tag{2.1}$$

where

$$\psi \in \mathcal{C}^\infty(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^N \times [0,1]; \mathbb{R}^d) \quad \text{and} \quad \forall (x,t) \in \mathbb{R}^d \times \pi^\delta, \psi(x,t,0,0) = x.$$

Let us now define the discrete time semigroup associated to $(X_t^\delta)_{t\in\pi^\delta}$. For every measurable function f from $\mathbb{R}^d$ to $\mathbb{R}$, and every $x \in \mathbb{R}^d$,

$$\forall t \in \pi^\delta, \qquad Q_t^\delta f(x) = \int_{\mathbb{R}^d} f(y) Q_t^\delta(x, \mathrm{d}y) := \mathbb{E}[f(X_t^\delta) | X_0^\delta = x].$$

We will obtain regularization properties for modifications of this discrete semigroup. Our approach relies on some hypothesis on ψ and Z^δ we now present.

Hypothesis on ψ. Polynomial growth and Hörmander property. We first consider a polynomial growth assumption concerning the derivatives of ψ: For $r \in \mathbb{N}^*$,

$\mathbf{A}_1^\delta(r)$. There exists $\mathfrak{D}, \mathfrak{D}_r \geqslant 1, \mathfrak{p}, \mathfrak{p}_r \in \mathbb{N}$ such that $\mathfrak{D} \geqslant \mathfrak{D}_2$, $\mathfrak{p} \geqslant \mathfrak{p}_2$ and for every $(x,t,z,y) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R}^N \times [0,1]$,

$$\sum_{|\alpha^x|+|\alpha^t|=0}^{r} \sum_{|\alpha^z|+|\alpha^y|=1}^{r-|\alpha^x|-|\alpha^t|} |\partial_x^{\alpha^x} \partial_t^{\alpha^t} \partial_z^{\alpha^z} \partial_y^{\alpha^y} \psi|_{\mathbb{R}^d}(x,t,z,y) \leqslant \mathfrak{D}_r(1 + |x|_{\mathbb{R}^d}^{\mathfrak{p}_r} + \delta^{-\frac{\mathfrak{p}_r}{2}} |z|_{\mathbb{R}^N}^{\mathfrak{p}_r}), \tag{2.2}$$

and

$$\{\sum_{l=1}^{d} |\partial_{x^l} \partial_y \psi|_{\mathbb{R}^d} + \sum_{i=1}^{N} |\partial_{x^l} \partial_{z^i} \psi|_{\mathbb{R}^d} + \sum_{i,j=1}^{N} |\partial_{x^l} \partial_{z^i} \partial_{z^j} \psi|_{\mathbb{R}^d}\}(x,t,z,y) \leqslant \mathfrak{D}(1 + \delta^{-\frac{\mathfrak{p}}{2}} |z|_{\mathbb{R}^N}^{\mathfrak{p}}) \tag{2.3}$$

Without loss of generality, we assume that the sequences $(\mathfrak{D}_r)_{r\in\mathbb{N}^*}$ and $(\mathfrak{p}_r)_{r\in\mathbb{N}^*}$ are non decreasing. We denote $\mathbf{A}_1^\delta(+\infty)$ when $\mathbf{A}_1^\delta(r)$ is satisfied for every $r \in \mathbb{N}^*$.
Notice also that, we obtain exactly the same results if we add $\mathfrak{D}\delta^{-1}|y|$ in the *r.h.s.* of (2.3), or if we add $\mathfrak{D}_r\delta^{-1}|y|$ in the *r.h.s.* of (2.2). This is due to the fact that the function ψ is only used for $y = \delta$ (or $y = C\delta$, $C \leqslant 1$) so the assumptions above are then satisfied replacing $\mathfrak{D}$ (respectively $\mathfrak{D}_r$) by $2\mathfrak{D}$ (respectively $2\mathfrak{D}_r$). Also, we do not give explicit dependence of the *r.h.s* of (2.2) or (2.3) *w.r.t.* the variable t because in our results, t is taken in a compact interval with form $[0,T]$.

At this point, let us observe that we can rely this assumption with the one in [20] where the authors directly study the existence of density of the solution of (1.3) by means of standard Malliavin calculus but when coefficients do not depend on time. Taking ψ linear in its third and fourth variable, and homogeneous, *i.e.* $\psi : (x,t,z,y) \mapsto x + V_0(x)y + \sum_{i=1}^N V_i(x)z^i$ then, exactly $\mathbf{A}_1^\delta(+\infty)$ is the regularity assumption made on $V_0, \ldots, V_N$ in [20] (combined with a weak local Hörmander property) to derive similar estimates as (2.1) in Corollary 2.1.

The second hypothesis we need on ψ is local weak Hörmander property on some vector fields we now introduce. We denote the Lie bracket of two $\mathcal{C}^1$ vector fields in $\mathbb{R}^d$, $[,] : (\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d))^2 \to \mathcal{C}^0(\mathbb{R}^d, \mathbb{R}^d)$, $f_1, f_2 \mapsto [f_1, f_2] := \nabla_x f_2 f_1 - \nabla_x f_1 f_2$.
We denote $V_0 = \partial_y \psi(.,.,0,0) + \frac{1}{2}\sum_{i=1}^N \partial_{z^i}^2 \psi(.,.,0,0)$, $V_i = \partial_{z^i}\psi(.,.,0,0)$, $i \in \mathbf{N}$, $\bar{V}_0 = V_0 - \frac{1}{2}\sum_{i=1}^N \nabla_x V_i V_i$. For a multi-index $\alpha \in \{0, \ldots, N\}^{\|\alpha\|}$ and $V : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, we define also $V^{[\alpha]}$ using the recurrence relation $V^{[(\alpha,0)]} = [\bar{V}_0, V^{[\alpha]}] + \partial_t V^{[\alpha]} + \frac{1}{2}\sum_{i=1}^N [V_i, [V_i, V^{[\alpha]}]]$ and $V^{[(\alpha,j)]} := [V_j, V^{[\alpha]}]$ if $j \in \{1, \ldots, N\}$ with the convention $V^{[\emptyset]} = V$. We are now in a position to introduce our Hörmander hypothesis on ψ: For $L \in \mathbb{N}$, the order of our Hörmander condition, let us define for every $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$,

$$\mathcal{V}_L(x,t) := 1 \wedge \inf_{\mathbf{b}\in\mathbb{R}^d, |\mathbf{b}|_{\mathbb{R}^d}=1} \sum_{\substack{\alpha\in\{0,\ldots,N\}^{\|\alpha\|};\\ \|\alpha\|\leqslant L}} \sum_{i=1}^{N} \langle V_i^{[\alpha]}(x,t), \mathbf{b}\rangle_{\mathbb{R}^d}^2. \tag{2.4}$$

We introduce, for $\mathrm{x} \in \mathbb{R}^d$:

$\mathbf{A}_2(\mathrm{x},L)$. Our local weak Hörmander property of order $L \in \mathbb{N}$,

$$\mathcal{V}_L(\mathrm{x},0) > 0. \tag{2.5}$$

Especially, this hypothesis is used at initial point for $\mathrm{x} = \mathrm{x}_0^\delta$. We will sometimes consider a uniform weak Hörmander property of order L,

$$\mathcal{V}_L^\infty := \inf_{t\in\mathbb{R}_+} \inf_{x\in\mathbb{R}^d} \mathcal{V}_L(x,t) > 0. \tag{2.6}$$

In this case, we denote $\mathbf{A}_2^\infty(L)$ instead of $\mathbf{A}_2(\mathrm{x},L)$. Also, we usually denote $\mathcal{V}_L(x) := \mathcal{V}_L(x,0)$.

It is worth noticing that, with the notations introduced in the Introduction, (2.5) is satisfied for some $L \in \mathbb{N}$ if and only if $\mathrm{span}(\cup_{n=0}^\infty \mathbf{V}_{*,n})(\mathrm{x}_0^\delta,0) = \mathbb{R}^d$, which is why, we refer to it as local weak Hörmander property. A similar observation holds for (2.6) in the uniform setting. The case $L=0$ corresponds to the elliptic case.

Hypothesis on Z^δ. Lebesgue lower bounded distributions. A first assumption concerns the finiteness of the moment of Z^δ: For $p \geqslant 0$,

$\mathbf{A}_3^\delta(p)$.

$$\mathfrak{M}_p(Z^\delta) := 1 \vee \sup_{t\in\pi^{\delta,*}} \mathbb{E}[|Z_t^\delta|_{\mathbb{R}^N}^p] < \infty. \tag{2.7}$$

.

We denote $\mathbf{A}_3^\delta(+\infty)$ the assumption such that $\mathbf{A}_3^\delta(p)$ is satisfied for every $p \geqslant 0$.

A second assumption is made on the distribution of Z^δ. We suppose that the distribution of Z^δ is Lebesgue lower bounded:

$\mathbf{A}_4^\delta$. There exists $z_* = (z_{*,t})_{t\in\pi^{\delta,*}}$ taking its values in $\mathbb{R}^N$ and $\varepsilon_*, r_* > 0$ such that for every Borel set $A \subset \mathbb{R}^N$ and every $t \in \pi^{\delta,*}$,

$$L_{z_*}^\delta(\varepsilon_*, r_*) \qquad \mathbb{P}(Z_t^\delta \in A) \geqslant \varepsilon_* \lambda_{\mathrm{Leb}}(A \cap B_{r_*}(z_{*,t})) \tag{2.8}$$

where λ_{Leb} is the Lebesgue measure on $\mathbb{R}^N$.

Let us comment assumption $\mathbf{A}_4^\delta$. First, notice that (2.8) holds if and only if there exists some non negative measures μ_t^δ with total mass $\mu_t^\delta(\mathbb{R}^N) < 1$ and a lower semi-continuous function $\varphi \geqslant 0$ such that $\mathbb{P}(Z_t^\delta \in dz) = \mu_t^\delta(dz) + \varphi(z - z_{*,t})dz$ for every $t \in \pi^{\delta,*}$. We also point out that the random variables $(Z_t^\delta)_{t\in\pi^{\delta,*}}$ are not assumed to be identically distributed. However, the fact that $r_* > 0$ and $\varepsilon_* > 0$ are the same for all k represents a mild substitute of this property. In order to construct φ we introduce the following function: For $v > 0$, set $\varphi_v : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\varphi_v(z) = \mathbf{1}_{|z|_{\mathbb{R}^N} \leqslant v} + \exp\Big(1 - \frac{v^2}{v^2 - (|z|_{\mathbb{R}^N} - v)^2}\Big)\mathbf{1}_{v<|z|_{\mathbb{R}^N}<2v}. \tag{2.9}$$

Then $\varphi_v \in \mathcal{C}_b^\infty(\mathbb{R}^N;\mathbb{R})$, $0 \leqslant \varphi_v \leqslant 1$ and we have the following crucial property: For every $p,q \in \mathbb{N}$, every $z \in \mathbb{R}^N$

$$|\sum_{\substack{\alpha^z\in\mathbb{N}^N \\ |\alpha^z|\in\{1,\ldots,q+1\}}} |\partial_z^{\alpha^z} \ln \varphi_v(z)|^2|^{\frac{p}{2}} \varphi_v(z) \leqslant \frac{C(q,p)N^{\frac{pq}{4}}}{v^{pq}}, \tag{2.10}$$

with the convention $\ln \varphi_v(z) = 0$ for $|z| \geqslant 2v$.
As an immediate consequence of (2.8), for every non negative function $f : \mathbb{R}^N \to \mathbb{R}_+$ and $t \in \pi^\delta$, $t > 0$,

$$\mathbb{E}[f(Z_t^\delta)] \geqslant \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(\ z - z_{*,t}\ ) f(z)\mathrm{d}z.$$

We denote

$$m_* = \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(z)dz = \varepsilon_* \int_{\mathbb{R}^N} \varphi_{r_*/2}(z - z_{*,t})\mathrm{d}z$$

We consider a sequence of independent random variables $\chi_t^\delta \in \{0,1\}$, $U_t^\delta, V_t^\delta \in \mathbb{R}^N$, $t \in \pi^{\delta,*}$, with laws given by

$$\begin{aligned}\mathbb{P}(\chi_t^\delta=1)=&m_*,\qquad \mathbb{P}(\chi_t^\delta=0)=1-m_*,\\ \mathbb{P}(\delta^{-\frac12}U_t^\delta\in \mathrm{d}z)=&\frac{\varepsilon_*}{m_*}\varphi_{r_*/2}(z-z_{*,t})\mathrm{d}z,\\ \mathbb{P}(\delta^{-\frac12}V_t^\delta\in \mathrm{d}z)=&\frac{1}{1-m_*}(\mathbb{P}(Z_t^\delta\in \mathrm{d}z)-\varphi_{\frac{r_*}{2}}(z-z_{*,t})\mathrm{d}z).\end{aligned}$$

where $\varphi_{\frac{r_*}{2}}$ satisfies (2.10) with $v=\frac{r_*}{2}$. Notice that $\mathbb{P}(V_t^\delta\in \mathrm{d}z)\geqslant 0$ and a direct computation shows that

$$\mathbb{P}(\chi_t^\delta U_t^\delta+(1-\chi_t^\delta)V_t^\delta\in dz)=\mathbb{P}(\delta^{\frac12}Z_t^\delta\in \mathrm{d}z).$$

This is the splitting procedure for Z_t^δ. Now on we will work with this representation of the law of Z_t^δ. Consequently, we always use the decomposition

$$\delta^{\frac12}Z_t^\delta=\chi_t^\delta U_t^\delta+(1-\chi_t^\delta)V_t^\delta.$$

The above splitting procedure has already been widely used in the literature and is usually referred to as the Nummelin splitting. In [25] and [21], it is used in order to prove convergence to equilibrium of Markov processes. In [9], [10] and [33], it is used to study the Central Limit Theorem. Also, in [23], the above splitting method (with $\mathbf{1}_{B_{r_*}(z_{*,t})}$ instead of $\varphi_{r_*/2}(z-z_{*,t})$) is used in a framework which is similar to the one in this paper. Finally in [7], it is used to prove regularization properties of Markov semigroup under the uniform ellipticity property: $\inf_{(x,t)\in\mathbb{R}^d\times\pi^\delta}\mathcal{V}_0(x,t)>0$.

We introduce a final structural assumption specifying that the time step δ needs to be small enough. For $\delta\in(0,1]$, when (2.3) holds, we define

$$\begin{aligned}\eta_1(\delta):=&\delta^{-d\frac{44}{91}}\min(1,\frac{10^d}{m_*^d|2^{10}(1+T^3)|^{\frac d2}})\quad\text{and}\\ \eta_2(\delta):=&\min(\delta^{-\frac12}\eta_1(\delta)^{-\frac1d},\frac12|\delta^{\frac12}8\mathfrak{D}|^{-\frac{1}{\mathfrak{p}+1}}).\end{aligned}\tag{2.11}$$

with $\mathfrak{p}$ given in (2.3). For $T\in\pi^{\delta,*}$, $\mathrm{x}\in\mathbb{R}^d$, we introduce the following assumption:

$\mathbf{A}_5^\delta(\mathrm{x},T)$. Assume that (2.3) and $\mathbf{A}_2(\mathrm{x},L)$ (see (2.5)) hold and that $\delta\in(0,1]$ is small enough so that

$$\begin{aligned}\eta_1(\delta)>&\max(1,\frac{2^{1-\frac d2}}{d^{-\frac d2}},2(\frac{40(L+1)N^{\frac{L(L+1)}{2}}}{T\mathcal{V}_L(\mathrm{x})m_*})^{d13^L},\\ &2\mathbf{1}_{L=0}+2\mathbf{1}_{L>0}|m_*|2^8(1+T)|^{143}10N^{\frac{L(L-1)}{2}}|^{d13^{L-1}})\quad\text{and}\\ \eta_2(\delta)>&1.\end{aligned}\tag{2.12}$$

Similarly as the assumption $\mathbf{A}_2(\mathrm{x},L)$, this hypothesis is used at initial point for $\mathrm{x}=\mathrm{x}_0^\delta$. Considering the lower bound of $\eta_1(\delta)$ in (2.12), it becomes apparent that while it remains independent of δ, it may assume excessively large values. This minimum could potentially be decreased with modifications to the proof structure, but at the expense of possibly higher upper bounds on the semigroup's derivatives. In this paper, we tailor our proof to minimize the reliance of $C(x,T)$ in (1.2) with respect to $\frac{1}{\mathcal{V}_L(x)}$ and $\frac1T$. Specifically, our proofs are designed so that the constant η appearing in Theorem 2.1, Corollary 2.1, and Theorem 2.2 are as small as possible. Explicit values for η are given in the proof of those results.

2.2. **An alternative regularization property.** In this section we provide the regularization property for a modified version of X^δ. We consider a d-dimensional standard (centered with covariance identity) Gaussian random variable G which is independent from $(Z_t^\delta)_{t\in\pi^{\delta,*}}$, and for $\theta>0$,

$$Q_T^{\delta,\theta}f(x)=\int_{\mathbb{R}^d}f(y)Q_T^{\delta,\theta}(x,\mathrm{d}y):=\mathbb{E}[f(X_T^\delta+\delta^\theta G)|X_0^\delta=x],\quad T\in\pi^\delta.\tag{2.13}$$

It can be seen as a regularization by convolution of the semigroup Q^δ. From a practical viewpoint, the modified version $X_T^\delta+\delta^\theta G$ is easily computable and then well adapted to simulation based approaches such as Monte Carlo methods.

Theorem 2.1. *Let $T \in \pi^{\delta,*}$, $\mathrm{x} \in \mathbb{R}^d$, $L \in \mathbb{N}$ and $f \in \mathcal{C}^\infty_{pol}(\mathbb{R}^d;\mathbb{R})$ satisfying: there exists $\mathfrak{D}_f \geqslant 0$ and $\mathfrak{p}_f \in \mathbb{N}$ such that for every $x \in \mathbb{R}^d$,*

$$|f(x)| \leqslant \mathfrak{D}_f(1+|x|^{\mathfrak{p}_f}_{\mathbb{R}^d}).$$

Then we have the following properties:

A. *Let $q \in \mathbb{N}$ and $\alpha,\beta \in \mathbb{N}^d$ such that $|\alpha|+|\beta| \leqslant q$. Assume that $\mathbf{A}^\delta_1(\max(q+3,2L+5))$ (see (2.2) and (2.3)), $\mathbf{A}_2(\mathrm{x},L)$ (see (2.5)), $\mathbf{A}^\delta_3(+\infty)$ (see (2.7)), $\mathbf{A}^\delta_4$ (see (2.8)) and $\mathbf{A}^\delta_5(\mathrm{x},T)$ (see 2.12)) hold. Then,*

$$|\partial^\alpha Q^{\delta,\theta}_T \partial^\beta f(\mathrm{x})| \leqslant \mathfrak{D}_f \frac{(1+|\mathrm{x}|^c_{\mathbb{R}^d})C\exp(CT)}{(\mathcal{V}_L(\mathrm{x})T)^\eta}, \tag{2.14}$$

where $\eta \geqslant 0$ depends on d,L,q and θ and $c,C \geqslant 0$ depend on $d,N,L,q,\mathfrak{D},\mathfrak{D}_{\max(q+3,2L+5)},\mathfrak{p}$, $\mathfrak{p}_{\max(q+3,2L+5)},\mathfrak{p}_f,\frac{1}{m_*},\frac{1}{r_*},\theta$ and on the moment of Z^δ and which may tend to infinity if one of those quantities tends to infinity.

B. *Assume that hypothesis from **A.** are satisfied with $\mathbf{A}^\delta_1(\max(q+3,2L+5))$ replaced by $\mathbf{A}^\delta_1(2L+5)$. Then,*

$$|Q^\delta_T f(\mathrm{x}) - Q^{\delta,\theta}_T f(\mathrm{x})| \leqslant \delta^\theta \mathfrak{D}_f \frac{(1+|\mathrm{x}|^c_{\mathbb{R}^d})C\exp(CT)}{(\mathcal{V}_L(\mathrm{x})T)^\eta} \tag{2.15}$$

where $\eta \geqslant 0$ depends on d,L and θ and $c,C \geqslant 0$ depend on $d,N,L,q,\mathfrak{D},\mathfrak{D}_{2L+5},\mathfrak{p}$, $\mathfrak{p}_{2L+5},\mathfrak{p}_f$, $\frac{1}{m_*},\frac{1}{r_*},\theta$ and on the moment of Z^δ and which may tend to infinity if one of those quantities tends to infinity.

Remark 2.1. *We point out that, in the case where $\mathfrak{p}_f = \mathfrak{p}_r = 0$ for every $r \in \mathbb{N}^*$, then $c = 0$ in (2.14) and (2.15). This remark remains valid in Corollary 2.1 (see (2.16)) and Theorem 2.2 (see (2.18) but not (2.19)) stated later in this Section. Assuming further that $\mathbf{A}^\infty_2(L)$ holds, the upper bounds established in Theroem 2.1 thus become uniform w.r.t. x.*

A consequence of Theorem 2.1 concerns the existence of a bounded density with bounded derivatives for $X^\delta_T + \delta^\theta G$. The proof of this result is given in Section 3.2. Notice that an explicit value is given for η. This type of result is usually referred to as hypoellipticity property of the operator $Q^{\delta,\theta}$.

Corollary 2.1. *Let $T \in \pi^{\delta,*}$, $\mathrm{x} \in \mathbb{R}^d$ and $L \in \mathbb{N}$. Let $q \in \mathbb{N}$, let $\alpha,\beta \in \mathbb{N}^d$ such that $|\alpha|+|\beta| \leqslant q$. Assume that $\mathbf{A}^\delta_1(\max(q+d+3,2L+5))$ (see (2.2) and (2.3)), $\mathbf{A}_2(\mathrm{x},L)$ (see (2.5)), $\mathbf{A}^\delta_3(+\infty)$ (see (2.7)), $\mathbf{A}^\delta_4$ (see (2.8)) and $\mathbf{A}^\delta_5(\mathrm{x},T)$ (see (2.12)) hold.*

Then, for every $y \in \mathbb{R}^d$, $Q^{\delta,\theta}_T(\mathrm{x},dy) = q^{\delta,\theta}_T(\mathrm{x},y)dy$ and $q^{\delta,\theta}_T \in \mathcal{C}^q(\mathbb{R}^d \times \mathbb{R}^d)$ satisfies, for every $p > 0$,

$$|\partial^\alpha_x \partial^\beta_y q^{\delta,\theta}_T(\mathrm{x},y)| \leqslant \frac{(1+|\mathrm{x}|^c_{\mathbb{R}^d})C\exp(CT)}{(\mathcal{V}_L(\mathrm{x})T)^\eta(1+|y|^p_{\mathbb{R}^d})}, \tag{2.16}$$

where $\eta \geqslant 0$ depends on d,L,q and θ and $c,C \geqslant 0$ depends on d,N,L,q, $\mathfrak{D}$, $\mathfrak{D}_{\max(q+d+3,2L+5)}$, $\mathfrak{p}$, $\mathfrak{p}_{\max(q+d+3,2L+5)}$, $\mathfrak{p}_f$, $\frac{1}{m_*},\frac{1}{r_*},\theta,p$ and on the moment of Z^δ and which may tend to infinity if one of those quantities tends to infinity.

Moreover, if $\mathfrak{p}_2 = 0$ (see hypothesis $\mathbf{A}^\delta_1$) and there exists $z^\infty \geqslant 1$ such that a.s. $\sup_{t\in\pi^{\delta,*}} |Z^\delta_t|_{\mathbb{R}^N} \leqslant z^\infty$, then,

$$|\partial^\alpha_x \partial^\beta_y q^{\delta,\theta}_T(\mathrm{x},y)| \leqslant \frac{C\exp(CT)}{(\mathcal{V}_L(\mathrm{x})T)^\eta} \exp(-\frac{|y-\mathrm{x}|^2_{\mathbb{R}^d}}{cT}), \tag{2.17}$$

where η is the same as in (2.16), $c > 0$ depends on $\mathfrak{D}_1$ and z^∞, and $C \geqslant 0$ depends on $d,N,L,q,\mathfrak{D}$, $\mathfrak{D}_{\max(q+d+3,2L+5)}$, $\mathfrak{p}$, $\mathfrak{p}_f$, $\frac{1}{m_*},\frac{1}{r_*},\theta$ and z^∞ and which may tend to infinity if one of those quantities tends to infinity.

2.3. **An invariance principle.** Let us consider $(X_t)_{t\geqslant 0}$ the $\mathbb{R}^d$-valued Itô process solution to the SDE (1.3).

In the following results, we show that, for a fixed $T > 0$, X^δ_T converges in total variation towards X_T. Notably, our result is stronger than the total variation convergence since we consider measurable test functions with polynomial growth. Moreover, X_T is endowed with a density which can be approximated by the one of $X^\delta_T + \delta^\theta G$. In an ideal situation, we would like to approximate the density of X_T using the one of X^δ_T. However, due to the absence of regularization properties for the random variable X^δ_T, we cannot offer any assurance regarding the existence of its density. Actually, since the random variables

$(Z_t^\delta)_{t\in\pi^{\delta,*}}$ do not necessarily have a density, we can easily build an example such that X_T^δ does not have a density, for instance by considering $X_T^\delta = \sum_{t\in\pi^{\delta,*};t\leqslant T} Z_t^\delta$. In contrast, since $X_T^\delta + \delta^\theta G$ satisfies the regularization property, we can guarantee the existence of its density together with an upper bound on this density.

Exploiting Theorem 2.1 and Corollary 2.1, we can deduce the convergence of the law of X_T^δ towards the one of X_T as δ tends to zero. We are, among others, interested by obtaining an upper bound for

$$|\mathbb{E}[f(X_T) - f(X_T^\delta)|X_0 = X_0^\delta = x]|$$

which writes $C(x)\delta^m \sup_{x\in\mathbb{R}^d}|f(x)|$ when $f \in \mathcal{M}_b(\mathbb{R}^d)$ (and similarly when f has polynomial growth). One main technical point is that the upper bound does not depend on the derivatives of f.

This result may be seen as an invariance principle under two aspects. First, the law of the limit X_T only depends on derivatives (of order one and two) of ψ evaluated at some points $(x,t,0,0)$ with $(x,t)\in\mathbb{R}^d\times\mathbb{R}_+$. As a consequence, if we replace ψ by any function $\tilde{\psi}$ giving the same evaluations of those derivatives, the limit of X_T^δ remains X_T. Another aspect is that the law of $(Z_t)_{t\in\pi^{\delta,*}}$ is not specified explicitly and can be chosen in a large set of probability measures. In particular, in the following result, we show that only $\mathbf{A}_3^\delta(+\infty)$ (see (2.7)) and $\mathbf{A}_4^\delta$ (see (2.8)) are assumed concerning the law of $(Z_t)_{t\in\pi^{\delta,*}}$.

Theorem 2.2. *Let $T\in\pi^\delta$, with $T\geqslant 2\delta$, $\mathrm{x}\in\mathbb{R}^d$, $L\in\mathbb{N}$ and $m>0$. We have the following properties:*

A. *Let $f\in\mathcal{M}(\mathbb{R}^d;\mathbb{R})$ satisfying: there exists $\mathfrak{D}_f\geqslant 0$ and $\mathfrak{p}_f\in\mathbb{N}$ such that for every $x\in\mathbb{R}^d$,*

$$|f(x)|\leqslant \mathfrak{D}_f(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_f}).$$

Assume that $\mathbf{A}_1^\delta(\max(6,2L+5))$ (see (2.2) and (2.3)), $\mathbf{A}_2(\mathrm{x},L)$ (see (2.5)), $\mathbf{A}_3^\delta(+\infty)$ (see (2.7)), $\mathbf{A}_4^\delta$ (see (2.8)) and $\mathbf{A}_5^\delta(\mathrm{x},T)$ (see (2.12)) hold. Then, for every $\epsilon>0$,

$$|\mathbb{E}[f(X_T) - f(X_T^\delta)|X_0 = X_0^\delta = \mathrm{x}]| \leqslant \delta^{\frac{1}{2}-\epsilon}\mathfrak{D}_f \frac{1+|\mathrm{x}|_{\mathbb{R}^d}^c}{(\mathcal{V}_L(\mathrm{x})T)^\eta} C\exp(CT), \tag{2.18}$$

where $\eta\geqslant 0$ depends on d,L and $\frac{1}{\epsilon}$ and $c,C\geqslant 0$ depend on $d,N,L,\mathfrak{D},\sup_{r\in\mathbb{N}^*}\mathfrak{D}_r$, $\mathfrak{p}$, $\sup_{r\in\mathbb{N}^*}\mathfrak{p}_r$, $\mathfrak{p}_f,\frac{1}{m_*},\frac{1}{r_*},\frac{1}{\epsilon}$ and on the moment of Z^δ and which may tend to infinity if one of those quantities tends to infinity.

B. *Assume that hypothesis from **A.** are satisfied.*
Then, X_T starting at point x has a density $y\in\mathbb{R}^d\mapsto p_T(\mathrm{x},y)$ with $p_T\in\mathcal{C}^\infty(\mathbb{R}^d\times\mathbb{R}^d)$.
Moreover, for every $\theta\geqslant\frac{3}{2}$, $q\in\mathbb{N}$, $\alpha,\beta\in\mathbb{N}^d$ with $|\alpha|+|\beta|\leqslant q$, $p\geqslant 0$, $\epsilon>0$ and every $y\in\mathbb{R}^d$,

$$|\partial_x^\alpha\partial_y^\beta p_T(\mathrm{x},y) - \partial_x^\alpha\partial_y^\beta q_T^{\delta,\theta}(\mathrm{x},y)| \leqslant \delta^{\frac{1}{2}-\epsilon}\frac{(1+|\mathrm{x}|_{\mathbb{R}^d}^c)C\exp(CT)}{(\mathcal{V}_L(\mathrm{x})T)^\eta(1+|y|_{\mathbb{R}^d}^p)}, \tag{2.19}$$

where $\eta\geqslant 0$ depends on d,L,q,θ and $\frac{1}{\epsilon}$ and $c,C\geqslant 0$ depend on $d,N,L,q,\mathfrak{D},\sup_{r\in\mathbb{N}^*}\mathfrak{D}_r$, $\mathfrak{p}$, $\sup_{r\in\mathbb{N}^*}\mathfrak{p}_r$, $\mathfrak{p}_f$, $\frac{1}{m_*}$, $\frac{1}{r_*}$, $\theta,p,\frac{1}{\epsilon}$ and on the moment of Z^δ and which may tend to infinity if one of those quantities tends to infinity.

Remark 2.2. *(1) Let us recall that for μ and ν two probability measure on $\mathbb{R}^d$, the total variation distance between μ and ν is given by*

$$\begin{aligned} d_{TV}(\mu,\nu) = \sup_{A\in\mathcal{B}(\mathbb{R}^d)}|\mu(A)-\nu(A)| &= \sup_{f\in\mathcal{M}(\mathbb{R}^d;\mathbb{R}),\|f\|_\infty\leqslant 1}\frac{1}{2}|\mu(f)-\nu(f)| \\ &= \sup_{f\in\mathcal{C}_K^\infty(\mathbb{R}^d;\mathbb{R}),\|f\|_\infty\leqslant 1}\frac{1}{2}|\mu(f)-\nu(f)| \end{aligned}$$

where $\mu(f)=\int_{\mathbb{R}^d}f(x)\mu(dx)$ and similarly for $\nu(f)$. The last equality above is a direct consequence of the Lusin's Theorem.

*In particular, (2.18) provides a bound on the total variation distance between the law of X_T starting from $\mathrm{x}\in\mathbb{R}^d$ (denoted $P_T(\mathrm{x},.)$) and the one of X_T^δ also starting from x (denoted $Q_T(\mathrm{x},.)$). In particular, under the hypothesis from **A.** in Theorem 2.2, then*

$$d_{TV}(P_T(\mathrm{x},.),Q_T(\mathrm{x},.)) \leqslant \delta^{\frac{1}{2}-\epsilon}\frac{1+|\mathrm{x}|_{\mathbb{R}^d}^c}{(\mathcal{V}_L(\mathrm{x})T)^\eta}C\exp(CT). \tag{2.20}$$

*(2) If we suppose in addition that $\theta \geqslant 2$ and for every $t \in \pi^{\delta,*}$, $i \in \mathbf{N}$, $\mathbb{E}[(Z^i_t)^3] = 0$ and we replace $\mathbf{A}^{\delta}_1(\max(6, 2L+5))$ by $\mathbf{A}^{\delta}_1(\max(7, 2L+5))$ in **A.**, then Theorem 2.2 (and also (2.20)) holds with $\delta^{\frac{1}{2}-\epsilon}$ replaced by $\delta^{1-\epsilon}$ and $(\mathfrak{D}_{\max(6,2L+5)}, \mathfrak{p}_{\max(6,2L+5)})$ replaced by $(\mathfrak{D}_{\max(7,2L+5)}, \mathfrak{p}_{\max(7,2L+5)})$ in the r.h.s. of (2.18) and (2.19).*

(3) More generally, let us suppose that, in addition to hypothesis from Theorem 2.2, the assumption $\mathbf{A}^{\delta}_1(+\infty)$ hold and, given $m > 0$, $\theta \geqslant m + 1$ and there exists $q(m) \in \mathbb{N}$ such that: For every $f \in \mathcal{C}^{\infty}_{pol}(\mathbb{R}^d; \mathbb{R})$ such that for every $\alpha \in \mathbb{N}^d$ and every $x \in \mathbb{R}^d$,

$$|\partial^{\alpha} f(x)| \leqslant \mathfrak{D}_{f,\alpha}(1 + |x|^{p(\alpha)}),$$

with $\mathfrak{D}_{f,\alpha} \geqslant 1$ and $p(\alpha) \geqslant 0$, then, for every $t \in \pi^{\delta}$,

$$|\mathbb{E}[f(X^{\delta}_{t+\delta}) - f(X_{t+\delta})|X_t = X^{\delta}_t = x]| \leqslant \delta^{m+1} \sum_{|\alpha| \leqslant q(m)} \mathfrak{D}_{f,\alpha} C(1 + |x|^p), \tag{2.21}$$

where C and p do not depend on $\mathfrak{D}_{f,\alpha}$ or δ. Then, Theorem 2.2 holds with $\delta^{\frac{1}{2}-\epsilon}$ replaced by $\delta^{m-\epsilon}$ and $(\mathfrak{D}_{\max(6,2L+5)}, \mathfrak{p}_{\max(6,2L+5)})$ replaced by $(\sup_{r\in\mathbb{N}^*} \mathfrak{D}_r, \sup_{r\in\mathbb{N}^*} \mathfrak{p}_r)$ in the r.h.s. of (2.18) and (2.19) (and also (2.20)). In this case η, c and C may depend on m.
When assuming simply that for every $t \in \pi^{\delta,*}$, $i \in \mathbf{N}$, $\mathbb{E}[(Z^i_t)^3] = 0$, we have automatically that (2.21) holds with $m = 1$, which leads to the previous remark.

*(4) By a straightforward application of Corollary 2.1 and Theorem 2.2, under the hypothesis from Theorem 2.2 point **B.**, we derive easily the following estimate of the density of X_T: Let $q \in \mathbb{N}$, let $\alpha, \beta \in \mathbb{N}^d$ such that $|\alpha| + |\beta| \leqslant q$ and let $p > 0$. Then, for every $y \in \mathbb{R}^d$,*

$$|\partial^{\alpha}_x \partial^{\beta}_y p_T(\mathrm{x}, y)| \leqslant \mathfrak{D}_f \frac{(1 + |\mathrm{x}|^c_{\mathbb{R}^d}) C \exp(CT)}{(\mathcal{V}_L(\mathrm{x})T)^{\eta}(1 + |y|^p_{\mathbb{R}^d})}.$$

(5) When uniform weak Hörmander property holds, that is $\mathbf{A}^{\infty}_2(L)$ (see (2.5)), then $\delta^{\frac{1}{2}-\epsilon}$ can be replaced by $\delta^{\frac{1}{2}}$ in (2.18) or (2.20) (but not in (2.19)). When we assume (2.21) holds, similar conclusions hold but with $\delta^{\frac{1}{2}-\epsilon}$ (respectively $\delta^{\frac{1}{2}}$) replaced by $\delta^{m-\epsilon}$ (resp. δ^m).

Example 2.1. *(1) Let us consider $X = (X^1, X^2)$, the solution of the 2-dimensional system of $\mathbb{R}$ valued SDE, starting at point $\mathrm{x} = (\mathrm{x}^1, \mathrm{x}^2) \in \mathbb{R}^2$ and given by*

$$\begin{aligned} dX^1_t &= b(X^1_t, t)dt + \sigma(X^1_t, t)dW_t \\ dX^2_t &= X^1_t dt \end{aligned}$$

where $(W_t)_{t\geqslant 0}$ is a one dimensional standard Brownian motion, b and σ are smooth with bounded derivatives of order one and polynomial bounds for higher orders. In the setting from (1.3), we have $V_0 : (x, t) \mapsto (b(x^1, t), x^1)$ and $V_1 : (x, t) \mapsto (\sigma(x^1, t), 0)$. In this example local ellipticity holds for X^1 as long as $\sigma(\mathrm{x}^1, t) \neq 0$. However ellipticity does not hold for X since $dim(span((\sigma, 0)))(x, 0) \leqslant 1$. Nevertheless, let us compute the Lie brackets. In particular

$$[V_0, V_1] : (x, t) \mapsto (\partial_{x^1}\sigma(x^1, t)b(x^1, t) - \partial_{x^1}b(x^1, t)\sigma(x^1, t), -\sigma(x^1, t)),$$

and, for $\sigma(\mathrm{x}^1, 0) \neq 0$, $span((\sigma, 0), (\partial_{x^1}\sigma b - \partial_{x^1}b\sigma + \partial_t\sigma, -\sigma)(\mathrm{x}, 0) = \mathbb{R}^2$ so that local weak Hörmander condition holds. Now, let us consider the Euler scheme of X, given by $(X^{\delta,1}_0, X^{\delta,2}_0) = \mathrm{x}$ and for $t \in \pi^{\delta}$,

$$\begin{aligned} X^{\delta,1}_{t+\delta} &= X^{\delta,1}_t + b(X^{\delta,1}_t, t)\delta + \sigma(X^1_t, t)\sqrt{\delta} Z^{\delta}_{t+\delta} \\ X^{\delta,2}_{t+\delta} &= X^{\delta,2}_t + X^{\delta,1}_t \delta, \end{aligned}$$

where $Z^{\delta}_t \in \mathbb{R}$, $t \in \pi^{\delta,*}$, are centered with variance one and Lebesgue lower bounded distribution and moment of order three equal to zero. With notations introduced in (2.4), for $\sigma(\mathrm{x}^1_0, 0) \neq 0$,

$$\begin{aligned} &\mathcal{V}_1(\mathrm{x}) \\ &= 1 \wedge \inf_{\mathbf{b}\in\mathbb{R}^d, |\mathbf{b}|_{\mathbb{R}^d}=1} \langle V_1(\mathrm{x}, 0), \mathbf{b}\rangle^2_{\mathbb{R}^d} + \langle [V_0 - \frac{1}{2}\nabla_x V_1 V_1, V_1](\mathrm{x}, 0) + \partial_t V_1(\mathrm{x}, 0), \mathbf{b}\rangle^2_{\mathbb{R}^d} \\ &= 1 \wedge \inf_{\mathbf{b}\in\mathbb{R}^d, |\mathbf{b}|_{\mathbb{R}^d}=1} \langle (\sigma, 0), \mathbf{b}\rangle^2_{\mathbb{R}^d} + \langle (\partial_{x^1}\sigma b - \partial_{x^1}b\sigma + \frac{1}{2}\sigma^2\partial^2_{x^1}\sigma + \partial_t\sigma, -\sigma), \mathbf{b}\rangle^2_{\mathbb{R}^d}(\mathrm{x}^1, 0) \\ &> 0, \end{aligned}$$

and for every $f \in \mathcal{M}(\mathbb{R}^d;\mathbb{R})$ *stafisfying hypothesis from Theorem 2.2,* $\boldsymbol{A}$*., we have, for* $T \in \pi^{\delta}$*,* $T \geqslant 2\delta$*,* $\epsilon \in (0,1]$*,*

$$|\mathbb{E}[f(X_T) - f(X^{\delta}_T)]| \leqslant \delta^{1-\epsilon}\mathfrak{D}_f \frac{1+|x|^{c}_{\mathbb{R}^d}}{(\mathcal{V}_1(x)T)^{\eta}} C\exp(CT).$$

where η, C, c *can explode if* ϵ *tends to zero.*

(2) In a similar but simpler way, we can give an extension of the central limit theorem in total variation distance, including the iterated time integrals of the Brownian motion.
We considere $Z^{\frac{1}{n}}_t \in \mathbb{R}$*,* $t \in \pi^{\delta,*}$*,* $n \in \mathbb{N}^*$*, which are centered with variance one and Lebesgue lower bounded distribution and we define* $S^{(0)}_l := \sum_{k=1}^{l} Z^{\frac{1}{n}}_{\frac{k}{n}}$*,* $l \in \mathbb{N}$*, and for* $h \in \mathbb{N}^*$*,* $S^{(h)}_l := \frac{1}{n}\sum_{k=1}^{l} S^{(h-1)}_k$*.*
Then $(S^{(0)}_n, \ldots, S^{(h)}_n)$*,* $h \in \mathbb{N}$*, converges in total variation distance, as* n *tends to infinity, toward the random variable* $(W_1, \int_0^1 W_s ds, \ldots, \int_0^1 \ldots \int_0^{s_2} W_{s_1} ds_1 \ldots ds_h)$ *where* $(W_t)_{t\geqslant 0}$ *is a one dimensional standard Brownian motion.*

3. A Malliavin-inspired approach to prove smoothing properties

Our strategy to obtain regularization properties is to establish some integration by parts formulas (Theorem 4.1, (4.3)) and then to bound the Malliavin weights appearing in those formulas (Theorem 4.1, (4.4)). These bounds on Malliavin weights are derived by bounding the Sobolev norms constructed with Malliavin derivatives (Theorem 4.2) and by bounding the moments of the inverse Malliavin covariance matrix (Theorem 4.3). In this section, we present the discrete Malliavin calculus tailored to our framework, and subsequently present our key regularization property results. Integration by parts formulas and estimates on the Malliavin weights will be derived in the next section.

3.1. **A generic discrete time Malliavin calculus.**

Since we are interested in random variables with form (2.1), where the laws of random variables Z^{δ} are arbitrary (and thus not only Gaussian) the standard Malliavin calculus is not adapted anymore. Therefore, we remain inspired by Malliavin calculus but we whether develop a discrete time differential calculus which happens to be well suited to our framework as soon as Z^{δ} involves a regular part *i.e.* is Lebesgue lower bounded. In this section, we always assume that $\mathbf{A}^{\delta}_4$ (see (2.8)) holds true.

In the following, we will denote $\chi^{\delta} = (\chi^{\delta}_t)_{t\in\pi^{\delta,*}}$, $U^{\delta} = (U^{\delta}_t)_{t\in\pi^{\delta,*}}$ and $V^{\delta} = (V^{\delta}_t)_{t\in\pi^{\delta,*}}$ and given a separable Hilbert space $(\mathcal{H}, \langle .,. \rangle_{\mathcal{H}})$ equipped with an orthonormal base $\mathfrak{H} := (\mathfrak{h}_n)_{n\in\mathbb{N}^*}$, we will consider the class of random variables:

$$\begin{aligned}\mathcal{S}^{\delta}(\mathcal{H}) = \{F = f(\chi^{\delta}, U^{\delta}, V^{\delta}) : \forall(\chi, v) \in \{0,1\}^{\pi^{\delta,*}} \times \mathbb{R}^{\pi^{\delta,*}\times\mathbf{N}},\\ u \mapsto f(\chi, u, v) \in \mathcal{C}^{\mathrm{F},\infty}(\mathbb{R}^{\pi^{\delta,*}\times\mathbf{N}};\mathcal{H}),\\ \partial^{\mathrm{F}}_{u_1,\ldots,u_l} f(\chi^{\delta}, U^{\delta}, V^{\delta}) \in \bigcap_{p=1}^{+\infty} \mathrm{L}^p(\Omega), \forall u_1,\ldots,u_l \in \mathbb{R}^{\pi^{\delta,*}\times\mathbf{N}}, l \in \mathbb{N}\}.\end{aligned}$$

In the previous definition, we have denoted by $\mathcal{C}^{\mathrm{F},\infty}(\mathbb{R}^{\pi^{\delta,*}\times\mathbf{N}};\mathcal{H})$, the set of functions defined on the vector space $\mathbb{R}^{\pi^{\delta,*}\times\mathbf{N}}$, that take values in $\mathcal{H}$ and which admit Fréchet directional derivatives of any order. When $\mathcal{H} = \mathbb{R}$, we simply denote $\mathcal{S}^{\delta}$.
We now construct a differential calculus based on the laws of the random variables U^{δ} which mimics the Malliavin calculus, following the ideas from [5], [2], [3] or [7]. We begin by introducing the basic element of our differential calculus.

Derivative operator and Malliavin covariance matrix. We consider the set of $\{0,1\}^{\pi^{\delta,*}\times\mathbf{N}}$-valued vectors $(u^i_t)_{(t,i)\in\pi^{\delta,*}\times\mathbf{N}}$ such that for every $t,s \in \pi^{\delta,*}$ and every $i,j \in \mathbf{N}$, $(u^i_t)_{s,j} = \mathbf{1}_{t,s}\mathbf{1}_{i,j}$. For $F \in \mathcal{S}^{\delta}(\mathcal{H})$, we define the Malliavin derivatives $D^{\delta}F := (D^{\delta}_{(t,i)}F)_{(t,i)\in\pi^{\delta,*}\times\mathbf{N}} \in \mathcal{S}^{\delta}(\mathcal{H})^{\pi^{\delta,*}\times\mathbf{N}}$ by

$$D^{\delta}_{(t,i)}F := \chi^{\delta}_t \partial^{\mathrm{F}}_{u^i_t} f(\chi^{\delta}, U^{\delta}, V^{\delta}), \quad (t,i) \in \pi^{\delta,*} \times \mathbf{N}.$$

For $\mathbf{T} \subset \pi^{\delta,*}$, we define $D^{\delta,\mathbf{T}}F = (D^\delta_{(t,i)}F)_{(t,i)\in\mathbf{T}\times\mathbf{N}} \in \mathcal{S}^\delta(\mathcal{H})^{\mathbf{T}\times\mathbf{N}}$. When $\mathbf{T} = \pi^{\delta,*}$ or when it is explicit enough, we simply denote $D^\delta F$. For $s \in (t-\delta, t]$, with $t \in \mathbf{T}$ we define also

$$D^\delta_{(s,i)}F := D^\delta_{(t,i)}F$$

and $D^\delta_{(s,i)} = 0$ otherwise. The higher order derivatives are defined by iterating D^δ. Let $\alpha = (\alpha^1, \ldots, \alpha^m) \in (\pi^{\delta,*} \times \mathbf{N})^m$, $m \in \mathbb{N}$. We define

$$D^\delta_\alpha F = D^\delta_{\alpha^1} \cdots D^\delta_{\alpha^m} F$$

when $m > 0$ and $D^\delta_\alpha F = D^\delta_\emptyset F = F$ if $m = 0$. We also introduce

$$D^{\delta,\mathbf{T},m}F = (D^\delta_\alpha F)_{\alpha\in(\mathbf{T}\times\mathbf{N})^q}.$$

The Malliavin covariance matrix of $F \in \mathcal{S}^\delta(\mathcal{H})$ on $\mathbf{T}$, is the matrix defined for every $\mathfrak{h}, \mathfrak{h}^\diamond \in \mathfrak{H}$ by

$$\begin{aligned} \sigma^\delta_{F,\mathbf{T}}[\mathfrak{h}, \mathfrak{h}^\diamond] &= \delta\langle D^{\delta,\mathbf{T}}\langle F, \mathfrak{h}\rangle_{\mathcal{H}}, D^{\delta,\mathbf{T}}\langle F, \mathfrak{h}^\diamond\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\mathbf{T}\times\mathbf{N}}} \\ &:= \delta \sum_{t\in\mathbf{T}} \sum_{l=1}^N D^\delta_{(t,l)}\langle F, \mathfrak{h}\rangle_{\mathcal{H}} D^\delta_{(t,l)}\langle F, \mathfrak{h}^\diamond\rangle_{\mathcal{H}} \end{aligned} \tag{3.1}$$

If $\mathbf{T} = (0,T] \cap \pi^\delta$ with $T \in \pi^{\delta,*}$ then

$$\sigma^\delta_{F,\mathbf{T}}[\mathfrak{h}, \mathfrak{h}^\diamond] = \int_0^T D^\delta_{(s,l)}\langle F, \mathfrak{h}\rangle_{\mathcal{H}} D^\delta_{(s,l)}\langle F, \mathfrak{h}^\diamond\rangle_{\mathcal{H}} \mathrm{d}s.$$

It is worth noticing that $\sigma^\delta_{F,\mathbf{T}}$ can be seen as a linear operator on $\mathcal{H}$ such that for every $h \in \mathcal{H}$, $\sigma^\delta_{F,\mathbf{T}}h := \sum_{\mathfrak{h},\mathfrak{h}^\diamond\in\mathfrak{H}} \sigma^\delta_{F,\mathbf{T}}[\mathfrak{h}, \mathfrak{h}^\diamond]\langle h, \mathfrak{h}^\diamond\rangle_{\mathcal{H}}\mathfrak{h}$. When $\mathcal{H}$ has finite dimension, this is the standard matrix product.

Now, we define, when it is possible, the inverse Malliavin covariance matrix. We consider the trace class norm of a bounded linear operator $\mathcal{L}$ on the Hilbert space $\mathcal{H}$ given by $|\mathcal{L}|_{tc} := \sum_{\mathfrak{h}\in\mathfrak{H}}\langle\sqrt{\mathcal{L}^*\mathcal{L}}\mathfrak{h}, \mathfrak{h}\rangle_{\mathcal{H}}$ where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$ for the scalar product $\langle , \rangle_{\mathcal{H}}$. We say that an operator is trace class if it is bounded, linear and $|\mathcal{L}|_{tc} < +\infty$.

When $\sigma^\delta_{F,\mathbf{T}} - I_{\mathcal{H}}$ (with $I_{\mathcal{H}}[\mathfrak{h}, \mathfrak{h}^\diamond] = \mathbf{1}_{\mathfrak{h},\mathfrak{h}^\diamond}, \mathfrak{h}, \mathfrak{h}^\diamond \in \mathfrak{H}$) is a trace class operator on $\mathcal{H}$, and the Fredholm determinant $\det \sigma^\delta_{F,\mathbf{T}}$ of $\sigma^\delta_{F,\mathbf{T}}$ (which is the standard determinant when $\mathcal{H}$ has finite dimension) is not zero, we define $\gamma^\delta_{F,\mathbf{T}} = (\sigma^\delta_{F,\mathbf{T}})^{-1}$, the inverse Malliavin covariance matrix of F.

Divergence and Ornstein Uhlenbeck operators. Let $G^\delta = \left(G^\delta_t\right)_{t\in\pi^{\delta,*}}$ with $G^\delta_t \in \mathcal{S}^\delta(\mathcal{H})^N$.The divergence operator is given by

$$\Delta^\delta_{\mathbf{T}} G^\delta = \delta \sum_{t\in\mathbf{T}} \sum_{i=1}^N G^{\delta,i}_t D^\delta_{(t,i)}\Gamma^\delta_t + D^\delta_{(t,i)}G^{\delta,i}_t \in \mathcal{S}^\delta(\mathcal{H}),$$

with, for $t \in \pi^{\delta,*}$,

$$\Gamma^\delta_t = \ln \varphi_{r_*/2}(\delta^{-\frac{1}{2}}U^\delta_t - z_{*,t}) \in \mathcal{S}^\delta(\mathbb{R}).$$

In particular, for $i \in \mathbf{N}$,

$$D^\delta_{(t,i)}\Gamma^\delta_t = \delta^{-\frac{1}{2}}\chi^\delta_t \partial_{z^i} \ln \varphi_{r_*/2}(\delta^{-\frac{1}{2}}U^\delta_t - z_{*,t}) \in \mathcal{S}^\delta(\mathbb{R}).$$

Finally, we define the Ornstein Uhlenbeck operator, for $F \in \mathcal{S}^\delta(\mathcal{H})$,

$$L^\delta_{\mathbf{T}}F = -\Delta^\delta_{\mathbf{T}} D^\delta F = -\delta \sum_{t\in\mathbf{T}} \sum_{i=1}^N D_{(t,i)}D_{(t,i)}F + D_{(t,i)}F D^\delta_{(t,i)}\Gamma^\delta_t \in \mathcal{S}^\delta(\mathcal{H}).$$

Notice that, if $\mathbf{T} = (0,T] \cap \pi^\delta$ with $T \in \pi^{\delta,*}$, then (denoting $t(s) = t$ for $s \in (t-\delta, t]$, $t \in \pi^{\delta,*}$),

$$L^\delta_{\mathbf{T}}F = -\int_0^T \sum_{i=1}^N D_{(s,i)}D_{(s,i)}F \mathrm{d}s - \delta \sum_{t\in\mathbf{T}} \sum_{i=1}^N D_{(t,i)}F D^\delta_{(t,i)}\Gamma^\delta_t \in \mathcal{S}^\delta(\mathcal{H})$$

Remark 3.1. *The basic random variables in our calculus are $Z_t^\delta, t \in \pi^{\delta,*}$ so we precise the way in which the differential operators act on them. Since $\delta^{\frac{1}{2}} Z_t^\delta = \chi_t^\delta U_t^\delta + \sqrt{n}(1-\chi_t^\delta)V_t^\delta$, it follows that for $w, t \in \pi^{\delta,*}$, $\mathbf{T} \subset \pi^\delta$, $i, j \in \mathbf{N}$,*

$$\delta^{\frac{1}{2}} D^\delta_{(t,i)} Z_w^{\delta,j} = \chi_w^\delta \mathbf{1}_{w,t} \mathbf{1}_{i,j}, \tag{3.2}$$

$$L^\delta_{\mathbf{T}} Z_t^{\delta,i} = \chi_t^\delta \partial_{z^i} \ln \varphi_{r_*/2}(\delta^{-\frac{1}{2}} U_t^\delta - z_{*,t}) \mathbf{1}_{t \in \mathbf{T}}. \tag{3.3}$$

3.2. **Regularization properties for approximations of the semigroup.** In the following, we will not work under $\mathbb{P}$, but under a localized measure which we define now. For $\mathbf{T} \subset \pi^{\delta,*}$, we denote $|\mathbf{T}| = \mathrm{Card}(\mathbf{T})$. When $|\mathbf{T}| > 0$ we define

$$\Lambda_{\mathbf{T}} = \left\{ \frac{1}{|\mathbf{T}|} \sum_{w \in \mathbf{T}} \chi_w^\delta \geqslant \frac{m_*}{2} \right\}.$$

Using the Hoeffding's inequality and the fact that $\mathbb{E}[\chi_t^\delta] = m_*$, it can be checked that for $\mathbf{T} = (s,t] \cap \pi^\delta$, $0 \leqslant s < t$,

$$\mathbb{P}(\Omega \setminus \Lambda_{\mathbf{T}}) \leqslant \exp(-\frac{m_*^2 |\mathbf{T}|}{2}).$$

The next step consists in localizing the random variables Z^δ and the Malliavin covariance matrix σ_F^δ. For the first one, we aim to control that the norm is not too high while for the latter, we aim to control that it is not too low. We first introduce a regularized version of the indicator function. For $v > 1$, we consider $\Psi_v \in \mathcal{C}_b^\infty(\mathbb{R}; [0,1])$ such that $\Psi_v(x) = 1$ if $|x| \leqslant v - \frac{1}{2}$ and 0 if $|x| \geqslant v$ and that the function $z \in \mathbb{R}^N \mapsto \Psi_v(|z|_{\mathbb{R}^N})$ belongs to $\mathcal{C}_b^\infty(\mathbb{R}^N; [0,1])$ (*e.g.* for $|x| \in (v - \frac{1}{2}, v)$, $\Psi_v(x) = \exp(1 - \frac{1}{1-(2|x|-2v+1)^2})$).

Given $\mathbf{T} \subset \pi^{\delta,*}$, we introduce

$$\Theta_{F,\eta,\mathbf{T}} = \Theta_{F,G,\eta_1,\mathbf{T}} \Theta_{\eta_2,\mathbf{T}} \mathbf{1}_{\Lambda_{\mathbf{T}}} \quad \text{with} \tag{3.4}$$

$$\Theta_{F,G,\eta_1,\mathbf{T}} = \Psi_{\eta_1}(G \det \gamma^\delta_{F,\mathbf{T}}), \quad \text{and} \quad \Theta_{\eta_2,\mathbf{T},t} = \prod_{w \in ((0,t] \cap \mathbf{T})} \Psi_{\eta_2}(|Z_w^\delta|_{\mathbb{R}^N}), \quad t \in \pi^\delta,$$

with $\Theta_{\eta_2,\mathbf{T}} = \Theta_{\eta_2,\mathbf{T},\infty}$.

3.2.1. *The regularization property for a modified measure.* We still fix $\delta > 0$ and we consider the Markov process $(X_t^\delta)_{t \in \pi^\delta}$, defined in (2.1). In order to state our results, we first introduce the tangent flow process $(\dot{X}_t)_{t \in \pi^\delta}$ defined by $\dot{X}_0 = I_{d \times d}$ and

$$\dot{X}_t := \partial_{\mathrm{x}_0^\delta} X_t^\delta, \tag{3.5}$$

the Jacobian matrix of derivatives of X^δ *w.r.t.* the initial value x_0^δ, which appears in our Malliavin weights.

We introduce $(Q_t^{\delta,\Theta})_{t \in \pi^\delta}$ such that,

$$\forall T \in \pi^\delta \quad Q_T^{\delta,\Theta} f(x) := \mathbb{E}[\Theta f(X_T^\delta) | X_0^\delta = x]. \tag{3.6}$$

where $\Theta = \Theta_{X_T^\delta, \det(\partial_{\mathrm{X}_0^\delta} X_T^\delta)^2, \eta, \mathbf{T}}$ following the definition (3.4) with $\mathbf{T} = (0,T] \cap \pi^\delta$, $\eta = (\eta_1(\delta), \eta_1(\delta))$ defined in (2.11).

Notice that $(Q_t^{\delta,\Theta})_{t \in \pi^\delta}$, is not a semigroup. We will not be able to prove the smoothing property for Q^δ but for $Q^{\delta,\Theta}$. The proof uses result established in Section 4. Our approach consists in demonstrating an integration by part formula in Theorem 4.1 built upon our finite disrete time Malliavin calculus, and then bounding the moments of the weights appearing in those formulas exploiting Theorem 4.2 and Theorem 4.3.

Theorem 3.1. *Let $T \in \pi^{\delta,*}$, $\mathbf{T} = (0,T] \cap \pi^\delta$, $\mathrm{x} \in \mathbb{R}^d$ and $f \in \mathcal{C}^\infty_{pol}(\mathbb{R}^d; \mathbb{R})$ satisfying: there exists $\mathfrak{D}_f \geqslant 0$ and $\mathfrak{p}_f \in \mathbb{N}$ such that for every $x \in \mathbb{R}^d$,*

$$|f(x)| \leqslant \mathfrak{D}_f (1 + |x|_{\mathbb{R}^d}^{\mathfrak{p}_f}).$$

Then we have the following properties:

A. *Let* $q\in\mathbb{N}$ *and* $\alpha,\beta\in\mathbb{N}^d$ *such that* $|\alpha|+|\beta|\leqslant q$. *Assume that* $\mathbf{A}_1^\delta(\max(q+3,2L+5))$ *(see (2.2), (2.3)),* $\mathbf{A}_2(\mathrm{x},L)$ *(see (2.5)),* $\mathbf{A}_3^\delta(+\infty)$ *(see (2.7)),* $\mathbf{A}_4^\delta$ *(see (2.8)) and* $\mathbf{A}_5^\delta(\mathrm{x},T)$ *(see (2.12)) hold. Then,*

$$
(3.7)\qquad |\partial^\alpha Q_T^{\delta,\Theta}\partial^\beta f(\mathrm{x})| \leqslant \mathfrak{D}_f \frac{1+\mathbf{1}_{\mathfrak{p}_{\max(q+3,2L+5)}+\mathfrak{p}_f>0}|\mathrm{x}|_{\mathbb{R}^d}^C}{(\mathcal{V}_L(\mathrm{x})T)^{13^L 3d(\frac{5}{4}q^2+2q+3)}} \times \mathfrak{D}^C_{\max(q+3,2L+5)}\exp(C(1+T)\mathfrak{M}_C(Z^\delta)\mathfrak{D}^4).
$$

with $C=C(d,N,L,q,\mathfrak{p},\mathfrak{p}_{\max(q+3,2L+5)},\mathfrak{p}_f,\frac{1}{m_*},\frac{1}{r_*})\geqslant 0$ *which may tend to infinity if one of the arguments tends to infinity.*

B. *Let* $h>0$. *Assume that hypothesis from* ***A.*** *are satisfied with* $\mathbf{A}_1^\delta(\max(q+3,2L+5))$ *replaced by* $\mathbf{A}_1^\delta(2L+5)$. *Then,*

$$
(3.8)\qquad |Q_T^\delta f(\mathrm{x})-Q_T^{\delta,\Theta}f(\mathrm{x})| \leqslant \delta^h\mathfrak{D}_f\frac{1+\mathbf{1}_{\mathfrak{p}_{2L+5}+\mathfrak{p}_f>0}|\mathrm{x}|^C_{\mathbb{R}^d}}{\mathcal{V}_L(\mathrm{x})^{13^L 3d\max(4,\frac{91h}{44d})}} \times \mathfrak{D}^C\mathfrak{D}^C_{2L+5}\mathfrak{M}_C(Z^\delta)C\exp(CT\mathfrak{M}_C(Z^\delta)\mathfrak{D}^4).
$$

with $C=C(d,N,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\mathfrak{p}_f,\frac{1}{m_*},h)\geqslant 0$ *which may tend to infinity if one of the arguments tends to infinity.*

Remark 3.2. *(1) In the case of uniform Hörmander hypothesis* $\mathbf{A}_2^\infty(L)$ *(see (2.5)), if we consider* $\delta\leqslant\delta_0$ *for some* δ_0 *small enough, then for any* $x\in\mathbb{R}^d$, $Q_T^{\delta,\Theta}f(x)$ *can be replaced by the localized probability measure* $\frac{1}{\mathbb{E}[\Theta|X_0^\delta=x]}\mathbb{E}[\Theta f(X_T^\delta)|X_0^\delta=x]$ *and the conclusion of Theorem 3.1 still hold. In case of non uniform Hörmander property,* δ_0 *would depend on* x *so it is not uniform anymore and we can not obtain the same result.*

(2) Using our approach, we can easily show that under uniform Hörmander hypothesis $\mathbf{A}_2^\infty(L)$ *(see (2.5)),* $(\mathcal{V}_L(\mathrm{x})T)^{-13^L 3d(\frac{5}{4}q^2+2q+3)}$ *can be replaced by* $(\mathcal{V}_L^\infty T)^{-13^L d(\frac{5}{4}q^2+2q+1)}$ *in the r.h.s. of (3.7) and* $\mathcal{V}_L(\mathrm{x})$ *can be replaced by* 1 *in the r.h.s. of (3.8).*

Proof. Let us prove **A.**. We have

$$
(3.9)\qquad \partial^\alpha Q_T^{\delta,\Theta}\partial^\beta f(\mathrm{x}) = \sum_{|\beta|\leqslant|\gamma|\leqslant q}\mathbb{E}[\Theta\partial^\gamma f(X_T^\delta)\mathcal{P}_\gamma(X_T^\delta)|X_0^\delta=\mathrm{x}],
$$

where $\mathcal{P}_\gamma(X_T^\delta)$ is a universal polynomial of $\partial^\rho_{\mathrm{X}_0^\delta}X_T^\delta, 1\leqslant|\rho|\leqslant q-|\gamma|+1$. Using the integration by parts formula (4.3) and the estimate (4.4 obtained in Theorem 4.1, we derive

$$
\begin{aligned}
|\mathbb{E}[\Theta\partial^\gamma f(X_T^\delta)\mathcal{P}_\gamma(X_T^\delta)|X_0^\delta=\mathrm{x}]| =& |\mathbb{E}[f(X_T^\delta)H^\delta_{\mathbf{T}}(X_T^\delta,\Theta\mathcal{P}_\gamma(X_T^\delta)[\gamma]|X_0^\delta=\mathrm{x}]|\\
\leqslant& \mathfrak{D}_f\mathbb{E}[(1+|X_T^\delta|^{\mathfrak{p}_f}_{\mathbb{R}^d})|H^\delta_{\mathbf{T}}(X_T^\delta,\Theta\mathcal{P}_\gamma(X_T^\delta))[\gamma]||X_0^\delta=\mathrm{x}]\\
\leqslant& C(d,q)\mathfrak{D}_f\times A_1\times A_2\times A_3\times A_4
\end{aligned}
$$

with, using Lemma 4.1 and Lemma 4.3 combined with the Cauchy-Schwarz inequality,

$$
\begin{aligned}
A_1 =& 1\vee\mathbb{E}[|\det\gamma^\delta_{X_T^\delta,\mathbf{T}}|^{\frac{5}{2}q^2+4q+2}\mathbf{1}_{\Theta>0}|X_0^\delta=\mathrm{x}]^{\frac12}\\
A_2 =& 1+\mathbb{E}[|X_T^\delta|^{10dq^2+24dq+8d}_{\mathbb{R}^d,\delta,\mathbf{T},1,q+1}|X_0^\delta=\mathrm{x}]^{\frac14}\\
&+\mathbb{E}[|L^\delta_{\mathbf{T}}X_T^\delta|^{16q}_{\mathbb{R}^d,\delta,\mathbf{T},q-1}|X_0^\delta=\mathrm{x}]^{\frac18}\mathbb{E}[|X_T^\delta|^{4d(q+2)^2}_{\mathbb{R}^d,\delta,\mathbf{T},1,q+1}|X_0^\delta=\mathrm{x}]^{\frac18}\\
A_3 =& \mathbb{E}[\sum_{m=0}^q|\det(\dot{X}_T^\delta)^2|^{8m}_{\mathbb{R}^d,\delta,\mathbf{T},q-m}|X_0^\delta=\mathrm{x}]^{\frac18}\\
A_4 =& \mathbb{E}[(1+|X_T^\delta|^{\mathfrak{p}_f}_{\mathbb{R}^d})^8|\mathcal{P}_\gamma(X_T^\delta)^8|^8_{\mathbb{R},\delta,\mathbf{T},|\gamma|}|X_0^\delta=\mathrm{x}]^{\frac18},
\end{aligned}
$$

with $\dot{X}_T^\delta$ defined in (3.5). Using Theorem 4.3 yields

$$
\begin{aligned}
A_1\leqslant& \frac{1+\mathbf{1}_{\mathfrak{p}_{2L+5}>0}|\mathrm{x}|^{C(d,L,q,\mathfrak{p}_{2L+5})}_{\mathbb{R}^d}}{(\mathcal{V}_L(\mathrm{x})T)^{-13^L 3d(\frac54 q^2+2q+3)}}\mathfrak{D}^{C(d,L,q)}_{2L+5}C(d,N,L,\frac{1}{m_*},p,\mathfrak{p}_{2L+5})\\
&\times\exp(C(d,L,q,\mathfrak{p}_{2L+5})(1+T)\mathfrak{M}_{C(d,L,q,\mathfrak{p},\mathfrak{p}_{2L+5},\mathfrak{q}^\delta_{\eta_2(\delta)})}(Z^\delta)\mathfrak{D}^4)),
\end{aligned}
$$

with $\mathfrak{q}^{\delta}_{\eta_2} := \lceil 1-\frac{\ln(\delta)}{2\ln(\eta_2(\delta))}\rceil$ which does not depend on δ.
Moreover, using the results from Theorem 4.2, we obtain

$$\begin{aligned}A_2\times A_3\times A_4 \leqslant&(|\mathrm{x}|_{\mathbb{R}^d}(\mathbf{1}_{\mathfrak{p}_{q+3}>0}+\mathbf{1}_{\mathfrak{p}_f>0})+\mathfrak{D}_{q+3})^{C(d,q,\mathfrak{p}_{q+3})}\\ &C(d,N,\frac{1}{r_*},q,\mathfrak{p}_{q+3},\mathfrak{p}_f)\\ &\times\exp(C(d,q,\mathfrak{p}_{q+3},\mathfrak{p}_f)(T+1)\mathfrak{M}_{C(p,q,\mathfrak{p},\mathfrak{p}_{q+3},\mathfrak{p}_f)}(Z^{\delta})\mathfrak{D}^2).\end{aligned}$$

We gather all the terms together and the proof of (3.7) is completed.
Now, let us prove **B.**. For every $x\in\mathbb{R}^d$, we have We have

$$\begin{aligned}|Q^{\delta}_T f(\mathrm{x})-Q^{\delta,\Theta}_T f(\mathrm{x})|\leqslant&\mathbb{E}[f(X^{\delta}_T)(1-\Theta)|X^{\delta}_0=\mathrm{x}]\\ \leqslant&\mathfrak{D}_f\mathbb{E}[(1+|X^{\delta}_T|^{\mathfrak{p}_f}_{\mathbb{R}^d})^2|X^{\delta}_0=\mathrm{x}]^{\frac12}\mathbb{E}[1-\Theta|X^{\delta}_0=\mathrm{x}]^{\frac12}\\ \leqslant&\mathfrak{D}_f 2\mathbb{E}[1+|X^{\delta}_T|^{2\mathfrak{p}_f}_{\mathbb{R}^d}|X^{\delta}_0=\mathrm{x}]^{\frac12}\mathbb{P}(\Theta<1|X^{\delta}_0=\mathrm{x})^{\frac12}.\end{aligned}$$

We obtain an upper bound for $\mathbb{P}(\Theta<1|X^{\delta}_0=x)$ by using (4.17). The upper bound of $\mathbb{E}[|X^{\delta}_t|^{2\mathfrak{p}_f}|X^{\delta}_0=x]$ is obtained using Lemma 4.5. It follows that, for every $a>0$ and every $p\geqslant 0$,

$$\begin{aligned}|Q^{\delta}_T f(\mathrm{x})-Q^{\delta,\Theta}_T f(\mathrm{x})|\leqslant&(\delta^{-1}\eta_2^{-a}\sup_{t\in\mathbf{T}}\mathfrak{M}_a(Z^{\delta})+\eta_1^{-(p+4)}(1+\mathcal{V}_L(\mathrm{x})^{-13^L 3d(p+4)}))\\ &\times\mathfrak{D}_f\mathfrak{D}^C\mathfrak{D}^C_{2L+5}\mathfrak{M}_C(Z^{\delta})(1+(\mathbf{1}_{\mathfrak{p}_{2L+5}>0}+\mathbf{1}_{\mathfrak{p}_f>0})|\mathrm{x}|^C_{\mathbb{R}^d})C\exp(CT\mathfrak{M}_C(Z^{\delta})\mathfrak{D}^4).\end{aligned}$$

with $C=C(d,N,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\mathfrak{p}_f,\frac{1}{m_*})$ which may tend to infinity if one of the arguments tends to infinity. We chose $p=p(h)=\max(0,\frac{91h}{44d}-4)$ so that $\eta_1(\delta)^{-(p(h)+4)}\leqslant\delta^h C(h)(1+T^{C(h)})$. Similarly we chose $a=a(h)=2(h+1)\max(\mathfrak{p}+1,\frac{91}{3})$ so that $\eta_2(\delta)^{-a(h)}\delta^{-1}\leqslant\delta^h C(\mathfrak{D},\mathfrak{p},h)(1+T^{C(h)})$ and

$$\begin{aligned}|Q^{\delta}_T f(x)-Q^{\delta,\Theta}_T f(\mathrm{x})|\leqslant&\delta^h(1+\mathcal{V}_L(\mathrm{x})^{-13^L 3d(p(h)+4)})\\ &\times\mathfrak{D}_f\mathfrak{D}^C\mathfrak{D}^C_{2L+5}\mathfrak{M}_C(Z^{\delta})(1+(\mathbf{1}_{\mathfrak{p}_{2L+5}>0}+\mathbf{1}_{\mathfrak{p}_f>0})|\mathrm{x}|^C_{\mathbb{R}^d})C\exp(CT\mathfrak{M}_C(Z^{\delta})\mathfrak{D}^4),\end{aligned}$$

with $C=C(d,N,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\mathfrak{p}_f,\frac{1}{m_*},h)$, and the proof of (3.8) is completed. □

From a practical viewpoint, an issue of this last result resides in the computation of $Q^{\delta,\Theta}$. Indeed, Θ is not simulable (at least easily) and then methods such as Monte Carlo do not seem to be applicable. A solution is provided by Theorem 2.1, where we show that the regularization properties are also satisfied by $Q^{\delta,\theta}$. In this case, Monte Carlo methods can be designed by simply simulating the sum of X^{δ}_T and of an independent Gaussian variable. The proof of this result exploits the one we just established in Theorem 3.1.

Proof of Theorem 2.1. Let us prove (2.14). As in (3.9), we write

$$\partial^{\alpha}Q^{\delta,\theta}_T\partial^{\beta}f(\mathrm{x})=\sum_{|\beta|\leqslant|\gamma|\leqslant q}\mathbb{E}[\partial^{\gamma}f(\delta^{\theta}G+X^{\delta}_T)\mathcal{P}_{\gamma}(X^{\delta}_T)|X^{\delta}_0=\mathrm{x}],$$

where $\mathcal{P}_{\gamma}(X^{\delta}_t)$ is a universal polynomial of $\partial^{\rho}_{\mathrm{X}^{\delta}_0}X^{\delta}_t, 1\leqslant|\rho|\leqslant q-|\gamma|+1$. We decompose

$$\mathbb{E}[\partial^{\gamma}f(\delta^{\theta}G+X^{\delta}_T)\mathcal{P}_{\gamma}(X^{\delta}_T)|X^{\delta}_0=\mathrm{x}]=A_1+A_2$$

with

$$\begin{aligned}A_1=&\mathbb{E}[\Theta\partial^{\gamma}f(\delta^{\theta}G+X^{\delta}_T)\mathcal{P}_{\gamma}(X^{\delta}_T)|X^{\delta}_0=\mathrm{x}],\\ A_2=&\mathbb{E}[\partial^{\gamma}f(\delta^{\theta}G+X^{\delta}_T)\mathcal{P}_{\gamma}(X^{\delta}_T)(1-\Theta)|X^{\delta}_0=\mathrm{x}].\end{aligned}$$

with $\Theta=\Theta_{X^{\delta}_T,\det(\dot{X}^{\delta}_T)^2,\eta,\mathbf{T}}$ defined in (3.6). The reasoning from the previous proof shows that

$$\begin{aligned}A_1\leqslant&\mathfrak{D}_f\frac{1+\mathbf{1}_{\mathfrak{p}_{\max(q+3,2L+5)}+\mathfrak{p}_f>0}|\mathrm{x}|^C_{\mathbb{R}^d}}{(\mathcal{V}_L(\mathrm{x})T)^{13^L 3d(\frac54 q^2+2q+3)}}\\ &\times\mathfrak{D}^C_{\max(q+3,2L+5)}\exp(C(1+T)\mathfrak{M}_C(Z^{\delta})\mathfrak{D}^4)\end{aligned}$$

with $C=C(d,N,L,q,\mathfrak{p},\mathfrak{p}_{\max(q+3,2L+5)},\mathfrak{p}_f,\frac{1}{m_*},\frac{1}{r_*})$. Moreover, since G follows the standard Gaussian distribution and is independent from X^{δ} and Θ, we have

$$A_2=\mathbb{E}[\mathcal{P}_{\gamma}(X^{\delta}_T)(1-\Theta)\int_{\mathbb{R}^d}\partial^{\gamma}f(\delta^{\theta}u+X^{\delta}_T)(2\pi)^{-\frac{d}{2}}e^{-\frac{|u|^2}{2}}du|X^{\delta}_0=\mathrm{x}].$$

Now, notice that

$$\partial^\gamma f(\delta^\theta u + X_T^\delta) = \delta^{-|\gamma|\theta} \partial_u^\gamma (f(\delta^\theta u + X_T^\delta)),$$

so that, using standard integration by parts, we have

$$A_2 = \delta^{-|\gamma|\theta} \mathbb{E}[\mathcal{P}_\gamma(X_T^\delta)(1-\Theta) \int_{\mathbb{R}^d} f(\delta^\theta u + X_T^\delta) H_\gamma(u)(2\pi)^{-\frac{d}{2}} e^{-\frac{|u|^2}{2}} \mathrm{d}u | X_0^\delta = \mathrm{x}],$$

where H_γ is the Hermite polynomial corresponding to the multi-index γ.
Finally, using the results from Theorem 4.2, we obtain

$$\begin{aligned}|A_2| \leqslant & \delta^{-|\gamma|\theta} \mathfrak{D}_f \mathbb{E}[1-\Theta | X_0^\delta = \mathrm{x}]^{\frac{1}{2}} (|\mathrm{x}|_{\mathbb{R}^d} (\mathbf{1}_{\mathfrak{p}_{q+3}>0} + \mathbf{1}_{\mathfrak{p}_f>0}) + \mathfrak{D}_{q+3})^{C(d,q,\mathfrak{p}_{q+3},\mathfrak{p}_f)} \\ & \times C(d, N, \frac{1}{r_*}, q, \mathfrak{p}_{q+3}, \mathfrak{p}_f) \\ & \times \exp(C(d,q,\mathfrak{p}_{q+3},\mathfrak{p}_f)(T+1) \mathfrak{M}_{C(p,q,\mathfrak{p},\mathfrak{p}_{q+3},\mathfrak{p}_f)}(Z^\delta) \mathfrak{D}^2).\end{aligned}$$

with, using Theorem 4.3 (see (4.17)) for every $a>0$ and every $p \geqslant 0$,

$$\begin{aligned}\mathbb{E}[1-\Theta | X_0^\delta = x] \leqslant & \mathbb{P}(\Theta < 1 | X_0^\delta = \mathrm{x}) \\ \leqslant & \delta^{-1} T \eta_2^{-a} \mathfrak{M}_a(Z^\delta) \\ & + \eta_1^{-(p+4)} \frac{1 + \mathbf{1}_{\mathfrak{p}_{2L+5}>0} |\mathrm{x}|_{\mathbb{R}^d}^C}{\mathcal{V}_L(\mathrm{x})^{13^L 3d(p+4)}} \\ & \times \mathfrak{D}^C \mathfrak{D}_{2L+5}^C \mathfrak{M}_C(Z^\delta) C \exp(CT \mathfrak{M}_C(Z^\delta) \mathfrak{D}^4).\end{aligned}$$

with $C = C(d, N, L, p, \mathfrak{p}, \mathfrak{p}_{2L+5}, \frac{1}{m_*})$. We chose $p = p(q\theta) = \max(0, \frac{91 q\theta}{44d} - 4)$ and $a = a(q\theta) = 2(q\theta+1)\max(\mathfrak{p}+1, \frac{91}{3})$. Therefore

$$\begin{aligned}|\partial^\alpha Q_T^{\delta,\theta} \partial^\beta f(\mathrm{x})| \leqslant & \mathfrak{D}_f \frac{1 + (\mathbf{1}_{\mathfrak{p}_{\max(q+3,2L+5)}>0} + \mathbf{1}_{\mathfrak{p}_f>0}) |\mathrm{x}|_{\mathbb{R}^d}^C}{(\mathcal{V}_L(\mathrm{x})T)^{13^L 3d \max(\frac{91 q\theta}{44d}, \frac{5}{4}q^2 + 2q + 3)}} \\ & \times \mathfrak{D}_{\max(q+3,2L+5)}^C \exp(C(1+T) \mathfrak{M}_C(Z^\delta) \mathfrak{D}^4).\end{aligned}$$

with $C = C(d, N, L, q, \mathfrak{p}, \mathfrak{p}_{\max(q+3,2L+5)}, \mathfrak{p}_f, \frac{1}{m_*}, \frac{1}{r_*}, \theta)$ and the proof of (2.14) is completed. Remark that with our approach, under the uniform Hörmander hypothesis $\mathbf{A}_2^\infty(L)$ (see (2.5)), we can show that $(\mathcal{V}_L(\mathrm{x})T)^{13^L 3d \max(\frac{91 q\theta}{44d}+2, \frac{5}{4}q^2+2q+3)}$ can be replaced by $(\mathcal{V}_L^\infty T)^{-13^L d(\frac{5}{4}q^2+2q+1)}$ in the *r.h.s.* above.

Let us prove (2.15). Since f has polynomial growth, it follows that

$$\begin{aligned}|Q_T^\delta f(\mathrm{x}) - Q_T^{\delta,\theta} f(\mathrm{x})| \leqslant & |\mathbb{E}[\Theta(f(X_T^\delta) - f(X_T^\delta + \delta^\theta G)) | X_0^\delta = \mathrm{x}]| \\ & + \mathfrak{D}_f C(\mathfrak{p}_f)(1 + \mathbb{E}[|X_T^\delta|_{\mathbb{R}^d}^{2\mathfrak{p}_f} | X_0^\delta = \mathrm{x}]^{\frac{1}{2}} + \delta^{\theta \mathfrak{p}_f} \mathbb{E}[|G|_{\mathbb{R}^d}^{2\mathfrak{p}_f}]^{\frac{1}{2}}) \mathbb{E}[1-\Theta | X_0^\delta = \mathrm{x}]^{\frac{1}{2}} \\ \leqslant & \delta^\theta \sum_{j=1}^d \int_0^1 |\mathbb{E}[\Theta(\partial_{x^j} f)(X_T^\delta + \lambda \delta^\theta G) G^j | X_0^\delta = \mathrm{x}]| \mathrm{d}\lambda \\ & + \mathfrak{D}_f C(\mathfrak{p}_f)(1 + |\mathrm{x}|_{\mathbb{R}^d}^{\mathfrak{p}_f}) \exp(T \mathfrak{D}^2 \mathfrak{M}_{C(\mathfrak{p},\mathfrak{p}_f)}(Z^\delta) C(\mathfrak{p}_f)) \\ & \times \mathbb{E}[1-\Theta | X_0^\delta = \mathrm{x}]^{\frac{1}{2}}.\end{aligned}$$

Using Theorem 3.1 (see (3.7) with $q=1$) and the estimate of $\mathbb{E}[1-\Theta | X_0^\delta = \mathrm{x}]$ obtained in the proof of 2.14 with $p = p(\theta) = \max(0, \frac{91\theta}{44d} - 4)$ and $a = a(\theta) = 2(\theta+1)\max(\mathfrak{p}+1, \frac{91}{3})$ we obtain

$$\begin{aligned}|Q_T^\delta f(\mathrm{x}) - Q_T^{\delta,\theta} f(\mathrm{x})| \leqslant & \delta^\theta \mathfrak{D}_f \frac{1 + \mathbf{1}_{\mathfrak{p}_{2L+5}+\mathfrak{p}_f>0} |\mathrm{x}|_{\mathbb{R}^d}^C}{(\mathcal{V}_L(\mathrm{x})T)^{13^L 3d \max(\frac{91\theta}{44d}, \frac{25}{4})}} \\ & \times \mathfrak{D}_{2L+5}^C \exp(C(1+T) \mathfrak{M}_C(Z^\delta) \mathfrak{D}^4),\end{aligned}$$

with $C = C(d, N, L, p, \mathfrak{p}, \mathfrak{p}_{2L+5}, \mathfrak{p}_f, \frac{1}{m_*}, \theta) \geqslant 0$. Notice that under the uniform Hörmander hypothesis $\mathbf{A}_2^\infty(L)$, $(\mathcal{V}_L(\mathrm{x})T)^{13^L 3d \max(\frac{91\theta}{44d}, \frac{25}{4})}$ can be replaced by $(\mathcal{V}_L^\infty T)^{13^L d \frac{17}{4}}$ in the *r.h.s.* above. □

We now show the existence as well as upper bounds for the density of X_T^δ. This result is mainly a consequence of Theorem 2.1. It is noteworthy that we also propose an Gaussian type bound when relying in a simplified framework. It is derived combining a representation formula for the density, Theorem 2.1 and the Azuma-Hoeffding inequality.

Proof of Corollary 2.1 . Since (2.14) holds, the existence of of the a density is due to Tanigushi (see [31], Lemma 3.1).
We first give a representation formula for $q_T^{\delta,\theta}$. Let $f \in \mathcal{C}_0^\infty(\mathbb{R}^d;\mathbb{R})$ (set of functions in $\mathcal{C}^\infty(\mathbb{R}^d;\mathbb{R})$ vanishing at infinity). Let us define $g : \mathbb{R}^d \to \mathbb{R}$ such that for every $x \in \mathbb{R}^d$,

$$g(x) := \int_{\mathbb{R}^d} f(y)\mathbf{1}_{x\geqslant y}\mathrm{d}y.$$

Then $g \in \mathcal{C}_{pol}^\infty(\mathbb{R}^d;\mathbb{R})$. In particular we can apply Theorem 4.1 with the test function g and for $\gamma_0 = (1,\ldots,1) \in \mathbb{N}^d$, since $\partial^{\gamma_0} g = f$, it follows that, with similar notations as in the proof of Theorem 2.1,

$$\begin{aligned}
\partial^\alpha Q_T^{\delta,\theta}\partial^\beta f(\mathrm{x}) =& \partial^\alpha Q_T^{\delta,\theta}\partial^{(\beta,\gamma_0)}g(\mathrm{x}) \\
=& \sum_{0\leqslant|\gamma|\leqslant q+d} \mathbb{E}[\Theta\partial^\gamma g(\delta^\theta G + X_T^\delta)\mathcal{P}_\gamma(X_T^\delta)|X_0^\delta = \mathrm{x}] \\
&+ \mathbb{E}[\partial^\gamma g(\delta^\theta G + X_T^\delta)\mathcal{P}_\gamma(X_T^\delta)(1-\Theta)|X_0^\delta = \mathrm{x}]. \\
=& \sum_{0\leqslant|\gamma|\leqslant q+d} \mathbb{E}[g(\delta^\theta G + X_t^\delta)H_{\mathbf{T}}^\delta(X_t^\delta, \Theta\mathcal{P}_\gamma(X_t^\delta)[\gamma]|X_0^\delta = \mathrm{x}] \\
&+ \mathbb{E}[\delta^{-|\gamma|\theta}\mathcal{P}_\gamma(X_T^\delta)(1-\Theta)H_\gamma(G))|X_0^\delta = \mathrm{x}] \\
=& \int_{y\in\mathbb{R}^d} f(y)\mathbb{E}[\mathbf{1}_{y\leqslant\delta^\theta G + X_t^\delta} H(\alpha,\beta)|X_0^\delta = \mathrm{x}]\mathrm{d}y,
\end{aligned}$$

with (using notation $\mathbf{T} = (0,T] \cap \pi^\delta$),

$$H(\alpha,\beta) = \sum_{0\leqslant|\gamma|\leqslant q+d} H_{\mathbf{T}}^\delta(X_t^\delta, \Theta\mathcal{P}_\gamma(X_t^\delta)[\gamma] + \delta^{-|\gamma|\theta}\mathcal{P}_\gamma(X_T^\delta)(1-\Theta)H_\gamma(G).$$

Moreover, following the same procedure as in the proof of Theorem 2.1, we have,

$$\mathbb{E}[|H(\alpha,\beta)|^2|X_0^\delta = \mathrm{x}]^{\frac{1}{2}} \leqslant \mathfrak{D}_f \frac{1 + \mathbf{1}_{\mathfrak{p}_{\max(q+d+3,2L+5)}>0}|\mathrm{x}|_{\mathbb{R}^d}^C}{|\mathcal{V}_L(\mathrm{x})T|^\eta} C\exp(CT)$$

Hence, using [31], Lemma 3.1, $\delta^\theta G + X_T^\delta$ has a smooth density $q_T^{\delta,\theta}$ and (2.17) holds. We can observe that we have the following representation formula for $q_T^{\delta,\theta}$ and its derivatives:

$$\partial_x^\alpha \partial_y^\alpha q_T^{\delta,\theta}(\mathrm{x},y) = (-1)^{|\beta|}\mathbb{E}[\mathbf{1}_{y\leqslant\delta^\theta G + X_t^\delta} H(\alpha,\beta)|X_0^\delta = \mathrm{x}].$$

The estimate (2.16) then follows from the Cauchy Schwarz inequality, Lemma 4.5 combined with Markov inequality and a similar approach as in the proof of the previous result to bound the moments of $H(\alpha,\beta)$. In particular

$$|\partial_x^\alpha \partial_y^\beta q_T^{\delta,\theta}(\mathrm{x},y)| \leqslant \frac{(1 + \mathbf{1}_{\mathfrak{p}_{\max(q+d+3,2L+5)}>0}|\mathrm{x}|_{\mathbb{R}^d}^C)C\exp(CT)}{|\mathcal{V}_L(\mathrm{x})T|^\eta(1+|y|_{\mathbb{R}^d}^p)},$$

where $\eta = 13^L 3d\max(\frac{91(d+q)\theta}{44d} + 2, \frac{5}{4}(d+q)^2 + 2(d+q) + 3)$ and $C \geqslant 0$ depends on $d, N, L, q, \mathfrak{D}$, $\mathfrak{D}_{\max(q+d+3,2L+5)}$, $\mathfrak{p}$, $\mathfrak{p}_{\max(q+d+3,2L+5)}$, $\mathfrak{p}_f$, $\frac{1}{m_*}$, $\frac{1}{r_*}$, θ, p and on the moment of Z^δ and which may tend to infinity if one of those quantities tends to infinity.

Now let us prove (2.17). Using Taylor expansions of ψ and recalling that $\psi(x,t,0,0) = x$ for every $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, and then the Azuma-Hoeffding inequality yields

$$
\begin{aligned}
\mathbb{P}(y \leqslant & \delta^\theta G + X^\delta_T | X^\delta_0 = \mathrm{x}) = \mathbb{P}(z - \delta^\theta G \leqslant X^\delta_T | X^\delta_0 = \mathrm{x}) \\
&\leqslant \mathbb{P}(y - x - \delta^\theta G \leqslant 3T\mathfrak{D}_2(1+|z^\infty|^2) + \delta^{\frac{1}{2}} \sum_{t\in\pi^\delta, t<T} \sum_{i=1}^N Z^{\delta,i}_{t+\delta} \partial_{z^i}\psi(X^\delta_t, t, 0, 0) | X^\delta_0 = \mathrm{x}) \\
&\leqslant \min_{j=1,\dots,d} \mathbb{P}(y^j - \mathrm{x}^j - \delta^\theta G^j - 3T\mathfrak{D}_2(1+|z^\infty|^2) \leqslant \\
&\qquad\qquad \delta^{\frac{1}{2}} \sum_{t\in\pi^\delta, t<T} \sum_{i=1}^N Z^{\delta,i}_{t+\delta} \partial_{z^i}\psi(X^\delta_t, t, 0, 0)^j | X^\delta_0 = \mathrm{x}) \\
&\leqslant \min_{j=1,\dots,d} \mathbb{E}[\exp(-\frac{(y^j - \mathrm{x}^j - \delta^\theta G^j - 3T\mathfrak{D}_2(1+|z^\infty|^2))^2}{3(3\mathfrak{D}_1 z^\infty)^2 T})] \\
&\leqslant \min_{j=1,\dots,d} \exp(-\frac{(y^j - \mathrm{x}^j - 3T\mathfrak{D}_2(1+|z^\infty|^2))^2}{3(3\mathfrak{D}_1 z^\infty)^2 T + \delta^{2\theta}})] \\
&\leqslant C\exp(CT - \frac{|y-\mathrm{x}|^2_{\mathbb{R}^d}}{cT}).
\end{aligned}
$$

where $c > 0$ depends on $\mathfrak{D}_1$ and z^∞ and C depends on $\mathfrak{D}_2$ and z^∞. Using the Cauchy-Schwarz inequality combined with the preceding estimate concludes the proof. □

We end this section with the proof of the invariance principle established in Theorem 2.2. Our strategy is to decompose the error using the Lindeberg approach and semigroup properties. Our focus is then on the short time estimate *i.e.* the error made on simply one time step of size δ. Then, we replace Q^δ by $Q^{\delta,\theta}$. Applying Taylor expansion techniques leads to a representation of the error involving some small variations of the process $X^{\delta,\theta}$ satisfying also regularization properties. Exploiting them leads to the expected result. A similar strategy can be designed to prove higher order convergence.

Proof of Theorem 2.2. For $x \in \mathbb{R}^d$, $s,t \in \pi^\delta$, $s \leqslant t$, we define $Q^\delta_{s,t}f(x) := \mathbb{E}[f(X^\delta_t)|X^\delta_s = x]$, $Q^{\delta,\theta}_{s,t}f(x) := \mathbb{E}[f(X^\delta_t + \delta^\theta G)|X^\delta_s = x]$,$P_{s,t}f(x) := \mathbb{E}[f(X_t)|X_s = x] = \mathbb{E}[f(X_t(s,x))]$ ($X_t(s,x)$, being the solution of (1.5) at time t and starting from x at time s), $\Delta f(x) := Q^\delta_{t,t+\delta}f(x) - P_{t,t+\delta}f(x)$ and $\Delta^\theta f(x) := Q^{\delta,\theta}_{t,t+\delta}f(x) - P_{t,t+\delta}f(x)$. It is straightforward to see that the results from Theorem 3.1 remains true replacing $(Q^{\delta,\theta}_t)_{t\geqslant 0}$ by $(Q^{\delta,\theta}_{s,t})_{t\geqslant 0}$ for any $s \geqslant 0$. For sake of clarity, we assume that P satisfies the same regularization property (2.14) as $Q^{\delta,\theta}$. Similar ideas as in [7] can be used to conclude under the actual hypothesis of Theorem 2.2.

We prove the result for $f \in \mathcal{C}^\infty_{pol}(\mathbb{R}^d)$. The extension to f simply measurable with polynomial growth follows from the Lusin's theorem. We provide the main key points avoiding heavy calculus which can be dealt with using similar arguments as the one we already developed to derive Theorem 2.1. Using the semigroup property satisfied by Q^δ and P, we have

$$
\begin{aligned}
Q^\delta_T f(\mathrm{x}) - P_T f(\mathrm{x}) &= \sum_{t\in\pi^\delta, t<T} Q^\delta_{0,t}\Delta_{t,t+\delta}P_{t+\delta,T}f(\mathrm{x}) \\
&= \sum_{t\in\pi^\delta, t<T} Q^{\delta,\theta}_{0,t}\Delta^\theta_{t,t+\delta}P_{t+\delta,T}f(\mathrm{x}) \\
&\quad + (Q^\delta_{0,t}(Q^\delta_{t,t+\delta} - Q^{\delta,\theta}_{t,t+\delta}) + (Q^\delta_{0,t} - Q^{\delta,\theta}_{0,t})Q^{\delta,\theta}_{t,t+\delta})P_{t+\delta,T}f(\mathrm{x}).
\end{aligned}
$$

Now, as a result of Taylor expansions, $\Delta^\theta_{t,t+\delta}f(\mathrm{x})$ can be written as a finite sum of term with form

$$
\mathbb{E}[\int_0^1 \partial^\alpha f(Y^\lambda_{t+\delta}(\mathrm{x}))B(\mathrm{x},t,\delta,\lambda)\mathrm{d}\lambda], \quad \alpha \in \mathbb{N}^d, |\alpha| \leqslant 3,
$$

where $Y^\lambda_{t+\delta}(\mathrm{x})$ takes values in $\{\mathrm{x}, \mathrm{x} + \lambda(\psi(\mathrm{x},t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta},\delta) - \mathrm{x} + \delta^\theta G), X_{t+\lambda\delta}(t,\mathrm{x})\}$ and, for any $p \geqslant 1$,

$$
\sup_{t\in\pi^\delta, t<T} \mathbb{E}[|B(X^{\delta,\theta}_t, t, \delta, \lambda)|^p_{\mathbb{R},\delta,\mathbf{T},|\alpha|} | X^{\delta,\theta}_0 = \mathrm{x}] \leqslant \delta^{\frac{3}{2}}(1 + |\mathrm{x}|^c_{\mathbb{R}^d})C\exp(CT),
$$

where $\mathbf{T}=(0,T]\cap\pi^{\delta}$ (we refer to (4.1) for the definition of our Sobolev-Malliavin norms $|.|_{\mathbb{R},\delta,\mathbf{T},|\alpha|}$). It follows that $Q^{\delta,\theta}_{0,t}\Delta^{\theta}_{t,t+\delta}P_{t+\delta,T}f(\mathrm{x})$ is a finite sum of terms with form

$$\mathbb{E}[\int_0^1 \partial^{\alpha}P_{t+\delta,T}f(Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t))B(X^{\delta,\theta}_t,t,\delta,\lambda)\mathrm{d}\lambda|X^{\delta}_0=\mathrm{x}].$$

At this point, we observe that a similar approach as the one developed in this paper ensures that the results from Theorem 2.1 remains true taking $T=t$ and replacing $X^{\delta,\theta}_t$ by $Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t)$. It hinges on the fact that our Malliavin derivatives of $Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t)-X^{\delta,\theta}_t$ can be bounded by a term of order δ. Moreover, $P_{t+\delta,T}f$ has polynomial growth. It follows that for $t\geqslant\frac{1}{3}T\delta^{\varepsilon}$, ε small enough, exploiting the integration by part from Theorem 4.1 (with $F=Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t)$) in a similar way as in the proof of Theorem 2.1 yields

$$\mathbb{E}[\int_0^1 \partial^{\alpha}P_{t+\delta,T}f(Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t))B(X^{\delta,\theta}_t,t,\delta,\lambda)\mathrm{d}\lambda|X^{\delta}_0=\mathrm{x}]\leqslant\delta^{\frac{3}{2}-\epsilon}\mathfrak{D}_f\frac{1+|\mathrm{x}|^{c}_{\mathbb{R}^d}}{|\mathcal{V}_L(x)T|^{\eta}}C\exp(CT).$$

Now let $t<\frac{1}{3}T\delta^{\varepsilon}$ so that $T-t-\delta>T(1-\frac{1}{3}\delta^{\varepsilon})-\delta\geqslant\frac{2}{3}T-\delta\geqslant\frac{1}{3}T$. We write $Q^{\delta,\theta}_{0,t}\Delta^{\theta}_{t,t+\delta}P_{t+\delta,T}f(\mathrm{x})$ as a finite sum of term with form

$$\begin{aligned}&\mathbb{E}[\int_0^1 \partial^{\alpha}(\phi_{\mathcal{V}_L(x)}P_{t+\delta,T}f)(Y^{\lambda}_{t+\delta}(X^{\delta,\theta}_t))B(X^{\delta,\theta}_t,t,\delta,\lambda)\mathrm{d}\lambda|X^{\delta}_0=\mathrm{x}]\\&+Q^{\delta,\theta}_{0,t}\Delta^{\theta}_{t,t+\delta}((1-\phi_{\mathcal{V}_L(\mathrm{x})})P_{t+\delta,T}f)(\mathrm{x}),\end{aligned}$$

where $\phi_{\mathcal{V}_L(\mathrm{X})}$ is a smooth localizing function satisfying, for every $y\in\mathbb{R}^d$,

$$\mathbf{1}_{|\mathcal{V}_L(y)-\mathcal{V}_L(\mathrm{X})|\leqslant\frac{\mathcal{V}_L(\mathrm{X})}{4}}\leqslant\phi_{\mathcal{V}_L(\mathrm{X})}(y)\leqslant\mathbf{1}_{|\mathcal{V}_L(y)-\mathcal{V}_L(\mathrm{X})|\leqslant\frac{\mathcal{V}_L(\mathrm{X})}{2}}$$

and having derivatives uniformly bounded by a polynomial of $\mathcal{V}_L(x)^{-1}$. Since $T-t-\delta>\frac{1}{3}T$, applying (2.14) for $P_{t+\delta,T}f$ enables to bound the first term of the *r.h.s.* above. To bound the second term, we remark that, since f has polynomial growth then so has $P_{t+\delta,T}f$ and we can show that $\mathfrak{D}_{P_{t+\delta,T}f}=\mathfrak{D}_f$ where C doest not depend on f. Hence

$$(1-\phi_{\mathcal{V}_L(x)}(y))P_{t+\delta,T}f(y)\leqslant C\mathfrak{D}_f(1+|y|^{c}_{\mathbb{R}^d})\mathbf{1}_{|\mathcal{V}_L(y)-\mathcal{V}_L(\mathrm{X})|>\frac{\mathcal{V}_L(\mathrm{X})}{4}}.$$

Therefore, using $\mathbf{A}^{\delta}_1(\max(6,2L+5))$ (see (2.2) and (2.3)) and $t<\frac{1}{3}T\delta^{\varepsilon}$, applications of Markov and Doob (see (4.31)) inequalities yields

$$Q^{\delta,\theta}_{0,t}\Delta^{\theta}_{t,t+\delta}(\mathbf{1}_{|\mathcal{V}_L(.)-\mathcal{V}_L(\mathrm{X})|>\frac{\mathcal{V}_L(\mathrm{X})}{4}})(\mathrm{x})\leqslant\delta^{\frac{3}{2}-\epsilon}\mathfrak{D}_f\frac{1+|\mathrm{x}|^{c}_{\mathbb{R}^d}}{|\mathcal{V}_L(\mathrm{x})T|^{\eta}}C\exp(CT)$$

and the bound on the second term follows from the Cauchy-Schwarz inequality and the proof of (2.18) is completed. If $\mathbf{A}^{\infty}_2(L)$ is assumed, the localization procedure with the function $\phi_{\mathcal{V}_L(\mathrm{X})}$ is not necessary anymore and the achieved convergence rate $\delta^{\frac{1}{2}-\epsilon}$ in (2.18) can be replaced by $\delta^{\frac{1}{2}}$.

Approximation (2.19) follows from an application of Theorem 2.6 in [4]. Notice that this application is also a reason why the convergence happens with rate $\delta^{\frac{1}{2}-\epsilon}$ instead of $\delta^{\frac{1}{2}}$ even in the uniform Hörmander setting $\mathbf{A}^{\infty}_2(L)$. □

4. Malliavin tools and estimates

In this Section we provide three main results which are crucial in the proof of regularization properties. First, we establish an integration by part formula in Theorem 4.1. The proof of regularization results then falls down to bound the weights appearing in those formulas. As a consequence of Proposition 4.1, it can be achieved by bounding the Sobolev norms of X^{δ} in Theorem 4.2 and by bounding the moments of the inverse Malliavin covariance matrix in Theorem 4.3.

4.1. **The integration by parts formula.** In this section, we aim to build some integration by parts formulas in order to prove the regularization properties. This kind of formulas is widely studied in Malliavin calculus for the Gaussian framework. In this section, we always assume that $\mathbf{A}^{\delta}_4$ (see (2.8)) holds true and consider $\mathbf{T}\subset\pi^{\delta,*}$. For $F\in\mathcal{S}^{\delta}(\mathcal{H})$ and $q\in\mathbb{N}$, we begin by introducing the Malliavin-Sobolev norms:

$$(4.1)\qquad |F|^2_{\mathcal{H},\delta,\mathbf{T},1,q}=\sum_{\substack{\alpha\in(\mathbf{T}\times\mathbf{N})^j\\ j\in\{1,\dots,q\}}}\delta^{j}|D^{\delta}_{\alpha}F|^2_{\mathcal{H}},\qquad |F|^2_{\mathcal{H},\delta,\mathbf{T},q}=|F|^2_{\mathcal{H}}+|F|^2_{\mathcal{H},\delta,\mathbf{T},1,q}$$

and for $p \geqslant 1$

$$\|F\|_{\mathcal{H},\delta,\mathbf{T},1,q,p} = \mathbb{E}[|F|^p_{\mathcal{H},\delta,\mathbf{T},1,q}]^{\frac{1}{p}} \qquad \|F\|_{\mathcal{H},\delta,\mathbf{T},q,p} = \mathbb{E}[|F|^p_{\mathcal{H}}]^{\frac{1}{p}} + \|F\|_{\mathcal{H},\delta,\mathbf{T},1,q,p}.$$

Below, we define the Malliavin weights that appear in our integration by parts formulas.
Let $F \in \mathcal{S}^\delta(\mathcal{H})$, $G \in \mathcal{S}^\delta$ and $\mathfrak{h} \in \mathfrak{H}$. We define

$$H^\delta_{\mathbf{T}}(F,G)[\mathfrak{h}] := -\langle G\gamma^\delta_{F,\mathbf{T}} L^\delta_{\mathbf{T}} F, \mathfrak{h}\rangle_{\mathcal{H}} - \delta \sum_{\mathfrak{h}^\diamond \in \mathfrak{H}} \langle D^{\delta,\mathbf{T}}(G\gamma^\delta_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^\diamond]), D^{\delta,\mathbf{T}}\langle F,\mathfrak{h}^\diamond\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\mathbf{T}\times\mathbf{N}}}.$$

Considering higher order integration by parts formulas, for $\bar{\mathfrak{h}} = (\mathfrak{h}_1,\dots,\mathfrak{h}^q) \in \mathfrak{H}^q$ we define $H^\delta_{\mathbf{T}}(F,G)[\bar{\mathfrak{h}}]$ by the recurrence

$$H^\delta_{\mathbf{T}}(F,G)[\bar{\mathfrak{h}}] := H^\delta_{\mathbf{T}}(F, H^\delta_{\mathbf{T}}(F,G)[\mathfrak{h}^1,\dots,\mathfrak{h}^{q-1}])[\mathfrak{h}^q]. \tag{4.2}$$

The purpose of this Section is to establish the following result which is a localized integration by parts formula together with an estimate of the Sobolev norms of the weights. In the following result we denote by $\mathcal{C}^{\mathrm{F},\infty}_{pol}$ the subset of functions f in $\mathcal{C}^{\mathrm{F},\infty}$, such that f and its Frechet derivatives of any order have polynomial growth.

Theorem 4.1. *Let $\mathbf{T} \subset \pi^{\delta,*}$, $q \in \mathbb{N}^*$, $\phi \in \mathcal{C}^{F,\infty}_{pol}(\mathcal{H};\mathbb{R})$ with $\mathfrak{d} := dim(\mathcal{H}) < \infty$. Let $F \in \mathcal{S}^\delta(\mathcal{H})$ and $G \in \mathcal{S}^\delta$ be such that $\mathbb{E}[|\det\gamma^\delta_{F,\mathbf{T}}|^p \mathbf{1}_{|G|_{\mathcal{H},\delta,\mathbf{T},q}>0}] < +\infty$ for every $p \geqslant 1$.*
Then, for every $\bar{\mathfrak{h}} = (\mathfrak{h}^1,\dots,\mathfrak{h}^q) \in \mathfrak{H}^q$,

$$\mathbb{E}[\partial^F_{\bar{\mathfrak{h}}}\phi(F)G] = \mathbb{E}[\phi(F)H^\delta_{\mathbf{T}}(F,G)[\bar{\mathfrak{h}}]] \tag{4.3}$$

with $H^\delta_{\mathbf{T}}(F,G)[\bar{\mathfrak{h}}]$ defined in (4.2). Moreover, for every $m \in \mathbb{N}$,

$$|H^\delta_{\mathbf{T}}(F,G)[\bar{\mathfrak{h}}]|_{\mathbb{R},\delta,\mathbf{T},m} \leqslant C(\mathfrak{d},q,m)\mathfrak{c}(\mathfrak{d},q,m,\mathbf{T},F,G) \tag{4.4}$$

with

$$\begin{aligned}\mathfrak{c}(\mathfrak{d},q,m,\mathbf{T},F,G) =&(1\vee\det\gamma^\delta_{F,\mathbf{T}})^{q(m+q+1)}\\ &\times(1+|F|^{2\mathfrak{d}q(m+q+2)}_{\mathcal{H},\delta,\mathbf{T},1,m+q+1}+|L^\delta_{\mathbf{T}}F|^{2q}_{\mathcal{H},\delta,\mathbf{T},m+q-1})|G|_{\mathbb{R},\delta,\mathbf{T},m+q}.\end{aligned}$$

First, we observe that in our framework, the duality formula eads as follows: For each $F,G \in \mathcal{S}^\delta(\mathcal{H})$,

$$\begin{aligned}\mathbb{E}[\langle F, L^\delta_{\mathbf{T}}G\rangle_{\mathcal{H}}] =&\mathbb{E}[\langle G, L^\delta_{\mathbf{T}}F\rangle_{\mathcal{H}}] = \delta\mathbb{E}[\langle D^{\delta,\mathbf{T}}F, D^{\delta,\mathbf{T}}G\rangle_{\mathcal{H}^{\mathbf{T}\times\mathbf{N}}}]\\ :=&\delta\sum_{t\in\mathbf{T}}\sum_{i=1}^N \mathbb{E}[\langle D^\delta_{(t,i)}F, D^\delta_{(t,i)}G\rangle_{\mathcal{H}}].\end{aligned} \tag{4.5}$$

This follows immediately using the independence structure and standard integration by parts on $\mathbb{R}^N$: Indeed, if $f,g \in \mathcal{C}^2(\mathbb{R}^N;\mathbb{R})$ and $t \in \pi^{\delta,*}$, then

$$\begin{aligned}&\sum_{i=1}^N \mathbb{E}[\partial_{u^i}f(U^\delta_t)\partial_{u^i}g(U^\delta_t)]\\ &=\frac{\varepsilon_*}{m_*}\sum_{i=1}^N\int_{\mathbb{R}^N}\partial_{u^i}f(u)\partial_{u^i}g(u)\delta^{-\frac{N}{2}}\varphi_{r_*/2}(\delta^{-\frac12}u-z_{*,t})du\\ &=-\frac{\varepsilon_*}{m_*}\sum_{i=1}^N\int_{\mathbb{R}^N}f(u)(\partial^2_{u^i}g(u)+\partial_{u^i}g(u)\frac{\partial_{u^i}\varphi_{r_*/2}(\delta^{-\frac12}u-z_{*,t})}{\varphi_{r_*/2}(\delta^{-\frac12}u-z_{*,t})})\delta^{-\frac{N}{2}}\varphi_{r_*/2}(\delta^{-\frac12}u-z_{*,t})du\\ &=-\mathbb{E}[f(U^\delta_t)\sum_{i=1}^N\partial^2_{u^i}g(U^\delta_t)+\partial_{u^i}g(U^\delta_t)\delta^{-\frac12}\partial_{z^i}\ln\varphi_{r_*/2}(\delta^{-\frac12}U^\delta_t-z_{*,t})].\end{aligned}$$

Now consider $F,G \in \mathcal{S}^\delta(\mathcal{H})$, so that $F = f(\chi^\delta,U^\delta,V^\delta)$ and $G = g(\chi^\delta,U^\delta,V^\delta)$ with for every $(\chi,v) \in \{0,1\}^{\pi^{\delta,*}} \times \mathbb{R}^{\pi^{\delta,*}\times\mathbf{N}}$, $u \mapsto f(\chi,u,v) \in \mathcal{C}^{\mathrm{F},\infty}(\mathbb{R}^{\pi^{\delta,*}\times\mathbf{N}};\mathcal{H})$ and similarly for g. Now, we introduce the functions $f_n := \langle f,\mathfrak{h}_n\rangle_{\mathcal{H}}, g_n := \langle g,\mathfrak{h}_n\rangle_{\mathcal{H}}$, $n \in \mathbb{N}^*$, which belong to $\mathcal{C}^{\mathrm{F},\infty}(\mathbb{R}^{\pi^{\delta,*}\times\mathbf{N}};\mathbb{R})$. It follows from the calculus above that

$$
\begin{aligned}
\mathbb{E}[\langle D^{\delta,\mathbf{T}}F, D^{\delta,\mathbf{T}}G\rangle_{\mathcal{H}^{\mathbf{T}\times\mathbf{N}}}] &= \sum_{n=1}^{\infty}\sum_{t\in\mathbf{T}}\sum_{i=1}^{N}\mathbb{E}[\chi_t^{\delta}\partial^{\mathrm{F}}_{u_t^i}f_n(\chi^{\delta},U^{\delta},V^{\delta})\partial^{\mathrm{F}}_{u_t^i}g_n(\chi^{\delta},U^{\delta},V^{\delta})] \\
&= -\sum_{n=1}^{\infty}\mathbb{E}[f_n(\chi^{\delta},U^{\delta},V^{\delta})\sum_{t\in\mathbf{T}}\chi_t^{\delta} \\
&\qquad \times\sum_{i=1}^{N}\partial^{\mathrm{F},2}_{u_t^i}g_n(\chi^{\delta},U^{\delta},V^{\delta}) + \partial^{\mathrm{F}}_{u_t^i}g_n(\chi^{\delta},U^{\delta},V^{\delta})\delta^{-\frac{1}{2}}\partial_{z^i}\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U_t^{\delta}-z_{*,t})] \\
&= -\mathbb{E}[\langle F, \sum_{t\in\mathbf{T}}\sum_{i=1}^{N}D^{\delta}_{(t,i)}D^{\delta}_{(t,i)}G + D^{\delta}_{(t,i)}GD^{\delta}_{(t,i)}\Gamma_t\rangle_{\mathcal{H}}] \\
&=\delta^{-1}\mathbb{E}[\langle F, L^{\delta}_{\mathbf{T}}G\rangle_{\mathcal{H}}],
\end{aligned}
$$

which is exactly (4.5). We have the following standard chain rule: Let $\phi\in\mathcal{C}^{\mathrm{F},1}(\mathcal{H};\mathcal{H}^{\diamond})$ with $\mathcal{H}^{\diamond}$ a Hilbert space and $F\in\mathcal{S}^{\delta}(\mathcal{H})$. Then

$$
D^{\delta,\mathbf{T}}\phi(F) = \partial^{\mathrm{F}}_{D^{\delta,\mathbf{T}}F}\phi(F)\in\mathcal{S}^{\delta}(\mathcal{H}^{\diamond})^{\mathbf{T}\times\mathbf{N}}, \tag{4.6}
$$

More particularly, when $\mathcal{H}^{\diamond}=\mathbb{R}$ we have

$$
D^{\delta,\mathbf{T}}\phi(F) = \langle \mathrm{d}^{\mathrm{F}}\phi(F), D^{\delta,\mathbf{T}}F\rangle_{\mathcal{H}}\in\mathcal{S}^{\delta}(\mathbb{R})^{\mathbf{T}\times\mathbf{N}}, \tag{4.7}
$$

Moreover, one can prove, using (4.6) and the duality relation (or direct computation), that

$$
L^{\delta}_{\mathbf{T}}\phi(F) = \langle \mathrm{d}^{\mathrm{F}}\phi(F), L^{\delta}_{\mathbf{T}}F\rangle_{\mathcal{H}} + \delta\sum_{\mathfrak{h},\mathfrak{h}^{\diamond}\in\mathfrak{H}}\partial^{\mathrm{F}}_{\mathfrak{h}^{\diamond}}\partial^{\mathrm{F}}_{\mathfrak{h}}\phi(F)\langle D^{\delta,\mathbf{T}}\langle F,\mathfrak{h}\rangle_{\mathcal{H}}, D^{\delta,\mathbf{T}}\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\mathbf{T}\times\mathbf{N}}} \tag{4.8}
$$

In order to prove Theorem 4.1, we will combine those identities with the following result.

Proposition 4.1. *Let $F\in\mathcal{S}^{\delta}(\mathcal{H})$ with $\mathfrak{d}:=dim(\mathcal{H})<\infty$, and $G\in\mathcal{S}^{\delta}(\mathbb{R})$. Let $m,q\in\mathbb{N}$, and $\bar{\mathfrak{h}}=(\mathfrak{h}^1,\ldots,\mathfrak{h}^l)\in\mathfrak{H}^l$ with $l\leqslant q$. Then*

$$
|H^{\delta}_{\mathbf{T}}(F,G)[\bar{\mathfrak{h}}]|_{\mathbb{R},\delta,\mathbf{T},m}\leqslant C(\mathfrak{d},q,m)\mathfrak{c}(\mathfrak{d},q,m,\mathbf{T},F,G)
$$

with

$$
\begin{aligned}
\mathfrak{c}(\mathfrak{d},q,m,\mathbf{T},F,G) =&(1\vee\det\gamma^{\delta}_{F,\mathbf{T}})^{q(m+q+1)} \\
&\times(1+|F|^{2\mathfrak{d}q(m+q+2)}_{\mathcal{H},\delta,\mathbf{T},1,m+q+1}+|L^{\delta}_{\mathbf{T}}F|^{2m}_{\mathcal{H},\delta,\mathbf{T},m+q-1})|G|_{\mathbb{R},\delta,\mathbf{T},m+q}.
\end{aligned}
$$

The reader can find the detailed proof of this result in [2], Theorem 3.4. (see also [5]).

Proof of Theorem 4.1. We prove the result for $m=1$. Then, a recurrence yields (4.3). Using the chain rule (4.7), we have for every $\mathfrak{h}^{\diamond}\in\mathfrak{H}$,

$$
\begin{aligned}
\langle D^{\delta,\mathbf{T}}\phi(F), D^{\delta,\mathbf{T}}\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\mathbf{T}\times\mathbf{N}}} =&\sum_{\mathfrak{h}\in\mathfrak{H}}\langle \mathrm{d}^{\mathrm{F}}\phi(F),\mathfrak{h}\rangle_{\mathcal{H}}\langle D^{\delta,\mathbf{T}}\langle F,\mathfrak{h}\rangle_{\mathcal{H}}, D^{\delta,\mathbf{T}}\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\mathbf{T}\times\mathbf{N}}} \\
=&\delta^{-1}\sum_{\mathfrak{h}\in\mathfrak{H}}\partial^{\mathrm{F}}_{\mathfrak{h}}\phi(F)\sigma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}]
\end{aligned}
$$

Using (4.8) with $F=(\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}},\phi(F))$, $\mathcal{H}=\mathbb{R}^2$ and $\phi:(x,y)\mapsto xy$, (4.5) with $F=\phi(F)\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}$ (respectively $F=G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}]\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}$), $G=G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}]$ (resp. $G=\phi(F)$) and $\mathcal{H}=\mathbb{R}$ (resp. $\mathcal{H}=\mathbb{R}$) and finally (4.8) with $F=(\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}},G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}])$, $\mathcal{H}=\mathbb{R}^2$ and $\phi:(x,y)\mapsto xy$, it follows that

$$
\begin{aligned}
\mathbb{E}[\partial_{\mathfrak{h}}^{\mathrm{F}}\phi(F)G] =& \delta\sum_{\mathfrak{h}^{\diamond}\in\mathfrak{H}}\mathbb{E}[G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}]\langle D^{\delta,\mathbf{T}}\phi(F),D^{\delta,\mathbf{T}}\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\mathbf{T}\times\mathbf{N}}}]\\
=&\frac{1}{2}\sum_{\mathfrak{h}^{\diamond}\in\mathfrak{H}}\mathbb{E}[G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}](L^{\delta}_{\mathbf{T}}(\phi(F)\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}})-\phi(F)L^{\delta}_{\mathbf{T}}\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}-\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}L^{\delta}_{\mathbf{T}}\phi(F))]\\
=&\frac{1}{2}\sum_{\mathfrak{h}^{\diamond}\in\mathfrak{H}}\mathbb{E}[\phi(F)\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}L^{\delta}_{\mathbf{T}}(G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}])-\phi(F)G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}]L^{\delta}_{\mathbf{T}}\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}\\
&-\phi(F)L^{\delta}_{\mathbf{T}}(G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}]\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}})]\\
=&-\sum_{\mathfrak{h}^{\diamond}\in\mathfrak{H}}\mathbb{E}[\phi(F)(G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}]L^{\delta}_{\mathbf{T}}\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}+\delta\langle D^{\delta,\mathbf{T}}(G\gamma^{\delta}_{F,\mathbf{T}}[\mathfrak{h},\mathfrak{h}^{\diamond}]),D^{\delta,\mathbf{T}}\langle F,\mathfrak{h}^{\diamond}\rangle_{\mathcal{H}}\rangle_{\mathbb{R}^{\mathbf{T}\times\mathbf{N}}})]
\end{aligned}
$$

which is exactly (4.3) for $q=1$. Iterating this formula, we obtain (4.3).

In order to obtain 4.4, we simply apply Proposition 4.1 and remark that $H^{\delta}_{\mathbf{T}}(F,G)[\bar{\mathfrak{h}}]$ and its Malliavin derivatives are equal to zero as soon as $G=0$. □

In the sequel we establish an estimate of the weights $H^{\delta}_{\mathbf{T}}$ which appear in the integration by parts formulas (4.3) when G is replaced by $G\Theta$ with $\Theta\in[0,1]$ the localizing random weight. The next result provides a bound on the Sobolev norms of $G\Theta$.

Lemma 4.1. *Let $q\in\mathbb{N}$. Let $G\in\mathcal{S}^{\delta}(\mathcal{H})$ and $\Theta\in\mathcal{S}^{\delta}$. Then*

$$
|G\Theta|_{\mathcal{H},\delta,\mathbf{T},q}\leqslant C(q)\sum_{m=0}^{q}|G|_{\mathcal{H},\delta,\mathbf{T},m}|\Theta|_{\mathbb{R},\delta,\mathbf{T},q-m}. \tag{4.9}
$$

Proof. We prove the result by recurrence. For $q\in\mathbb{N}$, we define $\mathcal{H}_0=\mathcal{H}$ and $\mathcal{H}_{q+1}=(\mathcal{H}_q)^{\mathbf{T}\times\mathbf{N}}$. The result is true for $q=0$. Assume it is true until some $q\in\mathbb{N}$ and let us show it still holds for $q+1$. We have

$$
|G\Theta|^2_{\mathcal{H},\delta,\mathbf{T},q+1}=|G\Theta|^2_{\mathcal{H}}+\sum_{l=0}^{q}\delta^{l+1}|D^{\delta,l}(\Theta D^{\delta}G+GD^{\delta}\Theta)|^2_{\mathcal{H}_{l+1}}
$$

with

$$
\begin{aligned}
|D^{\delta,l}(\Theta D^{\delta}G)|_{\mathcal{H}_{l+1}}\leqslant&\ \delta^{-\frac{l}{2}}|\Theta D^{\delta}G|_{\mathcal{H}^{\mathbf{T}\times\mathbf{N}},\delta,\mathbf{T},l}\\
\leqslant& C(l)\delta^{-\frac{l}{2}}\sum_{m=0}^{l}|\Theta|_{\mathbb{R},\delta,\mathbf{T},l-m}|D^{\delta}G|_{\mathcal{H}_1,\delta,\mathbf{T},m}=\delta^{-\frac{l+1}{2}}C(l)\sum_{m=0}^{l}|\Theta|_{\mathbb{R},\delta,\mathbf{T},l-m}|G|_{\mathcal{H},\delta,\mathbf{T},1,m+1},
\end{aligned}
$$

where we have applied (4.9) with G replaced by $D^{\delta}G$, $q=l$ and $\mathcal{H}=\mathcal{H}_1$. Similarly

$$
\begin{aligned}
|D^{\delta,l}(GD^{\delta}\Theta)|_{\mathcal{H}_{l+1}}=&|\sum_{|\alpha|=l}\sum_{|\beta|=1}|D^{\delta}_{\alpha}(GD^{\delta}_{\beta}\Theta)|^2_{\mathcal{H}}|^{\frac{1}{2}}=|\sum_{|\beta|=1}|D^{\delta,l}(GD^{\delta}_{\beta}\Theta)|^2_{\mathcal{H}_l}|^{\frac{1}{2}}\\
\leqslant&|\sum_{|\beta|=1}\delta^{-l}|GD^{\delta}_{\beta}\Theta|^2_{\mathcal{H},\delta,\mathbf{T},l}|^{\frac{1}{2}}\\
\leqslant&C(l)\delta^{-\frac{l}{2}}\sum_{m=0}^{l}|G|_{\mathcal{H},\delta,\mathbf{T},m}|\sum_{|\beta|=1}|D^{\delta}_{\beta}\Theta|^2_{\mathcal{H},\delta,\mathbf{T},l-m}|^{\frac{1}{2}}\\
\leqslant&C(l)\delta^{-\frac{l+1}{2}}\sum_{m=0}^{l}|G|_{\mathcal{H},\delta,\mathbf{T},m}|\Theta|_{\mathbb{R},\delta,\mathbf{T},l+1-m}
\end{aligned}
$$

and the proof is completed. □

The next result provides a bound on the Sobolev norms of $\phi(F)$ when $F\in\mathcal{S}^{\delta}(\mathbb{R}^{\mathfrak{d}})$.

Lemma 4.2. *Let $q\in\mathbb{N}$. Let $\mathfrak{d}\in\mathbb{N}^*$, let $F\in\mathcal{S}^{\delta}(\mathbb{R}^{\mathfrak{d}})$ and $\phi\in\mathcal{C}^q(\mathbb{R}^{\mathfrak{d}},\mathbb{R})$. Then*

$$
|\phi(F)|_{\mathcal{H},\delta,\mathbf{T},q}\leqslant C(q)\sum_{m=0}^{q}|F|^m_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},1,q+1-m}|\sum_{\alpha\in\mathbb{N}^{\mathfrak{d}};|\alpha|\leqslant m}|\partial^{\alpha}_x\phi(F)|^2_{\mathbb{R}}|^{\frac{1}{2}} \tag{4.10}
$$

Proof. We prove the result by recurrence. For $q \in \mathbb{N}$, we define $\mathcal{H}_0 = \mathbb{R}$ and $\mathcal{H}_{q+1} = (\mathcal{H}_q)^{\mathbf{T}\times\mathbf{N}}$. The result is true for $q = 0$. Assume it is true until some $q \in \mathbb{N}$ and let us show it still holds for $q+1$. We have

$$|\phi(F)|^2_{\mathcal{H},\delta,\mathbf{T},q+1} = |\phi(F)|^2_{\mathbb{R}} + \sum_{l=0}^{q} \delta^{l+1}|D^{\delta,l+1}\phi(F)|^2_{\mathcal{H}_{l+1}}.$$

Moreover, using Lemma 4.1, (4.10) with $\phi(F)$ replace by $\partial_x^{(j)}\phi(F)$ and the Cauchy-Schwarz inequality yields

$$\begin{aligned}
&|D^{\delta,l+1}\phi(F)|^2_{\mathcal{H}_{l+1}} = |D^{\delta,l}(D^{\delta}\phi(F))|^2_{\mathcal{H}_{l+1}} = \sum_{j=1}^{\mathfrak{d}} |D^{\delta,l}(\partial_x^{(j)}\phi(F)D^{\delta}F^j)|^2_{\mathcal{H}_{l+1}}\\
&\leqslant \delta^{-l}\sum_{j=1}^{\mathfrak{d}} |\partial_x^{(j)}\phi(F)D^{\delta}F^j|^2_{\mathcal{H}_1,\delta,\mathbf{T},l}\\
&\leqslant \delta^{-l}\sum_{j=1}^{\mathfrak{d}} |C(l)\sum_{m=0}^{l} |\partial_x^{(j)}\phi(F)|_{\mathbb{R},\delta,\mathbf{T},l-m}|D^{\delta}F^j|_{\mathcal{H}_1,\delta,\mathbf{T},m}|^2\\
&\leqslant C(l)\delta^{-l}\sum_{j=1}^{\mathfrak{d}} |\sum_{m=0}^{l}\sum_{m^{\diamond}=0}^{l-m} |F|^{m^{\diamond}}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},1,l+1-m-m^{\diamond}}| \sum_{\alpha\in\mathbb{N}^{\mathfrak{d}};|\alpha|\leqslant m^{\diamond}} |\partial_x^{\alpha}\partial_x^{(j)}\phi(F)|^2_{\mathbb{R}}|^{\frac{1}{2}}|D^{\delta}F^j|_{\mathcal{H}_1,\delta,\mathbf{T},m}|^2\\
&\leqslant C(l)\delta^{-l-1}|\sum_{m=0}^{l}\sum_{m^{\diamond}=0}^{l-m} |F|^{m^{\diamond}}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},1,l+1-m-m^{\diamond}}|F|_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},1,m+1}| \sum_{\alpha\in\mathbb{N}^{\mathfrak{d}};|\alpha|\leqslant m^{\diamond}+1} |\partial_x^{\alpha}\phi(F)|^2_{\mathbb{R}}|^{\frac{1}{2}}|^2\\
&\leqslant C(l)\delta^{-l-1}|\sum_{m=0}^{l+1} |F|^{m}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},1,l+2-m}| \sum_{\alpha\in\mathbb{N}^{\mathfrak{d}};|\alpha|\leqslant m} |\partial_x^{\alpha}\phi(F)|^2_{\mathbb{R}}|^{\frac{1}{2}}|^2,
\end{aligned}$$

and the proof is completed.

□

Lemma 4.3. *Let $q \in \mathbb{N}$. Let $\mathfrak{d} \in \mathbb{N}^*$, let $F \in \mathcal{S}^{\delta}(\mathbb{R}^{\mathfrak{d}})$ and $G \in \mathcal{S}^{\delta}$. Then*

$$\begin{aligned}
|\Psi_{\eta_1}(G\det\gamma^{\delta}_{F,\mathbf{T}})|_{\mathbb{R},\delta,\mathbf{T},q} \leqslant &C(q)\|\Psi_{\eta_1}\|_{\infty,q}(1\vee|\det\gamma^{\delta}_{F,\mathbf{T}}|^{\frac{(q+2)^2}{4}})(1+|F|^{\mathfrak{d}\frac{(q+2)^2}{2}}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},1,q+1})\\
&\times\sum_{m=0}^{q}|G|^{m}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},q-m}
\end{aligned}\tag{4.11}$$

and

$$|\ \Psi_{\eta_2}(|Z^{\delta}_{w}|_{\mathbb{R}^N})|_{\mathbb{R},\delta,\mathbf{T},q} \leqslant C(q)\|\Psi_{\eta_2}(|.|_{\mathbb{R}^N})\|_{\infty,q}\tag{4.12}$$

Proof. First let us recall that it is proved in [5], Proposition 2, that

$$|\det\gamma^{\delta}_{F,\mathbf{T}}|_{\mathbb{R},\delta,\mathbf{T},q} \leqslant C(q)|\det\gamma^{\delta}_{F,\mathbf{T}}|^{q+1}(1+|F|^{2\mathfrak{d}(q+1)}_{\mathcal{H},\delta,\mathbf{T},1,q+1}).$$

Using Lemma 4.1 and Lemma 4.2 and that $\Psi_{\eta_1} \in \mathcal{C}^{\infty}_b(\mathbb{R})$, we have

$$\begin{aligned}
&|\Psi_{\eta_1}(G\det\gamma^{\delta}_{F,\mathbf{T}})|_{\mathbb{R},\delta,\mathbf{T},q}\\
&\leqslant C(q)\sum_{m=0}^{q}|G\det\gamma^{\delta}_{F,\mathbf{T}}|^{m}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},1,q+1-m}|\sum_{\alpha\in\mathbb{N}^{\mathfrak{d}};|\alpha|\leqslant m}|\partial_x^{\alpha}\Psi_{\eta_1}(G\det\gamma^{\delta}_{F,\mathbf{T}})|^2_{\mathbb{R}}|^{\frac{1}{2}}\\
&\leqslant C(q)\sum_{m=0}^{q}|G|^{m}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},q+1-m}|\det\gamma^{\delta}_{F,\mathbf{T}}|^{m}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},q+1-m}|\sum_{\alpha\in\mathbb{N}^{\mathfrak{d}};|\alpha|\leqslant m}|\partial_x^{\alpha}\Psi_{\eta_1}(G\det\gamma^{\delta}_{F,\mathbf{T}})|^2_{\mathbb{R}}|^{\frac{1}{2}}\\
&\leqslant C(q)\|\Psi_{\eta_1}\|_{\infty,q}\sum_{m=0}^{q}|G|^{m}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},q+1-m}(1+|\det\gamma^{\delta}_{F,\mathbf{T}}|^{(q+2-m)m})(1+|F|^{2\mathfrak{d}m(q+2-m)}_{\mathcal{H},\delta,\mathbf{T},1,q+2-m})\\
&\leqslant C(q)\|\Psi_{\eta_1}\|_{\infty,q}(1+|\det\gamma^{\delta}_{F,\mathbf{T}}|^{\frac{(q+2)^2}{4}})(1+|F|^{\mathfrak{d}\frac{(q+2)^2}{2}}_{\mathcal{H},\delta,\mathbf{T},1,q+1})\sum_{m=0}^{q}|G|^{m}_{\mathbb{R}^{\mathfrak{d}},\delta,\mathbf{T},q-m},
\end{aligned}$$

and the proof of (4.11) is completed. In order to prove (4.12), we simply use (3.2) together with Lemma4.2. ☐

4.2. **Sobolev Norms.** Before we state our results, we recall that $\partial_{\mathrm{X}_0^\delta} X_t^\delta$, $t \in \pi^\delta$, is the tangent flow and is introduced in (3.5). In a similar way, for $\alpha \in \mathbb{N}^d$, $\partial^\alpha_{\mathrm{X}_0^\delta} X_t^\delta$ denotes the derivatives of X_t^δ of order $|\alpha|$ *w.r.t.* x_0^δ and is given by $\partial^{\alpha^1}_{(\mathrm{X}_0^\delta)^1} \dots \partial^{\alpha^d}_{(\mathrm{X}_0^\delta)^d} X_t^\delta$. The following result provides an upper bound for the Sobolev norms appearing in the upper bound of the Malliavin weights established in Theorem 4.1.

Theorem 4.2. *Let $T \in \pi^{\delta,*}$ and $\mathbf{T} = (0,T] \cap \pi^\delta$. Let $q \in \mathbb{N}$, $q^\diamond \in \{0,1\}$, $p \geqslant 1$ and $\alpha \in \mathbb{N}^d$ a multi-index. Assume that $\mathbf{A}_1^\delta(q+|\alpha|+2)$ (see (2.2) and (2.3)), $\mathbf{A}_3^\delta(+\infty)$ (see (2.7)) and $\mathbf{A}_4^\delta$ (see (2.8)) hold. Then*

$$
\begin{aligned}
\mathbb{E}[\sup_{t\in\mathbf{T}} |\partial^\alpha_{\mathrm{X}_0^\delta} X_t^\delta|^p_{\mathbb{R}^d,\delta,\mathbf{T},q^\diamond,q}]^{\frac{1}{p}} \leqslant & (|\mathrm{x}_0^\delta|_{\mathbb{R}^d}(\mathbf{1}_{\mathfrak{p}_{q+|\alpha|+2}>0} + \mathbf{1}_{q^\diamond=|\alpha|=0}) + \mathfrak{D}_{q+|\alpha|+2})^{C(q,\mathfrak{p}_{q+|\alpha|+2})} \\
& \times C(d,N,\frac{1}{r_*},q,\mathfrak{p}_{q+|\alpha|+2}) \\
& \times \exp(C(q,p,\mathfrak{p}_{q+|\alpha|+2})(T+1)\mathfrak{M}_{C(p,q,\mathfrak{p},\mathfrak{p}_{q+|\alpha|+2})}(Z^\delta)\mathfrak{D}^2).
\end{aligned}
\tag{4.13}
$$

Moreover, if we replace the assumption $\mathbf{A}_1^\delta(q+|\alpha|+2)$, by the assumption $\mathbf{A}_1^\delta(q+4)$, then

$$
\begin{aligned}
\mathbb{E}[\sup_{t\in\mathbf{T}} |L^\delta_{\mathbf{T}} X_t^\delta|^p_{\mathbb{R}^d,\delta,\mathbf{T},q}]^{\frac{1}{p}} \leqslant & (|\mathrm{x}_0^\delta|_{\mathbb{R}^d}\mathbf{1}_{\mathfrak{p}_{q+4}>0} + \mathfrak{D}_{q+4})^{C(q,\mathfrak{p}_{q+4})} \\
& \times C(d,N,\frac{1}{r_*},q,\mathfrak{p}_{q+4})\exp(C(q,p,\mathfrak{p}_{q+4})(T+1)\mathfrak{M}_{C(p,q,\mathfrak{p},\mathfrak{p}_{q+4})}(Z^\delta)\mathfrak{D}^2).
\end{aligned}
\tag{4.14}
$$

Remark 4.1. *This result was obtained in* [7] *(see Theorem 4.2) in the case $\mathfrak{p}_r = 0$ for r large enough in the assumption $\mathbf{A}_1^\delta(r)$ (see (2.2)).*

4.3. **Malliavin covariance matrix.** In this Section, we provide an upper bound for the localized moments of the inverse of the Malliavin covariance matrix of $(X_t^\delta)_{t\in\pi^\delta}$ defined in (3.1). In the statement of our result, we employ the following quantities

$$
\begin{aligned}
\overline{\eta}_1(\delta) &:= \min(\delta^{-d\frac{44}{91}}, \delta^{-d\frac{44}{91}} \frac{10^d}{m_*^d |2^{10}(1+T^3)|^{\frac{d}{2}}}), \\
\underline{\eta}_1 &:= \max(1, \frac{2^{1-\frac{d}{2}}}{d^{-\frac{d}{2}}}, 2(\frac{40(L+1)N^{\frac{L(L+1)}{2}}}{T\mathcal{V}_L(\mathrm{x}_0^\delta)m_*})^{d13^L}, 2\mathbf{1}_{L=0} + 2\mathbf{1}_{L>0}(m_*|2^8(1+T)|^{143}10N^{\frac{L(L-1)}{2}})^{d13^{L-1}}).
\end{aligned}
$$

We will also use the following assumption

$$
\delta^{\frac{1}{2}}\eta_2^{\mathfrak{p}+1}8\mathfrak{D} < 1. \tag{4.15}
$$

Theorem 4.3. *Let $T \in \pi^{\delta,*}$ and $\mathbf{T} = (0,T] \cap \pi^\delta$ and $p \geqslant 0$. Assume that $\eta_1 \in (\underline{\eta}_1, \overline{\eta}_1(\delta)]$, that $\eta_2 \in (1, \delta^{-\frac{1}{2}}\eta_1^{-\frac{1}{d}}]$ and that (4.15) holds. Also assume that $\mathbf{A}_1^\delta(2L+5)$ (see (2.2) and (2.3)), $\mathbf{A}_2(\mathrm{x}_0^\delta, L)$ (see (2.5)), $\mathbf{A}_3^\delta(+\infty)$ (see (2.7)) and $\mathbf{A}_4^\delta$ (see (2.8)) hold. Define also $\mathfrak{q}^\delta_{\eta_2} := \lceil 1 - \frac{\ln(\delta)}{2\ln(\eta_2)} \rceil$. Then*

$$
\begin{aligned}
\mathbb{E}[|\det \gamma^\delta_{X_T^\delta,\mathbf{T}}|^p \mathbf{1}_{\Theta_{X_T^\delta,\det(\dot{X}_T^\delta)^2,\eta,\mathbf{T}}>0}] \leqslant & \frac{1+\mathbf{1}_{\mathfrak{p}_{2L+5}>0}|\mathrm{x}_0^\delta|_{\mathbb{R}^d}^{C(d,L,p,\mathfrak{p}_{2L+5})}}{(\mathcal{V}_L(\mathrm{x}_0^\delta)T)^{13^L 3d(p+4)}} \mathfrak{D}_{2L+5}^{C(d,L,p)} \\
& \times C(d,N,L,\frac{1}{m_*},p,\mathfrak{p}_{2L+5})\exp(C(d,L,p,\mathfrak{p}_{2L+5})(1+T)\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\mathfrak{q}^\delta_{\eta_2})}(Z^\delta)\mathfrak{D}^4).
\end{aligned}
\tag{4.16}
$$

and, for every $a > 0$,

$$
\begin{aligned}
\mathbb{P}(\Theta_{X_T^\delta,\det(\dot{X}_T^\delta)^2,\eta,\mathbf{T}} < 1) \leqslant & \delta^{-1}T\eta_2^{-a}\mathfrak{M}_a(Z^\delta) \\
& + \eta_1^{-(p+4)}\frac{1+\mathbf{1}_{\mathfrak{p}_{2L+5}>0}|\mathrm{x}_0^\delta|_{\mathbb{R}^d}^{C(d,L,p,\mathfrak{p}_{2L+5})}}{\mathcal{V}_L(\mathrm{x}_0^\delta)^{13^L 3d(p+4)}} \\
& \times \mathfrak{D}^{C(d,L,p)}\mathfrak{D}_{2L+5}^{C(d,L,p)}\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2L+5})}(Z^\delta)C(d,N,L,\frac{1}{m_*},p,\mathfrak{p}_{2L+5}) \\
& \times \exp(C(d,L,p,\mathfrak{p}_{2L+5})T\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\mathfrak{q}^\delta_{\eta_2})}(Z^\delta)\mathfrak{D}^4).
\end{aligned}
\tag{4.17}
$$

Remark 4.2. *We have the following observations concerning the result above.*

(1) Under the assumption $\mathbf{A}_5^\delta(x_0^\delta,T)$ (see 2.12)), we have $\eta_1 \in (\underline{\eta}_1,\overline{\eta}_1(\delta)]$, $\delta^{-\frac{1}{2}}\overline{\eta}_1(\delta)^{-\frac{1}{d}} > 1$ and (4.15) holds.
(2) The terms 13^L in the r.h.s. of both (4.16) and (4.17) can be replaced by $(12+a)^L$, $a>0$, but the miscellaneous constants $C(.)$ may explode when a tends to zero or to infinity.
(3) When the uniform Hörmander hypothesis $\mathbf{A}_2^\infty(L)$ (see (2.5)) holds, the estimates (4.16) and (4.17) can be improved. In particular the term $(T\mathcal{V}_L(x_0^\delta))^{-13^L 3d(p+4)}$ in the r.h.s. of (4.16) may be replaced by $(\mathcal{V}_L^\infty T)^{-13^L dp}$ and $\mathcal{V}_L(x_0^\delta)^{-13^L 3d(p+4)}$ may be replaced by 1 in the r.h.s. of (4.17) In this uniform elliptic setting (L = 0) we thus recover the results from [7] Proposition 4.4.

4.4. **Proof of Theorem 4.2.** We begin by introducing for every $(x,t,z,y)\in\mathbb{R}^d\times\pi^\delta\times\mathbb{R}^N\times[0,1]$ and $(i,j)\in\{1,\ldots,N\}$,

$$A_1^i(x,t)=\partial_{z^i}\psi(x,t,0,0),\quad A_2^{i,j}(x,t,z)=\int_0^1(1-\lambda)\partial_{z^i}\partial_{z^j}\psi(x,t,\lambda z,0)\mathrm{d}\lambda \tag{4.18}$$

$$A_3(x,t,z,y)=\int_0^1\partial_y\psi(x,t,z,\lambda y)\mathrm{d}\lambda$$

We will also denote $A_1:=(A_1^i)_{i\in\mathbf{N}}$ and $A_2:=(A_2^{i,j})_{i,j\in\mathbf{N}^2}$. Before we treat the Sobolev norms of X^δ and $L_{\mathbf{T}}^\delta X^\delta$ we establish some preliminary results. The first one gives an estimate of the Sobolev norms of $L_{\mathbf{T}}^\delta Z^\delta$.

Lemma 4.4. *Let $\mathbf{T}\subset\pi^{\delta,*}$ and $t\in\pi^\delta$, $t>0$. We have the following properties.*

A. *For every $i=1,\ldots,N$, we have*

$$\mathbb{E}[L_{\mathbf{T}}^\delta Z_t^{\delta,i}]=0. \tag{4.19}$$

B. *Assume that (2.10) holds for $v=\frac{r_*}{2}$. Then, for every $q\in\mathbb{N}$ and $p\geqslant1$,*

$$\|L_{\mathbf{T}}^\delta Z_t^\delta\|_{\mathbb{R}^N,\delta,\mathbf{T},q,p}\leqslant\frac{C(N,p,q)m_*^{\frac{1}{p}}}{r_*^{q+1}}\mathbf{1}_{t\in\mathbf{T}}. \tag{4.20}$$

Proof. We prove **A.**. Using the duality relation (4.5) with $\mathcal{H}=\mathbb{R}$, we obtain immediatly $\mathbb{E}[L_{\mathbf{T}}^\delta Z_t^{\delta,i}]=\sum_{(w,j)\in\in\mathbf{T}\times\mathbf{N}}\mathbb{E}[D_{(w,j)}^\delta 1 D_{(w,j)}^\delta Z_t^{\delta,i}]=0$. In order to prove **B.** we recall (see (3.3)) that

$$L_{\mathbf{T}}^\delta Z_t^{\delta,i}=\chi_t^\delta\partial_{z^i}\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U_t^\delta-z_{*,t})\mathbf{1}_{t\in\mathbf{T}}$$

and

$$L_{\mathbf{T}}^\delta Z_t^\delta=\chi_t^\delta\nabla_z\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U_t^\delta-z_{*,t})\mathbf{1}_{t\in\mathbf{T}}.$$

For a multi-index $\alpha=(\alpha^1,\ldots,\alpha^q)$ with $\alpha^j=(t_j,i_j)$, $t_j\in\pi^\delta,t_j>0$, $i_j\in\{1,\ldots,N\}$,

$$D_\alpha^\delta L_{\mathbf{T}}^\delta Z_t^{\delta,i}=\delta^{-\frac{|\alpha|}{2}}\chi_t^\delta\partial_u^{\alpha_i^u}\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U_t^\delta-z_{*,t})\mathbf{1}_{t\in\mathbf{T}}\mathbf{1}_{\cap_{j=1}^q\{t=t_j\}}$$

with $\alpha_i^u:=((\alpha_i^u)^j)_{j\in\mathbf{N}}$, $(\alpha_i^u)^j=\mathbf{1}_{i=j}+\sum_{l=1}^q\mathbf{1}_{i_l=j}$. In particular,

$$\sum_{\substack{\alpha\in(\mathbf{T}\times\mathbf{N})^j\\ j\leqslant q}}\delta^j|D_\alpha^\delta L_{\mathbf{T}}^\delta Z_t^\delta|_{\mathbb{R}^N}^2=\chi_t^\delta\sum_{\substack{\alpha^u\in\mathbb{N}^N\\ |\alpha^u|\in\{1,\ldots,q+1\}}}|\partial_u^{\alpha^u}\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U_t^\delta-z_{*,t})|^2\mathbf{1}_{t\in\mathbf{T}}$$

Since the function $\varphi_{r_*/2}$ is constant on $B_{r_*/2}(0)$ and on $\mathbb{R}^d\setminus\overline{B}_{r_*}(0)$, using (2.10), we obtain

$$\begin{aligned}
&\mathbb{E}[|\sum_{\substack{\alpha\in(\mathbf{T}\times\mathbf{N})^j\\ j\leqslant q}}\delta^j|D_\alpha L_{\mathbf{T}}^\delta Z_t^\delta|_{\mathbb{R}^N}^2|^{\frac{p}{2}}]\\
=&\mathbf{1}_{t\in\mathbf{T}}\frac{\varepsilon_*\mathbb{E}\left[|\chi_t^\delta|^p\right]}{m_*}\int_{\mathbb{R}^N}|\sum_{\substack{\alpha^u\in\mathbb{N}^N\\ |\alpha^u|\in\{1,\ldots,q+1\}}}|\partial_u^{\alpha^u}\ln\varphi_{\frac{r_*}{2}}(\delta^{-\frac{1}{2}}u-z_{*,t})|^2|^{\frac{p}{2}}\delta^{\frac{N}{2}}\varphi_{\frac{r_*}{2}}(\delta^{-\frac{1}{2}}u-z_{*,t})du\\
=&\mathbf{1}_{t\in\mathbf{T}}\varepsilon_*\int_{r_*/2\leqslant|u|\leqslant r_*}|\sum_{\substack{\alpha^u\in\mathbb{N}^N\\ |\alpha^u|\in\{1,\ldots,q+1\}}}|\partial_u^{\alpha^u}\ln\varphi_{\frac{r_*}{2}}(u)|^2|^{\frac{p}{2}}\varphi_{\frac{r_*}{2}}(u)du\\
\leqslant&\frac{C(N,p,q)\delta^{\frac{p}{2}}\varepsilon_*|\pi^{\frac{1}{2}}r_*|^N}{r_*^{p(q+1)}}\mathbf{1}_{t\in\mathbf{T}}.
\end{aligned}$$

In order to derive (4.20), we observe that $m_* \geqslant \varepsilon_* \lambda_{\text{Leb}}(B(0, \frac{r_*}{2}))$ so that $\varepsilon_* |\pi^{\frac{1}{2}} \frac{r_*}{2}|^N \leqslant C m_*$. □

Now, we establish a bound on the moments of $(X^\delta_t)_{t \in \pi^\delta}$.

Lemma 4.5. *Let* $T > 0$, $\mathbf{T} = [0,T] \cap \pi^\delta$ *and* $p \geqslant 1$. *Assume that* $\mathbf{A}^\delta_1(2)$ *(see (2.2) and (2.3)) and* $\mathbf{A}^\delta_3((\mathfrak{p}+1)(p \vee 2))$ *(see (2.7)) hold. Then,*

$$\mathbb{E}[\sup_{t \in \mathbf{T}} |X^\delta_t|^p_{\mathbb{R}^d}]^{\frac{1}{p}} \leqslant (1 + |X^\delta_0|_{\mathbb{R}^d}) \exp(C(p) T \mathfrak{D}^{\frac{2}{p} \vee 1} \mathfrak{M}_{(\mathfrak{p}+1)(p \vee 2)}(Z^\delta)^{\frac{1}{p}}). \tag{4.21}$$

Proof. Consider $t \in \pi^{\delta,*}$. Using the Taylor expansion yields

$$\begin{aligned}
|X^\delta_t|^p_{\mathbb{R}^d} =& |X^\delta_{t-\delta}|^p_{\mathbb{R}^d} + p |X^\delta_{t-\delta}|^{p-2}_{\mathbb{R}^d} \sum_{i=1}^d X^{\delta,i}_{t-\delta}(X^{\delta,i}_t - X^{\delta,i}_{t-\delta}) \\
&+ \sum_{i,j=1}^d (X^\delta_t - X^\delta_{t-\delta})_{i \otimes j} \\
&\times p \int_0^1 (1-\lambda) |X^\delta_{t-\delta} + \lambda(X^\delta_t - X^\delta_{t-\delta})|^{p-2} \mathbf{1}_{i=j} \\
&+ (p-2)(1-\lambda) |X^\delta_{t-\delta} + \lambda(X^\delta_t - X^\delta_{t-\delta})|^{p-4}_{\mathbb{R}^d} (X^\delta_{t-\delta} + \lambda(X^\delta_t - X^\delta_{t-\delta}))_{i \otimes j} \mathrm{d}\lambda
\end{aligned}$$

with notation $\mathbf{x}_{i \otimes j} = \mathbf{x}^i \mathbf{x}^j$ for $\mathbf{x} \in \mathbb{R}^d$, $i,j \in \{1, \ldots, d\}$ and, with notations from (4.18),

$$\begin{aligned}
X^\delta_t =& X^\delta_{t-\delta} + \delta^{\frac{1}{2}} \sum_{i=1}^N Z^{\delta,i}_t \int_0^1 \partial_{z^i} \psi(X^\delta_{t-\delta}, t-\delta, \lambda \delta^{\frac{1}{2}} Z^\delta_t, 0) \mathrm{d}\lambda + \delta A_3(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, \delta) \\
=& X^\delta_{t-\delta} + \delta^{\frac{1}{2}} \sum_{i=1}^N Z^{\delta,i}_t A^i_1(X^\delta_{t-\delta}, t-\delta, 0, 0) + \delta \sum_{i,j=1}^N Z^{\delta,i}_t Z^{\delta,j}_t A^{i,j}_2(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, 0) \\
&+ \delta A_3(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, \delta),
\end{aligned}$$

Moreover, for every $(x,t,z,y) \in \mathbb{R}^d \times \pi^\delta \times \mathbb{R}^N \times [0,1]$, we have

$$\partial_y \psi(x,z,t,y) = \partial_y \psi(0,z,t,y) + \sum_{l=1}^d x^l \int_0^1 \partial_{x^l} \partial_y \psi(\lambda x, z, t, y) \mathrm{d}\lambda$$

with similar formulas for the derivatives *w.r.t.* z. Moreover, it follows from assumption $\mathbf{A}^\delta_1(2)$, (2.2) that

$$\{|\partial_y \psi|_{\mathbb{R}^d} + \sum_{i=1}^N |\partial_{z^i} \psi|_{\mathbb{R}^d} + \sum_{i,j=1}^N |\partial_{z^i} \partial_{z^j} \psi|_{\mathbb{R}^d}\}(0,t,z,y) \leqslant \mathfrak{D}_2(1 + \delta^{\frac{\mathfrak{p}_2}{2}} |z|^{\mathfrak{p}_2}_{\mathbb{R}^N})$$

Combining the previous inequality with $\mathbf{A}^\delta_1(2)$, (2.3) yields

$$\begin{aligned}
\{|\partial_y \psi|_{\mathbb{R}^d} + &\sum_{i=1}^N |\partial_{z^i} \psi|_{\mathbb{R}^d} + \sum_{i,j=1}^N |\partial_{z^i} \partial_{z^j} \psi|_{\mathbb{R}^d}\}(x,t,z,y) \leqslant \mathfrak{D}_2(1 + \delta^{\frac{\mathfrak{p}_2}{2}} |z|^{\mathfrak{p}_2}_{\mathbb{R}^N}) \\
&+ \sum_{l=1}^d x^l \int_0^1 \{|\partial_{x^l} \partial_y \psi|_{\mathbb{R}^d} + \sum_{i=1}^N |\partial_{x^l} \partial_{z^i} \psi|_{\mathbb{R}^d} + \sum_{i,j=1}^N |\partial_{x^l} \partial_{z^i} \partial_{z^j} \psi|_{\mathbb{R}^d}\}(\lambda x, t, z, y) \mathrm{d}\lambda \\
&\leqslant \mathfrak{D}_2(1 + \delta^{\frac{\mathfrak{p}_2}{2}} |z|^{\mathfrak{p}_2}_{\mathbb{R}^N}) + \mathfrak{D} |x|_{\mathbb{R}^d}(1 + \delta^{-\frac{\mathfrak{p}}{2}} |z|^{\mathfrak{p}}_{\mathbb{R}^N}) =: D(x,z,\delta)
\end{aligned}$$

In particular, since $\mathfrak{D} \geqslant \mathfrak{D}_2$ and $\mathfrak{p} \geqslant \mathfrak{p}_2$, for $p \geqslant 2$

$$\begin{aligned}
|\mathbb{E}[|X^\delta_t|^p_{\mathbb{R}^d}] - \mathbb{E}[|X^\delta_{t-\delta}|^p_{\mathbb{R}^d}]| \leqslant& p\delta \mathbb{E}[\, |X^\delta_{t-\delta}|^{p-1}_{\mathbb{R}^d} D(X^\delta_{t-\delta}, \delta^{\frac{1}{2}} Z^\delta_t, \delta)(1 + |Z^\delta_t|^2_{\mathbb{R}^N})] \\
&+ p(p-1)\delta 2^{p-2} \mathbb{E}[|X^\delta_{t-\delta}|^{p-2}_{\mathbb{R}^d} D(X^\delta_{t-\delta}, \delta^{\frac{1}{2}} Z^\delta_t, \delta)^2 (1 + |Z^\delta_t|_{\mathbb{R}^N})^2 \\
&+ \delta^{\frac{p}{2}} D(X^\delta_{t-\delta}, \delta^{\frac{1}{2}} Z^\delta_t, \delta)^p (1 + |Z^\delta_t|_{\mathbb{R}^N})^p] \\
\leqslant& C(p) \mathfrak{M}_{(\mathfrak{p}+1)(p \vee 2)}(Z^\delta) \mathfrak{D}^{p \vee 2} \delta \mathbb{E}[1 + |X^\delta_{t-\delta}|^p_{\mathbb{R}^d}]
\end{aligned}$$

and (4.21) follows from the Gronwall lemma. For $p \in [1, 2)$, it simply remains to use the Cauchy-Schwarz inequality.

□

In order to obtain estimates of the Sobolev norms which appear in Theorem 4.2, we derive some estimates for a generic class of processes which involves the Malliavin derivatives of $\partial^{\alpha}_{X^{\delta}_0} X^{\delta}$ and $L^{\delta}_{\mathbf{T}} X^{\delta}_t$. We first write, for $t \in \pi^{\delta}$,

$$\begin{aligned} X^{\delta}_{t+\delta} =& X^{\delta}_t + \delta^{\frac{1}{2}} \sum_{i=1}^{N} Z^{\delta,i}_{t+\delta} A^i_1(X^{\delta}_t, t) + \delta \sum_{i,j=1}^{N} Z^{\delta,i}_{t+\delta} Z^{\delta,j}_{t+\delta} A^{i,j}_2(X^{\delta}_t, t, \delta^{\frac{1}{2}} Z^{\delta}_{t+\delta}) \\ &+ A_3(X^{\delta}_t, t, \delta^{\frac{1}{2}} Z^{\delta}_{t+\delta}, \delta), \end{aligned}$$

with A_1, A_2, and A_3 defined in (4.18). We introduce the $\mathbb{R}^{d\times d}$-valued process $(B_t)_{t\in\pi^{\delta}}$ such that for every $t \in \pi^{\delta}$,

$$B_t = \delta^{\frac{1}{2}} \sum_{i=1}^{N} Z^{\delta,i}_{t+\delta} \nabla_x A^i_1(X^{\delta}_t, t) + \delta \sum_{i,j=1}^{N} Z^{\delta,i}_{t+\delta} Z^{\delta,j}_{t+\delta} \nabla_x A^{i,j}_2(X^{\delta}_t, t, \delta^{\frac{1}{2}} Z^{\delta}_{t+\delta}) + \delta \nabla_x A_3(X^{\delta}_t, t, \delta^{\frac{1}{2}} Z^{\delta}_{t+\delta}, \delta).$$

We now consider a Hilbert space $\mathcal{H}$ and introduce some $\mathcal{H}^d$-valued processes $(B^{1,i}_t)_{t\in\pi^{\delta}}$, $(B^{2,i}_t)_{t\in\pi^{\delta}}$, which are both adapted to the filtration $(\sigma(Z^{\delta}_{\delta}, \ldots, Z^{\delta}_t))_{t\in\pi^{\delta}}$ and $(B^3_t)_{t\in\pi^{\delta}}$ which is adapted to the filtration $(\sigma(Z^{\delta}_{\delta}, \ldots, Z^{\delta}_{t+\delta}))_{t\in\pi^{\delta}}$ and for every $h \in \mathcal{H}$, $\langle B^{l,i}, h\rangle_{\mathcal{H}}$, $l = 1, 2$, and $\langle B^3, h\rangle_{\mathcal{H}}$, all belong to $(\mathcal{S}^{\delta})^d$. In this proof we will consider a $\mathcal{H}^d$-valued generic process $(Y_t)_{t\in\pi^{\delta}}$ which satisfies,for every $t \in \pi^{\delta}$,

$$Y_{t+\delta} = Y_t + B_t Y_t + \delta^{\frac{1}{2}} \sum_{i=1}^{N} Z^{\delta,i}_{t+\delta} B^{1,i}_t + \delta^{\frac{1}{2}} \sum_{i=1}^{N} L^{\delta}_{\mathbf{T}} Z^{\delta,i}_{t+\delta} B^{2,i}_t + B^3_t \tag{4.22}$$

$$\begin{aligned} \mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q,p}(B^1, B^2, B^3) =& 1 \\ &+ \sup_{t\in\mathbf{T}} (\|B^{1,\cdot}_{t-\delta}\|_{(\mathcal{H}^d)^{\mathbf{N}},\delta,\mathbf{T},q,p} + \|B^{2,\cdot}_{t-\delta}\|_{(\mathcal{H}^d)^{\mathbf{N}},\delta,\mathbf{T},q,p} + \|\sum_{\substack{w\in\pi^{\delta} \\ w<t}} B^3_w\|_{\mathcal{H}^d,\delta,\mathbf{T},q,p}). \end{aligned}$$

where for $(B(i,l))_{(i,l)\in\mathbf{N}\times\{1,\ldots,d\}}$ taking values in $\mathcal{H}$, $|B|_{(\mathcal{H}^d)^{\mathbf{N}}} = |\sum_{i=1}^{N}\sum_{l=1}^{d} |B(i,l)|^2_{\mathcal{H}}|^{\frac{1}{2}}$. Before we estimate the Sobolev norms, we recall the Burkholder inequality for Hilbert space. We consider a separable Hilbert space $\mathcal{H}$, we denote $|.|_{\mathcal{H}}$ the norm of $\mathcal{H}$ and, for a random variable $F \in \mathcal{H}$, we denote $\|F\|_{\mathcal{H},p} = \mathbb{E}[|F|^p_{\mathcal{H}}]^{\frac{1}{p}}$. Moreover we consider a martingale $\mathcal{M}_n \in \mathcal{H}$, $n \in \mathbb{N}$ and we recall Burkholder inequality in this framework: For each $p \geqslant 2$ there exists a constant $\mathfrak{b}_p \geqslant 1$ such that

$$\forall n \in \mathbb{N}, \quad \|\sup_{k\in\{0,\ldots,n\}} \mathcal{M}_k\|_{\mathcal{H},p} \leqslant \mathfrak{b}_p \mathbb{E}[(\sum_{k=1}^{n} |\mathcal{M}_k - \mathcal{M}_{k-1}|^2_{\mathcal{H}})^{\frac{p}{2}}]^{\frac{1}{p}}. \tag{4.23}$$

As an immediate consequence

$$\|\sup_{k\in\{0,\ldots,n\}} \mathcal{M}_k\|_{\mathcal{H},p} \leqslant \mathfrak{b}_p |\sum_{k=1}^{n} \|\mathcal{M}_k - \mathcal{M}_{k-1}\|^2_{\mathcal{H},p}|^{\frac{1}{2}}. \tag{4.24}$$

This first result gives an estimate of the Sobolev norms of $(X^{\delta}_t)_{t\in\mathbf{T}}$, $(Y_t)_{t\in\mathbf{T}}$ *w.r.t.* the quatity above.

Proposition 4.2. *Let $T > 0$, $\mathbf{T} = (0, T] \cap \pi^{\delta}$. Let $q \in \mathbb{N}$ and $p \geqslant 1$. Assume that $\mathbf{A}^{\delta}_1(q + 2)$ (see (2.2) and (2.3)), $\mathbf{A}^{\delta}_3(+\infty)$ (see (2.7)) and $\mathbf{A}^{\delta}_4$ (see (2.8)) hold. Then*

$$\begin{aligned} \mathbb{E}[\sup_{t\in\mathbf{T}} |X^{\delta}_t|^p_{\mathbb{R}^d,1,q}]^{\frac{1}{p}} \leqslant& (|X^{\delta}_0|_{\mathbb{R}^d} \mathbf{1}_{\mathfrak{p}_{q+2}>0} + \mathfrak{D}_{q+2})^{C(q,\mathfrak{p}_{q+2})} \\ &\times \exp(C(q, p, \mathfrak{p}_{q+2})(T + 1)\mathfrak{M}_{C(p,q,\mathfrak{p},\mathfrak{p}_{q+2})}(Z^{\delta})\mathfrak{D}^2). \end{aligned} \tag{4.25}$$

when $q \geqslant 1$. Moreover, for $(Y_t)_{t\in\pi^{\delta}}$ satisfying (4.22), if we assume that $\mathbf{A}^{\delta}_1(q + 2)$ holds, then

$$\begin{aligned} &\mathbb{E}[\sup_{t\in\mathbf{T}} |Y_t|^p_{\mathcal{H}^d,\delta,\mathbf{T},q}]^{\frac{1}{p}} \\ &\leqslant (\mathbb{E}[|Y_0|^{2^q p}_{\mathcal{H}^d,\delta,\mathbf{T},q}]^{\frac{1}{2^q p}} + \mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q,2^q p}(B^1, B^2, B^3))(|X^{\delta}_0|_{\mathbb{R}^d} \mathbf{1}_{\mathfrak{p}_{q+3}>0} + \mathfrak{D}_{q+3})^{C(q,\mathfrak{p}_{q+3})} \\ &\quad\times C(d, N, m_*, \frac{1}{r_*}, q, \mathfrak{p}_{q+3}) \exp(C(N, q, p, \mathfrak{p}_{q+3})(T + 1)\mathfrak{M}_{C(p,q,\mathfrak{p},\mathfrak{p}_{q+3})}(Z^{\delta})\mathfrak{D}^2). \end{aligned} \tag{4.26}$$

Proof. **Step 1.** Let $q=0$. We first prove that

$$\begin{aligned}\mathbb{E}[\sup_{t\in\mathbf{T}}|Y_t|^p_{\mathcal{H}^d}]^{\frac{1}{p}} \leqslant&(\mathbb{E}[|Y_0|^p_{\mathcal{H}^d}]^{\frac{1}{p}}+\mathfrak{b}_p\mathfrak{M}_p(Z^\delta)^{\frac{1}{p}}T^{\frac{1}{2}}\mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},0,p}(B^1,0,0)\\ &+\mathfrak{b}_p\frac{C(N,p)m_*^{\frac{1}{p}}}{r_*}T^{\frac{1}{2}}\mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},0,p}(0,B^2,0)+\mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},0,p}(0,0,B^3))\\ &\times\exp(C(p)(T+1)\mathfrak{M}_{p(\mathfrak{p}+2)}(Z^\delta)^{\frac{2}{p}}\mathfrak{D}^2).\end{aligned}\tag{4.27}$$

We study the terms which appear in the right hand side of (4.22). We consider $i,j\in\mathbf{N}$. Notice that for every $t\in\pi^\delta$, $\mathbb{E}[L^\delta_{\mathbf{T}}Z^{\delta,i}_{t+\delta}]=0$ (see (4.19)) and $B^{2,i}_t$ is $\mathcal{F}^{Z^\delta}_t$-measurable. It follows from (4.24) (with $\mathcal{H}$ replaced by $\mathcal{H}^d$) and (4.20) that

$$\begin{aligned}\mathbb{E}[\sup_{t\in\mathbf{T}}|\delta^{\frac{1}{2}}\sum_{i=1}^N\sum_{\substack{w\in\pi^\delta\\ w<t}}L^\delta_{\mathbf{T}}Z^{\delta,i}_{w+\delta}B^{2,i}_w|^p_{\mathcal{H}^d}]^{\frac{2}{p}}\leqslant&\mathfrak{b}_p^2\delta\sum_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[|\sum_{i=1}^N L^\delta_{\mathbf{T}}Z^{\delta,i}_{t+\delta}B^{2,i}_t|^p_{\mathcal{H}^d}]^{\frac{2}{p}}\\ \leqslant&\mathfrak{b}_p^2\frac{C(N,p)m_*^{\frac{2}{p}}}{r_*^2}\delta\sum_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[|\sum_{i=1}^N|B^{2,i}_t|^2_{\mathcal{H}^d}|^{\frac{p}{2}}]^{\frac{2}{p}}\\ =&\mathfrak{b}_p^2\frac{C(N,p)m_*^{\frac{2}{p}}}{r_*^2}T\sup_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[|B^{2,\cdot}_t|^p_{(\mathcal{H}^d)^{\mathbf{N}}}]^{\frac{2}{p}}.\end{aligned}$$

In the same way,

$$\mathbb{E}[\sup_{t\in\mathbf{T}}|\delta^{\frac{1}{2}}\sum_{i=1}^N\sum_{\substack{w\in\mathbf{T}\\ w<t}}Z^{\delta,i}_{w+\delta}B^{1,i}_w|^p_{\mathcal{H}^d}]^{\frac{2}{p}}\leqslant\mathfrak{b}_p^2\mathfrak{M}_p(Z^\delta)^{\frac{2}{p}}T\sup_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[|B^{1,\cdot}_t|^p_{(\mathcal{H}^d)^{\mathbf{N}}}]^{\frac{2}{p}}.$$

Using $\mathbf{A}_1$ (see (2.3)) together with (4.24) (with $\mathcal{H}$ replaced by $\mathcal{H}^d$) yields

$$\begin{aligned}\mathbb{E}[\sup_{t\in\mathbf{T}}|\delta^{\frac{1}{2}}\sum_{i=1}^N\sum_{\substack{w\in\pi^\delta\\ w<t}}Z^{\delta,i}_{w+\delta}\nabla_xA^i_1(X^\delta_w,w)Y_w|^p_{\mathcal{H}^d}]^{\frac{2}{p}}\leqslant&\mathfrak{b}_p^2\delta\sum_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[|\sum_{i=1}^N Z^{\delta,i}_{t+\delta}\nabla_xA^i_1(X^\delta_t,t)Y_t|^p_{\mathcal{H}^d}]^{\frac{2}{p}}\\ \leqslant&\mathfrak{b}_p^2\mathfrak{M}_p(Z^\delta)^{\frac{2}{p}}\mathfrak{D}^2\delta\sum_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[|Y_t|^p_{\mathcal{H}^d}]^{\frac{2}{p}}.\end{aligned}$$

Applying $\mathbf{A}^\delta_1$ (see (2.3)) with the triangle inequality also gives

$$\begin{aligned}&\mathbb{E}[\sup_{t\in\mathbf{T}}|\delta\sum_{\substack{w\in\pi^\delta\\ w<t}}Z^{\delta,i}_{w+\delta}Z^{\delta,j}_{w+\delta}\nabla_xA^{i,j}_2(X^\delta_w,w,\delta^{\frac{1}{2}}Z^\delta_{w+\delta})Y_w|^p_{\mathcal{H}^d}]^{\frac{1}{p}}\\ \leqslant&\delta\sum_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[|Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}\nabla_xA^{i,j}_2(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta})Y_t|^p_{\mathcal{H}^d}]^{\frac{1}{p}}\\ \leqslant&2\mathfrak{M}_{p(\mathfrak{p}+2)}(Z^\delta)^{\frac{1}{p}}\mathfrak{D}\delta\sum_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[|Y_t|^p_{\mathcal{H}^d}]^{\frac{1}{p}},\end{aligned}$$

and similarly

$$\begin{aligned}\mathbb{E}[\sup_{t\in\mathbf{T}}|\delta\sum_{\substack{w\in\pi^\delta\\ w<t}}\nabla_xA_3(X^\delta_w,w,\delta^{\frac{1}{2}}Z^\delta_{w+\delta},\delta)Y_w|^p_{\mathcal{H}^d}]^{\frac{1}{p}}\leqslant&\delta\sum_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[\nabla_xA_3(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta},\delta)Y_t|^p_{\mathcal{H}^d}]^{\frac{1}{p}}\\ \leqslant&2\mathfrak{M}_{p\mathfrak{p}}(Z^\delta)^{\frac{1}{p}}\mathfrak{D}\delta\sum_{\substack{t\in\pi^\delta\\ t<T}}\mathbb{E}[|Y_t|^p_{\mathcal{H}^d}]^{\frac{1}{p}}.\end{aligned}$$

We gather all the terms and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\mathbb{E}[\sup_{\substack{t\in\pi^\delta\\t\leqslant T}}|Y_t|^p_{\mathcal{H}^d}]^{\frac{1}{p}} \leqslant &\mathbb{E}[|Y_0|^p_{\mathcal{H}^d}]^{\frac{1}{p}}+\mathfrak{b}_p\mathfrak{M}_p(Z^\delta)^{\frac{1}{p}}T^{\frac{1}{2}}\sup_{\substack{t\in\pi^\delta\\t<T}}\mathbb{E}[B^{1,\cdot}_t|^p_{(\mathcal{H}^d)^{\mathbf{N}}}]^{\frac{1}{p}}\\&+\mathfrak{b}_p\frac{C(N,p)m_*^{\frac{1}{p}}}{r_*}T^{\frac{1}{2}}\sup_{\substack{t\in\pi^\delta\\t<T}}\mathbb{E}[|B^{2,\cdot}_t|^p_{(\mathcal{H}^d)^{\mathbf{N}}}]^{\frac{1}{p}}+\sup_{\substack{t\in\pi^\delta\\t<T}}\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\w<t}}|B^3_w|_{\mathcal{H}^d}|^p]^{\frac{1}{p}}\\&+(4T^{\frac{1}{2}}+\mathfrak{b}_p)\mathfrak{M}_{p(\mathfrak{p}+2)}(Z^\delta)^{\frac{1}{p}}\mathfrak{D}(\delta\sum_{\substack{t\in\pi^\delta\\t<T}}\mathbb{E}[|Y_t|^p_{\mathcal{H}^d}]^{\frac{2}{p}})^{\frac{1}{2}}\end{aligned}$$

Hence, using the Gronwall lemma yields (4.27).

Step 2. Let us prove (4.25). For $q\in\mathbb{N}$, we define $\mathcal{R}_0=\mathbb{R}$ and $\mathcal{R}_{q+1}=(\mathcal{R}_q)^{\mathbf{T}\times\mathbf{N}}$ and we have

$$\mathbb{E}[\sup_{t\in\mathbf{T}}|X^\delta_t|^p_{\mathbb{R}^d,\delta,\mathbf{T},1,q}]^{\frac{1}{p}}=\mathbb{E}[\sup_{t\in\mathbf{T}}\sum_{q^\diamond=1}^{q}|D^{\delta,q^\diamond}X^\delta_t|^p_{\mathcal{R}^d_{q^\diamond}}]^{\frac{1}{p}}.$$

First, we focus on the case $q=1$ and prove that

$$\begin{aligned}\delta^{\frac{1}{2}}\mathbb{E}[\sup_{t\in\mathbf{T}}|D^\delta X^\delta_t|^p_{\mathcal{R}^d_1}]^{\frac{1}{p}}=&\delta^{\frac{1}{2}}\mathbb{E}[\sup_{t\in\mathbf{T}}|\sum_{w\in\mathbf{T}}\sum_{i=1}^{N}|D^\delta_{(w,i)}X^\delta_t|^2_{\mathbb{R}^d}|^{\frac{p}{2}}]^{\frac{1}{p}}\\\leqslant&\mathfrak{D}_3(1+|\mathrm{x}^\delta_0|^{\mathfrak{p}_3}_{\mathbb{R}^d})\exp((T+1)\mathfrak{D}^2\mathfrak{M}_{p(\mathfrak{p}+1)(\mathfrak{p}_3\vee2)}(Z^\delta)^2C(p,\mathfrak{p}_3)).\end{aligned}\tag{4.28}$$

We remark that for every $t\in\pi^\delta$, $w\in\mathbf{T}$, and every $i\in\mathbf{N}$.

$$\delta^{\frac{1}{2}}D^\delta_{(w,i)}X^\delta_{t+\delta}=(I_{d\times d}+B_t)\delta^{\frac{1}{2}}D^\delta_{(w,i)}X^\delta_t+(B^3_{1,t})_{w,i},$$

with, for $(w,i)\in\mathbf{T}\times\mathbf{N}$,

$$\begin{aligned}(B^3_{1,t})_{w,i}=&\chi^\delta_{t+\delta}\mathbf{1}_{w=t+\delta}(\delta^{\frac{1}{2}}A^i_1(X^\delta_t,t)+\delta\sum_{j=1}^{N}Z^{\delta,j}_{t+\delta}(1+\mathbf{1}_{i=j})A^{i,j}_2(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta})\\&+\delta^{\frac{3}{2}}\sum_{j,l=1}^{N}Z^{\delta,j}_{t+\delta}Z^{\delta,l}_{t+\delta}\partial_{z^i}A^{j,l}_2(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta})+\delta^{\frac{3}{2}}\partial_{z^i}A_3(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta},\delta)).\end{aligned}$$

In particular, $\delta^{\frac{1}{2}}D^\delta X^\delta_t=(\delta^{\frac{1}{2}}D^\delta_{(w,i)}X^\delta_t)_{(w,i)\in\mathbf{T}\times\mathbf{N}}$ is a $\mathcal{R}^d_1$-valued random variable and, for $t\in\pi^\delta$, we have

$$\delta^{\frac{1}{2}}D^\delta X^\delta_{t+\delta}=(I_{d\times d}+B_t)\delta^{\frac{1}{2}}D^\delta X^\delta_t+B^3_{1,t}.$$

Then, (4.28) follows from Lemma 4.5 (see (4.21)) and (4.27) with $Y=\delta^{\frac{1}{2}}D^\delta X^\delta$, $\mathcal{H}=\mathcal{R}_1$, and B^3 thus defined since the assumption $\mathbf{A}^\delta_1(3)$ (see (2.2)) implies that

$$\begin{aligned}&\mathfrak{S}_{\mathcal{R}^d_1,\delta,\mathbf{T},0,p}(0,0,B^3_{1,\cdot})\\&=1+\sup_{\substack{t\in\pi^\delta\\t<T}}\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\w<t}}B^3_{1,w}|^p_{\mathcal{R}^d_1}]^{\frac{1}{p}}=1+\sup_{\substack{t\in\pi^\delta\\t<T}}\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\w<t}}\sum_{i=1}^{N}|(B^3_{1,w})_{w+\delta,i}|^2_{\mathbb{R}^d}|^{\frac{p}{2}}]^{\frac{1}{p}}\\&\leqslant1+\mathbb{E}[|\sum_{\substack{t\in\pi^\delta\\t<T}}|(B^3_{1,t})_{t+\delta,\cdot}|^2_{(\mathbb{R}^d)^{\mathbf{N}}}|^{\frac{p}{2}}]^{\frac{1}{p}}\\&\leqslant1+T^{\frac{1}{2}}\delta^{-\frac{1}{2}}\sup_{\substack{t\in\pi^\delta\\t<T}}\mathbb{E}[|(B^3_{1,t})_{t+\delta,\cdot}|^p_{(\mathbb{R}^d)^{\mathbf{N}}}]^{\frac{1}{p}}\\&\leqslant1+5T^{\frac{1}{2}}\mathfrak{D}_3(\mathfrak{M}_{2p}(Z^\delta)^{\frac{1}{p}}+\mathfrak{M}_{2p}(Z^\delta)^{\frac{1}{p}}\mathbb{E}[\sup_{t\in\mathbf{T}}|X^\delta_{t-\delta}|^{p\mathfrak{p}_3}_{\mathbb{R}^d}]^{\frac{1}{p}}+\mathfrak{M}_{p(\mathfrak{p}_3+2)}(Z^\delta)^{\frac{1}{p}}).\end{aligned}$$

Now let us focus on the case $q\in\mathbb{N}$, $q\geqslant2$. Similarly as in the case $q=1$, $\delta^{\frac{q}{2}}D^{\delta,q}X^\delta_t$ is a $\mathcal{R}^d_q$-valued random variable and, for $t\in\pi^\delta$, we have

$$\delta^{\frac{q}{2}}D^{\delta,q}X^\delta_{t+\delta}=(I_{d\times d}+B_t)\delta^{\frac{q}{2}}D^{\delta,q}X^\delta_t+\delta^{\frac{1}{2}}\sum_{i=1}^{N}Z^{\delta,i}_{t+\delta}B^{1,i}_{q,t}+B^3_{q,t},$$

with, $B^{1,i}_{1,.}=0$, $B^3_{1,.}$ defined in the beginning of **Step 2**, and for $q\geqslant 2$,

$$B^{1,i}_{q,t} =\delta^{\frac{q}{2}}(D^\delta X^\delta_t)^T\mathbf{H}_x A^i_1(X^\delta_t,t)D^{\delta,q-1}X^\delta_t+\delta^{\frac12}D^\delta B^{1,i}_{q-1,t}$$
$$B^3_{q,t} =\delta^{\frac{q-1}{2}}(B^{3,1}_t+B^{3,2}_t)D^{\delta,q-1}X^\delta_t+\delta^{\frac12}D^\delta B^3_{q-1,t}+\delta\sum_{i=1}^N B^{1,i}_{q-1,t}D^\delta Z^{\delta,i}_{t+\delta},$$

with, for $(w,v)\in\mathbf{T}\times\mathbf{N}$,

$$\begin{aligned}B^{3,1}_t =&\delta\sum_{i,j=1}^N Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}(\delta^{\frac12}D^\delta X^\delta_t)^T\mathbf{H}_x A^{i,j}_2(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta})\\&+\delta(\delta^{\frac12}D^\delta X^\delta_t)^T\mathbf{H}_x A_3(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta},\delta)\\(B^{3,2}_t)_{w,v} =& \chi^\delta_{t+\delta}\mathbf{1}_{w=t+\delta}(\delta^{\frac12}\nabla_x A^v_1(X^\delta_t,t)+\delta\sum_{j=1}^N Z^{\delta,j}_{t+\delta}(1+\mathbf{1}_{v=j})\nabla_x A^{v,j}_2(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta})\\&+\delta^{\frac32}\sum_{i,j=1}^N Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}\partial_{z^v}\nabla_x A^{i,j}_2(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta})+\delta^{\frac32}\partial_{z^i}\nabla_x A_3(X^\delta_t,t,\delta^{\frac12}Z^\delta_{t+\delta},\delta)).\end{aligned}$$

First, we remark that, since $B^1_{1,.}=0$, it follows from Lemma 4.1 and (4.25) that, for $l\in\mathbb{N}$, if assumption $\mathbf{A}^\delta_1(q+l+1)$ (see (2.2)) holds, then

$$\begin{aligned}&\mathfrak{S}_{\mathcal{R}^d_q,\delta,\mathbf{T},l,p}(B^1_{q,.},0,0)\\&\quad\leqslant\mathfrak{S}_{\mathcal{R}^d_q,\delta,\mathbf{T},l,p}(\delta^{\frac{q}{2}}(D^\delta X^\delta)^T\mathbf{H}_x A_1(X^\delta,.)D^{\delta,q-1}X^\delta,0,0)+\mathfrak{S}_{\mathcal{R}^d_{q-1},\delta,\mathbf{T},l+1,p}(B^1_{q-1,.},0,0)\\&\quad\leqslant\sum_{q^\diamond=1}^{q-1}\mathfrak{S}_{\mathcal{R}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(\delta^{\frac{q-q^\diamond+1}{2}}(D^\delta X^\delta)^T\mathbf{H}_x A_1(X^\delta,.)D^{\delta,q-q^\diamond}X^\delta,0,0)\\&\quad\leqslant C(d,q,l)\mathfrak{D}_{q+l+1}\mathbb{E}[\sup_{t\in\mathbf{T}}|1+|X^\delta_t|^{q+l}_{\mathbb{R}^d,1,q+l-1}|^p|1+|X^\delta_t|^{\mathfrak{p}_{q+l+1}}_{\mathbb{R}^d}|^p]^{\frac1p}.\end{aligned}$$

Moreover

$$\begin{aligned}\mathfrak{S}_{\mathcal{R}^d_q,\delta,\mathbf{T},l,p}(0,0,B^3_{q,.})\leqslant&\mathfrak{S}_{\mathcal{R}^d_q,\delta,\mathbf{T},l,p}(0,0,\delta^{\frac{q-1}{2}}(B^{3,1}+B^{3,2})D^{\delta,q-1}X^\delta)\\&+\mathfrak{S}_{\mathcal{R}^d_{q-1},\delta,\mathbf{T},l+1,p}(0,0,B^3_{q-1,.})\\&+\mathfrak{S}_{\mathcal{R}^d_q,\delta,\mathbf{T},l,p}(0,0,\delta\sum_{i=1}^N B^{1,i}_{q-1,.}D^\delta Z^{\delta,i}_{.+\delta})\\\leqslant&\sum_{q^\diamond=1}^{q-1}\mathfrak{S}_{\mathcal{R}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(0,0,\delta^{\frac{q-q^\diamond}{2}}(B^{3,1}+B^{3,2})D^{\delta,q-q^\diamond}X^\delta)\\&+\sum_{q^\diamond=1}^{q-1}\mathfrak{S}_{\mathcal{R}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(0,0,\delta\sum_{i=1}^N B^{1,i}_{q-q^\diamond,.}D^\delta Z^{\delta,i}_{.+\delta})\\&+\mathfrak{S}_{\mathcal{R}^d_1,\delta,\mathbf{T},q+l-1,p}(0,0,B^3_{1,.}).\end{aligned}$$

Using a similar approach as for the case $q = 1$, assuming $\mathbf{A}_1^\delta(q+l+2)$ holds (see (2.2)), then

$$\begin{aligned}\mathfrak{S}_{\mathcal{R}_1^d,\delta,\mathbf{T},q+l-1,p}(0,0,B^3_{1,.}) = 1 + \sup_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}[|\sum_{q^\diamond=0}^{q+l-1} \delta^{q^\diamond} | \sum_{\substack{w\in\pi^\delta\\ w<t}} D^{\delta,q^\diamond} B^3_{1,w}|^2_{\mathcal{R}^d_{q^\diamond+1}}|^{\frac{p}{2}}]^{\frac{1}{p}} \\ =1 + \sup_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}} \sum_{i=1}^{N} \sum_{q^\diamond=0}^{q+l-1} \delta^{q^\diamond} |D^{\delta,q^\diamond}(B^3_{1,w})_{w+\delta,i}|^2_{\mathcal{R}_{q^\diamond}}|^{\frac{p}{2}}]^{\frac{1}{p}} \\ \leqslant 1 + T^{\frac{1}{2}}\delta^{-\frac{1}{2}}|q+l|^{\frac{1}{2}} \sup_{\substack{t\in\pi^\delta\\ t<T}} \sup_{q^\diamond\in\{0,\ldots,q+l-1\}} \mathbb{E}[|\delta^{\frac{q^\diamond}{2}} D^{\delta,q^\diamond}(B^3_{1,t})_{t+\delta,.}|^p_{(\mathcal{R}^d_{q^\diamond})^{\mathbf{N}}}]^{\frac{1}{p}} \\ \leqslant 1 + T^{\frac{1}{2}} C(d,q,l)\mathfrak{D}_{q+l+2}\mathfrak{M}_{p(\mathfrak{p}_{q+l+2}+2)}(Z^\delta)^{\frac{1}{p}} \\ \times \mathbb{E}[\sup_{t\in\mathbf{T}} |1 + |X^\delta_{t-\delta}|^{q+l-1}_{\mathbb{R}^d,1,q+l-1}|^p |1 + |X^\delta_{t-\delta}|^{\mathfrak{p}_{q+l+2}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}.\end{aligned}$$

Moreover, for $q^\diamond \in \{1,\ldots,q-1\}$,

$$\begin{aligned}&\mathfrak{S}_{\mathcal{R}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(0,0,\delta^{\frac{q-1}{2}}(B^{3,1}+B^{3,2})D^{\delta,q-q^\diamond}X^\delta) \\ &=1 + \sup_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}} \delta^{\frac{q-1}{2}}(B^{3,1}_w + B^{3,2}_w)D^{\delta,q-q^\diamond}X^\delta_w|^p_{\mathcal{R}^d_{q-q^\diamond+1},q^\diamond+l-1}]^{\frac{1}{p}} \\ &\leqslant 1 + \sum_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}[|\delta^{\frac{q-1}{2}} B^{3,1}_t D^{\delta,q-q^\diamond}X^\delta_t|^p_{\mathcal{R}^d_{q-q^\diamond+1},q^\diamond+l-1}]^{\frac{1}{p}} \\ &\quad + \sup_{\substack{t\in\pi^\delta\\ t<T}} \mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}} \delta^{\frac{q-1}{2}} B^{3,2}_w D^{\delta,q-q^\diamond}X^\delta_w|^p_{\mathcal{R}^d_{q-q^\diamond+1},q^\diamond+l-1}]^{\frac{1}{p}},\end{aligned}$$

with, since $\mathbf{A}_1^\delta(q+l+2)$ (see (2.2) holds,

$$\begin{aligned}\mathbb{E}[|\delta^{\frac{q-1}{2}} B^{3,1}_t D^{\delta,q-q^\diamond}X^\delta_t|^p_{\mathcal{R}^d_{q-q^\diamond+1},q^\diamond+l-1}]^{\frac{1}{p}} \leqslant C(d,q,l)\delta\mathfrak{M}_{p(\mathfrak{p}_{q+l+2}+2)}(Z^\delta)^{\frac{1}{p}}\mathfrak{D}_{q+l+2} \\ \times \mathbb{E}[\sup_{t\in\mathbf{T}} |1 + |X^\delta_t|^{q+l}_{\mathbb{R}^d,1,q+l-1}|^p |1 + |X^\delta_t|^{\mathfrak{p}_{q+l+2}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}} \delta^{\frac{q-1}{2}} B^{3,2}_w D^{\delta,q-q^\diamond}X^\delta_w|^p_{\mathcal{R}^d_{q-q^\diamond},q^\diamond+l-1}]^{\frac{1}{p}} \\ &=\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}} \sum_{i=1}^{N} |\delta^{\frac{q-1}{2}}(B^{3,2}_w)_{w+\delta,i} D^{\delta,q-q^\diamond}X^\delta_w|^2_{\mathcal{R}^d_{q-q^\diamond},q^\diamond+l-1}|^{\frac{p}{2}}]^{\frac{1}{p}} \\ &\leqslant |\delta \sum_{\substack{w\in\pi^\delta\\ w<t}} \sum_{i=1}^{N} \mathbb{E}[|\delta^{\frac{q-2}{2}}(B^{3,2}_w)_{w+\delta,i} D^{\delta,q-q^\diamond}X^\delta_w|^p_{\mathcal{R}^d_{q-q^\diamond},q^\diamond+l-1}]^{\frac{2}{p}}|^{\frac{1}{2}},\end{aligned}$$

together with the estimate

$$\begin{aligned}\mathbb{E}[|\delta^{\frac{q-2}{2}}(B^{3,2}_w)_{w,i} D^{\delta,q-q^\diamond}X^\delta_w|^p_{\mathcal{R}^d_{q-q^\diamond},q^\diamond+l-1}]^{\frac{1}{p}} \leqslant C(d,q,l)\delta\mathfrak{M}_{p(\mathfrak{p}_{q+l+2}+2)}(Z^\delta)^{\frac{1}{p}}\mathfrak{D}_{q+l+1} \\ \times \mathbb{E}[\sup_{t\in\mathbf{T}} |1 + |X^\delta_t|^{q+l-2}_{\mathbb{R}^d,1,q+l-1}|^p |1 + |X^\delta_t|^{\mathfrak{p}_{q+l+1}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}.\end{aligned}$$

.

Finally, for $q^\diamond \in \{1,\dots,q-1\}$, assuming $\mathbf{A}_1^\delta(q+l)$ (see (2.2)) yields

$$\begin{aligned}
&\mathfrak{S}_{\mathcal{R}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(0,0,\delta\sum_{i=1}^N B^{1,i}_{q-q^\diamond,.}D^\delta Z^{\delta,i}_{.+\delta})\\
&\qquad\leqslant 1+\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}}\sum_{i=1}^N \delta^2 |B^{1,i}_{q-q^\diamond,w}D^\delta_{(w+\delta,i)}Z^{\delta,i}_{w+\delta}|^2_{\mathcal{R}^d_{q-q^\diamond},q^\diamond+l-1}|^{\frac{p}{2}}]^{\frac{1}{p}}\\
&\qquad\leqslant 1+\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}}\delta|B^1_{q-q^\diamond,w}|^2_{(\mathcal{R}^d_{q-q^\diamond})^{\mathbf{N}},q^\diamond+l-1}|^{\frac{p}{2}}]^{\frac{1}{p}}\\
&\qquad\leqslant 1+T^{\frac{1}{2}}\sup_{t\in\mathbf{T}}\mathbb{E}[|\sum_{i=1}^N B^{1,i}_{q-q^\diamond,t}|^p_{(\mathcal{R}^d_{q-q^\diamond})^{\mathbf{N}},q^\diamond+l-1}]^{\frac{1}{p}}\\
&\qquad\leqslant 1+T^{\frac{1}{2}}\mathfrak{S}_{\mathcal{R}^d_{q-q^\diamond},\delta,\mathbf{T},q^\diamond+l-1,p}(B^1_{q-q^\diamond,.},0,0)\\
&\qquad\leqslant 1+T^{\frac{1}{2}}C(d,q,l)\mathfrak{D}_{q+l}(1+\mathbb{E}[\sup_{t\in\mathbf{T}}|1+|X^\delta_t|^{q+l-1}_{\mathbb{R}^d,1,q+l-1}|^p|1+|X^\delta_t|^{\mathfrak{p}_{q+l}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}).
\end{aligned}$$

More specifically, we have shown that

$$\begin{aligned}
\mathfrak{S}_{\mathcal{R}^d_q,\delta,\mathbf{T},l,p}(0,0,B^3_{q,.})\leqslant& C(d,q,l)(1+T)\mathfrak{M}_{p(\mathfrak{p}_{q+l+2}+2)}(Z^\delta)^{\frac{1}{p}}\mathfrak{D}_{q+l+2}\\
&\times(1+\mathbb{E}[\sup_{t\in\mathbf{T}}|1+|X^\delta_t|^{q+l}_{\mathbb{R}^d,1,q+l-1}|^p|1+|X^\delta_t|^{\mathfrak{p}_{q+l+2}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}).
\end{aligned}$$

Since $\mathbf{A}_1^\delta(q+2)$ holds, taking $l=0$ and applying (4.27) yields, for $q\geqslant 2$,

$$\begin{aligned}
\mathbb{E}[\sup_{t\in\mathbf{T}}|X^\delta_t|^p_{\mathbb{R}^d,\delta,\mathbf{T},1,q}]^{\frac{1}{p}}\leqslant& C(d,q,p)(1+T)\mathfrak{M}_{p((\mathfrak{p}_{q+2}\vee\mathfrak{p})+2)}(Z^\delta)^{\frac{1}{p}}\mathfrak{D}_{q+2}\\
&\times\exp(C(p)(T+1)\mathfrak{M}_{p(\mathfrak{p}+2)}(Z^\delta)^{\frac{2}{p}}\mathfrak{D}^2)\\
&\times\mathbb{E}[\sup_{t\in\mathbf{T}}|1+|X^\delta_t|^q_{\mathbb{R}^d,1,q-1}|^p|1+|X^\delta_t|^{\mathfrak{p}_{q+2}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}.
\end{aligned}$$

Using a recursive approach cimbined with (4.28) yields (4.25).

Step 3. In this last step, we prove (4.26). For $q\in\mathbb{N}$, we define $\mathcal{H}_0=\mathcal{H}$ and $\mathcal{H}_{q+1}=(\mathcal{H}_q)^{\mathbf{T}\times\mathbf{N}}$. For Y satisfying (4.22),we have (remember that $D^{\delta,q}Y_t$, $t\in\pi^\delta$, belongs to $\mathcal{H}^d_q$), for every $t\in\pi^\delta$

$$\delta^{\frac{q}{2}}D^{\delta,q}Y_{t+\delta}=\delta^{\frac{q}{2}}D^{\delta,q}Y_t+B_t\delta^{\frac{q}{2}}D^{\delta,q}Y_t+\delta^{\frac{1}{2}}\sum_{i=1}^N Z^{\delta,i}_{t+\delta}B^{q,i}_{1,t}+\delta^{\frac{1}{2}}\sum_{i=1}^N L^\delta_{\mathbf{T}}Z^{\delta,i}_{t+\delta}B^{2,i}_{q,t}+B^3_{q,t}$$

with

$$\begin{aligned}
B^{1,i}_{q,t}=&\delta^{\frac{1}{2}}D^\delta B^{1,i}_{q-1,t}+\delta^{\frac{q}{2}}(D^\delta X^\delta_t)^T\mathbf{H}_xA^i_1(X^\delta_t,t)D^{\delta,q-1}Y_t\\
B^{2,i}_{q,t}=&\delta^{\frac{1}{2}}D^\delta B^{2,i}_{q-1,t}\\
B^3_{q,t}=&\delta^{\frac{1}{2}}D^\delta B^3_{q-1,t}+\delta^{\frac{q}{2}}\sum_{i=1}^N\nabla_xA^i_1(X^\delta_t,t)D^\delta(\delta^{\frac{1}{2}}Z^{\delta,i}_{t+\delta})D^{\delta,q-1}Y_t\\
&+\delta^{\frac{q}{2}}\frac{1}{2}\sum_{i,j=1}^N D^\delta(\delta Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}\nabla_xA^{i,j}_2(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta},\delta))D^{\delta,q-1}Y_t\\
&+\delta^{\frac{q}{2}}D^\delta(\delta\nabla_xA_3(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta},\delta)D^{\delta,q-1}Y_t\\
&+\delta^{\frac{1}{2}}\sum_{i=1}^N B^{1,i}_{q-1,t}D^\delta(\delta^{\frac{1}{2}}Z^{\delta,i}_{t+\delta})+B^{2,i}_{q-1,t}D^\delta L^\delta_{\mathbf{T}}(\delta^{\frac{1}{2}}Z^{\delta,i}_{t+\delta}).
\end{aligned}$$

Now, we remark that for $l \in \mathbb{N}$, it follows from (4.9) that

$$\begin{aligned}
\mathfrak{S}_{\mathcal{H}^d_q,\delta,\mathbf{T},l,p}(B^1_{q,.},0,0) \leqslant & \mathfrak{S}_{\mathcal{H}^d_q,\delta,\mathbf{T},l,p}(\delta^{\frac{q}{2}}(D^\delta X^\delta)^T \mathbf{H}_x A_1(X^\delta,.)D^{\delta,q-1}Y,0,0) \\
& + \mathfrak{S}_{\mathcal{H}^d_{q-1},\delta,\mathbf{T},l+1,p}(B^1_{q-1,.},0,0) \\
\leqslant & \sum_{q^\diamond=1}^{q} \mathfrak{S}_{\mathcal{H}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(\delta^{\frac{q-q^\diamond+1}{2}}(D^\delta X^\delta)^T \mathbf{H}_x A_1(X^\delta,.)D^{\delta,q-q^\diamond}Y^\delta,0,0) \\
& + \mathfrak{S}_{\mathcal{H}^d\delta,\mathbf{T},q+l,p}(B^1,0,0) \\
\leqslant & \mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q+l,p}(B^1,0,0) \\
& + C(d,q,l)\mathfrak{D}_{q+l+2}\mathbb{E}[\sup_{t\in\mathbf{T}}|1+|X^\delta_t|^{q+l}_{\mathbb{R}^d,1,q+l}|^{2p}|1+|X^\delta_t|^{\mathfrak{p}_{q+l+2}}_{\mathbb{R}^d}|^{2p}]^{\frac{1}{2p}} \\
& \times (1+\mathbb{E}[\sup_{t\in\mathbf{T}}|Y|^{2p}_{\mathcal{H}^d,q+l-1}]^{\frac{1}{2p}})
\end{aligned}$$

and similarly $\mathfrak{S}_{\mathcal{H}^d_q,\delta,\mathbf{T},l,p}(0,B^2_{q,.},0) \leqslant \mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q+l,p}(0,B^2,0)$.
Now, similarly as in **Step 2**, we denote for $t \in \pi^\delta$ and $(w,v) \in \mathbf{T}\times\mathbf{N}$,

$$\begin{aligned}
B^{3,1}_t =& \delta \sum_{i,j=1}^{N} Z^{\delta,i}_{t+\delta} Z^{\delta,j}_{t+\delta}(\delta^{\frac{1}{2}} D^\delta X^\delta_t)^T \mathbf{H}_x A^{i,j}_2(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta}) \\
& + \delta(\delta^{\frac{1}{2}} D^\delta X^\delta_t)^T \mathbf{H}_x A_3(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta},\delta) \\
(B^{3,2}_t)_{w,v} =& \chi^\delta_{t+\delta}\mathbf{1}_{w=t+\delta}(\delta^{\frac{1}{2}}\nabla_x A^v_1(X^\delta_t,t) + \delta\sum_{j=1}^{N} Z^{\delta,j}_{t+\delta}(1+\mathbf{1}_{v=j})\nabla_x A^{v,j}_2(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta}) \\
& + \delta^{\frac{3}{2}} \sum_{i,j=1}^{N} Z^{\delta,i}_{t+\delta} Z^{\delta,j}_{t+\delta}\partial_{z^v}\nabla_x A^{i,j}_2(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta}) + \delta^{\frac{3}{2}}\partial_{z^i}\nabla_x A_3(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta},\delta)),
\end{aligned}$$

and we have

$$\begin{aligned}
& \mathfrak{S}_{\mathcal{H}^d_q,\delta,\mathbf{T},l,p}(0,0,B^3_{q,.}) \\
& \leqslant \mathfrak{S}_{\mathcal{H}^d_q,\delta,\mathbf{T},l,p}(0,0,\delta^{\frac{q-1}{2}}(B^{3,1}+B^{3,2})D^{\delta,q-1}Y) + \mathfrak{S}_{\mathcal{H}^d_{q-1},\delta,\mathbf{T},l+1,p}(0,0,B^3_{q-1,.}) \\
& + \mathfrak{S}_{\mathcal{H}^d_q,\delta,\mathbf{T},l,p}(0,0,\delta^{\frac{1}{2}}\sum_{i=1}^{N} B^{1,i}_{q-1,.}D^\delta(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta}) + B^{2,i}_{q-1,.}D^\delta L^\delta_{\mathbf{T}}(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta})) \\
\leqslant & \sum_{q^\diamond=1}^{q} \mathfrak{S}_{\mathcal{H}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(0,0,\delta^{\frac{q-q^\diamond}{2}}(B^{3,1}+B^{3,2})D^{\delta,q-q^\diamond}Y) \\
& + \sum_{q^\diamond=1}^{q} \mathfrak{S}_{\mathcal{H}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(0,0,\delta^{\frac{1}{2}}\sum_{i=1}^{N} B^{1,i}_{q-q^\diamond,.}D^\delta(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta}) + B^{2,i}_{q-q^\diamond,.}D^\delta L^\delta_{\mathbf{T}}(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta})) \\
& + \mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q+l,p}(0,0,B^3)
\end{aligned}$$

Moreover, for $q^\diamond \in \{1,\ldots,q\}$,

$$\begin{aligned}
& \mathfrak{S}_{\mathcal{H}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(0,0,\delta^{\frac{q-q^\diamond}{2}}(B^{3,1}+B^{3,2})D^{\delta,q-q^\diamond}Y) \\
& \qquad = 1+\sup_{\substack{t\in\pi^\delta \\ t<T}}\mathbb{E}[|\sum_{\substack{w\in\pi^\delta \\ w<t}}(B^{3,1}_w+B^{3,2}_w)\delta^{\frac{q-q^\diamond}{2}}D^{\delta,q-q^\diamond}Y_w|^p_{\mathcal{H}^d_{q-q^\diamond+1},q^\diamond+l-1}]^{\frac{1}{p}} \\
& \qquad \leqslant 1+\sum_{\substack{t\in\pi^\delta \\ t<T}}\mathbb{E}[|B^{3,1}_t\delta^{\frac{q-q^\diamond}{2}}D^{\delta,q-q^\diamond}Y_t|^p_{\mathcal{H}^d_{q-q^\diamond+1},q^\diamond+l-1}]^{\frac{1}{p}} \\
& \qquad + \sup_{\substack{t\in\pi^\delta \\ t<T}}\mathbb{E}[|\sum_{\substack{w\in\pi^\delta \\ w<t}}B^{3,2}_w\delta^{\frac{q-q^\diamond}{2}}D^{\delta,q-q^\diamond}Y_w|^p_{\mathcal{H}^d_{q-q^\diamond+1},q^\diamond+l-1}]^{\frac{1}{p}},
\end{aligned}$$

with, using (4.9) and assuming that $\mathbf{A}_1^\delta(q+l+3)$ (see (2.2)) holds,

$$\mathbb{E}[|B_t^{3,1}\delta^{\frac{q-q^\diamond}{2}}D^{\delta,q-q^\diamond}Y_t|^p_{\mathcal{R}^d_{q-q^\diamond+1},q^\diamond+l-1}]^{\frac{1}{p}} \leqslant C(d,q,l)\delta\mathfrak{M}_{p(\mathfrak{p}_{q+l+3}+2)}(Z^\delta)^{\frac{1}{p}}\mathfrak{D}_{q+l+3}$$
$$\times \mathbb{E}[\sup_{t\in\mathbf{T}}|1+|X_t^\delta|^{q+l}_{\mathbb{R}^d,1,q+l}|^{2p}|1+|X_t^\delta|^{\mathfrak{p}_{q+l+3}}_{\mathbb{R}^d}|^{2p}]^{\frac{1}{2p}}(1+\mathbb{E}[\sup_{t\in\mathbf{T}}|Y_t|^{2p}_{\mathbb{R}^d,q+l-1}]^{\frac{1}{2p}})$$

and

$$\begin{aligned}&\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}}B_w^{3,2}\delta^{\frac{q-q^\diamond}{2}}D^{\delta,q-q^\diamond}Y_w|^p_{\mathcal{R}^d_{q-q^\diamond},q^\diamond+l-1}]^{\frac{1}{p}}\\
=&\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}}\sum_{i=1}^N|(B_w^{3,2})_{w+\delta,i}\delta^{\frac{q-q^\diamond}{2}}D^{\delta,q-q^\diamond}Y_w|^2_{\mathcal{R}^d_{q-q^\diamond},q^\diamond+l-1}|^{\frac{p}{2}}]^{\frac{1}{p}}\\
\leqslant&|\delta\sum_{\substack{w\in\pi^\delta\\ w<t}}\sum_{i=1}^N\mathbb{E}[|\delta^{-\frac{1}{2}}(B_w^{3,2})_{w+\delta,i}\delta^{\frac{q-q^\diamond}{2}}D^{\delta,q-q^\diamond}Y_w|^p_{\mathcal{R}^d_{q-q^\diamond},q^\diamond+l-1}]^{\frac{2}{p}}|^{\frac{1}{2}}\end{aligned}$$

together with the estimate

$$\mathbb{E}[|\delta^{-\frac{1}{2}}(B_w^{3,2})_{w,.}\delta^{\frac{q-q^\diamond}{2}}D^{\delta,q-q^\diamond}Y_w|^p_{(\mathcal{H}^d_{q-q^\diamond})^{\mathbf{N}},q^\diamond+l-1}]^{\frac{1}{p}} \leqslant C(d,q,l)\delta\mathfrak{M}_{p(\mathfrak{p}_{q+l+3}+2)}(Z^\delta)^{\frac{1}{p}}\mathfrak{D}_{q+l+3}$$
$$\times \mathbb{E}[\sup_{t\in\mathbf{T}}|1+|X_t^\delta|^{q+l-1}_{\mathbb{R}^d,1,q+l-1}|^{2p}|1+|X_t^\delta|^{\mathfrak{p}_{q+l+3}}_{\mathbb{R}^d}|^{2p}]^{\frac{1}{2p}}(1+\mathbb{E}[\sup_{t\in\mathbf{T}}|Y_t|^{2p}_{\mathcal{H}^d,q+l-1}]^{\frac{1}{2p}}).$$

Finally, for $q^\diamond\in\{1,\ldots,q\}$,

$$\begin{aligned}&\mathfrak{S}_{\mathcal{R}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(0,0,\delta^{\frac{1}{2}}\sum_{i=1}^N B^{1,i}_{q-q^\diamond,.}D^\delta(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta}))\\
&\leqslant 1+T^{\frac{1}{2}}\mathfrak{S}_{\mathcal{R}^d_{q-q^\diamond},\delta,\mathbf{T},q^\diamond+l-1,p}(B^1_{q-q^\diamond,.},0,0)\\
&\leqslant 1+T^{\frac{1}{2}}\mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q+l-1,p}(B^1,0,0)\\
&\quad+T^{\frac{1}{2}}C(d,q,l)\mathfrak{D}_{q+l+1}\mathbb{E}[\sup_{t\in\mathbf{T}}|1+|X_t^\delta|^{q+l-1}_{\mathbb{R}^d,1,q+l-1}|^{2p}|1+|X_t^\delta|^{\mathfrak{p}_{q+l+1}}_{\mathbb{R}^d}|^{2p}]^{\frac{1}{2p}}\\
&\quad\times(1+\mathbb{E}[\sup_{t\in\mathbf{T}}|Y|^{2p}_{\mathcal{H}^d,q+l-2}]^{\frac{1}{2p}}).\end{aligned}$$

Moreover, recall that for a multi-index $\alpha=(\alpha^1,\ldots,\alpha^q)$ with $\alpha^j=(t_j,i_j)$, $t_j\in\pi^\delta, t_j>0$, $i_j\in\mathbf{N}$,

$$D^\delta_\alpha L^\delta_{\mathbf{T}}Z^{\delta,i}_t=\delta^{-\frac{|\alpha|}{2}}\chi^\delta_t\partial_u^{\alpha_i^u}\ln\varphi_{r_*/2}(\delta^{-\frac{1}{2}}U^\delta_t-z_{*,t})\mathbf{1}_{t\in\mathbf{T}}\mathbf{1}_{\cap_{j=1}^q\{t=t_j\}},$$

with $\alpha_i^u:=(\alpha_i^u)^j)_{j\in\mathbf{N}}$, $(\alpha^u)^j=\mathbf{1}_{i=j}+\sum_{l=1}^q\mathbf{1}_{i_l=j}$. Using (4.9) with the estimate (4.20) from Lemma 4.4 yields, for every $q^\diamond\in\{1,\ldots,q\}$,

$$\begin{aligned}&\mathfrak{S}_{\mathcal{H}^d_{q-q^\diamond+1},\delta,\mathbf{T},q^\diamond+l-1,p}(0,0,\delta^{\frac{1}{2}}\sum_{i=1}^N B^{2,i}_{q-q^\diamond,.}D^\delta L^\delta_{\mathbf{T}}(\delta^{\frac{1}{2}}Z^{\delta,i}_{.+\delta}))\\
&\leqslant 1+\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}}\sum_{i=1}^N\delta|B^{2,i}_{q-q^\diamond,w}D^\delta_{(w+\delta,i)}L^\delta_{\mathbf{T}}(\delta^{\frac{1}{2}}Z^{\delta,i}_{w+\delta})|^2_{\mathcal{H}^d_{q-q^\diamond},q^\diamond+l-1}|^{\frac{p}{2}}]^{\frac{1}{p}}\\
&\leqslant 1+\mathbb{E}[|\sum_{\substack{w\in\pi^\delta\\ w<t}}\sum_{i=1}^N\delta|B^{2,i}_{q-q^\diamond,w}D^\delta_{(w+\delta,i)}L^\delta_{\mathbf{T}}(\delta^{\frac{1}{2}}Z^{\delta,i}_{w+\delta})|^2_{\mathcal{H}^d_{q-q^\diamond},q^\diamond+l-1}|^{\frac{p}{2}}]^{\frac{1}{p}}\\
&\leqslant 1+C(q)T^{\frac{1}{2}}\sup_{t\in\mathbf{T}}\mathbb{E}[|B^2_{q-q^\diamond,t}|^{2p}_{(\mathcal{H}^d_{q-q^\diamond})^{\mathbf{N}},q^\diamond+l-1}]^{\frac{1}{2p}}\|L^\delta_{\mathbf{T}}Z^\delta_t\|_{\mathbb{R}^N,\delta,\mathbf{T},q^\diamond+l,2p}\\
&\leqslant 1+T^{\frac{1}{2}}C(N,q,p)\frac{m_*^{\frac{1}{2p}}}{r_*^{q+l+1}}\mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q+l-1,2p}(0,B^2,0).\end{aligned}$$

In particular, we have shown that

$$\begin{aligned}\mathfrak{S}_{\mathcal{H}^d_q,\delta,\mathbf{T},l,p}(0,0,B^3_{q,.}) \leqslant & C(d,q,l,p)(1+T^{\frac{1}{2}})\mathfrak{M}_{p(\mathfrak{p}_{q+l+3}+2)}(Z^\delta)^{\frac{1}{p}}+\mathfrak{D}_{q+l+3} \\ & \times \mathbb{E}[\sup_{t\in\mathbf{T}}|1+|X^\delta_t|^{q+l+\mathfrak{p}_{q+l+3}}_{\mathbb{R}^d,q+l}|^{2p}|1+|X^\delta_t|^{2p\mathfrak{p}_{q+l+3})}_{\mathbb{R}^d,q+l}]^{\frac{1}{2p}}|(1+\mathbb{E}[\sup_{t\in\mathbf{T}}|Y_t|^{2p}_{\mathbb{R}^d,q+l-1}]^{\frac{1}{2p}}) \\ & + T^{\frac{1}{2}}\mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q+l-1,p}(B^1,0,0) \\ & + T^{\frac{1}{2}}C(N,q,p)\frac{m_*^{\frac{1}{2p}}}{r_*^{q+l+1}}\mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q+l-1,2p}(0,B^2,0)+\mathfrak{S}_{\mathcal{H}^d,\delta,\mathbf{T},q+l,p}(0,0,B^3).\end{aligned}$$

Since $\mathbf{A}^\delta_1(q+3)$ (see (2.2)) holds, taking $l=0$ and applying (4.27) and (4.25) concludes the proof of (4.26)

□

Now, we are in a position to prove Theorem 4.2.

Proof of Theorem 4.2. We do not treat the case $(\mathfrak{p}_n)_{n\in\mathbb{N}^*}\equiv 0$ which is similar but simpler. The result is a consequence of the fact that we do not use Lemma 4.5 in this case. Let us focus on the case $(\mathfrak{p}_n)_{n\in\mathbb{N}^*}\not\equiv 0$. We treat the Sobolev norms of $\partial^\alpha_{\mathrm{X}^\delta_0}X^\delta_t$. In the case $|\alpha|=1$, (4.13) is a direct consequence of Proposition 4.2, since

$$\partial^\alpha_{\mathrm{X}^\delta_0}X^\delta_{t+\delta}=\partial^\alpha_{\mathrm{X}^\delta_0}X^\delta_t+B_t\partial^\alpha_{\mathrm{X}^\delta_0}X^\delta_t.$$

For $\alpha=(\alpha^1,\ldots,\alpha^d)\in\mathbb{N}^d$ with $|\alpha|\in\mathbb{N}^*$, we consider $i_0\in\{1,\ldots,d\}$ such that $\alpha^{i_0}\in\mathbb{N}^*$ and $\alpha^-=\{\alpha^1,\ldots,\alpha^{i_0-1},\alpha^{i_0}-1,\alpha^{i_0+1},\ldots,\alpha^d\}$. Then

$$\partial^\alpha_{\mathrm{X}^\delta_0}X^\delta_{t+\delta}=\partial^\alpha_{\mathrm{X}^\delta_0}X^\delta_t+B_t\partial^\alpha_{\mathrm{X}^\delta_0}X^\delta_t+\delta^{\frac{1}{2}}\sum_{i=1}^N Z^{\delta,i}_{t+\delta}B^{1,i}_{\alpha,t}+B^3_{\alpha,t},$$

with $B^1_\alpha=B^3_\alpha=0$ if $|\alpha|=1$ and for $|\alpha|\geqslant 2$,

$$\begin{aligned}B^{1,i}_{\alpha,t}=&(\partial_{\mathrm{X}^{\delta,i_0}_0}X^\delta_t)^T\mathbf{H}_xA^i_1(X^\delta_t,t)\partial^{\alpha^-}_{\mathrm{X}^\delta_0}X^\delta_t+\partial_{\mathrm{X}^{\delta,i_0}}B^{1,i}_{\alpha^-,t}\\ B^3_{\alpha,t}=&\dot{B}^{i_0}_t\partial^{\alpha^-}_{\mathrm{X}^\delta_0}X^\delta_t+\partial_{\mathrm{X}^{\delta,i_0}_0}B^3_{\alpha^-,t},\end{aligned}$$

with

$$\begin{aligned}\dot{B}^{i_0}_t=&\delta\sum_{i,j=1}^N Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}(\partial_{\mathrm{X}^{\delta,i_0}}X^\delta_t)^T\mathbf{H}_xA^{i,j}_2(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta})\\ &+\delta(\partial_{\mathrm{X}^{\delta,i_0}}X^\delta_t)^T\mathbf{H}_xA_3(X^\delta_t,t,\delta^{\frac{1}{2}}Z^\delta_{t+\delta},\delta)\end{aligned}$$

In particular, if we assume that $\mathbf{A}^\delta_1(q+|\beta|+3)$ (see (2.2)) holds, for every $p\geqslant 1$, and every $i\in\mathbf{N}$, and every multi-index $\beta\in\mathbb{N}^d$, using a recursive approach, we obtain

$$\begin{aligned}&\|\partial^\beta_{\mathrm{X}^\delta_0}B^1_{\alpha,t}\|_{(\mathbb{R}^d)^{\mathbf{N}},\delta,\mathbf{T},q,p}\\ \leqslant & C(d,q,|\beta|)\mathfrak{D}_{q+|\beta|+3}\sup_{t\in\mathbf{T}}\sum_{\substack{q^\diamond\in\{0,1\},\alpha^\diamond\in\mathbb{N}^d\\ 1-q^\diamond\leqslant|\alpha^\diamond|<|\alpha|+|\beta|}}\mathbb{E}[|1+|\partial^{\alpha^\diamond}_{\mathrm{X}^\delta_0}X^\delta_t|^{q+|\beta|+2}_{\mathbb{R}^d,\delta,\mathbf{T},q^\diamond,q}|^p|1+|X^\delta_t|^{\mathfrak{p}_{q+|\beta|+3}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}\\ &+\|\partial^\beta_{\mathrm{X}^\delta_0}\partial_{\mathrm{X}^{\delta,i_0}_0}B^{1,i}_{\alpha^-,t}\|_{\mathbb{R}^d,\delta,\mathbf{T},q,p}\\ \leqslant & C(d,q,|\alpha|,|\beta|)\mathfrak{D}_{q+|\alpha|+|\beta|+1}\\ &\times\sup_{t\in\mathbf{T}}\sum_{\substack{q^\diamond\in\{0,1\},\alpha^\diamond\in\mathbb{N}^d\\ 1-q^\diamond\leqslant|\alpha^\diamond|<|\alpha|+|\beta|}}\mathbb{E}[|1+|\partial^{\alpha^\diamond}_{\mathrm{X}^\delta_0}X^\delta_t|^{q+|\alpha|+|\beta|}_{\mathbb{R}^d,\delta,\mathbf{T},q^\diamond,q}|^p|1+|X^\delta_t|^{\mathfrak{p}_{q+|\alpha|+|\beta|+1}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}.\end{aligned}$$

Since $\mathbf{A}^\delta_1(q+|\alpha|+2)$ (see (2.2)) holds, applying this estimate to the case $\beta=\emptyset$ yields

$$\begin{aligned}\|B^1_{\alpha,t}\|_{(\mathbb{R}^d)^{\mathbf{N}},\delta,\mathbf{T},q,p}\leqslant & C(d,q,|\alpha|)\mathfrak{D}_{q+|\alpha|+1}\\ &\times\sup_{t\in\mathbf{T}}\sum_{\substack{q^\diamond\in\{0,1\},\alpha^\diamond\in\mathbb{N}^d\\ 1-q^\diamond\leqslant|\alpha^\diamond|<|\alpha|}}\mathbb{E}[|1+|\partial^{\alpha^\diamond}_{\mathrm{X}^\delta_0}X^\delta_t|^{q+|\alpha|}_{\mathbb{R}^d,\delta,\mathbf{T},q^\diamond,q}|^p|1+|X^\delta_t|^{\mathfrak{p}_{q+|\alpha|+1}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}.\end{aligned}$$

and similarly,

$$\|\sum_{\substack{w\in\pi^{\delta}\\ w<t}} B^3_{\alpha,w}\|_{\mathbb{R}^d,\delta,\mathbf{T},q,p} \leqslant C(d,q,|\alpha|)(1+T)\mathfrak{D}_{q+|\alpha|+2}\mathfrak{M}_{p(\mathfrak{p}_{q+|\alpha|+2}+2)}(Z^{\delta})^{\frac{1}{p}}$$

$$\times \sup_{t\in\mathbf{T}} \sum_{\substack{q^{\diamond}\in\{0,1\},\alpha^{\diamond}\in\mathbb{N}^d\\ 1-q^{\diamond}\leqslant|\alpha^{\diamond}|<|\alpha|}} \mathbb{E}[|1+|\partial^{\alpha^{\diamond}}_{\mathrm{X}^{\delta}_0} X^{\delta}_t|^{q+|\alpha|}_{\mathbb{R}^d,\delta,\mathbf{T},q^{\diamond},q}|^p|1+|X^{\delta}_t|^{\mathfrak{p}_{q+|\alpha|+2}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}.$$

Then (4.13) follows from Proposition 4.2 combined with a recursive approach. We now study the Sobolev norms of $L^{\delta}_{\mathbf{T}}X^{\delta}_t$. We have

$$L^{\delta}_{\mathbf{T}}X^{\delta}_{t+\delta} = L^{\delta}_{\mathbf{T}}X^{\delta}_t + B_t L^{\delta}_{\mathbf{T}}X^{\delta}_t + \delta^{\frac{1}{2}}\sum_{i=1}^{N} Z^{\delta,i}_{t+\delta}B^{1,i}_t + \delta^{\frac{1}{2}}\sum_{i=1}^{N} L^{\delta}_{\mathbf{T}}Z^{\delta,i}_{t+\delta}B^{2,i}_t + B^3_t,$$

with

$$B^{1,i}_t = \sum_{l,r=1}^{d}\partial_{x^l}\partial_{x_r}A^i_1(X^{\delta}_t,t)\langle D^{\delta}X^{\delta,r}_t, D^{\delta}X^{\delta,l}_t\rangle_{\mathbb{R}^{\mathbf{T}\times\mathbf{N}}} = \mathrm{Tr}(\sigma^{\delta}_{X^{\delta}_t,\mathbf{T}}\mathbf{H}_x A^i_1(X^{\delta}_t,t)),$$

$$B^{2,i}_t = A^i_1(X^{\delta}_t,t)$$

$$\begin{aligned} B^3_t =& \delta\sum_{i,j=1}^{N}(Z^{\delta,i}_{t+\delta}L^{\delta}_{\mathbf{T}}Z^{\delta,j}_{t+\delta} + Z^{\delta,j}_{t+\delta}L^{\delta}_{\mathbf{T}}Z^{\delta,i}_{t+\delta} + \chi^{\delta}_{t+\delta}\mathbf{1}_{i,j})A^{i,j}_2(X^{\delta}_t,t,\delta^{\frac{1}{2}}Z^{\delta}_{t+\delta})\\ &+ Z^{\delta,i}_{t+\delta}Z^{\delta,j}_{t+\delta}(\mathrm{Tr}(\sigma^{\delta}_{X^{\delta}_t,\mathbf{T}}\mathbf{H}_x A^{i,j}_2(X^{\delta}_t,t,\delta^{\frac{1}{2}}Z^{\delta}_{t+\delta})) + \delta^{\frac{1}{2}}\sum_{l=1}^{N}\partial_{z^l}A^{i,j}_2(X^{\delta}_t,t,\delta^{\frac{1}{2}}Z^{\delta}_{t+\delta})L^{\delta}_{\mathbf{T}}Z^{\delta,l}_{t+\delta}\\ &+ \chi^{\delta}_{t+\delta}\delta\sum_{l=1}^{N}\partial^2_{z^l}A^{i,j}_2(X^{\delta}_t,t+\delta,\delta^{\frac{1}{2}}Z^{\delta}_{t+\delta}))\\ &+ \mathrm{Tr}(\sigma^{\delta}_{X^{\delta}_t,\mathbf{T}}\mathbf{H}_x A_3(X^{\delta}_t,t,\delta^{\frac{1}{2}}Z^{\delta}_{t+\delta},\delta) + \delta^{\frac{1}{2}}\sum_{l=1}^{N}\partial_{z^l}A_3(X^{\delta}_t,t,\delta^{\frac{1}{2}}Z^{\delta}_{t+\delta},\delta)L^{\delta}_{\mathbf{T}}Z^{\delta,l}_{t+\delta}\\ &+ \chi^{\delta}_{t+\delta}\delta\sum_{l=1}^{N}\partial^2_{z^l}A_3(X^{\delta}_t,t,\delta^{\frac{1}{2}}Z^{\delta}_{t+\delta},\delta). \end{aligned}$$

Moreover, for every $p\geqslant 1$, and every $i\in\mathbf{N}$, using $\mathbf{A}^{\delta}_1(q+4)$ (see (2.2)),

$$\|B^{1,i}_t\|_{(\mathbb{R}^d)^{\mathbf{N}},\delta,\mathbf{T},q,p} \leqslant C(d,q)\mathfrak{D}_{q+3}\sup_{t\in\mathbf{T}}\mathbb{E}[|1+|X^{\delta}_t|^{q+2}_{\mathbb{R}^d,1,q+1}|^p|1+|X^{\delta}_t|^{\mathfrak{p}_{q+3}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}},$$

$$\|B^{2,i}_t\|_{(\mathbb{R}^d)^{\mathbf{N}},\delta,\mathbf{T},q,p} \leqslant C(d,q)\mathfrak{D}_{q+1}\sup_{t\in\mathbf{T}}\mathbb{E}[|1+|X^{\delta}_t|^{q}_{\mathbb{R}^d,1,q}|^p|1+|X^{\delta}_t|^{\mathfrak{p}_{q+1}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}},$$

and

$$\begin{aligned}\|\sum_{\substack{w\in\pi^{\delta}\\ w<t}} B^3_w\|_{\mathbb{R}^d,\delta,\mathbf{T},q,p} \leqslant& C(d,q)(1+T)\mathfrak{D}_{q+4}\mathfrak{M}_{2p(\mathfrak{p}_{q+4}+2)}(Z^{\delta})^{\frac{1}{2p}}\\ &\times\sup_{t\in\mathbf{T}}\mathbb{E}[|1+|X^{\delta}_t|^{q+2}_{\mathbb{R}^d,1,q+1}|^p|1+|X^{\delta}_t|^{\mathfrak{p}_{q+4}}_{\mathbb{R}^d}|^p]^{\frac{1}{p}}\\ &\times(1+\sup_{t\in\mathbf{T}}\|L^{\delta}_{\mathbf{T}}Z^{\delta}_t\|_{\mathbb{R}^N,\delta,\mathbf{T},q,2p}).\end{aligned}$$

We finally use (4.20) from Lemma 4.4 and Proposition 4.2 to complete the proof of (4.14).

□

4.5. **Proof of Theorem 4.3.**

4.5.1. *Preliminaries.* Before we focus on the proof of Theorem 4.3, we provide a representation formula for the Malliavin derivatives using the variation of constant formula and some technical results we will employ in our proof.

Representations formula. Let $w,t\in\pi^{\delta,*}, i\in\mathbf{N}$. Then $D^{\delta}_{(w,i)}X^{\delta}_t(x)=0$ for every $w>t$ and for $w\leqslant t$,

$$D^{\delta}_{(w,i)}X^{\delta}_t = \chi^{\delta}_t \partial_{z^i}\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \delta)\mathbf{1}_{w=t} + \nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \delta)D^{\delta}_{(w,i)}X^{\delta}_{t-\delta}(x).$$

We consider the tanget flow process $(\dot{X}_t)_{t\in\pi^\delta}$ defined by $\dot{X}_0 = I_{d\times d}$ and

$$\dot{X}_t := \partial_{X^{\delta}_0} X^{\delta}_t = \nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \delta)\dot{X}_{t-\delta}.$$

We now define the inverse tangent flow. To prove the invertibility, we consider the Hilbert space $(\mathbb{R}^{d\times d}, \langle , \rangle_{\mathbb{R}^{d\times d}})$, with the Frobenius scalar product defined by $\langle M, M^{\diamond}\rangle_{\mathbb{R}^{d\times d}} := \mathrm{Trace}(M^{\diamond}M^T) = \sum_{i=1}^d (M^{\diamond}M^T)_{i,i}$, $M, M^{\diamond} \in \mathbb{R}^{d\times d}$. Notice that for $M \in \mathbb{R}^{d\times d}$, $\|M\|_{\mathbb{R}^d} \leqslant |M|_{\mathbb{R}^{d\times d}} \leqslant d^{\frac{1}{2}}\|M\|_{\mathbb{R}^d}$. Also, for $k \in \mathbb{N}^*$, $|M^k|_{\mathbb{R}^{d\times d}} \leqslant \|M\|_{\mathbb{R}^d}|M^{k-1}|_{\mathbb{R}^{d\times d}} \leqslant |M|^k_{\mathbb{R}^{d\times d}}$ (with $M^0 = I_{d\times d}$ and $M^l = MM^{l-1}$, $l \in \{1, \ldots, k\}$).
Now, since $\nabla_x\psi(x, t, 0, 0) = I_{d\times d}$ for every $(x, t) \in \mathbb{R}^d \times \pi^\delta$, it follows from the Taylor expansion of $\nabla_x\psi$, that

$$\begin{aligned}\nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \delta) =& I_{d\times d} + \delta^{\frac{1}{2}}\sum_{l=1}^N Z^{\delta,l}_t \int_0^1 \partial_{z^l}\nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \lambda\delta^{\frac{1}{2}}Z^{\delta}_t, 0)\mathrm{d}\lambda. \\ &+ \delta\int_0^1 \partial_y\nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \lambda\delta)\mathrm{d}\lambda,\end{aligned}$$

and using the assumption $\mathbf{A}_1$ (see (2.3)) yields

$$|I_{d\times d} - \nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \delta)|_{\mathbb{R}^{d\times d}} \leqslant \delta^{\frac{1}{2}}4\mathfrak{D}\max(|Z^{\delta}_t|^{\mathfrak{p}+1}_{\mathbb{R}^N}, 1). \quad (4.29)$$

In particular, under the assumption (4.15), we remark that, on the set $\{|Z^{\delta}_t|_{\mathbb{R}^N} \leqslant \eta_2\}$, we have

$$\begin{aligned}|\det\nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \delta)|^{\frac{2}{d}} \geqslant& \inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1} |\nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \delta)\xi|_{\mathbb{R}^d} \\ \geqslant& 1 - \|I_{d\times d} - \nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \delta)\|_{\mathbb{R}^d} \\ \geqslant& 1 - \delta^{\frac{1}{2}}2\mathfrak{D}(1 + \eta_2^{\mathfrak{p}+1}) \geqslant \frac{1}{2}.\end{aligned}$$

The matrix $\nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, \delta^{\frac{1}{2}}Z^{\delta}_t, \delta)$ is thus invertible on the set $\{|Z^{\delta}_t|_{\mathbb{R}^N} \leqslant \eta_2\}$. We are now in a position to introduce the inverse tangent flow, namely $(\mathring{X}_t)_{t\in\pi^\delta}$ satisfying $\mathring{X}_0 = I_{d\times d}$ and which is well defined for every $t \in \pi^{\delta,*}$ as soon as we are on the set $\{\Theta_{\eta_2,\pi^{\delta,*},t} > 0\}$. In this case

$$\mathring{X}_t := \dot{X}_t^{-1} = \mathring{X}_{t-\delta}\nabla_x\psi(X^{\delta}_{t-\delta}, t-\delta, Z^{\delta}_t, \delta)^{-1}.$$

In particular we introduce $\mathring{X}_{\eta_2,t} := \mathring{X}_t\mathbf{1}_{\Theta_{\eta_2,\pi^{\delta,*},t}>0}$ which is well defined for every $t \in \pi^\delta$.
We conclude this introduction observing that we have the so-called variation of constant formula. On the set $\{\Theta_{\eta_2,\pi^{\delta,*},t} > 0\}$, for every $(w, i) \in \pi^{\delta,*} \cap (0, t] \times \mathbf{N}$,

$$D^{\delta}_{(w,i)}X^{\delta}_t = \chi^{\delta}_w \dot{X}_t\mathring{X}_w\partial_{z^i}\psi(X^{\delta}_{w-\delta}, w-\delta, \delta^{\frac{1}{2}}Z^{\delta}_w, \delta). \quad (4.30)$$

Before we give the proof Theorem 4.3, we start with some preliminary results which are crucial in the study of the determinant of the inverse of the Malliavin covariance matrix.

Preliminary resuls. Two standard results will be used in our approach, namely the Burkholder inequality (see (4.23)) and an exponential martigale inequality, we recall thereafter. First, let us introduce some notations. Given a $\mathbb{R}$-valued process $(Y_t)_{t\in\pi^\delta}$ progressively measurable *w.r.t.* a filtration $(\mathcal{F}^Y_t)_{t\in\pi^\delta}$, we denote $\tilde{\Delta}^Y_t = \delta^{-\frac{1}{2}}(Y_{t+\delta} - \mathbb{E}[Y_{t+\delta}|\mathcal{F}^Y_t])$, $\bar{\Delta}^J_t = \delta^{-1}\mathbb{E}[Y_{t+\delta} - Y_t|\mathcal{F}^Y_t]$.
Let $(M_t)_{t\in\pi^\delta}$ be a $\mathbb{R}$-valued local square integrable $(\mathcal{F}_t)_{t\in\pi^\delta}$-martingale. We denote $[M]_t = |M_0|^2 + \delta\sum_{w\in\pi^\delta, w<t}|\tilde{\Delta}^M_w|^2$ and $\langle M\rangle_t = \mathbb{E}[|M_0|^2] + \delta\sum_{w\in\pi^\delta, w<t}\mathbb{E}[|\tilde{\Delta}^M_w|^2|\mathcal{F}^M_w]$. Then (see [14] Corollary 3.4 or [15]), we have the following extension of the Freedman inequality [16]: For $a, b > 0$ and $t \in \pi^\delta$,

$$\mathbb{P}(\sup_{\substack{w\in\pi^\delta\\ w\leqslant t}} |M_w| \geqslant a, [M]_t + \langle M\rangle_t < b) \leqslant 2\exp(-\frac{a^2}{2b}) \quad (4.31)$$

Now, let us give some additional intermediate results which are proved in the Appendix 4.5.2. The first one is a technical result that is used to bound the probability that the determinant of a random matrix $\mathfrak{C}$ is under some threshold by studying $\mathbb{P}(\xi^T\mathfrak{C}\xi \leqslant \epsilon)$ for $\xi \in \mathbb{R}^d$.

Lemma 4.6. *Let Σ be a $\mathbb{R}^{d\times d}$-valued random variable and $\epsilon \in (0, \frac{2^{\frac{1}{2}}}{d^{\frac{1}{2}}})$, Then*

$$\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1} \xi^T\Sigma\xi \leqslant \frac{1}{2}\epsilon) \leqslant C(d)\epsilon^{-2d} \sup_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1} \mathbb{P}(\xi^T\Sigma\xi \leqslant \epsilon) + \mathbb{P}(\|\Sigma\|_{\mathbb{R}^d} > \frac{1}{3\epsilon}). \tag{4.32}$$

The second result provides an estimate of the moments of the inverse tangent flow.

Lemma 4.7. *Let $T>0$, $\mathbf{T}=(0,T]\cap\pi^\delta$, let $p\geqslant 2$ and let $\eta_2>1$. Assume that (2.3) from $\mathbf{A}_1$ and $\mathbf{A}_3^\delta(p(\mathfrak{q}^\delta_{\eta_2}\vee(2\mathfrak{p}+2)))$ (see (2.7)) hold and that (4.15) holds. Then,*

$$\mathbb{E}[\sup_{t\in\mathbf{T}}\|\mathring{X}_t\|^p_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t}>0}]^{\frac{1}{p}} \leqslant C(d)\exp(C(p)T\mathfrak{M}_{p(\mathfrak{q}^\delta_{\eta_2}\vee(2\mathfrak{p}+2))}(Z^\delta)^{\frac{2}{p}}\mathfrak{D}^4). \tag{4.33}$$

with $\mathfrak{q}^\delta_{\eta_2} := \lceil 1-\frac{\ln(\delta)}{2\ln(\eta_2)}\rceil$ introduced in Theorem 4.3.

The next result is a discrete time Lie expansion satisfied by our process X^δ together with a control of the remainder appearing.

Lemma 4.8 (Discrete time Lie expansion)**.** *Let $V\in\mathcal{C}^2(\mathbb{R}^d\times\mathbb{R}_+)$ and let $\eta_2>1$. Assume that $\psi\in\mathcal{C}^3(\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{R}^N\times[0,1])$. Then for every $t\in\pi^{\delta,*}$,*

$$\begin{aligned}\mathring{X}_{\eta_2,t}V(X^\delta_t,t) =& \mathring{X}_{\eta_2,t-\delta}V(X^\delta_{t-\delta},t-\delta) + \delta^{\frac{1}{2}}\sum_{i=1}^N Z^{\delta,i}_t\mathring{X}_{\eta_2,t-\delta}V^{[i]}(X^\delta_{t-\delta},t-\delta)\\ &+\delta\mathring{X}_{\eta_2,t-\delta}V^{[0]}(X^\delta_{t-\delta},t-\delta)) + \mathring{X}_{\eta_2,t-\delta}\mathbf{R}^\delta V(X^\delta_{t-\delta},t-\delta,Z^\delta_t)\end{aligned}$$

Moreover, let us introduce the $\mathbb{R}^d$-valued functions defined for every $(x,t,z)\in\mathbb{R}^d\times\pi^{\delta,*}\times\in\mathbb{R}^N$ by

$$\begin{aligned}\tilde{\mathbf{R}}^\delta V(x,t-\delta,z) =& \mathbf{R}^\delta V(x,t-\delta,z) - \mathbb{E}[\mathbf{R}^\delta V(x,t-\delta,Z^\delta_t)]\\ \overline{\mathbf{R}}^\delta V(x,t-\delta) =& \mathbb{E}[\mathbf{R}^\delta V(x,t-\delta,Z^\delta_t)].\end{aligned}$$

Let $\alpha^x\in\mathbb{N}^d$ and assume that $\mathbf{A}_1^\delta(|\alpha^x|+4)$ (see (2.2) and (2.3)) and $\mathbf{A}_3^\delta(2\max(3\mathfrak{p}+(\mathfrak{p}_{|\alpha^x|+4}+2)(\max(|\alpha^x|,2)+3)+4,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2))$ (see (2.7)) hold, that $V\in\mathcal{C}^{|\alpha^x|+3}_{pol}(\mathbb{R}^d\times\mathbb{R}_+;\mathbb{R}^d) := \{f\in\mathcal{C}^{|\alpha^x|+3}(\mathbb{R}^d\times\mathbb{R}_+;\mathbb{R}^d), \exists\mathfrak{D}_{f,|\alpha^x|+3}\geqslant 1, \mathfrak{p}_{f,|\alpha^x|+3}\in\mathbb{N}, \forall(x,t)\in\mathbb{R}^d\times\mathbb{R}_+, |f(x,t)|_{\mathbb{R}^d}\leqslant\mathfrak{D}_{f,|\alpha^x|+3}(1+|x|^{\mathfrak{p}_{f,|\alpha^x|+3}}_{\mathbb{R}^d})\}$ and that (4.15) holds.
Then, for every $(x,t,z)\in\mathbb{R}^d\times\pi^{\delta,*}\times\mathbb{R}^N$,

$$\begin{aligned}|\tilde{\mathbf{R}}(x,t-\delta,z)|_{\mathbb{R}^d} \leqslant& \delta C\mathfrak{M}_{\max(6\mathfrak{p}+10\mathfrak{p}_4+28,\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1)}(Z^\delta)\\ &\times\mathfrak{D}^3\mathfrak{D}_4^7\mathfrak{D}^2_{V,3}(1+|x|^{2\max(7\mathfrak{p}_4,1)+4\mathfrak{p}_{V,3}}_{\mathbb{R}^d}+|z|^{4\max(6\mathfrak{p}+10\mathfrak{p}_4+28,\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1)}_{\mathbb{R}^N}).\end{aligned} \tag{4.34}$$

and

$$\begin{aligned}|\partial^{\alpha^x}_x\overline{\mathbf{R}}(x,t-\delta)|_{\mathbb{R}^d} \leqslant& \delta^{\frac{3}{2}}C(|\alpha^x|)\mathfrak{M}_{2\max(3\mathfrak{p}+(\mathfrak{p}_{|\alpha^x|+4}+2)(\max(|\alpha^x|,2)+3)+4,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)\\ &\times\mathfrak{D}^3\mathfrak{D}^{2\max(|\alpha^x|,2)+3}_{|\alpha^x|+4}\mathfrak{D}^2_{V,|\alpha^x|+3}(1+|x|^{\mathfrak{p}_{|\alpha^x|+4}(2\max(|\alpha^x|,2)+3)+2\mathfrak{p}_{V,|\alpha^x|+3}}_{\mathbb{R}^d}).\end{aligned} \tag{4.35}$$

The last result is a Norris Lemma adapted to discrete time processes. In the continuous case, this lemma can be found in [24], Lemma 2.3.2. Before giving this result, we introduce some notations. Let $q>0$ and $\mathbf{T}\subset\pi^{\delta,*}$. Given a $\mathbb{R}$-valued process $(Y_t)_{t\in\pi^\delta}$ progressively measurable *w.r.t.* a filtration $(\mathcal{F}^Y_t)_{t\in\pi^\delta}$, we denote,

$$\begin{aligned}\mathfrak{N}_{Y,\mathbf{T}}(q) :=& 1+\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^q] + \mathbb{E}[\sup_{t\in\mathbf{T}}|\bar{\Delta}^Y_{t-\delta}|^q] + \mathbb{E}[\sup_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]]\\ &+\mathbb{E}[\sup_{t\in\mathbf{T}}|\bar{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^q] + \mathbb{E}[\sup_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]].\end{aligned} \tag{4.36}$$

Lemma 4.9 (Discrete time Norris Lemma)**.** *Let $T\geqslant\delta$, $\mathbf{T}=(0,T]\cap\pi^\delta$. Let $(Y_t)_{t\in\pi^\delta}$ be a $\mathbb{R}$-valued random process progressively measurable with respect to a filtration $(\mathcal{F}^Y_t)_{t\in\pi^\delta}$, let $r\in(0,\frac{1}{12})$ and let $p>0$. Let us introduce $q(r,p)=\max(4,\frac{44p}{1-12r})$ and assume that*

$$\mathfrak{N}_{Y,\mathbf{T}}(q(r,p)) < +\infty.$$

Then, for every $\epsilon \in [|2^{10}(1+T^3)\delta|^{\frac{44}{91-36r}}, |2^8(1+T)|^{-\frac{11}{1-12r}}]$, *then*

$$\begin{aligned}(4.37)\qquad &\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2<\epsilon,\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]+|\bar{\Delta}^Y_{t-\delta}|^2\geqslant\epsilon^r)\\ &\leqslant\epsilon^p(1+T^{2q(r,p)})2^{5q(r,p)+5}\mathfrak{N}_{Y,\mathbf{T}}(q(r,p))+12\exp(-\frac{\epsilon^{-\frac{1-12r}{22}}}{2^{11}(1+T^2)}).\end{aligned}$$

4.5.2. *Proof of Theorem 4.3.*

Proof of Theorem 4.3. **Step 1.** For every $i\in\mathbf{N}$, we introduce the $\mathbb{R}^d$-valued process $(\mathring{\Psi}_{i,t})_{t\in\mathbf{T}}$ defined for every $t\in\mathbf{T}$ by $\mathring{\Psi}_{i,t}=\mathring{X}_{t-\delta}\nabla_x\psi^{-1}\partial_{z^i}\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)$. Notice that, for every $t\in\mathbf{T}$,

$$\mathring{X}_t\partial_{z^i}\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)=\mathring{\Psi}_{i,t}.$$

We introduce the notation $\dot{v}^2=\dot{v}\dot{v}^T\in\mathbb{R}^{d\times d}$ for a vector $\dot{v}\in\mathbb{R}^d$. Using the variation of constant formula (4.30), denoting $\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}=\delta\sum_{(t,i)\in\mathbf{T}\times\mathbf{N}}\chi^\delta_t(\mathring{\Psi}_{i,t})^2$, on the set $\{\Theta_{\eta_2,\mathbf{T},t}>0\}$, we have

$$\begin{aligned}\sigma^\delta_{X^\delta_T,\mathbf{T}}=&\delta\sum_{(t,i)\in\mathbf{T}\times\mathbf{N}}(D^\delta_{(t,i)}X^\delta_T)^2=\delta\sum_{(t,i)\in\mathbf{T}\times\mathbf{N}}\chi^\delta_t(\dot{X}_T\mathring{X}_t\partial_{z^i}\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta))^2\\ =&\delta\sum_{(t,i)\in\mathbf{T}\times\mathbf{N}}\chi^\delta_t(\dot{X}_T\mathring{\Psi}_{i,t})^2=\dot{X}_T\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}\dot{X}_T^T.\end{aligned}$$

We first show that the proof of (4.16), boils down to prove that there exists $\bar{\mathfrak{e}}\in(\eta_1^{-\frac{1}{d}},\frac{2^{\frac{1}{2}}}{d^{\frac{1}{2}}}]$ and $\mathbf{C}\geqslant1$ (which do not depend on δ and will be made explicit in the sequel) such that, for every $\epsilon\in(\eta_1^{-\frac{1}{d}},\bar{\mathfrak{e}})$,

$$(4.38)\qquad \sup_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\mathbb{P}(\xi^T\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}\xi\leqslant2\epsilon,\Theta_{\eta_2,\mathbf{T}}>0)\leqslant\mathbf{C}\epsilon^{d(p+4)}.$$

and

$$(4.39)\qquad \mathbb{P}(\|\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}\|_{\mathbb{R}^d}>\frac{1}{6\epsilon},\Theta_{\eta_2,\mathbf{T}}>0)\leqslant\mathbf{C}\epsilon^{d(p+2)}.$$

In this case

$$\mathbb{E}[|\det\tilde{\gamma}^\delta_{X^\delta_T,\mathbf{T}}|^p\mathbf{1}_{\Theta_{X^\delta_T,\eta,\mathbf{T}}>0}]\leqslant C(d,p)\mathbf{C}+\lceil\bar{\mathfrak{e}}^{-d}\rceil^p.$$

where $\tilde{\gamma}^\delta_{X^\delta_T,\mathbf{T}}=\dot{X}_T^T\gamma^\delta_{X^\delta_T\mathbf{T}}\dot{X}_T$ and (4.16) follows from the Cauchy-Schwarz inequality together with Lemma 4.7.. The result of **Step 1** is mainly a consequence of Lemma 4.6. We begin by noticing that

$$\mathbb{P}(|\det\tilde{\gamma}^\delta_{X^\delta_T,\mathbf{T}}|\mathbf{1}_{\Theta_{X^\delta_T,\eta,\mathbf{T}}>0}\geqslant\epsilon^{-d})=\mathbb{P}(|\det\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}|\leqslant\epsilon^d,\Theta_{X^\delta_T,\eta,\mathbf{T}}>0)$$

Since $|\det\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}|>\eta_1^{-1}$ on $\{\Theta_{X^\delta_T,\eta_1,\mathbf{T}}>0\}$, the quantity above is equal to zero as soon as $\epsilon^d\leqslant\eta_1^{-1}$ and for every $\epsilon^d>\eta_1^{-1}$,

$$\begin{aligned}\mathbb{P}(|\det\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}|\leqslant\epsilon^d,\Theta_{X^\delta_T,\eta,\mathbf{T}}>0)\leqslant&\mathbb{P}(|\det\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}|\leqslant\epsilon^d,\Theta_{\eta_2,\mathbf{T}}>0)\\ \leqslant&\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^T\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}\xi\leqslant\epsilon,\Theta_{\eta_2,\mathbf{T}}>0).\end{aligned}$$

Applying Lemma 4.6 (with (4.38) and (4.39)), for every $\epsilon\in(\eta_1^{-\frac{1}{d}},\bar{\mathfrak{e}})$,

$$\mathbb{P}(|\det\tilde{\sigma}^\delta_{X^\delta_T,\mathbf{T}}|\leqslant\epsilon^d,\Theta_{X^\delta_T,\eta,\mathbf{T}}>0)\leqslant C(d)\mathbf{C}\epsilon^{d(p+2)}.$$

Therefore

$$\begin{aligned}\mathbb{E}[|\det\tilde{\gamma}^\delta_{X^\delta_T,\mathbf{T}}|^p\mathbf{1}_{\Theta_{X^\delta_T,\eta,\mathbf{T}}>0}]\leqslant&C(d)\mathbf{C}\sum_{k=\lceil\bar{\mathfrak{e}}^{-d}\rceil}^{\lceil\eta_1\rceil-1}\frac{(k+1)^p}{k^{p+2}}+\lceil\bar{\mathfrak{e}}^{-d}\rceil^p\\ \leqslant&C(d)\mathbf{C}\sum_{k=1}^{+\infty}\frac{(k+1)^p}{k^{p+2}}+\lceil\bar{\mathfrak{e}}^{-d}\rceil^p\leqslant C(d)\mathbf{C}2^p\frac{\pi^2}{6}+\lceil\bar{\mathfrak{e}}^{-d}\rceil^p.\end{aligned}$$

and the proof of **Step 1** is completed.

Step 2. In this part, we focus on te proof of (4.38). More particularly, we demonstrate that, if $\eta_1 \in (1, \delta^{-d\frac{44}{91-36r}} \min(1, \frac{10^d}{m_*^d |2^{10}(1+T^3)|^{d\frac{44}{91-36r}}})]$ and $\eta_2 \in (1, \delta^{-\frac{1}{2}}\eta_1^{-\frac{1}{d}}]$, then for every $r \in (0, \frac{1}{12})$, if we fix,

$$\bar{\mathfrak{e}} \in [\eta_1^{-\frac{1}{d}}, \min(\frac{2^{\frac{1}{2}}}{d^{\frac{1}{2}}}, (\frac{T\mathcal{V}_L(\mathrm{x}_0^\delta, 0)m_*}{40(L+1)N^{\frac{L(L+1)}{2}}})^{r^{-L}}, \mathbf{1}_{L=0} + \mathbf{1}_{L>0}|m_* \frac{|2^8(1+T)|^{-\frac{11}{1-12r}}}{10N^{\frac{L(L-1)}{2}}}|^{r^{-L+1}}))$$

then, for every $\epsilon \in [\eta_1^{-\frac{1}{d}}, \bar{\mathfrak{e}})$,

$$\begin{aligned}(4.40) \qquad \sup_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1} & \mathbb{P}(\xi^T \tilde{\sigma}^\delta_{X^\delta_t,\mathbf{T}}\xi \leqslant 2\epsilon, \Theta_{\eta_2,\mathbf{T}} > 0) \\ \leqslant & \epsilon^{d(p+4)}(1+\mathcal{V}_L(\mathrm{x}_0^\delta)^{-\frac{3d(p+4)}{r^L}})(1 + \mathbf{1}_{\mathfrak{p}_{2L+5}>0}|\mathrm{x}_0^\delta|_{\mathbb{R}^d}^{C(d,L,p,\mathfrak{p}_{2L+5},\frac{1}{r},\frac{1}{1-12r})}) \\ & \times \mathfrak{D}^{C(d,L,p,\frac{1}{r},\frac{1}{1-12r})}\mathfrak{D}_{2L+5}^{C(d,L,p,\frac{1}{r},\frac{1}{1-12r})}\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\frac{1}{r},\frac{1}{1-12r})}(Z^\delta) \\ & \times C(d,N,L,\frac{1}{m_*},p,\mathfrak{p}_{2L+5},\frac{1}{r},\frac{1}{1-12r}) \\ & \times \exp(C(d,L,p,\mathfrak{p}_{2L+5},\frac{1}{r},\frac{1}{1-12r})T\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\mathfrak{q}^\delta_{\eta_2},\frac{1}{r},\frac{1}{1-12r})}(Z^\delta)\mathfrak{D}^4)).\end{aligned}$$

Notice that (4.16) is obtained by taking $r = \frac{1}{13}$

Step 2.1. For every $l \in \{0,\ldots,L\}$ and $\xi \in \mathbb{R}^d$, we introduce the $\mathbb{R}_+$-valued process $(\mathring{V}_{\xi,l,t})_{t\in\mathbf{T}}$ defined for every $t \in \mathbf{T}$ by $\mathring{V}_{\xi,l,t} = \sum_{\alpha\in\mathbf{N}^l}\sum_{i\in\mathbf{N}}\langle \xi, \mathring{X}_{t-\delta}V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta)\rangle^2_{\mathbb{R}^d}$. Let $r \in (0, \frac{1}{12})$, and denote $N_{l,r} = (\frac{10}{m_*})^{r^l} 4^{\frac{1-r^l}{1-r}} \prod_{j=1}^l N^{jr^{l-j}}$. Assume that $\eta_1 \in (1, \delta^{-\frac{d}{2+v}}]$ and $\eta_2 \in (1, \delta^{-\frac{1}{2}}\eta_1^{-\frac{1+\tilde{v}}{2d}}]$ with $v, \tilde{v} > 0$. Then, for every $\xi \in \mathbb{R}^d$ with $|\xi| = 1$ and every $\epsilon \in [\eta_1^{-\frac{1}{d}}, 1)$

$$\begin{aligned}(4.41) \qquad & \mathbb{P}(\delta \sum_{(t,i)\in\mathbf{T}\times\mathbf{N}} \chi^\delta_t \langle \xi, \mathring{\Psi}_{i,t}\rangle^2_{\mathbb{R}^d} \leqslant 2\epsilon, \Theta_{\eta_2,\mathbf{T}} > 0) \\ & \leqslant \sum_{l=0}^{L-1} \mathbb{P}(\delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l,t} \leqslant N_{l,r}\epsilon^{r^l}, \delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l+1,t} > N_{l+1,r}\epsilon^{r^{l+1}}, \Theta_{\eta_2,\mathbf{T}} > 0) \\ & \quad + \mathbb{P}(\delta\sum_{t\in\mathbf{T}}\sum_{l=0}^{L}\mathring{V}_{\xi,l,t} \leqslant (L+1)\frac{10}{m_*}N^{\frac{L(L+1)}{2}}\epsilon^{r^L}, \Theta_{\eta_2,\mathbf{T}} > 0) \\ & \quad + \epsilon^{d(p+4)}\mathfrak{D}_3^{C(d,\frac{1}{v},\frac{1}{\tilde{v}})}(1 + \mathbf{1}_{\mathfrak{p}_3>0}|\mathrm{x}^\delta_0|_{\mathbb{R}^d}^{C(d,p,\mathfrak{p}_3,\frac{1}{v},\frac{1}{\tilde{v}})})\mathfrak{M}_{C(d,p,\mathfrak{p},\mathfrak{p}_3,\frac{1}{v},\frac{1}{\tilde{v}})}(Z^\delta) \\ & \quad \times C(d,p,\mathfrak{p}_3,\frac{1}{v},\frac{1}{\tilde{v}})\exp(C(d,p,\mathfrak{p}_3,\frac{1}{v},\frac{1}{\tilde{v}})T\mathfrak{M}_{C(d,p,\mathfrak{p},\mathfrak{p}_3,\mathfrak{q}^\delta_{\eta_2},\frac{1}{v},\frac{1}{\tilde{v}})}(Z^\delta)\mathfrak{D}^4) \\ & \quad + 2\exp(-\epsilon^{-\frac{v}{2}}).\end{aligned}$$

First, we notice that

$$\begin{aligned}& \mathbb{P}(\delta \sum_{(t,i)\in\mathbf{T}\times\mathbf{N}} \chi^\delta_t\langle\xi,\mathring{\Psi}_{i,t}\rangle^2_{\mathbb{R}^d} \leqslant 2\epsilon, \Theta_{\eta_2,\mathbf{T}} > 0) \\ \leqslant & \mathbb{P}(\delta\sum_{t\in\mathbf{T}}\chi^\delta_t\mathring{V}_{\xi,0,t} \leqslant 8\epsilon, \Theta_{\eta_2,\mathbf{T}} > 0) \\ & + \mathbb{P}(\delta\sum_{(t,i)\in\mathbf{T}\times\mathbf{N}}\langle\xi,\mathring{\Psi}_{i,t} - \mathring{X}_{t-\delta}V_i(X^\delta_{t-\delta},t-\delta)\rangle^2_{\mathbb{R}^d} > 2\epsilon, \Theta_{\eta_2,\mathbf{T}} > 0),\end{aligned}$$

with

$$\begin{aligned}\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\chi^\delta_t\mathring{V}_{\xi,0,t} \leqslant 4\epsilon, \Theta_{\eta_2} > 0) \leqslant & \mathbb{P}(\delta|\sum_{t\in\mathbf{T}}(\chi^\delta_t - m_*)\mathring{V}_{\xi,0,t}| > 2\epsilon, \Theta_{\eta_2,\mathbf{T}} > 0) \\ & + \mathbb{P}(\delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,0,t} \leqslant \frac{10}{m_*}\epsilon, \Theta_{\eta_2,\mathbf{T}} > 0).\end{aligned}$$

Now we have

$$\begin{aligned}&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\mathring{V}_{0,t}\leqslant\frac{10}{m_*}\epsilon,\Theta_{\eta_2,\mathbf{T}}>0)\\ \leqslant&\sum_{l=0}^{L-1}\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l,t}\leqslant N_{l,r}\epsilon^{r^l},\delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l+1,t}>N_{l+1,r}\epsilon^{r^{l+1}},\Theta_{\eta_2,\mathbf{T}}>0)\\ &+\mathbb{P}(\bigcap_{l=0}^{L}\delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l,t}\leqslant N_{l,r}\epsilon^{r^l},\Theta_{\eta_2,\mathbf{T}}>0),\end{aligned}$$

with, since $\sup_{l\in\{0,\ldots,L\}}N_{l,r}\leqslant N_{0,r}N^{\frac{L(L+1)}{2}}=\frac{10}{m_*}N^{\frac{L(L+1)}{2}}$,

$$\mathbb{P}(\bigcap_{l=0}^{L}\delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l,t}\leqslant N_{l,r}\epsilon^{r^l},\Theta_{\eta_2,\mathbf{T}}>0)\leqslant\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\sum_{l=0}^{L}\mathring{V}_{\xi,l,t}\leqslant(L+1)\frac{10}{m_*}N^{\frac{L(L+1)}{2}}\epsilon^{r^L},\Theta_{\eta_2,\mathbf{T}}>0).$$

Moreover, for $v^\diamond\in(0,v)$,

$$\begin{aligned}&\mathbb{P}(\delta|\sum_{t\in\mathbf{T}}(\chi^\delta_t-m_*)\mathring{V}_{\xi,0,t}|>2\epsilon,\Theta_{\eta_2,\mathbf{T}}>0)\\ \leqslant&\mathbb{P}(\delta|\sum_{t\in\mathbf{T}}(\chi^\delta_t-m_*)\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}\mathring{V}_{\xi,0,t}|>2\epsilon,\\ &\quad\delta^2\sum_{t\in\mathbf{T}}(m_*(1-m_*)+(\chi^\delta_t-m_*)^2)\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}|\mathring{V}_{\xi,0,t}|^2\leqslant2\epsilon^{2+v-v^\diamond})\\ &+\mathbb{P}(\delta^2|\sum_{t\in\mathbf{T}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}|\mathring{V}_{\xi,0,t}|^2>2\epsilon^{2+v-v^\diamond}).\end{aligned}$$

Using (4.31), with $M_t=\sum_{\substack{w\in\pi^{\delta,*}\\ w\leqslant t}}\delta(\chi^\delta_t-m_*)\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}\mathring{V}_{\xi,0,t}$, the first term of the *r.h.s.* of the inequality above is bounded by $2\exp(-\epsilon^{-(v-v^\diamond)})$. In order to treat the second term, we remark that, $\mathring{V}_{\xi,0,t}=\sum_{i\in\mathbf{N}}\langle\xi,\mathring{X}_{t-\delta}V_i(X^\delta_{t-\delta})\rangle^2_{\mathbb{R}^d}$ and using the Markov inequality, for every $a>0$,

$$\begin{aligned}&\mathbb{P}(\delta^2\sum_{t\in\mathbf{T}}|\sum_{i=1}^{N}\langle\xi,\mathring{X}_{t-\delta}V_i(X^\delta_{t-\delta},t-\delta)\rangle^2_{\mathbb{R}^d}|^2\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}>2\epsilon^{2+v-v^\diamond})\\ \leqslant&\delta^a\epsilon^{-a(v-v^\diamond+2)}\mathfrak{D}_1^{4a}T^a\mathbb{E}[\sup_{t\in\mathbf{T}}\|\mathring{X}_{t-\delta}\|^{4a}_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}(1+\sup_{t\in\mathbf{T}}|X_{t-\delta}|^{\mathfrak{p}_1}_{\mathbb{R}^d})^{4a}]\end{aligned}$$

In particular we chose $a=\frac{d(p+4)\ln(\eta_1)}{-(v-v^\diamond+2)\ln(\eta_1)-d\ln(\delta)}$ (remember that $\delta\leqslant\eta_1^{-\frac{2+v}{d}}$ so that $a\leqslant\frac{d(p+4)}{v^\diamond}$) and apply Lemma 4.7 (see (4.33)) and Lemma 4.5 (when $4a<2$ we also use the Hölder inequality).

Now, we study $\mathbb{P}(\delta\sum_{(t,i)\in\mathbf{T}\times\mathbf{N}}\langle\xi,\mathring{\Psi}_{i,t}-\mathring{X}_{t-\delta}V_i(X^\delta_{t-\delta},t-\delta)\rangle^2_{\mathbb{R}^d}>2\epsilon,\Theta_{\eta_2,\mathbf{T}}>0)$. Recall that, on the set $\{\Theta_{\eta_2,\mathbf{T}}>0\}$, we have $|Z^{\delta,j}_t|\leqslant\eta_2$. We denote $\mathbf{D}_{\eta_2}=\{z\in\mathbb{R}^N,|z^i|\leqslant\delta^{\frac12}\eta_2,i\in\{1,\ldots,N\}\}$. We fix $(x,t,z,y)\in\mathbb{R}^d\times\mathbf{T}\times\mathbf{D}_{\eta_2}\times(0,1]$. Using the Taylor expansion yields

$$\begin{aligned}|\nabla_x\psi^{-1}\partial_{z^i}\psi(x,t-\delta,z,y)-V_i(x,t-\delta)|_{\mathbb{R}^d}\leqslant&\delta^{\frac12}\eta_2\sum_{j\in\mathbb{N}}|\partial_{z^j}(\nabla_x\psi^{-1}\partial_{z^i}\psi)(x,t-\delta,z,y)|_{\mathbb{R}^d}\\ &+\delta|\partial_y(\nabla_x\psi^{-1}\partial_{z^i}\psi)(x,t-\delta,z,y)|_{\mathbb{R}^d},\end{aligned}$$

with

$$\begin{aligned}\partial_y(\nabla_x\psi^{-1}\partial_{z^i}\psi)&=\nabla_x\psi^{-1}\partial_y\nabla_x\psi\nabla_x\psi^{-1}\partial_{z^i}\psi+\nabla_x\psi^{-1}\partial_y\partial_{z^i}\psi\\ \partial_{z^j}(\nabla_x\psi^{-1}\partial_{z^i}\psi)&=\nabla_x\psi^{-1}\partial_{z^j}\nabla_x\psi\nabla_x\psi^{-1}\partial_{z^i}\psi+\nabla_x\psi^{-1}\partial_{z^j,z^i}\psi.\end{aligned}$$

We focus on the study of the second term above. The study of the first one is similar and left to the reader. Remark that

$$\begin{aligned}\sum_{i,j\in\mathbf{N}}|\partial_{z^j}(\nabla_x\psi^{-1}\partial_{z^i}\psi)|_{\mathbb{R}^d}\leqslant&\|\nabla_x\psi^{-1}\|^2_{\mathbb{R}^d}\sum_{j\in\mathbf{N}}\|\partial_{z^j}\nabla_x\psi\|_{\mathbb{R}^d}\sum_{i\in\mathbf{N}}|\partial_{z^i}\psi|_{\mathbb{R}^d}\\ &+\|\nabla_x\psi^{-1}\|_{\mathbb{R}^d}\sum_{i,j\in\mathbf{N}}|\partial_{z^j,z^i}\psi|_{\mathbb{R}^d}.\end{aligned}$$

We show that, the function $\|\nabla_x \psi^{-1}\|_{\mathbb{R}^d}$ is bounded on $\mathbb{R}^d \times \mathbf{T} \times \mathbf{D}_{\eta_2} \times (0,1]$. We consider the following decomposition

$$\nabla_x \psi^{-1}(x, t-\delta, z, \delta) = I_{d\times d} - (\nabla_x \psi(x, t-\delta, z, \delta) - I_{d\times d})\nabla_x \psi^{-1}(x, t-\delta, z, \delta).$$

Now, assumption $\mathbf{A}_1$ (see (2.3)) implies that (4.29) holds. It follows that under the assumptions (4.15), for every $(x,t,z) \in \mathbb{R}^d \times \mathbf{T} \times \mathbf{D}_{\eta_2}$, $\|\nabla_x \psi(x, t-\delta, z, \delta) - I_{d\times d}\|_{\mathbb{R}^d} \leqslant \frac{1}{2}$ and then $\|\nabla_x \psi^{-1}\|_{\mathbb{R}^d} \leqslant 2$. Moreover

$$\begin{aligned}\sum_{j\in\mathbf{N}} \|\partial_{z^j} \nabla_x \psi\|_{\mathbb{R}^d} \leqslant& \sum_{j\in\mathbf{N}} |\sum_{l=1}^{d} |\partial_{z^j} \partial_{x^l} \psi|^2_{\mathbb{R}^d}|^{\frac{1}{2}} \\ \leqslant& \sum_{j\in\mathbf{N}} \sum_{l=1}^{d} |\partial_{z^j} \partial_{x^l} \psi|_{\mathbb{R}^d}.\end{aligned}$$

Using similar estimates for the term $\partial_y(\nabla_x \psi^{-1} \partial_{z^i} \psi)$ together with $\mathbf{A}_1(3)$ (see (2.2)), we obtain, for every $a \geqslant \frac{1}{2}$,

$$\begin{aligned}&\mathbb{P}(\delta \sum_{(t,i)\in\mathbf{T}\times\mathbf{N}} \langle \xi, \mathring{\Psi}_{i,t} - \mathring{X}_{t-\delta} V_i(X^\delta_{t-\delta}, t-\delta)\rangle^2_{\mathbb{R}^d} > 2\epsilon, \Theta_{\eta_2,\mathbf{T}} > 0) \\ \leqslant& C(a)\delta^a \eta_2^{2a} \epsilon^{-a} \mathfrak{D}_3^{4a} T^a (\mathbb{E}[\sup_{t\in\mathbf{T}} \|\mathring{X}_{t-\delta}\|^{2a}_{\mathbb{R}^d} \mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0} (1 + \sup_{t\in\mathbf{T}} |X_{t-\delta}|^{4a\mathfrak{p}_3}_{\mathbb{R}^d})] \\ &+ C(a)\delta^a \eta_2^{2a} \epsilon^{-a} \mathbb{E}[\sup_{t\in\mathbf{T}} \|\mathring{X}_{t-\delta}\|^{2a}_{\mathbb{R}^d} \mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0} |\delta \sum_{t\in\mathbf{T}} |Z^\delta_t|^{4\mathfrak{p}_3}_{\mathbb{R}^N}|^a].\end{aligned}$$

Moreover, the Hölder inequality (since $2a \geqslant 1$) yields

$$\mathbb{E}[|\delta \sum_{t\in\mathbf{T}} |Z^\delta_t|^{4\mathfrak{p}_3}_{\mathbb{R}^N}|^{2a}] \leqslant T^{2a-1} \mathbb{E}[\delta \sum_{t\in\mathbf{T}} |Z^\delta_t|^{8a\mathfrak{p}_3}_{\mathbb{R}^N}|] \leqslant T^{2a} \mathfrak{M}_{8a\mathfrak{p}_3}(Z^\delta).$$

We chose $a = \max(\frac{1}{2}, \lceil \frac{d(p+4)\ln(\eta_1)}{-\ln(\eta_1) - d\ln(\delta) - 2d\ln(\eta_2)} \rceil)$ (remember that $\delta \leqslant \eta_2^{-2} \eta_1^{\frac{1+\bar{v}}{d}}$ so that $a \leqslant \lceil \frac{d(p+4)}{\bar{v}} \rceil$) and conclude using Cauchy-Schwarz inequality, Lemma 4.7 (see (4.33)) and Lemma 4.5. Gathering all the upper bounds together, (take $v^\diamond = \frac{v}{2}$), we obtain (4.41).

Step 2.2. Let us show that, for every $\epsilon \in (0, (\frac{T\mathcal{V}_L(\mathrm{X}^\delta_0, 0) m_*}{40(L+1) N^{\frac{L(L+1)}{2}}})^{r^{-L}}]$,

$$\begin{aligned}&\mathbb{P}(\delta \sum_{t\in\mathbf{T}} \sum_{l=0}^{L} \mathring{V}_{\xi,l,t} \leqslant (L+1)\frac{10}{m_*} N^{\frac{L(L+1)}{2}} \epsilon^{r^L}, \Theta_{\eta_2,\mathbf{T}} > 0) \\ \leqslant& \epsilon^{d(p+4)} \mathcal{V}_L(\mathrm{X}^\delta_0)^{-\frac{3d(p+4)}{r^L}} (1 + \mathbf{1}_{\mathfrak{p}_{4+2L}>0} |\mathrm{X}^\delta_0|^{C(d,L,p,\mathfrak{p}_{4+2L},\frac{1}{r})}_{\mathbb{R}^d}) \\ &\times \mathfrak{D}^{C(d,L,p,\frac{1}{r})} \mathfrak{D}_{4+2L}^{C(d,L,p,\frac{1}{r})} \mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{4+2L},\frac{1}{r})}(Z^\delta) \\ &\times C(d, N, L, \frac{1}{m_*}, p, \mathfrak{p}_{4+2L}, \frac{1}{r}) \exp(C(d, L, p, \mathfrak{p}_{4+2L}, \frac{1}{1-v}, \frac{1}{r}) T \mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{4+2L},\mathfrak{q}^\delta_{\eta_2},\frac{1}{r})}(Z^\delta)\mathfrak{D}^4) \\ &+ 2\exp(-\frac{\mathcal{V}_L(\mathrm{X}^\delta_0)}{32\epsilon^{\frac{rL}{3}} N\binom{N+L}{N}}).\end{aligned}$$

It is worth noting that, in case of uniform Hörmander properties, we have a similar result but with $\mathcal{V}_L(\mathrm{X}^\delta_0)$ replaced by 1 in the r.h.s. above.

Now let us focus on the proof. of **Step 2.2**. Let us denote $\epsilon_{r,L} = (L+1)\frac{10}{m_*} N^{\frac{L(L+1)}{2}} \epsilon^{r^L}$. Let $\mathbf{S} := \{\delta, \ldots, \lceil \frac{4\epsilon_{r,L}}{\delta \mathcal{V}_L(\mathrm{X}^\delta_0, 0)} \rceil \delta\}$. Since $\epsilon \leqslant (\frac{T\mathcal{V}_L(\mathrm{X}^\delta_0, 0) m_*}{40(L+1) N^{\frac{L(L+1)}{2}}})^{r^{-L}}$, then $\mathbf{S} \subset \mathbf{T}$. Therefore,

$$
\begin{aligned}
&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\sum_{l=0}^{L}\mathring{V}_{\xi,l,t}\leqslant (L+1)N_{L,r}\epsilon^{r^L},\Theta_{\eta_2,\mathbf{T}}>0)\\
&\quad\leqslant\mathbb{P}(\delta\sum_{t\in\mathbf{S}}\sum_{l=0}^{L}\mathring{V}_{\xi,l,t}\leqslant \epsilon_{r,L},\Theta_{\eta_2,\mathbf{T}}>0)\\
&\quad\leqslant\mathbb{P}(\frac{1}{2}\delta|\mathbf{S}|\sum_{|\alpha|\leqslant L}\sum_{i=1}^{N}\langle\xi,V_i^{[\alpha]}(\mathrm{x}_0^\delta)\rangle_{\mathbb{R}^d}^2-\epsilon_{r,L}\\
&\qquad\quad\leqslant\delta\sum_{t\in\mathbf{S}}\sum_{|\alpha|\leqslant L}\sum_{i=1}^{N}|\langle\xi,\mathring{X}_{t-\delta}V_i^{[\alpha]}(X_{t-\delta}^\delta,t-\delta)-V_i^{[\alpha]}(\mathrm{x}_0^\delta)\rangle_{\mathbb{R}^d}|^2,\Theta_{\eta_2,\mathbf{T}}>0)\\
&\quad\leqslant\mathbb{P}(\sup_{t\in\mathbf{S}}\sum_{|\alpha|\leqslant L}\sum_{i=1}^{N}|\langle\xi,\mathring{X}_{\eta_2,t-\delta}V_i^{[\alpha]}(X_{t-\delta}^\delta,t-\delta)-V_i^{[\alpha]}(\mathrm{x}_0^\delta,0)\rangle_{\mathbb{R}^d}|^2\geqslant\frac{\mathcal{V}_L(\mathrm{x}_0^\delta)}{4})\\
&\quad\leqslant\mathbb{P}(\sup_{t\in\mathbf{S}}\sum_{|\alpha|\leqslant L}\sum_{i=1}^{N}|M_{\alpha,i,t-\delta}|^2\geqslant\frac{\mathcal{V}_L(\mathrm{x}_0^\delta)}{8}-\sup_{t\in\mathbf{S}}\sum_{|\alpha|\leqslant L}\sum_{i=1}^{N}|B_{\alpha,i,t-\delta}|^2)\\
&\quad\leqslant\sum_{|\alpha|\leqslant L}\sum_{i=1}^{N}\mathbb{P}(\sup_{t\in\mathbf{S}}|M_{\alpha,i,t-\delta}|^2\geqslant\frac{\mathcal{V}_L(\mathrm{x}_0^\delta)}{8N\binom{N+L}{N}}-\sup_{t\in\mathbf{S}}|B_{\alpha,i,t-\delta}|^2)
\end{aligned}
$$

with for every $t\in\mathbf{T}$,

$$
M_{\alpha,i,t}=\delta^{\frac{1}{2}}\sum_{w\in\mathbf{T};0<w\leqslant t}\tilde{\Delta}_{w-\delta}^{Y_{\alpha,i}},\qquad B_{\alpha,i,t}=\delta\sum_{w\in\mathbf{T};0<w\leqslant t}\bar{\Delta}_{w-\delta}^{Y_{\alpha,i}},
$$

where $Y_{\alpha,i,0}=0$ and for every $t\in\mathbf{T}$,

$$
Y_{\alpha,i,t}=\langle\xi,\mathring{X}_{\eta_2,t}V_i^{[\alpha]}(X_t^\delta,t)-V_i^{[\alpha]}(\mathrm{x}_0^\delta,0)\rangle_{\mathbb{R}^d}.
$$

Now we decompose our estimate in the following way

$$
\begin{aligned}
&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\sum_{l=0}^{L}\mathring{V}_{\xi,l,t}\leqslant (L+1)N_{L,r}\epsilon^{r^L},\Theta_{\eta_2,\mathbf{T}}>0)\\
&\quad\leqslant\sum_{|\alpha|\leqslant L}\sum_{i=1}^{N}\mathbb{P}(\sup_{t\in\mathbf{S}}|M_{\alpha,i,t-\delta}|^2\geqslant\frac{\mathcal{V}_L(\mathrm{x}_0^\delta)}{16N\binom{N+L}{N}},\Theta_{\eta_2,\mathbf{T}}>0)\\
&\qquad+\mathbb{P}(\sup_{t\in\mathbf{S}}|B_{\alpha,i,t-\delta}|^2>\frac{\mathcal{V}_L(\mathrm{x}_0^\delta)}{16N\binom{N+L}{N}},\Theta_{\eta_2,\mathbf{T}}>0).
\end{aligned}
$$

We study the second term of the *r.h.s.* above. Using the Markov inequality, for every $a>0$, we have

$$
\begin{aligned}
\mathbb{P}(\sup_{t\in\mathbf{S}}|B_{\alpha,i,t-\delta}|^2>\frac{\mathcal{V}_L(\mathrm{x}_0^\delta,0)}{16N\binom{N+L}{N}})\leqslant&\mathbb{P}(|\delta\sum_{t\in\mathbf{S}}|\bar{\Delta}_{t-\delta}^{Y_{\alpha,i}}||^2>\frac{\mathcal{V}_L(\mathrm{x}_0^\delta)}{16N\binom{N+L}{N}},\Theta_{\eta_2,\mathbf{T}}>0)\\
\leqslant&4^a\delta^a|\mathbf{S}|^a\sup_{t\in\mathbf{S}}\mathbb{E}[|\bar{\Delta}_{t-\delta}^{Y_{\alpha,i}}|^a]|\frac{N\binom{N+L}{N}}{\mathcal{V}_L(\mathrm{x}_0^\delta)}|^{\frac{a}{2}}.
\end{aligned}
$$

In particular, we chose $a=\frac{d(p+4)}{r^L}$ so that $\delta^a|\mathbf{S}|^a\leqslant C(a,N,L,\frac{1}{m_*},\frac{1}{r})\mathcal{V}_L(\mathrm{x}_0^\delta)^{-a}\epsilon^{d(p+4)}$.
As a consequence of Lemma 4.8 with $V=V_i^{[\alpha]}$ and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
\mathbb{E}[|\bar{\Delta}_{t-\delta}^{Y_{\alpha,i}}|^a]\leqslant&C(d,a)\mathfrak{D}^{3a}\mathfrak{D}_4^{7a}\mathfrak{D}_{V_i^{[\alpha]},3}^{2a}\mathfrak{M}_{2\max(3\mathfrak{p}+5\mathfrak{p}_4+14,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)^a\\
&\times\mathbb{E}[\|\mathring{X}_{t-\delta}\|_{\mathbb{R}^d}^{2a}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T}}>0}]^{\frac{1}{2}}(1+\mathbb{E}[|X_{t-\delta}^\delta|_{\mathbb{R}^d}^{2a(7\mathfrak{p}_4+2\mathfrak{p}_{V_i^{[\alpha]},3})}]^{\frac{1}{2}}),
\end{aligned}
$$

and we bound the *r.h.s.* above using Lemma 4.7 (see (4.33)) and Lemma 4.5 (when $4a<2$ we also use the Hölder inequality to conclude).

Moreover, for $v' > 0$

$$
\begin{aligned}
&\mathbb{P}(\sup_{t\in\mathbf{S}} |M_{\alpha,i,t-\delta}|^2 \geqslant \frac{\mathcal{V}_L(\mathrm{x}_0^\delta)}{16N\binom{N+L}{N}})\\
\leqslant&\mathbb{P}(\sup_{t\in\mathbf{S}} |M_{\alpha,i,t}|^2 \geqslant \frac{\mathcal{V}_L(\mathrm{x}_0^\delta)}{16N\binom{N+L}{N}}, \delta\sum_{t\in\mathbf{S}}\mathbb{E}[|\tilde{\Delta}^{M_{\alpha,i}}_{t-\delta}|^2|\mathcal{F}^{X^\delta}_{t-\delta}] + |\tilde{\Delta}^{M_{\alpha,i}}_{t-\delta}|^2 < \epsilon^{\frac{r^L}{2+v'}})\\
&+\mathbb{P}(\delta\sum_{t\in\mathbf{S}}\mathbb{E}[|\tilde{\Delta}^{M_{\alpha,i}}_{t-\delta}|^2|\mathcal{F}^{X^\delta}_{t-\delta}] + |\tilde{\Delta}^{M_{\alpha,i}}_{t-\delta}|^2 \geqslant \epsilon^{\frac{r^L}{2+v'}}).
\end{aligned}
$$

Using the Doob exponential inequality (4.31), the first term is bounded by $2\exp(-\frac{\mathcal{V}_L(\mathrm{X}_0^\delta)}{32\epsilon^{\frac{r^L}{2+v'}}N\binom{N+L}{N}})$. In order to bound the second term we take $a \geqslant 1$ and using again the Markov and Hölder inequalities and that $\tilde{\Delta}^{Y_{\alpha,i}} = \tilde{\Delta}^{M_{\alpha,i}}$, yields

$$
\mathbb{P}(\delta\sum_{t\in\mathbf{S}}\mathbb{E}[|\tilde{\Delta}^{M_{\alpha,i}}_{t-\delta}|^2|\mathcal{F}^{X^\delta}_{t-\delta}] + |\tilde{\Delta}^{M_{\alpha,i}}_{t-\delta}|^2) \geqslant \epsilon^{\frac{r^L}{2+v'}}) \leqslant \delta^a|\mathbf{S}|^a\epsilon^{-a\frac{r^L}{(2+v')}}2^{a+1}\sup_{t\in\mathbf{S}}\mathbb{E}[|\tilde{\Delta}^{Y_{\alpha,i}}_{t-\delta}|^{2a}].
$$

At this point, we chose $a = \frac{(2+v')d(p+4)}{(1+v')r^L}$ so that $\delta^a|\mathbf{S}|^a\epsilon^{-a\frac{r^L}{2+v'}} \leqslant C(a,N,L,\frac{1}{m_*},\frac{1}{r})L^a\mathcal{V}_L(\mathrm{x}_0^\delta,0)^{-a}\epsilon^{d(p+4)}$. Remark that $a \leqslant \frac{2d(p+4)}{r^L}$. In order to bound the *r.h.s.* above we use Lemma 4.8. Hence

$$
\begin{aligned}
\mathbb{E}[|\tilde{\Delta}^{Y_{\alpha,i}}_{t-\delta}|^{2a}] \leqslant& C(a,N,L,\frac{1}{r})\mathfrak{D}^{6a}\mathfrak{D}_4^{14a}\mathfrak{D}^{4a}_{V_i^{[\alpha]},3}\mathfrak{M}_{8\max(2a,\frac{1}{4})\max(3\mathfrak{p}+5\mathfrak{p}_4+14,\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+2)}(Z^\delta)^2\\
&\times\mathbb{E}[\|\mathring{X}_{t-\delta}\|^{4a}_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T}}>0}]^{\frac{1}{2}}(1+\mathbb{E}[|X^\delta_{t-\delta}|_{\mathbb{R}^d}^{8a(7\mathfrak{p}_4+2\mathfrak{p}_{V_i^{[\alpha]},3})}]^{\frac{1}{2}}),
\end{aligned}
$$

and then use Lemma 4.7 (see (4.33)) and Lemma 4.5. Remarking that $\mathfrak{D}_{V_i^{[\alpha]},3} \leqslant C(|\alpha|)\mathfrak{D}_{4+2|\alpha|}^{C(|\alpha|)}$ and $\mathfrak{p}_{V_i^{[\alpha]},3} \leqslant C(|\alpha|)\mathfrak{p}_{4+2|\alpha|}$ and taking $v' = 1$ concludes the proof of **Step 2.2**.

Step 2.3. Consider the case $L \in \mathbb{N}^*$. Let $l \in \{0,\ldots,L-1\}$. Assume that $\eta_1 \in (1,\delta^{-\frac{d}{2}}]$. Let us show that for every

$$
\epsilon \in [\max(\eta_1^{-\frac{1}{d}}, |\frac{|2^{10}(1+T^3)\delta|^{\frac{44}{91-36r}}}{N_{l,r}}|^{r^{-l}}), |\frac{|2^8(1+T)|^{-\frac{11}{1-12r}}}{N_{l,r}}|^{r^{-l}}],
$$

then

$$
\begin{aligned}
&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l,t} \leqslant N_{l,r}\epsilon^{r^l}, \delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l+1,t} > N_{l+1,r}\epsilon^{r^{l+1}}, \Theta_{\eta_2,\mathbf{T}} > 0)\\
\leqslant&\epsilon^{d(p+4)}\mathfrak{D}^{C(d,L,p,\frac{1}{1-v},\frac{1}{r},\frac{1}{1-12r})}\mathfrak{D}_{2l+7}^{C(d,L,p,\frac{1}{r},\frac{1}{1-12r})}\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2l+7},\frac{1}{r})}(Z^\delta)\\
&\times(1+\mathbf{1}_{\mathfrak{p}_{2l+7}>0}|\mathrm{x}_0^\delta|_{\mathbb{R}^d}^{C(d,L,p,\mathfrak{p}_{2l+7},\frac{1}{r},\frac{1}{1-12r})})\\
&\times C(d,N,L,\frac{1}{m_*},p,\mathfrak{p}_{2l+7},\frac{1}{r},\frac{1}{1-12r})\\
&\times\exp(C(d,L,p,\mathfrak{p}_{2l+7},\frac{1}{r},\frac{1}{1-12r})T\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2l+7},\mathfrak{q}^\delta_{\eta_2},\frac{1}{r},\frac{1}{1-12r})}(Z^\delta)\mathfrak{D}^4)\\
&+12\exp(-\frac{|N_{l,r}\epsilon^{r^{-l}}|^{-\frac{1-12r}{22}}}{2^{11}(1+T^2)}).
\end{aligned}
$$

First, for $\alpha \in \mathbf{N}^l$ and $i \in \mathbf{N}$, we introduce the $\mathbb{R}$-valued process $(Y^\diamond_{\alpha,i,t})_{t\in\pi^{\delta\delta}}$ such that $Y^\diamond_{\alpha,i,0} = 0$ and $Y^\diamond_{\alpha,i,t} = \langle\xi,\mathring{X}_{\eta_2,t-\delta}V_i^{[\alpha]}(X^\delta_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}$, $t \in \pi^{\delta,*}$. In particular, on the set $\{\Theta_{\eta_2,\mathbf{T}} > 0\}$, $\mathring{V}_{\xi,l,t} = \sum_{\alpha\in\mathbf{N}^l}\sum_{i\in\mathbf{N}}|Y^\diamond_{\alpha,i,t}|^2$, $t \in \pi^\delta$. In particular, it follows from Lemma 4.8 with $V = V_i^{[\alpha]}$, that, for $t \in \pi^{\delta,*}$,

$$
\begin{aligned}
Y^\diamond_{\alpha,i,t+\delta}-Y^\diamond_{\alpha,i,t} =& \delta^{\frac{1}{2}}\sum_{j=1}^N Z_t^{\delta,j}\langle\xi,\mathring{X}_{\eta_2,t-\delta}V_i^{[(\alpha,j)]}(X^\delta_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}\\
&+\delta\langle\xi,\mathring{X}_{\eta_2,t-\delta}V_i^{[(\alpha,0)]}(X^\delta_{t-\delta},t-\delta))\rangle_{\mathbb{R}^d} + \langle\xi,\mathring{X}_{\eta_2,t-\delta}\mathbf{R}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta},t-\delta,Z^\delta_t)\rangle_{\mathbb{R}^d}\\
=&\delta^{\frac{1}{2}}\sum_{j=1}^N Z_t^{\delta,j}Y^\diamond_{(\alpha,j),i,t} + \delta Y^\diamond_{(\alpha,0),i,t} + \langle\xi,\mathring{X}_{\eta_2,t-\delta}\mathbf{R}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta},t-\delta,Z^\delta_t)\rangle_{\mathbb{R}^d}
\end{aligned}
$$

and

$$\mathring{V}_{\xi,l+1,t} = \sum_{\alpha\in\mathbf{N}^l}\sum_{i\in\mathbf{N}} \mathbb{E}[|\tilde{\Delta}_t^{Y^\diamond_{\alpha,i}} - \delta^{-\frac{1}{2}}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\tilde{\mathbf{R}}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta, Z^\delta_t)\rangle_{\mathbb{R}^d}|^2|\mathcal{F}_{t-\delta}^{Y^\diamond_{\alpha,i}}]$$
$$+ |\bar{\Delta}_t^{Y^\diamond_{\alpha,i}} - \delta^{-1}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\overline{\mathbf{R}}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta)\rangle_{\mathbb{R}^d}|^2.$$

Therefore,

$$\begin{aligned}
&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l,t} \leqslant N_{l,r}\epsilon^{r^l}, \delta\sum_{t\in\mathbf{T}}\mathring{V}_{\xi,l+1,t} > N_{l+1,r}\epsilon^{(1-v)r^{l+1}}, \Theta_{\eta_2,\mathbf{T}} > 0)\\
\leqslant&\mathbb{P}(\delta\sum_{t\in\mathbf{T}^-}\sum_{\alpha\in\mathbf{N}^l}\sum_{i\in\mathbf{N}}|Y^\diamond_{\alpha,i,t}|^2 \leqslant N_{l,r}\epsilon^{r^l},\\
&\delta\sum_{t\in\mathbf{T}}\sum_{\alpha\in\mathbf{N}^l}\sum_{i\in\mathbf{N}}\mathbb{E}[|\tilde{\Delta}_t^{Y^\diamond_{\alpha,i}} - \delta^{-\frac{1}{2}}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\tilde{\mathbf{R}}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta, Z^\delta_t)\rangle_{\mathbb{R}^d}|^2|\mathcal{F}_{t-\delta}^{Y^\diamond_{\alpha,i}}]\\
&+ |\bar{\Delta}_t^{Y^\diamond_{\alpha,i}} - \delta^{-1}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\overline{\mathbf{R}}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta)\rangle_{\mathbb{R}^d}|^2 > N_{l+1,r}\epsilon^{r^{l+1}})\\
\leqslant&\sum_{\alpha\in\mathbf{N}^l}\sum_{i\in\mathbf{N}}\mathbb{P}(\delta\sum_{t\in\mathbf{T}^-}|Y^\diamond_{\alpha,i,t}|^2 \leqslant N_{l,r}\epsilon^{(1-v)r^l}, \delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}_t^{Y^\diamond_{\alpha,i}}|^2] + |\bar{\Delta}_t^{Y^\diamond_{\alpha,i}}|^2 > \frac{1}{4}N^{-l-1}N_{l+1,r}\epsilon^{r^{l+1}})\\
&+\sum_{\alpha\in\mathbf{N}^l}\sum_{i\in\mathbf{N}}\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\delta^{-\frac{1}{2}}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\tilde{\mathbf{R}}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta, Z^\delta_t)\rangle_{\mathbb{R}^d}|^2]\\
&+ |\delta^{-1}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\overline{\mathbf{R}}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta)\rangle_{\mathbb{R}^d}|^2 > \frac{1}{4}N^{-l-1}N_{l+1,r}\epsilon^{r^{l+1}})
\end{aligned}$$

where $\mathbf{T}^- = \mathbf{T}\setminus\{\sup\{t, t\in\mathbf{T}\}\}$. We bound the the first term of the *r.h.s.* above. Since $N_{l+1,r} = 4N^{l+1}N^r_{l,r}$, $r\in(0,\frac{1}{12})$, and $N_{l,r}\epsilon^{r^l}\in[|2^{10}(1+T^3)\delta|^{\frac{44}{91-36r}}, |2^8(1+T)|^{-\frac{11}{1-12r}}]$, this bound is obtained by applying Lemma 4.9 with $Y^\diamond = Y^\diamond_{\alpha,i}$, $\mathbf{T} = \mathbf{T}^-$, $\epsilon = N_{l,r}\epsilon^{r^l}$, and $p = \frac{d(p+6)}{r^l}$. In particular we have to bound $\mathfrak{N}_{Y^\diamond_{\alpha,i},\mathbf{T}^-}(q(d,r,l,p))$ (this quantity being defined in (4.36)) with $q(d,r,l,p) = \max(4, \frac{44d(p+4)}{r^l-12r^{l+1}})$. We notice that $\bar{\Delta}_0^{\bar{\Delta}^{Y^\diamond_{\alpha,i}}} = \langle\xi, V_i^{[\alpha]}(\mathrm{x}^\delta_0, 0)\rangle_{\mathbb{R}^d}$, $\tilde{\Delta}_0^{\bar{\Delta}^{Y^\diamond_{\alpha,i}}} = 0$ and that, for $t\in\pi^{\delta,*}$, as a consequence of Lemma 4.8,

$$\begin{aligned}
\bar{\Delta}_t^{\bar{\Delta}^{Y^\diamond_{\alpha,i}}} =&Y^\diamond_{(\alpha,0,0),i,t} + \delta^{-1}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\overline{\mathbf{R}}^\delta V_i^{[(\alpha,0)]}(X^\delta_{t-\delta}, t-\delta)\\
&+ \delta^{-1}\langle\xi, \mathring{X}_{\eta_2,t-\delta}(\overline{\mathbf{R}}^\delta V_i^{[\alpha]})^{[0]}(X^\delta_{t-\delta}, t-\delta)\rangle_{\mathbb{R}^d} + \delta^{-2}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\overline{\mathbf{R}}^\delta\overline{\mathbf{R}}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta)\rangle_{\mathbb{R}^d},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\Delta}_t^{\bar{\Delta}^{Y^\diamond_{\alpha,i}}} =&\sum_{j=1}^N Z^{\delta,j}_t Y^\diamond_{(\alpha,0,j),i,t} + \delta^{-\frac{1}{2}}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\tilde{\mathbf{R}}^\delta V_i^{[(\alpha,0)]}(X^\delta_{t-\delta}, t-\delta, Z^\delta_t)\\
&+ \delta^{-1}Z^{\delta,j}_t\langle\xi, \mathring{X}_{\eta_2,t-\delta}(\overline{\mathbf{R}}^\delta V_i^{[\alpha]})^{[j]}(X^\delta_{t-\delta}, t-\delta)\rangle_{\mathbb{R}^d}\\
&+ \delta^{-\frac{3}{2}}\langle\xi, \mathring{X}_{\eta_2,t-\delta}\tilde{\mathbf{R}}^\delta\overline{\mathbf{R}}^\delta V_i^{[\alpha]}(X^\delta_{t-\delta}, t-\delta, Z^\delta_t)\rangle_{\mathbb{R}^d}.
\end{aligned}$$

Applying (4.35) and (4.34), we obtain

$$\begin{aligned}
\mathfrak{N}_{Y^\diamond_{\alpha,i},\mathbf{T}^-}(q(d,r,l,p)) \leqslant& C(d,l,p,\frac{1}{1-v},\frac{1}{1-12r},\frac{1}{r})\mathfrak{M}_{4q(d,r,l,p)\max(3\mathfrak{p}+6\mathfrak{p}_7+16,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)\\
&\times\mathfrak{D}^{6q(d,r,l,p)}\mathfrak{D}_{2l+7}^{C(l)q(d,r,l,p)}\\
&\times\mathbb{E}[\sup_{t\in\mathbf{T}}\|\mathring{X}_{\eta_2,t-\delta}\|^{2q(d,r,l,p)}_{\mathbb{R}^d}]^{\frac{1}{2}}(1+\mathbb{E}[\sup_{t\in\mathbf{T}}|X^\delta_{t-\delta}|^{2C(l)q(d,r,l,p)\mathfrak{p}_{2l+7}}_{\mathbb{R}^d}]^{\frac{1}{2}}).
\end{aligned}$$

Using the Markov and Cauchy-Schwarz inequalities gives also, for every $a > 0$,

$$\begin{aligned}&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\delta^{-\frac{1}{2}}\langle\xi,\mathring{X}_{\eta_2,t-\delta}\tilde{\mathbf{R}}^{\delta}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta,Z^{\delta}_t)\rangle_{\mathbb{R}^d}|^2]\\&\quad+|\delta^{-1}\langle\xi,\mathring{X}_{\eta_2,t-\delta}\overline{\mathbf{R}}^{\delta}V_i^{[\alpha]}(X^{\delta}_{t-\delta},t-\delta)\rangle_{\mathbb{R}^d}|^2>\frac{1}{4}N^{-l-1}N_{l+1,r}\epsilon^{r^{l+1}})\\&\leqslant\delta^{\frac{a}{2}}\epsilon^{-ar^{l+1}}T^aC(N,\frac{1}{m_*},l,r,a)\\&\quad\times\mathfrak{M}_{4a\max(6\mathfrak{p}+10\mathfrak{p}_4+28,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^{\delta})\\&\quad\times\mathfrak{D}^{3a}\mathfrak{D}_{2l+4}^{C(l)a}\\&\quad\times\mathbb{E}[\sup_{t\in\mathbf{T}}\|\mathring{X}_{\eta_2,t-\delta}\|^{2a}_{\mathbb{R}^d}]^{\frac{1}{2}}(1+\mathbb{E}[\sup_{t\in\mathbf{T}}|X^{\delta}_{t-\delta}|^{2aC(l)\mathfrak{p}_{2l+4})}_{\mathbb{R}^d}]^{\frac{1}{2}})\end{aligned}$$

In particular, we chose $a=\frac{d(p+4)\ln(\eta_1)}{-r^{l+1}\ln(\eta_1)-\frac{d}{2}\ln(\delta)}$ so that $\delta^{\frac{a}{2}}\epsilon^{-ar^{l+1}}\leqslant\epsilon^{d(p+4)}$ (notice that since $\delta\leqslant\eta_1^{-\frac{2}{d}}$ and $r\in(0,\frac{1}{12})$, then $a\leqslant 2d(p+4)$) and then apply Lemma 4.7 (see (4.33)) and Lemma 4.5 to conclude the proof of **Step 2.3** (when $4a<2$ we also use the Hölder inequality).

Step 2.4 We are now in a position to conclude the proof of **Step 2**. Gathering the estimates obtained in **Step 2.1**, **2.2** and **2.3**, we have proved that, if $\eta_1\in(1,\delta^{-\frac{d}{2+v}}]$ and $\eta_2\in(1,\delta^{-\frac{1}{2}}\eta_1^{\frac{1+\tilde{v}}{2d}}]$ with $v,\tilde{v}>0$, for every $r\in(0,\frac{1}{12})$ and for every

$$\begin{aligned}\epsilon\in&[\max(\eta_1^{-\frac{1}{d}},\mathbf{1}_{L>0}\frac{m_*|2^{10}(1+T^3)\delta|^{\frac{44}{91-36r}}}{10}),\\&\min(\frac{2^{\frac{1}{2}}}{d^{\frac{1}{2}}},(\frac{T\mathcal{V}_L(\mathrm{x}^{\delta}_0,0)m_*}{40(L+1)N^{\frac{L(L+1)}{2}}})^{r^{-L}},\mathbf{1}_{L=0}+\mathbf{1}_{L>0}|m_*\frac{|2^8(1+T)|^{-\frac{11}{1-12r}}}{10N^{\frac{L(L-1)}{2}}}|^{r^{-L+1}})),\end{aligned}$$

then

$$\begin{aligned}&\sup_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\mathbb{P}(\xi^T\tilde{\sigma}^{\delta}_{X^{\delta}_t,\mathbf{T}}\xi\leqslant 2\epsilon,\Theta_{\eta_2,\mathbf{T}}>0)\\&\leqslant\epsilon^{d(p+4)}(1+\mathcal{V}_L(\mathrm{x}^{\delta}_0)^{-\frac{3d(p+4)}{r^L}})(1+\mathbf{1}_{\mathfrak{p}_{2L+5}>0}|\mathrm{x}^{\delta}_0|_{\mathbb{R}^d}^{C(d,L,p,\mathfrak{p}_{2L+5},\frac{1}{v},\frac{1}{\tilde{v}},\frac{1}{r},\frac{1}{1-12r})})\\&\quad\times\mathfrak{D}^{C(d,L,p,\frac{1}{v},\frac{1}{\tilde{v}},\frac{1}{r},\frac{1}{1-12r})}\mathfrak{D}_{2L+5}^{C(d,L,p,\frac{1}{v},\frac{1}{\tilde{v}},\frac{1}{r},\frac{1}{1-12r})}\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\frac{1}{v},\frac{1}{\tilde{v}},\frac{1}{r},\frac{1}{1-12r})}(Z^{\delta})\\&\quad\times C(d,N,L,\frac{1}{m_*},p,\mathfrak{p}_{2L+5},\frac{1}{v},\frac{1}{\tilde{v}},\frac{1}{r},\frac{1}{1-12r})\\&\quad\times\exp(C(d,L,p,\mathfrak{p}_{2L+5},\frac{1}{v},\frac{1}{\tilde{v}},\frac{1}{r},\frac{1}{1-12r})T\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\mathfrak{q}^{\delta}_{\eta_2},\frac{1}{v},\frac{1}{\tilde{v}},\frac{1}{r},\frac{1}{1-12r})}(Z^{\delta})\mathfrak{D}^4))\\&\quad+2C(d)(\exp(-\epsilon^{-\frac{v}{2}})+\exp(-\frac{\mathcal{V}_L(\mathrm{x}^{\delta}_0)}{32\epsilon^{\frac{rL}{3}}N\binom{N+L}{N}})+6\exp(-\frac{|\frac{10}{m_*}\epsilon|^{-\frac{1-12r}{22}}}{2^{11}(1+T^2)})).\end{aligned}$$

We fix $v=\frac{3-36r}{44}$ and $\tilde{v}=1$ and the proof of **Step 2** is completed.

Step 3. We now focus on the proof of (4.39). In particular, we show that for every $\epsilon\in\mathbb{R}^*$,

$$\begin{aligned}\mathbb{P}(\|\tilde{\sigma}^{\delta}_{X^{\delta}_t,\mathbf{T}}\|_{\mathbb{R}^d}>\frac{1}{6\epsilon})\leqslant&\epsilon^{d(p+2)}(|\mathrm{x}^{\delta}_0|_{\mathbb{R}^d}\mathbf{1}_{\mathfrak{p}_3>0}+\mathfrak{D}_3)^{C(\mathfrak{p}_3)}\\&\times\exp(C(d,p,\mathfrak{p}_3)(T+1)\mathfrak{M}_{C(d,p,\mathfrak{p},\mathfrak{p}_3,\mathfrak{q}^{\delta}_{\eta_2})}(Z^{\delta})\mathfrak{D}^4).\end{aligned}\tag{4.42}$$

First, we notice that, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|\tilde{\sigma}^{\delta}_{X^{\delta}_t,\mathbf{T}}\|_{\mathbb{R}^d}\leqslant&\|\sigma^{\delta}_{X^{\delta}_t,\mathbf{T}}\|_{\mathbb{R}^d}\|\mathring{X}_T\|^2_{\mathbb{R}^d}\\\leqslant&|X^{\delta}_t|^2_{\mathbb{R}^d,\delta,\mathbf{T},1,1}\|\mathring{X}_T\|^2_{\mathbb{R}^d}.\end{aligned}$$

As a consequence of the Markov inequality and again the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\mathbb{P}(\|\tilde{\sigma}^{\delta}_{X^{\delta}_t,\mathbf{T}}\|_{\mathbb{R}^d}>\frac{1}{6\epsilon},\Theta_{\eta_2,\mathbf{T}}>0)\\&\leqslant\epsilon^{d(p+2)}6^{d(p+2)}\|X^{\delta}_t\|^2_{\mathbb{R}^d,\delta,\mathbf{T},1,1,4d(p+2)}\mathbb{E}[\sup_{t\in\mathbf{T}}\|\mathring{X}_t\|^{4d(p+2)}_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t}>0}]^{\frac{1}{2}}.\end{aligned}$$

To conclude the proof of **Step 3**, we then apply Proposition 4.2 (see (4.25)) and Lemma 4.7 and obtain (4.42).

Step 4. In order to complete the proof of Theorem 4.3, it remains to show that (4.17) holds. Similarly as in **Step 1**, we have

$$\begin{aligned}\mathbb{P}(|\det\tilde{\gamma}^{\delta}_{X^{\delta}_T,\mathbf{T}}|\geqslant\frac{\eta_1}{2},\Theta_{X^{\delta}_T,\eta,\mathbf{T}}>0)\leqslant&\mathbb{P}(|\det\tilde{\sigma}^{\delta}_{X^{\delta}_T,\mathbf{T}}|\leqslant 2\eta_1^{-1},\Theta_{\eta_2,\mathbf{T}}>0)\\ \leqslant&\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^T\tilde{\sigma}^{\delta}_{X^{\delta}_T,\mathbf{T}}\xi\leqslant 2^{\frac{1}{d}}\eta_1^{-\frac{1}{d}},\Theta_{\eta_2,\mathbf{T}}>0).\end{aligned}$$

Using the result from **Step 2**, (see (4.40) with $\epsilon=2^{\frac{1}{d}}\eta_1^{-\frac{1}{d}}$ and $r=\frac{1}{13}$), for $p\geqslant 0$, we have

$$\begin{aligned}&\mathbb{P}(|\det\tilde{\gamma}^{\delta}_{X^{\delta}_T,\mathbf{T}}|\geqslant\frac{\eta_1}{2},\Theta_{\eta_2,\mathbf{T}}>0)\\ \leqslant&\eta_1^{-(p+4)}(1+\mathcal{V}_L(\mathrm{x}^{\delta}_0)^{-13^L6d(p+4)})(1+\mathbf{1}_{\mathfrak{p}_{2L+5}>0}|\mathrm{x}^{\delta}_0|^{C(d,L,p,\mathfrak{p}_{2L+5})}_{\mathbb{R}^d})\\ &\times\mathfrak{D}^{C(d,L,p)}\mathfrak{D}_{2L+5}^{C(d,L,p)}\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2L+5})}(Z^{\delta})\\ &\times C(d,N,L,\frac{1}{m_*},p,\mathfrak{p}_{2L+5})\exp(C(d,L,p,\mathfrak{p}_{2L+5})T\mathfrak{M}_{C(d,L,p,\mathfrak{p},\mathfrak{p}_{2L+5},\mathfrak{q}^{\delta}_{\eta_2})}(Z^{\delta})\mathfrak{D}^4)).\end{aligned}$$

To conclude the proof, we simply observe that

$$\begin{aligned}\mathbb{P}(\Theta_{X^{\delta}_T,\eta,\mathbf{T}}<1)\leqslant&\mathbb{P}(|\det\gamma^{\delta}_{X^{\delta}_t,\mathbf{T}}|\geqslant\eta_1-\frac{1}{2})+\sum_{t\in\mathbf{T}}\mathbb{P}(|Z^{\delta}_t|_{\mathbb{R}^N}\geqslant\eta_2-\frac{1}{2})\\ \leqslant&\mathbb{P}(|\det\gamma^{\delta}_{X^{\delta}_t,\mathbf{T}}|\geqslant\frac{\eta_1}{2},\Theta_{\eta_2,\mathbf{T}}>0)+\sum_{t\in\mathbf{T}}\mathbb{P}(|Z^{\delta}_t|_{\mathbb{R}^N}\geqslant\eta_2)+\sum_{t\in\mathbf{T}}\mathbb{P}(|Z^{\delta}_t|_{\mathbb{R}^N}\geqslant\frac{\eta_2}{2})\\ \leqslant&\mathbb{P}(|\det\gamma^{\delta}_{X^{\delta}_t,\mathbf{T}}|\geqslant\frac{\eta_1}{2},\Theta_{\eta_2,\mathbf{T}}>0)+2\sum_{t\in\mathbf{T}}\mathbb{P}(|Z^{\delta}_t|_{\mathbb{R}^N}\geqslant\frac{\eta_2}{2}).\end{aligned}$$

□

Appendix A. Proof of technical lemmas

A.1. Proof of Lemma 4.6.

Proof. First notice that, since $\epsilon\in(0,\sqrt{\frac{2}{d}})$, there exists $\{\xi_1,\ldots,\xi_{N(\epsilon)}\}$ with $\xi_i\in\mathbb{R}^d$, $N(\epsilon)\leqslant 7d^32^d\epsilon^{-2d}$ (see *e.g.* [32] Theorem 1.1 or [28] Theorem 2 for a refined constant) such that $\{\xi\in\mathbb{R}^d,|\xi|_{\mathbb{R}^d}=1\}\subset\cup_{i=1}^{N(\epsilon)}\{\xi\in\mathbb{R}^d,|\xi_i-\xi|_{\mathbb{R}^d}\leqslant\frac{\epsilon^2}{2}\}$. Moreover

$$\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^T\Sigma\xi\leqslant\frac{1}{2}\epsilon)=\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^T\Sigma\xi\leqslant\frac{1}{2}\epsilon,\|\Sigma\|_{\mathbb{R}^d}\leqslant\frac{1}{3\epsilon})+\mathbb{P}(\|\Sigma\|_{\mathbb{R}^d}>\frac{1}{3\epsilon}).$$

In particular for every $\xi\in\mathbb{R}^d,|\xi|_{\mathbb{R}^d}=1$,

$$\begin{aligned}\xi^T\Sigma\xi=&\xi_i^T\Sigma\xi_i+(\xi-\xi_i)^T(\Sigma\xi_i+\Sigma^T\xi)\\ \geqslant&\xi_i^T\Sigma\xi_i-2|\xi_i-\xi|_{\mathbb{R}^d}\|\Sigma\|_{\mathbb{R}^d}-|\xi_i-\xi|^2_{\mathbb{R}^d}\|\mathfrak{C}\|_{\mathbb{R}^d}.\end{aligned}$$

Therefore

$$\mathbb{P}(\inf_{\xi\in\mathbb{R}^d;|\xi|_{\mathbb{R}^d}=1}\xi^T\Sigma\xi\leqslant\frac{1}{2}\epsilon,\|\Sigma\|_{\mathbb{R}^d}\leqslant\frac{1}{3\epsilon})\leqslant\mathbb{P}(\cup_{i=1}^{N(\epsilon)}\xi_i^T\Sigma\xi_i\leqslant\epsilon)$$

and the proof of (4.32) is completed taking $C(d)=7d^32^d$. □

A.2. Proof of Lemma 4.7.

In this proof, we are going to use the Burkholder inequality (see (4.23)) on the Hilbert space $(\mathbb{R}^{d\times d},\langle,\rangle_{\mathbb{R}^{d\times d}})$, with the scalar product defined by $\langle M,M^{\diamond}\rangle_{\mathbb{R}^{d\times d}}:=\mathrm{Trace}(M^{\diamond}M^T)=\sum_{i=1}^d(M^{\diamond}M^T)_{i,i}$, $M,M^{\diamond}\in\mathbb{R}^{d\times d}$. Recall that for $M\in\mathbb{R}^{d\times d}$, $\|M\|_{\mathbb{R}^d}\leqslant|M|_{\mathbb{R}^{d\times d}}$.

Proof. **Step 1.** First we show that

$$\begin{aligned}\mathbb{E}[\sup_{t\in\mathbf{T}}|\mathring{X}_t|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t}>0}]^{\frac{1}{p}}\leqslant&d+\mathbb{E}[\sup_{t\in\mathbf{T}}|\sum_{w\in\pi^{\delta}\cap(0,t]}\hat{\Upsilon}_w|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}}\\ &+\mathbb{E}[\sup_{t\in\mathbf{T}}|\sum_{w\in\pi^{\delta}\cap(0,t]}\tilde{\Upsilon}_w|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}}.\end{aligned}$$

where we have introduced $\Upsilon_t=\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t}>0}\mathring{X}_{t-\delta}(I_{d\times d}-\nabla_x\psi^{-1})(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)$, $\hat{\Upsilon}_t=\mathbb{E}[\Upsilon_t|\mathcal{F}^{Z^\delta}_{t-\delta}]$ and $\tilde{\Upsilon}_t=\Upsilon_t-\hat{\Upsilon}_t$, $t\in\pi^{\delta,*}$. On the set $\{\Theta_{\eta_2,\mathbf{T},t}>0\}$, we have

$$\mathring{X}_t=I_{d\times d}-\sum_{w\in\pi^\delta\cap(0,t]}\mathring{X}_{w-\delta}(I_{d\times d}-\nabla_x\psi^{-1}(X^\delta_{w-\delta},w-\delta,\delta^{\frac{1}{2}}Z^\delta_w,\delta)).$$

Now, using the triangle inequality yields

$$\begin{aligned}\mathbb{E}[\sup_{t\in\mathbf{T}}|\mathring{X}_t|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T}}>0}]^{\frac{1}{p}}\leqslant&\sqrt{d}+\mathbb{E}[\sup_{t\in\mathbf{T}}|\sum_{w\in\pi^\delta\cap(0,t]}\Upsilon_w|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T}}>0}]^{\frac{1}{p}}\\ \leqslant&\sqrt{d}+\mathbb{E}[\sup_{t\in\mathbf{T}}|\sum_{w\in\pi^\delta\cap(0,t]}\Upsilon_w|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}},\end{aligned}$$

and, using the triangle inequality once again, the proof of **Step 1** is completed.

Step 2. Let us show that, for $t\in\mathbf{T}$,

$$|\hat{\Upsilon}_t|_{\mathbb{R}^{d\times d}}\leqslant\delta|\mathring{X}_{t-\delta}|_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}39\mathfrak{D}^2\mathfrak{M}_{\mathfrak{q}^\delta_{\eta_2}\vee(2\mathfrak{p}+2)}(Z^\delta)$$

We begin by noticing that, since $\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t}>0}=\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}$ (with $\mathbf{D}_{\eta_2}=\{z\in\mathbb{R}^N,|z^i|\leqslant\delta^{\frac{1}{2}}\eta_2,i\in\mathbf{N}\}$ introduced in the proof of Theorem 4.3), for every $t\in\pi^{\delta,*}$,

$$|\hat{\Upsilon}_t|_{\mathbb{R}^{d\times d}}=|\mathring{X}_{t-\delta}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}\mathbb{E}[I_{d\times d}-\nabla_x\psi^{-1}(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]|_{\mathbb{R}^{d\times d}}$$

Now we remark that,using the Neumann series, we have, on the set $\{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}\}$

$$\begin{aligned}|(\nabla_x\psi^{-1}-2I_{d\times d}+\nabla_x\psi)(X^\delta_{t-\delta},t-\delta,&\delta^{\frac{1}{2}}Z^\delta_t,\delta)|_{\mathbb{R}^{d\times d}}\\ &\leqslant\sum_{k=2}^\infty|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^k_{\mathbb{R}^{d\times d}}\end{aligned}$$

so that

$$\begin{aligned}|\hat{\Upsilon}_t|_{\mathbb{R}^{d\times d}}\leqslant&|\mathring{X}_{t-\delta}|_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}(|\mathbb{E}[(I_{d\times d}-\nabla_x\psi)(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]|_{\mathbb{R}^{d\times d}}\\ &+\mathbb{E}[\sum_{k=2}^\infty|I_{d\times d}-\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)|^k_{\mathbb{R}^{d\times d}}\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}].)\end{aligned}$$

On the one hand, using the Taylor expansion of $\nabla_x\psi$,

$$\begin{aligned}\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)=&I_{d\times d}+\delta^{\frac{1}{2}}\sum_{i=1}^N Z^{\delta,i}_t\nabla_xV_i(X^\delta_{t-\delta},t-\delta)\\ &+\delta\int_0^1\partial_y\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\lambda\delta)\mathrm{d}\lambda\\ &+\delta\sum_{i,l=1}^N Z^{\delta,i}_tZ^{\delta,l}_t\int_0^1(1-\lambda)\partial_{z^i}\partial_{z^l}\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\lambda\delta^{\frac{1}{2}}Z^\delta_t,0)\mathrm{d}\lambda.\end{aligned}$$

Now, we remakr that

$$\mathbb{E}[\delta^{\frac{1}{2}}\sum_{l=1}^N Z^{\delta,l}_t\nabla_xV_l(X^\delta_{t-\delta},t-\delta)(\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}+\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\notin\mathbf{D}_{\eta_2}})|\mathcal{F}^{Z^\delta}_{t-\delta}]=0.$$

The Markov inequality, combined with (2.3) implies that,

$$\mathbb{E}[\delta^{\frac{1}{2}}|\sum_{l=1}^N Z^{\delta,l}_t\nabla_xV_l(X^\delta_{t-\delta},t-\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\notin\mathbf{D}_{\eta_2}}|_{\mathbb{R}^{d\times d}}\ |\mathcal{F}^{Z^\delta}_{t-\delta}]\leqslant\delta\mathfrak{D}\mathbb{E}[|Z^\delta_t|^{\mathfrak{q}^\delta_{\eta_2}}_{\mathbb{R}^N}].$$

In particular

$$\begin{aligned}|\mathbb{E}[(I_{d\times d}-\nabla_x\psi)(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}|\mathcal{F}^{Z^\delta}_{t-\delta}]|_{\mathbb{R}^{d\times d}}&\\ \leqslant\delta\mathfrak{D}\mathbb{E}[|Z^\delta_t|^{\mathfrak{q}^\delta_{\eta_2}}_{\mathbb{R}^N}]+\delta6\mathfrak{D}\mathbb{E}[1+|Z^\delta_t|^{\mathfrak{p}+2}_{\mathbb{R}^N}].&\end{aligned}$$

On the other hand, using (4.29), for every $k \in \mathbb{N}, k \geqslant 2$, we have

$$\mathbb{E}[|I_{d\times d} - \nabla_x \psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, \delta)|^k_{\mathbb{R}^{d\times d}} \mathbf{1}_{\delta^{\frac{1}{2}} Z^\delta_t \in \mathbf{D}_{\eta_2}} |\mathcal{F}^{Z^\delta}_{t-\delta}] \\ \leqslant \delta^{\frac{k}{2}} \eta_2^{(k-2)(\mathfrak{p}+1)} 4^k \mathfrak{D}^k \mathbb{E}[\max(|Z^\delta_t|^{2(\mathfrak{p}+1)}_{\mathbb{R}^N}, 1)].$$

Since $\delta^{\frac{1}{2}} \eta_2^{\mathfrak{p}+1} 4\mathfrak{D} < \frac{1}{2}$, the geometric series converge and

$$\mathbb{E}[\sum_{k=2}^{\infty} |I_{d\times d} - \nabla_x \psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, \delta)|^k_{\mathbb{R}^{d\times d}} \mathbf{1}_{\delta^{\frac{1}{2}} Z^\delta_t \in \mathbf{D}_{\eta_2}} |\mathcal{F}^{Z^\delta}_{t-\delta}] \\ \leqslant \delta 32 \mathfrak{D}^2 \mathfrak{M}_{2(\mathfrak{p}+1)}(Z^\delta).$$

We gather all the terms together and the proof of **Step 2** is completed.
Step 3. Let us show that

$$\mathbb{E}[|\tilde{\Upsilon}_t|^p_{\mathbb{R}^{d\times d}} \mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}]^{\frac{1}{p}} \leqslant \delta^{\frac{1}{2}} \mathbb{E}[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}} 101 \mathfrak{D}^2 \mathfrak{M}_{p(\mathfrak{q}^\delta_{\eta_2} \vee (2\mathfrak{p}+2))}(Z^\delta)^{\frac{1}{p}}).$$

First, we remark that

$$|\tilde{\Upsilon}_t|_{\mathbb{R}^{d\times d}} \leqslant |\Upsilon_t|_{\mathbb{R}^{d\times d}} + |\hat{\Upsilon}_t|_{\mathbb{R}^{d\times d}}.$$

We have already studied the second term of the *r.h.s.* in **Step 2** so we focus on the first one. Proceeding similarly as in **Step 2**, we have

$$|\Upsilon_t|_{\mathbb{R}^{d\times d}} \leqslant |\mathring{X}_{t-\delta}|_{\mathbb{R}^{d\times d}} \mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0} (|(I_{d\times d} - \nabla_x \psi)(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, \delta) \mathbf{1}_{\delta^{\frac{1}{2}} Z^\delta_t \in \mathbf{D}_{\eta_2}}|_{\mathbb{R}^{d\times d}} \\ + \sum_{k=2}^{\infty} |I_{d\times d} - \nabla_x \psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, \delta)|^k_{\mathbb{R}^{d\times d}} \mathbf{1}_{\delta^{\frac{1}{2}} Z^\delta_t \in \mathbf{D}_{\eta_2}}).$$

Using (4.29), it follows that

$$\mathbb{E}[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}} \mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0} |I_{d\times d} - \nabla_x \psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, \delta)|^p_{\mathbb{R}^{d\times d}} \mathbf{1}_{\delta^{\frac{1}{2}} Z^\delta_t \in \mathbf{D}_{\eta_2}}] \\ \leqslant \delta^{\frac{p}{2}} \mathbb{E}[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}} \mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}] \mathfrak{D}^p 4^p 2 \mathfrak{M}_{p(\mathfrak{p}+1)}(Z^\delta).$$

Moreover, since $\delta^{\frac{1}{2}} \eta_2^{\mathfrak{p}+1} 4\mathfrak{D} < \frac{1}{2}$, on the space $\{\delta^{\frac{1}{2}} Z^\delta_t \in \mathbf{D}_{\eta_2}\}$, we have

$$\sum_{k=2}^{\infty} |I_{d\times d} - \nabla_x \psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, \delta)|^k_{\mathbb{R}^{d\times d}} \leqslant \delta 32 \mathfrak{D}^2 (1 \vee |Z^\delta_t|^{2(\mathfrak{p}+1)}_{\mathbb{R}^N})$$

and

$$\mathbb{E}[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}} \mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0} |\sum_{k=2}^{\infty} |I_{d\times d} - \nabla_x \psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac{1}{2}} Z^\delta_t, \delta)|^k_{\mathbb{R}^{d\times d}}|^p \mathbf{1}_{\delta^{\frac{1}{2}} Z^\delta_t \in \mathbf{D}_{\eta_2}}] \\ \leqslant \delta^p \mathbb{E}[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}} \mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}] 32^p \mathfrak{D}^{2p} 2 \mathfrak{M}_{2p(\mathfrak{p}+1)}(Z^\delta).$$

Gathering all the terms concludes the proof of **Step 3**.
Step 4. We are now in a position to conclude the proof. First, employing the Burkholder inequality (4.23), we have for every $p \geqslant 2$,

$$\mathbb{E}[\sup_{t\in\mathbf{T}} |\sum_{w\in\pi^\delta\cap(0,t]} \tilde{\Upsilon}_t|^p_{\mathbb{R}^{d\times d}}] \leqslant \mathfrak{b}_p \mathbb{E}[(\sum_{t\in\mathbf{T}} |\tilde{\Upsilon}_t|^2_{\mathbb{R}^{d\times d}})^{\frac{p}{2}}] \\ \leqslant \mathfrak{b}_p (\sum_{t\in\mathbf{T}} \mathbb{E}[|\tilde{\Upsilon}_t|^p_{\mathbb{R}^{d\times d}}]^{\frac{2}{p}})^{\frac{p}{2}}.$$

We deduce from **Step 1,2,3** that

$$\begin{aligned}
&\mathbb{E}[\sup_{t\in\mathbf{T}\cup\{0\}}|\mathring{X}_t|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t}>0}]^{\frac{1}{p}}\\
&\quad\leqslant d+\mathbb{E}[\sup_{t\in\mathbf{T}}|\sum_{w\in\pi^\delta\cap(0,t]}\hat{\Upsilon}_w|^p_{\mathbb{R}^{d\times d}}]^{\frac{1}{p}}+\mathfrak{b}_p(\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Upsilon}_t|^p_{\mathbb{R}^{d\times d}}]^{\frac{2}{p}})^{\frac{1}{2}}\\
&\quad\leqslant d+39\mathfrak{D}^2\mathfrak{M}_{\mathfrak{q}^\delta_{\eta_2}\vee(2\mathfrak{p}+2)}(Z^\delta)\mathbb{E}[|\sum_{t\in\mathbf{T}}\delta|\mathring{X}_{t-\delta}|_{\mathbb{R}^{d\times d}}|^p\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}]^{\frac{1}{p}}\\
&\qquad+\mathfrak{b}_p101\mathfrak{D}^2\mathfrak{M}_{p(\mathfrak{q}^\delta_{\eta_2}\vee(2\mathfrak{p}+2))}(Z^\delta)^{\frac{1}{p}}(\sum_{t\in\mathbf{T}}\delta\mathbb{E}[|\mathring{X}_{t-\delta}|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}]^{\frac{2}{p}})^{\frac{1}{2}}\\
&\quad\leqslant d+\mathfrak{b}_p140\mathfrak{D}^2\mathfrak{M}_{p(\mathfrak{q}^\delta_{\eta_2}\vee(2\mathfrak{p}+2))}(Z^\delta)^{\frac{1}{p}}(\sum_{t\in\mathbf{T}}\delta\mathbb{E}[\sup_{w\in\mathbf{T}\cup\{0\},w<t}|\mathring{X}_w|^p_{\mathbb{R}^{d\times d}}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t-\delta}>0}]^{\frac{2}{p}})^{\frac{1}{2}}.
\end{aligned}$$

Therefore, as a consequence of the Gronwall lemma,

$$\mathbb{E}[\sup_{t\in\mathbf{T}}\|\mathring{X}_t\|^p_{\mathbb{R}^d}\mathbf{1}_{\Theta_{\eta_2,\mathbf{T},t}>0}]^{\frac{1}{p}}\leqslant\sqrt{2}d\exp(\mathfrak{b}_p^2140^2\mathfrak{D}^4T\mathfrak{M}_{p(\mathfrak{q}^\delta_{\eta_2}\vee(2\mathfrak{p}+2))}(Z^\delta)^{\frac{2}{p}}),$$

with $\mathfrak{b}_p$ defined in (4.23) and the proof of (4.33) is completed.

□

A.3. **Proof of Lemma 4.8.**

Proof. **Step 1.** Let us show that for every $t\in\pi^{\delta,*}$,

$$\begin{aligned}
V(X^\delta_t,t)-V(X^\delta_{t-\delta},t-\delta)=&\,\delta^{\frac{1}{2}}\sum_{l=1}^N Z^{\delta,i}_t\nabla_x V(X^\delta_{t-\delta},t-\delta)V_l(X^\delta_{t-\delta},t-\delta)\\
&+\delta\nabla_x V(X^\delta_{t-\delta},t-\delta)V_0(X^\delta_{t-\delta},t-\delta)+\delta\partial_t V(X^\delta_{t-\delta},t-\delta)\\
&+\delta\frac{1}{2}\sum_{l=1}^N V_l(X^\delta_{t-\delta},t-\delta)^T\mathbf{H}_x V(X^\delta_{t-\delta},t-\delta)V_l(X^\delta_{t-\delta},t-\delta)\\
&+R^{\delta,1}(X^\delta_{t-\delta},t-\delta,Z^\delta_t),
\end{aligned}$$

with for every $(x,t,z)\in\mathbb{R}^d\times\pi^\delta\times\mathbb{R}^N$,

$$\begin{aligned}
R^{\delta,1}(x,t,z)=&R^{\delta,1,3}(x,t,z)+\nabla_x V(x,t)R^{\delta,1,2}(x,t,z)\\
&+\frac{1}{2}\delta\sum_{i,l=1}^N(z^iz^l-\mathbf{1}_{i=l})V_i(x,t)^T\mathbf{H}_xV(x,t)V_l(x,t)\\
&+2\delta^{\frac{1}{2}}\sum_{l=1}^N z^lV_l(x,t)^T\mathbf{H}_xV(x,t)R^{\delta,1,1}(x,t,z)\\
&+R^{\delta,1,1}(x,t,z)^T\mathbf{H}_xV(x,t)R^{\delta,1,1}(x,t,z)
\end{aligned}$$

where

$$R^{\delta,1,1}(x,t,z)=\delta\int_0^1\partial_y\psi(x,t,\delta^{\frac{1}{2}}z,\lambda\delta)\mathrm{d}\lambda+\delta\sum_{i,l=1}^N z^iz^l\int_0^1(1-\lambda)\partial_{z^i}\partial_{z^l}\psi(x,t,\lambda\delta^{\frac{1}{2}}z,0)\mathrm{d}\lambda,$$

$$\begin{aligned}
R^{\delta,1,2}(x,t,z)=&\delta\frac{1}{2}\sum_{i,l=1}^N(z^iz^l-\mathbf{1}_{i=l})\partial_{z^i}\partial_{z^l}\psi(x,t,0,0)+\delta^2\int_0^1(1-\lambda)\partial_y^2\psi(x,t,\delta^{\frac{1}{2}}z,\lambda\delta)\mathrm{d}\lambda\\
&+\delta^{\frac{3}{2}}\sum_{l=1}^N z^l\int_0^1\partial_{z^l}\partial_y\psi(x,t,\lambda\delta^{\frac{1}{2}}z,0)\mathrm{d}\lambda\\
&+\delta^{\frac{3}{2}}\frac{1}{2}\sum_{i,j,l=1}^N z^iz^lz^l\int_0^1(1-\lambda)^2\partial_{z^i}\partial_{z^j}\partial_{z^l}\psi(x,t,\lambda\delta^{\frac{1}{2}}z,0)\mathrm{d}\lambda,
\end{aligned}$$

and

$$
\begin{aligned}
R^{\delta,1,3}(x,t,z) = \delta^2 \int_0^1 \partial_t^2 V(x,t+\lambda\delta)\mathrm{d}\lambda, \\
+ \sum_{i=1}^d \int_0^1 \partial_{x^i}\mathcal{T}V(x+\lambda R^{\delta,1,0}(x,t,z),t)\mathrm{d}\lambda R^{\delta,1,0}(x,t,z)_i \\
+ \frac{1}{2}\sum_{i,j,k=1}^d R^{\delta,1,0}(x,t,z,y)_{i\otimes j\otimes k}\int_0^1 (1-\lambda)^2\partial_{x^i}\partial_{x^j}\partial_{x_k}V(x+\lambda R^{\delta,1,0}(x,t,z),t)\mathrm{d}\lambda
\end{aligned}
$$

with

$$
R^{\delta,1,0}(x,t,z) = \delta\int_0^1 \partial_y\psi(x,t,z,\lambda\delta)\mathrm{d}\lambda + \delta^{\frac{1}{2}}\sum_{i=1}^N z^i\int_0^1 (1-\lambda)\partial_{z^i}\psi(x,t,\lambda z,0)\mathrm{d}\lambda,
$$

and

$$
\mathcal{T}V(x,t) := \delta\int_0^1 \partial_t V(x,t+\lambda\delta)\mathrm{d}\lambda = \delta\partial_t V(x,t) + \delta^2\int_0^1 \partial_t^2 V(x,t+\lambda\delta)\mathrm{d}\lambda.
$$

We begin by noticing that, using the Taylor expansion of ψ with respect to its third and fourth variables, we have

$$
\begin{aligned}
\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac{1}{2}}Z^\delta_t,\delta) =& X^\delta_{t-\delta} + R^{\delta,1,0}(X^\delta_{t-\delta},t-\delta,Z^\delta_t) \\
=& X^\delta_{t-\delta} + \delta^{\frac{1}{2}}\sum_{l=1}^N Z^{\delta,l}_t V_l(X^\delta_{t-\delta},t-\delta) + R^{\delta,1,1}(X^\delta_{t-\delta},t-\delta,Z^\delta_t), \\
=& X^\delta_{t-\delta} + \delta^{\frac{1}{2}}\sum_{l=1}^N Z^{\delta,l}_t V_l(X^\delta_{t-\delta},t-\delta) + \delta V_0(X^\delta_{t-\delta},t-\delta) + R^{\delta,1,2}(X^\delta_{t-\delta},Z^\delta_t).
\end{aligned}
$$

Now, using again the Taylor expansion on the function V *w.r.t.* its second variable,

$$
\begin{aligned}
V(X^\delta_t,t) - V(X^\delta_{t-\delta},t-\delta) =& \mathcal{T}V(X^\delta_{t-\delta},t-\delta) \\
&+ (\mathcal{T}V+V)(X^\delta_t,t-\delta) - (\mathcal{T}V+V)(X^\delta_{t-\delta},t-\delta).
\end{aligned}
$$

The Taylor expansion on the function $\mathcal{T}V$ *w.r.t* its first variable yields

$$
\begin{aligned}
\mathcal{T}V(X^\delta_t,t-\delta) =& \mathcal{T}V(X^\delta_{t-\delta},t-\delta) \\
&+ \sum_{i=1}^d R^{\delta,1,0}(X^\delta_t,t-\delta,z)_i\int_0^1 \partial_{x^i}\mathcal{T}V(X^\delta_t+\lambda R^{\delta,1,0}(X^\delta_t,t-\delta,z),t)\mathrm{d}\lambda.
\end{aligned}
$$

Finally, from the the Taylor expansion on the function V *w.r.t.* its first variable, we have also

$$
\begin{aligned}
V(X^\delta_t,t-\delta) =& V(X^\delta_{t-\delta},t-\delta) \\
&+ \nabla_x V(X^\delta_{t-\delta},t-\delta)(X^\delta_t - X^\delta_{t-\delta}) \\
&+ \frac{1}{2}(X^\delta_t - X^\delta_{t-\delta})^T\mathbf{H}_x V(X^\delta_{t-\delta},t-\delta)(X^\delta_t - X^\delta_{t-\delta}) \\
&+ \frac{1}{2}\sum_{i,j,k=1}^d R^{\delta,1,0}(X^\delta_{t-\delta},t-\delta,Z^\delta_t)_{i\otimes j\otimes k} \\
&\times\int_0^1 (1-\lambda)^2\partial_{x^i}\partial_{x^j}\partial_{x_k}V(X^\delta_{t-\delta}+\lambda R^{\delta,1,0}(X^\delta_{t-\delta},t-\delta,Z^\delta_t,\delta))\mathrm{d}\lambda,
\end{aligned}
$$

and gathering the terms completes the proof of **Step 1**.

Step 2. Let us show that for every $t\in\pi^{\delta,*}$, on the set $\{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}\}$ (with $\mathbf{D}_{\eta_2} = \{z\in\mathbb{R}^N, |z^i|\leqslant \delta^{\frac{1}{2}}\eta_2, i\in\mathbf{N}\}$ introduced in the proof of Theorem 4.3), we have

$$
\begin{aligned}
\nabla_x \psi^{-1}(X^\delta_{t-\delta}, t-\delta, \delta^{\frac12} Z^\delta_t, \delta) = I_{d\times d} &- \delta^{\frac12} \sum_{i=1}^N Z^{\delta,i}_t \nabla_x V_l(X^\delta_{t-\delta}, t-\delta) \\
&- \delta \left(\nabla_x V_0(X^\delta_{t-\delta}, t-\delta) - \sum_{i=1}^N \nabla_x V_i(X^\delta_{t-\delta}, t-\delta)^2 \right) \\
&+ \mathcal{R}^{\delta,2}(X^\delta_{t-\delta}, t-\delta, Z^\delta_t),
\end{aligned}
$$

with, for every $(x,t,z) \in \mathbb{R}^d \times \pi^\delta \times \mathbb{R}^N$,

$$
\begin{aligned}
\mathcal{R}^{\delta,2}(x,t,z) = \mathcal{R}^{\delta,2,3}(x,t,z) - \mathcal{R}^{\delta,2,2}(x,t,z) + \delta \sum_{i,l=1}^N (z^i z^l - \mathbf{1}_{i=l}) \nabla_x V_i(x,t) \nabla_x V_l(x,t) \\
- \delta^{\frac12} \sum_{l=1}^N z^l (\nabla_x V_l(x,t) R^{\delta,2,1}(x,t,z) + R^{\delta,2,1}(x,t,z) \nabla_x V_l(x,t)) - \mathcal{R}^{\delta,2,1}(x,t,z)^2
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{R}^{\delta,2,1}(x,t,z) =& \delta \int_0^1 \nabla_x \partial_y \psi(x,t,\delta^{\frac12} z, \lambda\delta) \mathrm{d}\lambda \\
&+ \delta \sum_{i,l=1}^N z^i z^l \int_0^1 (1-\lambda) \nabla_x \partial_{z^i} \partial_{z^l} \psi(x,t,\lambda\delta^{\frac12} z, 0) \mathrm{d}\lambda
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{R}^{\delta,2,2}(x,t,z) =& \delta^2 \int_0^1 (1-\lambda) \nabla_x \partial_y^2 \psi(x,t,\delta^{\frac12} z, \lambda\delta) \mathrm{d}\lambda \\
&+ \delta \frac12 \sum_{i,l=1}^N (z^i z^l - \mathbf{1}_{i=l}) \nabla_x \partial_{z^i} \partial_{z^l} \psi(x,t,0,0) \\
&+ \delta^{\frac32} \frac12 \sum_{i,j,l=1}^N (z^i z^j z^l \int_0^1 (1-\lambda)^2 \nabla_x \partial_{z^i} \partial_{z^j} \partial_{z^l} \psi(x,t,\lambda\delta^{\frac12} z, 0) \mathrm{d}\lambda \\
&+ \delta^{\frac32} \sum_{l=1}^N z^l \int_0^1 \nabla_x \partial_{z^l} \partial_y \psi(x,t,\lambda\delta^{\frac12} z, 0) \mathrm{d}\lambda
\end{aligned}
$$

and

$$
\mathcal{R}^{\delta,2,3}(x,t,z) = (\nabla_x \psi^{-1} - I_{d\times d} - (I_{d\times d} - \nabla_x \psi) - (I_{d\times d} - \nabla_x \psi)^2)(x,t,\delta^{\frac12} z, \delta)
$$

where for a matrix $M \in \mathbb{R}^{d\times d}$, $M^2 = MM$. The proof simply boils down to notice that we have both

$$
\nabla_x \psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac12} Z^\delta_t, \delta) = I_{d\times d} + \delta^{\frac12} \sum_{l=1}^N Z^{\delta,l}_t \nabla_x V_l(X^\delta_{t-\delta}, t-\delta) + \mathcal{R}^{\delta,2,1}(X^\delta_{t-\delta}, t-\delta, Z^\delta_t)
$$

and

$$
\begin{aligned}
\nabla_x \psi(X^\delta_{t-\delta}, t-\delta, \delta^{\frac12} Z^\delta_t, \delta) = I_{d\times d} + \delta^{\frac12} \sum_{l=1}^N Z^{\delta,l}_t \nabla_x V_l(X^\delta_{t-\delta}, t-\delta) + \delta \nabla_x V_0(X^\delta_{t-\delta}, t-\delta) \\
+ \mathcal{R}^{\delta,2,2}(X^\delta_{t-\delta}, t-\delta, Z^\delta_t).
\end{aligned}
$$

We gather all the terms together and the proof of **Step 2** is completed.

Step 3. Let us show that for every $t\in\pi^{\delta,*}$, on the set $\{\Theta_{\eta_2,\mathbf{T},t}>0\}$, we have

$$\begin{aligned}\mathring{X}_t V(X^\delta_t,t) =& \mathring{X}_{t-\delta} V(X^\delta_{t-\delta},t-\delta)\\ &+\delta^{\frac12}\sum_{i=1}^N Z^{\delta,i}_t \mathring{X}_{t-\delta} V^{[i]}(X^\delta_{t-\delta},t-\delta)+\delta\mathring{X}_{t-\delta}V^{[0]}(X^\delta_{t-\delta},t-\delta))\\ &+\mathring{X}_{t-\delta}R^\delta V(X^\delta_{t-\delta},t-\delta,Z^\delta_t)\end{aligned}$$

with, for every $(x,t,z)\in\mathbb{R}^d\times\pi^\delta\times\mathbb{R}^N$,

$$R^\delta V(x,t,z)=R^{\delta,1}(x,t,z)+R^{\delta,2}(x,t,z)+R^{\delta,3}(x,t,z),$$

with $R^{\delta,2}(x,t,z)=\mathcal{R}^{\delta,2}(x,t,z)V(x,t)$ and

$$\begin{aligned}R^{\delta,3}(x,t,z)=&-\delta\sum_{i,l=1}^N(z^iz^l-\mathbf{1}_{i=l})\nabla_xV_i(x,t)\nabla_xV(x,t)V_l(x,t)\\ &+(-\delta(\nabla_xV_0(x,t)-\sum_{l=1}^N(\nabla_xV_l(x,t))^2)+\mathcal{R}^{\delta,2}(x,t,z))\\ &\times(\delta^{\frac12}\sum_{i=1}^N z^i\nabla_xV(x,t)V_l(x,t)+\delta\nabla_xV(x,t)V_0(x,t)+\delta\partial_tV(x,t)\\ &+\delta\frac12\sum_{i=1}^N V_i(x,t)^T\mathbf{H}_xV(x,t)V_i(x,t)+R^{\delta,1}(x,t,z))\\ &-(\delta^{\frac12}\sum_{i=1}^N z^i\nabla_xV_i(x,t))\\ &\times(\delta\nabla_xV(x,t)V_0(x,t)+\delta\partial_tV(x,t,t)+\delta\frac12\sum_{i=1}^N V_i(x,t)^T\mathbf{H}_xV(x,t)V_i(x,t)+R^{\delta,1}(x,t,z)).\end{aligned}$$

First, we write

$$\begin{aligned}&\mathring{X}_tV(X^\delta_t,t)-\mathring{X}_{t-\delta}V(X^\delta_{t-\delta},t-\delta)\\ =&\mathring{X}_{t-\delta}\nabla_x\psi^{-1}(X^\delta_{t-\delta},t-\delta,\delta^{\frac12}Z^\delta_t,\delta)\left(V(X^\delta_t,t)-V(X^\delta_{t-\delta},t-\delta)\right)\\ &+\mathring{X}_{t-\delta}\left(\nabla_x\psi(X^\delta_{t-\delta},t-\delta,\delta^{\frac12}Z^\delta_t,\delta)^{-1}-I_{d\times d}\right)V(X^\delta_{t-\delta},t-\delta),\end{aligned}$$

Using **Step 1** and **Step 2**,

$$\begin{aligned}&\nabla_x\psi^{-1}(X^\delta_{t-\delta},t-\delta,Z^\delta_t,\delta)(V(X^\delta_t,t)-V(X^\delta_{t-\delta},t-\delta))\\ =&\delta^{\frac12}\sum_{l=1}^N Z^{\delta,l}_t\nabla_xV(X^\delta_{t-\delta},t-\delta)V_l(X^\delta_{t-\delta},t-\delta)\\ &+\delta\nabla_xV(X^\delta_{t-\delta},t-\delta)V_0(X^\delta_{t-\delta},t-\delta)\\ &+\delta\partial_tV(X^\delta_{t-\delta},t-\delta)\\ &+\delta\frac12\sum_{l=1}^N V_l(X^\delta_{t-\delta},t-\delta)^T\mathbf{H}_xV(X^\delta_{t-\delta},t-\delta)V_l(X^\delta_{t-\delta},t-\delta)\\ &-\delta\sum_{l=1}^N\nabla_xV_l(X^\delta_{t-\delta},t-\delta)\nabla_xV(X^\delta_{t-\delta},t-\delta)V_l(X^\delta_{t-\delta},t-\delta)\\ &+R^{\delta,3}(X^\delta_{t-\delta},t-\delta,Z^\delta_t)+R^{\delta,1}(X^\delta_{t-\delta},t-\delta,Z^\delta_t).\end{aligned}$$

The study of the other term was done in **Step 2** and the proof of **Step 3** is completed.

Step 4. Let us prove (4.35) and (4.34). In the sequel, for $i\in\{1,2,3\}$, $t\in\pi^{\delta,*}$, we introduce the functions defined for every $x\in\mathbb{R}^d$ by $\overline{R}^i_t(x)=\mathbb{E}[R^{\delta,i}(x,t-\delta,Z^\delta_t)\mathbf{1}_{\delta^{\frac12}Z^\delta_t\in\mathbf{D}_{\eta_2}}]$ and for $i\in\{1,2\}$, $j\in\{1,2,3\}$, $\overline{R}^{i,j}_t(x)=\mathbb{E}[R^{\delta,i,j}(x,t-\delta,Z^\delta_t)\mathbf{1}_{\delta^{\frac12}Z^\delta_t\in\mathbf{D}_{\eta_2}}]$ (with the notation $R^{\delta,2,j}=\mathcal{R}^{\delta,2,j}V$). In particular, since $\{\Theta_{\eta_2,\mathbf{T},t}>0\}=\{\Theta_{\eta_2,\mathbf{T},t-\delta}>0\}\cap\{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}\}$, then $\overline{\mathbf{R}}^\delta V(x,t-\delta)=\sum_{i=1}^5\overline{R}^i_t(x)=\mathbb{E}[R^\delta(x,t-\delta,Z^\delta_t)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}]+\overline{R}^4_t(x)+\overline{R}^5_t(x)$ with

$$\begin{aligned}\overline{R}^4_t(x)=&-\delta^{\frac{1}{2}}\sum_{i=1}^N V^{[i]}(x,t-\delta)\mathbb{E}[Z^{\delta,i}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\notin\mathbf{D}_{\eta_2}}],\\ \overline{R}^5_t(x)=&-\delta V^{[0]}(x,t-\delta)\mathbb{P}(\delta^{\frac{1}{2}}Z^\delta_t\notin\mathbf{D}_{\eta_2}).\end{aligned}$$

We first study $\partial_x^{\alpha^x}\overline{R}^1_t$ for $\alpha^x\in\mathbb{N}^d$.

We observe that, for every $t\in\pi^{\delta,*}$,

$$\begin{aligned}\sum_{i,l=1}^N&(Z^{\delta,i}_tZ^{\delta,l}_t-\mathbf{1}_{i=l})\partial_{z^i}\partial_{z^l}\psi(x,t-\delta,0,0)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}\\ &=\sum_{i,l=1}^N(Z^{\delta,i}_tZ^{\delta,l}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}-\mathbb{E}[Z^{\delta,i}_tZ^{\delta,l}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}])\partial_{z^i}\partial_{z^l}\psi(x,t,0,0)\\ &\quad-\sum_{i,l=1}^N\mathbb{E}[Z^{\delta,i}_tZ^{\delta,l}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\notin\mathbf{D}_{\eta_2}}]\partial_{z^i}\partial_{z^l}\psi(x,t,0,0),\end{aligned}$$

with $|\sum_{i,l=1}^N\mathbb{E}[Z^{\delta,i}_tZ^{\delta,l}_t\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\notin\mathbf{D}_{\eta_2}}]|\leqslant\eta_2^{-q}\mathbb{E}[|Z^\delta_t|^{2+q}_{\mathbb{R}^N}]$, for every $q>0$. In paritcular we take $q=\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil$ (recall that we have necessarily $\frac{3\ln(\delta)}{2\ln(\eta_2)}<0$). Using standard calculus together with hypothesis $\mathbf{A}_1(|\alpha^x|+3)$ (see (2.2)) and $\mathbf{A}^\delta_3(\max(\mathfrak{p}_{|\alpha^x|+3}+3,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2))$ (see (2.7)), we obtain, for every $x\in\mathbb{R}^d$,

$$|\partial_x^{\alpha^x}\overline{R}^{1,2}_t(x)|_{\mathbb{R}^d}\leqslant\delta^{\frac{3}{2}}C\mathfrak{M}_{\max(\mathfrak{p}_{|\alpha^x|+3}+3,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)\mathfrak{D}_{|\alpha^x|+3}(1+|x|^{\mathfrak{p}_{|\alpha^x|+3}}_{\mathbb{R}^d})$$

By similar arguments, it follows from $\mathbf{A}_1(|\alpha^x|+2)$ (see (2.2)), that

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}^{1,3}_t(x)|_{\mathbb{R}^d}\leqslant&\delta^{\frac{3}{2}}C(|\alpha^x|)\mathfrak{M}_{\max(\mathfrak{p}_{|\alpha^x|+2}(|\alpha^x|+3),\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)\\ &\times\mathfrak{D}^{|\alpha^x|+3}_{|\alpha^x|+2}\mathfrak{D}_{V,|\alpha^x|+3}(1+|x|^{\mathfrak{p}_{|\alpha^x|+2}(|\alpha^x|+3)+\mathfrak{p}_{V,|\alpha^x|+3}}_{\mathbb{R}}).\end{aligned}$$

At this point, we remark that

$$\overline{R}^1_t=\overline{R}^{1,3}_t+\nabla_xV\overline{R}^{1,2}_t+\overline{R}^{1,4}_t,$$

with, for every $x\in\mathbb{R}^d$ and $t\in\pi^{\delta,*}$,

$$\begin{aligned}\overline{R}^{1,4}_t(x)=&\mathbb{E}[2\delta^{\frac{1}{2}}\sum_{l=1}^NZ^{\delta,l}_tV_l(x,t-\delta)^T\mathbf{H}_xV(x,t-\delta)R^{\delta,1,1}(x,t-\delta,Z^\delta_t)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}]\\ &+\mathbb{E}[R^{\delta,1,1}(x,t-\delta,Z^\delta_t)^T\mathbf{H}_xV(x,t-\delta)R^{\delta,1,1}(x,t-\delta,Z^\delta_t)\mathbf{1}_{\delta^{\frac{1}{2}}Z^\delta_t\in\mathbf{D}_{\eta_2}}],\end{aligned}$$

which satisfies, using hypothesis $\mathbf{A}_1(|\alpha^x|+2)$ (see (2.2)) and $\mathbf{A}^\delta_3(2\mathfrak{p}_{|\alpha^x|+2}+4)$ (see (2.7)),

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}^{1,4}_t(x)|_{\mathbb{R}^d}\leqslant&\delta^{\frac{3}{2}}C(|\alpha^x|)\mathfrak{M}_{2\mathfrak{p}_{|\alpha^x|+2}+4}(Z^\delta)\\ &\times\mathfrak{D}^2_{|\alpha^x|+2}\mathfrak{D}_{V,|\alpha^x|+2}(1+|x|^{2\mathfrak{p}_{|\alpha^x|+2}+\mathfrak{p}_{V,|\alpha^x|+2}}_{\mathbb{R}}).\end{aligned}$$

We conclude that, under the assumptions $\mathbf{A}_1(|\alpha^x|+3)$ (see (2.2)) and $\mathbf{A}^\delta_3(\max(\mathfrak{p}_{|\alpha^x|+3}+3,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2))$ (see (2.7)), then, for every $x\in\mathbb{R}^d$,

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}^1_t(x)|_{\mathbb{R}^d}\leqslant&\delta^{\frac{3}{2}}C(|\alpha^x|)\mathfrak{M}_{\max(\mathfrak{p}_{|\alpha^x|+3}(|\alpha^x|+3)+4,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)\\ &\times\mathfrak{D}^{|\alpha^x|+3}_{|\alpha^x|+3}\mathfrak{D}_{V,|\alpha^x|+3}(1+|x|^{\mathfrak{p}_{|\alpha^x|+3}(|\alpha^x|+3)+\mathfrak{p}_{V,|\alpha^x|+3}}_{\mathbb{R}^d}).\end{aligned}$$

Now, we focus on the study of $\overline{R}^2_t$.

Using similar arguments as in the study of $\partial_x^{\alpha^x}\overline{R}_t^{1,2}$, under the assumptions $\mathbf{A}_1(|\alpha^x|+4)$ (see (2.2)) and $\mathbf{A}_3^\delta(\max(\mathfrak{p}_{|\alpha^x|+4}+3,\lceil -\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2))$ (see (2.7)), then, for every $x\in\mathbb{R}^d$,

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}_t^{2,2}(x)|_{\mathbb{R}^d}\leqslant&\delta^{\frac{3}{2}}C(d,|\alpha^x|)\mathfrak{M}_{\max(\mathfrak{p}_{|\alpha^x|+4}+3,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)\mathfrak{D}_{|\alpha^x|+4}\mathfrak{D}_{V,|\alpha^x|}\\&\times(1+|x|_{\mathbb{R}}^{\mathfrak{p}_{|\alpha^x|+4}+\mathfrak{p}_{V,|\alpha^x|}})\end{aligned}$$

We then bound the derivatives of $\overline{R}_t^{2,3}$. For every $x\in\mathbb{R}^d$,

$$\begin{aligned}\mathcal{R}^{\delta,2,3}(x,t,z)V(x,t-\delta)=&(\nabla_x\psi^{-1}-I_{d\times d}-(I_{d\times d}-\nabla_x\psi)-(I_{d\times d}-\nabla_x\psi)^2)(x,t,\delta^{\frac{1}{2}}z,\delta)\\=&\sum_{k=3}^{\infty}(I_{d\times d}-\nabla_x\psi(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta))^k,\end{aligned}$$

where for a matrix $M\in\mathbb{R}^{d\times d}$, $M^{k+1}=MM^k$, $k\in\mathbb{N}$. If $|\alpha^x|=1$, then

$$\begin{aligned}&\partial_x^{\alpha^x}\overline{R}_t^{2,3}(x)\\&=-\mathbb{E}[\sum_{k=3}^{\infty}\sum_{l=1}^{k}((I_{d\times d}-\nabla_x\psi)^{l-1}\partial_x^{\alpha^x}\nabla_x\psi(I_{d\times d}-\nabla_x\psi)^{k-l})(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta)]V(x,t-\delta)\\&\quad+\mathbb{E}[\sum_{k=3}^{\infty}(I_{d\times d}-\nabla_x\psi(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta))^k]\partial_x^{\alpha^x}V(x,t-\delta).\end{aligned}$$

We consider now $\alpha^x\in\mathbb{N}^d$, with $|\alpha^x|\in\mathbb{N}^*$. Iterating the formula above and observing that we have also

$$\begin{aligned}|\partial_x^{\alpha^x}\nabla_x\psi(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta)|_{\mathbb{R}^{d\times d}}=&\delta^{\frac{1}{2}}|\sum_{l=1}^{N}Z_t^{\delta,l}\partial_x^{\alpha^x}\nabla_xV_l(x,t-\delta)+\partial_x^{\alpha^x}\mathcal{R}^{\delta,2,1}(x,t-\delta,Z_t^\delta)|_{\mathbb{R}^{d\times d}}\\\leqslant&\delta^{\frac{1}{2}}\mathfrak{D}_{|\alpha^x|+2}(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+2}}+|Z_t^\delta|_{\mathbb{R}^N}^{\mathfrak{p}_{|\alpha^x|+2}})|Z_t^\delta|_{\mathbb{R}^N}\\&+\delta\mathfrak{D}_{|\alpha^x|+3}(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+3}}+|Z_t^\delta|_{\mathbb{R}^N}^{\mathfrak{p}_{|\alpha^x|)+3}})(1+|Z_t^\delta|_{\mathbb{R}^N}^2)\\\leqslant&2\delta^{\frac{1}{2}}\mathfrak{D}_{|\alpha^x|+3}(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+3}}+|Z_t^\delta|_{\mathbb{R}^N}^{\mathfrak{p}_{|\alpha^x|+3}})(1+|Z_t^\delta|_{\mathbb{R}^N}^2).\end{aligned}$$

Therefore,

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}_t^{2,3}(x)|_{\mathbb{R}^d}\leqslant&C(d,|\alpha^x|)\mathbb{E}[\mathfrak{D}_{|\alpha^x|+3}^{|\alpha^x|}(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+3}|\alpha^x|}+|Z_t^\delta|_{\mathbb{R}^N}^{\mathfrak{p}_{|\alpha^x|+3}|\alpha^x|})\\&\times(1+|Z_t^\delta|_{\mathbb{R}^N}^{2|\alpha^x|})\mathfrak{D}_{V,|\alpha^x|}(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{V,|\alpha^x|}}|)\\&\times\sum_{k=0}^{\infty}\delta^{\frac{\max(3-k,0)}{2}}(k+1)^{|\alpha^x|}|I_{d\times d}-\nabla_x\psi|_{\mathbb{R}^{d\times d}}^k(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\in\mathbf{D}_{\eta_2}}].\end{aligned}$$

Using $\mathbf{A}_1^\delta$ (see (2.3)), we have (4.29). Moreover, when $k\geqslant 3$, we use $|Z_t^\delta|_{\mathbb{R}^N}^k\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\in\mathbf{D}_{\eta_2}}\leqslant|Z_t^\delta|_{\mathbb{R}^N}^3\eta_2^{k-3}$ and we obtain

$$\begin{aligned}&\mathbb{E}[\delta^{\frac{\max(3-k,0)}{2}}(k+1)^{|\alpha^x|}(1+|Z_t^\delta|_{\mathbb{R}^N}^{(\mathfrak{p}_{|\alpha^x|+3}+2)|\alpha^x|})|I_{d\times d}-\nabla_x\psi|_{\mathbb{R}^{d\times d}}^k(x,t-\delta,\delta^{\frac{1}{2}}Z_t^\delta,\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\in\mathbf{D}_{\eta_2}}]\\&\leqslant\delta^{\frac{\max(k,3)}{2}}\eta_2^{\max(k-3,0)(\mathfrak{p}+1)}(k+1)^{|\alpha^x|}4^k\mathfrak{D}^k\mathbb{E}[1+|Z_t^\delta|_{\mathbb{R}^N}^{3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+3}+2)|\alpha^x|}].\end{aligned}$$

Since $\delta^{\frac{1}{2}}\eta_2^{\mathfrak{p}+1}4\mathfrak{D}<\frac{1}{2}$ (see (4.15)), we obtain the estimate

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}_t^{2,3}(x)|_{\mathbb{R}^d}\leqslant&\delta^{\frac{3}{2}}C(d,|\alpha^x|)\mathfrak{M}_{3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+3}+2)|\alpha^x|}(Z^\delta)\\&\times\mathfrak{D}^3\mathfrak{D}_{|\alpha^x|+3}^{|\alpha^x|}\mathfrak{D}_{V,|\alpha^x|}(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+3}|\alpha^x|+\mathfrak{p}_{V,|\alpha^x|}}).\end{aligned}$$

At this point, we observe that,

$$\overline{R}_t^2=\overline{R}_t^{2,3}-\overline{R}_t^{2,2}-\overline{R}_t^{2,4},$$

where we have introduced the function $\overline{R}_t^{2,4}$ defined for every $x \in \mathbb{R}^d$ by

$$\begin{aligned}\overline{R}_t^{2,4}(x) =& \mathbb{E}[\mathcal{R}^{\delta,2,1}(x,t-\delta,Z_t^{\delta,})^2 V(x,t-\delta)\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathbf{D}_{\eta_2}} \\ &+ \delta^{\frac{1}{2}} \sum_{l=1}^N Z_t^{\delta,l} \nabla_x V_l(x,t-\delta)\mathcal{R}^{\delta,2,1}(x,t-\delta,Z_t^\delta)V(x,t-\delta)\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathbf{D}_{\eta_2}}] \\ &+ \delta^{\frac{1}{2}} \sum_{l=1}^N Z_t^{\delta,l} \mathcal{R}^{\delta,2,1}(x,t-\delta,Z_t^\delta)\nabla_x V_l(x,t-\delta))V(x,t-\delta)\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathbf{D}_{\eta_2}}],\end{aligned}$$

which satisfies, using hypothesis $\mathbf{A}_1^\delta(|\alpha^x|+3)$ (see (2.2)) and $\mathbf{A}_3^\delta(2\mathfrak{p}_{|\alpha^x|+3}+4)$ (see (2.7)),

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}_t^{2,4}(x)|_{\mathbb{R}^d} \leqslant& \delta^{\frac{3}{2}} C(|\alpha^x|)\mathfrak{M}_{2\mathfrak{p}_{|\alpha^x|+3}+4}(Z^\delta) \\ &\times \mathfrak{D}^2_{|\alpha^x|+3}\mathfrak{D}_{V,|\alpha^x|}(1+|x|_{\mathbb{R}^d}^{2\mathfrak{p}_{|\alpha^x|+3}+\mathfrak{p}_{V,|\alpha^x|}}).\end{aligned}$$

We conclude that, under the assumptions $\mathbf{A}_1^\delta(|\alpha^x|+4)$ (see (2.2) and (2.3)) and $\mathbf{A}_3^\delta(\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)\max(|\alpha^x|,2)+1,\lceil -\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2))$ (see (2.7)), and $\delta^{\frac{1}{2}}\eta_2^{\mathfrak{p}+1}4\mathfrak{D} < \frac{1}{2}$, then, for every $x \in \mathbb{R}^d$,

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}_t^{2}(x)|_{\mathbb{R}^d} \leqslant& \delta^{\frac{3}{2}} C(d,|\alpha^x|)\mathfrak{M}_{\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)\max(|\alpha^x|,2)+1,\lceil -\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta) \\ &\times \mathfrak{D}^3\mathfrak{D}^{\max(|\alpha^x|,2)}_{|\alpha^x|+4}\mathfrak{D}_{V,|\alpha^x|}(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+4}\max(|\alpha^x|,2)+\mathfrak{p}_{V,|\alpha^x|}}).\end{aligned}$$

We now focus on the study of $\overline{R}_t^3$.

$$\overline{R}_t^3 = \overline{R}_t^{3,1} - \overline{R}_t^{3,2} + \overline{R}_t^{3,3} - \overline{R}_t^{3,4}$$

where we have introduced

$$\begin{aligned}\overline{R}_t^{3,1}(x) =& \delta^{\frac{1}{2}} \sum_{l=1}^N \mathbb{E}[Z_t^{\delta,l}\mathcal{R}^{\delta,2}(x,t-\delta,Z_t^\delta)\nabla_x V(x,t-\delta)V_l(x,t-\delta)\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathbf{D}_{\eta_2}}] \\ \overline{R}_t^{3,2}(x) =& \delta^{\frac{1}{2}} \sum_{l=1}^N \mathbb{E}[Z_t^{\delta,l}\nabla_x V_l(x,t-\delta)R^{\delta,1}(x,t-\delta,Z_t^\delta))\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathbf{D}_{\eta_2}}] \\ \overline{R}_t^{3,3}(x) =& \mathbb{E}[\mathbf{1}_{\delta^{\frac{1}{2}} Z_t^\delta \in \mathbf{D}_{\eta_2}}\mathcal{R}^{\delta,2}(x,t-\delta,Z_t^\delta) \\ &\times(\delta\nabla_x V(x,t-\delta)V_0(x,t-\delta)+\delta\partial_t V(x,t-\delta) \\ &+\delta\frac{1}{2}\sum_{l=1}^N V_l(x,t-\delta)^T\mathbf{H}_x V(x,t-\delta)V_l(x,t-\delta)+R^{\delta,1}(x,t-\delta,Z_t^\delta))] \\ \overline{R}_t^{3,4}(x) =& \delta^2(\nabla_x V_0(x,t-\delta)-\sum_{l=1}^N\nabla_x V_l(x,t-\delta)^2) \\ &\times(\nabla_x V(x,t-\delta)V_0(x,t-\delta)+\partial_t V(x,t-\delta) \\ &+\frac{1}{2}\sum_{l=1}^N V_l(x,t-\delta)^T\mathbf{H}_x V(x,t-\delta)V_l(x,t-\delta)).\end{aligned}$$

Using standard computations together with hypothesis $\mathbf{A}_1^\delta(|\alpha^x|+2)$ (see (2.2)) yields

$$|\partial_x^{\alpha^x}\overline{R}_t^{3,4}(x)|_{\mathbb{R}^d} \leqslant \delta^2 C(|\alpha^x|)\mathfrak{D}^4_{|\alpha^x|+2}\mathfrak{D}_{V,|\alpha^x|+2}(1+|x|_{\mathbb{R}^d}^{4\mathfrak{p}_{|\alpha^x|+2}+\mathfrak{p}_{V,|\alpha^x|+2}}).$$

Using a similar approach as in the study of $\overline{R}_t^1$, as a consequence of $\mathbf{A}_1^\delta(|\alpha^x|+3)$ (see (2.2)) and $\mathbf{A}_3^\delta(\max(\mathfrak{p}_{|\alpha^x|+3}(|\alpha^x|+3)+4,\lceil -\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)+1)$, we derive

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}_t^{3,2}(x)|_{\mathbb{R}^d} \leqslant& \delta^{\frac{3}{2}} C(d,|\alpha^x|)\mathfrak{M}_{\max(\mathfrak{p}_{|\alpha^x|+3}(|\alpha^x|+3)+4,\lceil -\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)+1}(Z^\delta) \\ &\times \mathfrak{D}^{|\alpha^x|+4}_{|\alpha^x|+3}\mathfrak{D}_{V,|\alpha^x|+3}(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+3}(|\alpha^x|+4)+\mathfrak{p}_{V,|\alpha^x|+3}}).\end{aligned}$$

From the same reasonning as in the study of $\overline{R}_t^2$, since (4.15) holds, it follows from $\mathbf{A}_1^\delta(|\alpha^x|+4)$ (see (2.2) and (2.3)) and $\mathbf{A}_3^\delta(2\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)(\max(|\alpha^x|,2)+3)+1,\lceil -\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2))$ (see (2.7)) that

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}_t^{3,3}(x)|_{\mathbb{R}^d} \leqslant&\delta^{\frac{5}{2}}C(d,|\alpha^x|)\mathfrak{M}_{2\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)(\max(|\alpha^x|,2)+3)+1,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)\\&\times\mathfrak{D}^3\mathfrak{D}_{|\alpha^x|+4}^{2\max(|\alpha^x|,2)+3}\mathfrak{D}_{V,|\alpha^x|+3}^2(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+4}(2\max(|\alpha^x|,2)+3)+2\mathfrak{p}_{V,|\alpha^x|+3}}).\end{aligned}$$

Similarly, since (4.15) holds, it follows from $\mathbf{A}_1^\delta(|\alpha^x|+4)$ (see (2.2) and (2.3)) and $\mathbf{A}_3^\delta(\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)\max(|\alpha^x|,2)+1,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)+1)$ (see (2.7)) that

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}_t^{3,1}(x)|_{\mathbb{R}^d} \leqslant&\delta^{\frac{3}{2}}C(d,|\alpha^x|)\mathfrak{M}_{\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)\max(|\alpha^x|,2)+1,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)+1}(Z^\delta)\\&\times\mathfrak{D}^3\mathfrak{D}_{|\alpha^x|+4}^{\max(|\alpha^x|,2)+1}\mathfrak{D}_{V,|\alpha^x|+1}^2(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+4}(\max(|\alpha^x|,2)+1)+2\mathfrak{p}_{V,|\alpha^x|+1}}).\end{aligned}$$

We conclude that under the assumptions (4.15) it follows from $\mathbf{A}_1^\delta(|\alpha^x|+4)$ (see (2.2) and (2.3)) and $\mathbf{A}_3^\delta(2\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)(\max(|\alpha^x|,2)+3)+1,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2))$ (see (2.7)) that

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{R}_t^{3}(x)|_{\mathbb{R}^d} \leqslant&\delta^{\frac{3}{2}}C(d,|\alpha^x|)\mathfrak{M}_{2\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)(\max(|\alpha^x|,2)+3)+1,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)\\&\times\mathfrak{D}^3\mathfrak{D}_{|\alpha^x|+4}^{2\max(|\alpha^x|,2)+3}\mathfrak{D}_{V,|\alpha^x|+3}^2(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+4}(2\max(|\alpha^x|,2)+3)+2\mathfrak{p}_{V,|\alpha^x|+3}}).\end{aligned}$$

To complete the proof, it remains to study $\overline{R}_t^4$ and $\overline{R}_t^5$. As a direct consequence of the Markov inequality,

$$\mathbb{E}[\sum_{i=1}^N Z_t^{\delta,i}\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\notin\mathbf{D}_{\eta_2}}]\leqslant\delta^{\frac{3}{2}}\mathfrak{M}_{\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1}(Z^\delta)$$

and

$$\mathbb{P}(\delta^{\frac{1}{2}}Z_t^\delta\notin\mathbf{D}_{\eta_2})\leqslant\delta^{\frac{3}{2}}\mathfrak{M}_{\lceil-\frac{\ln(\delta)}{2\ln(\eta_2)}\rceil}(Z^\delta).$$

Consequently

$$|\partial_x^{\alpha^x}\overline{R}_t^{4}(x)|_{\mathbb{R}^d} \leqslant\delta^{\frac{3}{2}}C(|\alpha^x|)\mathfrak{M}_{\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1}(Z^\delta)\mathfrak{D}_{|\alpha^x|+2}\mathfrak{D}_{V,|\alpha^x|+1}(1+|x|_{\mathbb{R}}^{\mathfrak{p}_{|\alpha^x|+2}+\mathfrak{p}_{V,|\alpha^x|+1}})$$

and

$$|\partial_x^{\alpha^x}\overline{R}_t^{5}(x)|_{\mathbb{R}^d} \leqslant\delta^{\frac{3}{2}}C(|\alpha^x|)\mathfrak{M}_{\lceil-\frac{\ln(\delta)}{2\ln(\eta_2)}\rceil+1}(Z^\delta)\mathfrak{D}_{|\alpha^x|+3}^2\mathfrak{D}_{V,|\alpha^x|+2}(1+|x|_{\mathbb{R}}^{2\mathfrak{p}_{|\alpha^x|+3}+\mathfrak{p}_{V,|\alpha^x|+2}}).$$

We conclude that under the assumptions (4.15), it follows from $\mathbf{A}_1^\delta(|\alpha^x|+4)$ (see (2.2) and (2.3)) and $\mathbf{A}_3^\delta(2\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)(\max(|\alpha^x|,2)+3)+1,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2))$ (see (2.7)) that

$$\begin{aligned}|\partial_x^{\alpha^x}\overline{\mathbf{R}}(x,t-\delta)|_{\mathbb{R}^d} \leqslant&\delta^{\frac{3}{2}}C(d,|\alpha^x|)\mathfrak{M}_{2\max(3(\mathfrak{p}+1)+(\mathfrak{p}_{|\alpha^x|+4}+2)(\max(|\alpha^x|,2)+3)+1,\lceil-\frac{3\ln(\delta)}{2\ln(\eta_2)}\rceil+2)}(Z^\delta)\\&\times\mathfrak{D}^3\mathfrak{D}_{|\alpha^x|+4}^{2\max(|\alpha^x|,2)+3}\mathfrak{D}_{V,|\alpha^x|+3}^2(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_{|\alpha^x|+4}(2\max(|\alpha^x|,2)+3)+2\mathfrak{p}_{V,|\alpha^x|+3}}).\end{aligned}$$

Finally, let us remark that $\tilde{\mathbf{R}}^\delta V(x,t-\delta)=(R^\delta(x,t-\delta,Z_t^\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\in\mathbf{D}_{\eta_2}}-\mathbb{E}[R^\delta(x,t-\delta,Z_t^\delta)\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\in\mathbf{D}_{\eta_2}}])+\tilde{R}_t^4(x)+\tilde{R}_t^5(x)$, with

$$\begin{aligned}\tilde{R}_t^4(x,z)=&-\delta^{\frac{1}{2}}\sum_{i=1}^N V^{[i]}(x,t-\delta)(z^i\mathbf{1}_{\delta^{\frac{1}{2}}z\notin\mathbf{D}_{\eta_2}}-\mathbb{E}[z^i\mathbf{1}_{\delta^{\frac{1}{2}}z\notin\mathbf{D}_{\eta_2}}]),\\\tilde{R}_t^5(x,z)=&-\delta V^{[0]}(x,t-\delta)(\mathbf{1}_{\delta^{\frac{1}{2}}z\notin\mathbf{D}_{\eta_2}}-\mathbb{P}(\delta^{\frac{1}{2}}Z_t^\delta\notin\mathbf{D}_{\eta_2})).\end{aligned}$$

Using $\mathbf{A}_1^\delta(2)$ (see (2.2)) and $\mathbf{A}_3^\delta(\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1)$ (see (2.7)),

$$\begin{aligned}|\tilde{R}_t^4(x,z)|_{\mathbb{R}^d}=&|\delta^{\frac{1}{2}}\sum_{i=1}^N V^{[i]}(x,t-\delta)(z^i\mathbf{1}_{\delta^{\frac{1}{2}}z\notin\mathbf{D}_{\eta_2}}-\mathbb{E}[Z_t^{\delta,i}\mathbf{1}_{\delta^{\frac{1}{2}}Z_t^\delta\notin\mathbf{D}_{\eta_2}}])|_{\mathbb{R}^d}\\\leqslant&\delta C\mathfrak{D}_2\mathfrak{D}_{V,1}(1+|x|_{\mathbb{R}^d}^{\mathfrak{p}_2+\mathfrak{p}_{V,1}})(|z|_{\mathbb{R}^N}^{\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1}+\mathfrak{M}_{\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1}(Z^\delta))\\\leqslant&\delta C\mathfrak{D}_2\mathfrak{D}_{V,1}\mathfrak{M}_{\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1}(Z^\delta)(1+|x|_{\mathbb{R}^d}^{2(\mathfrak{p}_2+\mathfrak{p}_{V,1})}+|z|_{\mathbb{R}^N}^{2\lceil-\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+2}).\end{aligned}$$

and using $\mathbf{A}_1^\delta(3)$ (see (2.2)),

$$|\tilde{R}_t^5(x,z)|_{\mathbb{R}^d} \leqslant \delta C \mathfrak{D}_3^2 \mathfrak{D}_{V,2}(1+|x|_{\mathbb{R}^d}^{2\mathfrak{p}_3+\mathfrak{p}_{V,2}}).$$

We treat the other terms by a similar but simpler (since it does not involves derivatives) method used to study $\overline{\mathbf{R}}$, we finally obtain

$$\begin{aligned}|\tilde{\mathbf{R}}(x,t-\delta,z)|_{\mathbb{R}^d} \leqslant& \delta C \mathfrak{M}_{2\max(3\mathfrak{p}+5\mathfrak{p}_4+14,\lceil -\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1)}(Z^\delta) \\ &\times \mathfrak{D}^3 \mathfrak{D}_4^7 \mathfrak{D}_{V,3}^2 (1+|x|_{\mathbb{R}^d}^{14\mathfrak{p}_4+4\mathfrak{p}_{V,3}} + |z|_{\mathbb{R}^N}^{4\max(3\mathfrak{p}+5\mathfrak{p}_4+14,\lceil -\frac{\ln(\delta)}{\ln(\eta_2)}\rceil+1)}).\end{aligned}$$

□

A.4. **Proof of Lemma 4.9.**

Proof. **Step 1.** First we show that for every $\epsilon \in [\underline{\epsilon}_1(\delta), \overline{\epsilon}_1(\delta)]$, every $s \in (3r, \frac{1}{2})$, $u \in (0, \frac{1}{2}-s)$, every $v, v^\diamond > 0$, and every $q \geqslant 4$,

$$\begin{aligned}&\mathbb{P}(\delta \sum_{t\in\mathbf{T}} |Y_t|^2 < \epsilon, \delta \sum_{t\in\mathbf{T}} \mathbb{E}[|\tilde{\Delta}_{t-\delta}^Y|^2 | \mathcal{F}_{t-\delta}^Y] + |\bar{\Delta}_{t-\delta}^Y|^2 \geqslant \epsilon^r, \mathcal{A}_{Y,u,q}) \\ &\leqslant \epsilon^p \mathbb{E}[|Y_0|^{\frac{p}{v}}] + \mathbb{P}(\delta |Y_0|^2 \geqslant \epsilon) \\ &\quad + \delta^{\frac{q}{4}} (\delta^{\frac{q}{4}} \epsilon^{-q(s+2u)} + \epsilon^{-q(s+u)} + \epsilon^{-q\frac{(2+v^\diamond)}{4}}) 2^{\frac{3q}{2}+2} (1+T^{2q})(1+\sup_{t\in\mathbf{T}} \mathbb{E}[|Y_{t-\delta}|^q]) \\ &\quad + 2\exp(-\frac{\epsilon^{-4s}}{16}) + 2\exp(-\frac{\epsilon^{-v^\diamond}}{2}) + 2\exp(-\frac{\epsilon^{2(s+u)-1}}{2^{11}T^2}) \\ &\quad + \mathbb{P}(\delta \sum_{t\in\mathbf{T}} |Y_t|^2 < \epsilon, \delta \sum_{t\in\mathbf{T}} \mathbb{E}[|\tilde{\Delta}_{t-\delta}^Y|^2 | \mathcal{F}_{t-\delta}^Y] + |\bar{\Delta}_{t-\delta}^Y|^2 \geqslant \epsilon^r, \mathfrak{A}_1, |Y_0|^2 < \frac{\epsilon^s}{\delta|\mathbf{T}|}, \mathcal{A}_{Y,u,q}),\end{aligned}$$

with

$$\underline{\epsilon}_1(\delta) = \max(|16\delta T^2|^{\frac{1}{s+2u}}, |2^{10}\delta T^3|^{\frac{1}{2u+2s+2v}}), \quad \overline{\epsilon}_1(\delta) = \min(|32 T^{\frac{3}{2}}|^{-\frac{1}{\frac{1}{2}-s-u}}, 2^{-\frac{1}{1-s}}),$$

and

$$\begin{aligned}\mathcal{A}_{Y,u,q} =& \{\sup_{t\in\mathbf{T}} |\bar{\Delta}_{t-\delta}^Y| \leqslant \epsilon^{-u}\} \cap \{\sup_{t\in\mathbf{T}} \mathbb{E}[|\tilde{\Delta}_{t-\delta}^Y|^q | \mathcal{F}_{t-\delta}^Y] \leqslant \epsilon^{-qu}\} \\ &\cap \{\sup_{t\in\mathbf{T}} |\bar{\Delta}_{t-\delta}^{\bar{\Delta}^Y}| \leqslant \epsilon^{-u}\} \cap \{\sup_{t\in\mathbf{T}} \mathbb{E}[|\tilde{\Delta}_{t-\delta}^{\bar{\Delta}^Y}|^q | \mathcal{F}_{t-\delta}^Y] \leqslant \epsilon^{-qu}\}, \\ \mathfrak{A}_1 :=& \{\delta^2 \sum_{\substack{w,t\in\mathbf{T} \\ w\leqslant t}} \mathbb{E}[|\tilde{\Delta}_{w-\delta}^Y|^2 | \mathcal{F}_{w-\delta}^Y] < \epsilon^s\} \cap \{\delta^2 \sum_{\substack{w,t\in\mathbf{T} \\ w\leqslant t}} |\tilde{\Delta}_{w-\delta}^Y|^2 < \epsilon^s\}.\end{aligned}$$

We begin by writing that, for every $t \in \mathbf{T}$, we have

$$\begin{aligned}Y_t^2 =& Y_{t-\delta}^2 + \delta^{\frac{1}{2}} 2\tilde{\Delta}_{t-\delta}^Y Y_{t-\delta} + \delta(2\bar{\Delta}_{t-\delta}^Y Y_{t-\delta} + |\tilde{\Delta}_{t-\delta}^Y|^2) + \delta^{\frac{3}{2}} 2\tilde{\Delta}_{t-\delta}^Y \bar{\Delta}_{t-\delta}^Y + \delta^2 |\bar{\Delta}_{t-\delta}^Y|^2 \\ =& Y_0^2 + \sum_{\substack{w\in\mathbf{T} \\ w\leqslant t}} \delta^{\frac{1}{2}} 2\tilde{\Delta}_{w-\delta}^Y Y_{w-\delta} + \delta(2\bar{\Delta}_{w-\delta}^Y Y_{w-\delta} + |\tilde{\Delta}_{w-\delta}^Y|^2) \\ &\qquad + \delta^{\frac{3}{2}} 2\tilde{\Delta}_{w-\delta}^Y \bar{\Delta}_{w-\delta}^Y + \delta^2 |\bar{\Delta}_{w-\delta}^Y|^2.\end{aligned}$$

and we introduce

$$\begin{aligned}\mathfrak{A}_2 :=& \{\delta^{\frac{3}{2}} |\sum_{\substack{w,t\in\mathbf{T} \\ w\leqslant t}} 2\tilde{\Delta}_{w-\delta}^Y Y_{w-\delta}| < \frac{\epsilon^s}{8}\} \cap \{|\delta^2 \sum_{\substack{w,t\in\mathbf{T} \\ w\leqslant t}} 2\bar{\Delta}_{w-\delta}^Y Y_{w-\delta}| < \frac{\epsilon^s}{8}\} \\ &\cap \{|\delta^2 \sum_{\substack{w,t\in\mathbf{T} \\ w\leqslant t}} |\tilde{\Delta}_{w-\delta}^Y|^2 - \mathbb{E}[|\tilde{\Delta}_{w-\delta}^Y|^2 | \mathcal{F}_w^Y]| < \frac{\epsilon^s}{8}\} \\ &\cap \{\delta^3 \sum_{\substack{w,t\in\mathbf{T} \\ w\leqslant t}} |\delta^{\frac{1}{2}} 2\tilde{\Delta}_{w-\delta}^Y \bar{\Delta}_{w-\delta}^Y + \delta |\bar{\Delta}_{w-\delta}^Y|^2| < \frac{\epsilon^s}{8}\}.\end{aligned}$$

In the sequel, for $t \in \pi^\delta$ we will denote $\mathfrak{n}_{\mathbf{T},\delta,t} = (|\mathbf{T}| - t\delta^{-1})$. Now we notice that, for every $s \in (3r, \frac{1}{2})$, $u \in (0, \frac{1}{2} - s)$, we have

$$
\begin{aligned}
&\mathbb{P}(|\delta^{\frac{3}{2}} \sum_{\substack{w,t \in \mathbf{T} \\ w \leqslant t}} 2\tilde{\Delta}^Y_{w-\delta} Y_{w-\delta}| \geqslant \frac{\epsilon^s}{8}, \delta \sum_{t\in\mathbf{T}} |Y_t|^2 < \epsilon, \mathcal{A}_{Y,u,q}) \\
\leqslant &\mathbb{P}(|\delta^{\frac{3}{2}} \sum_{t\in\mathbf{T}} \mathfrak{n}_{\mathbf{T},\delta,t-\delta} \tilde{\Delta}^Y_{t-\delta} Y_{t-\delta}| \geqslant \frac{\epsilon^s}{16}, \delta \sum_{t\in\mathbf{T}} |Y_{t-\delta}|^2 < 2\epsilon, \mathcal{A}_{Y,u,q}) + \mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon) \\
\leqslant &\mathbb{P}(|\delta^{\frac{3}{2}} \sum_{t\in\mathbf{T}} \mathfrak{n}_{\mathbf{T},\delta,t-\delta} \tilde{\Delta}^Y_{t-\delta} Y_{t-\delta}| \geqslant \frac{\epsilon^s}{16}, \\
&\qquad \delta^3 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2 (|\tilde{\Delta}^Y_{t-\delta}|^2 + \mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}]) |Y_{t-\delta}|^2 < 8|\delta|\mathbf{T}||^2 \epsilon^{1-2u}) \\
&\quad + \mathbb{P}(\delta^3 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2 (|\tilde{\Delta}^Y_{t-\delta}|^2 + \mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}]) |Y_{t-\delta}|^2 \geqslant 8|\delta|\mathbf{T}||^2 \epsilon^{1-2u}, \\
&\qquad\quad \delta \sum_{t\in\mathbf{T}} |Y_{t-\delta}|^2 < 2\epsilon, \mathcal{A}_{Y,u,q}) + \mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon).
\end{aligned}
$$

Using the martingale exponential inequality (4.31), the first term of the *r.h.s.* above is bounded by $2\exp(-\frac{\epsilon^{2(s+u)-1}}{2^{11}|\delta|\mathbf{T}||^2})$. We now study the second term of the *r.h.s.* above. Let us denote $H_t = |\tilde{\Delta}^Y_t|^2 - \mathbb{E}[|\tilde{\Delta}^Y_t|^2 | \mathcal{F}^Y_t]$ so that $(H_t)_{t\in\pi^\delta}$ is a martingale. We have

$$
\begin{aligned}
&\mathbb{P}(\delta^3 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2 (|\tilde{\Delta}^Y_{t-\delta}|^2 + \mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}]) |Y_{t-\delta}|^2 \geqslant 8|\delta|\mathbf{T}||^2 \epsilon^{1-2u}, \\
&\qquad \delta \sum_{t\in\mathbf{T}} |Y_{t-\delta}|^2 < 2\epsilon, \mathcal{A}_{Y,u,q}) \\
\leqslant &\mathbb{P}(\delta^3 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2 H_{t-\delta} |Y_{t-\delta}|^2 \geqslant 4|\delta|\mathbf{T}||^2 \epsilon^{1-2u}, \mathcal{A}_{Y,u,q}) \\
&\quad + \mathbb{P}(\delta^3 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2 |\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}] |Y_{t-\delta}|^2 \geqslant 2|\delta|\mathbf{T}||^2 \epsilon^{1-2u}, \delta \sum_{t\in\mathbf{T}} |Y_{t-\delta}|^2 < 2\epsilon, \mathcal{A}_{Y,u,q}).
\end{aligned}
$$

Since since $\mathfrak{n}_{\mathbf{T},\delta,t} \leqslant |\mathbf{T}|$ for every $t \in \mathbf{T}$, the second term of the *r.h.s.* above is equal to zero. We then focus to the first term of the *r.h.s.* above. Let $v^\diamond > 0$. Then

$$
\begin{aligned}
&\mathbb{P}(\delta^3 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2 H_{t-\delta} |Y_{t-\delta}|^2 \geqslant 4|\delta|\mathbf{T}||^2 \epsilon^{1-2u}, \mathcal{A}_{Y,u,q}) \\
\leqslant &\mathbb{P}(\delta^3 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2 H_{t-\delta} |Y_{t-\delta}|^2 \geqslant 4|\delta|\mathbf{T}||^2 \epsilon^{1-2u}, \\
&\quad \delta^6 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^4 (|H_{t-\delta}|^2 + \mathbb{E}[|H_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}]) |Y_{t-\delta}|^4 < \epsilon^{2+v^\diamond-4u}, \mathcal{A}_{Y,u,q}) \\
&\quad + \mathbb{P}(\delta^6 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^4 (|H_{t-\delta}|^2 + \mathbb{E}[|H_{t-\delta}|^2 | \mathcal{F}^Y_{t-\delta}]) |Y_{t-\delta}|^4 \geqslant \epsilon^{2+v^\diamond-4u}, \mathcal{A}_{Y,u,q})
\end{aligned}
$$

Using (4.31), the first term of the *r.h.s.* above is bounded by $2\exp(-\frac{\epsilon^{-v^\diamond}}{2}$. To study the second term, we use the Markov and the Hölder inequalities and for every $q^\diamond \geqslant 1$ (more specifically, triangle inequality when $q^\diamond = 1$), we obtain

$$\begin{aligned}
&\mathbb{P}(\delta^6 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^4(|H_{t-\delta}|^2 + \mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}])|Y_{t-\delta}|^4 \geqslant \epsilon^{2+v^\diamond-4u}, \mathcal{A}_{Y,u,q})\\
\leqslant&\mathbb{P}(|\delta|\mathbf{T}||^{q^\diamond-1}\delta \sum_{t\in\mathbf{T}} ||H_{t-\delta}|^2 + \mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]|^{q^\diamond}|Y_{t-\delta}|^{4q^\diamond} \geqslant \delta^{-q^\diamond}\frac{\epsilon^{q^\diamond(2+v^\diamond-4u)}}{|\delta|\mathbf{T}||^{4q^\diamond}}, \mathcal{A}_{Y,u,q})\\
\leqslant&\delta^{q^\diamond}\epsilon^{-q^\diamond(2+v^\diamond-4u)}|\delta|\mathbf{T}||^{5q^\diamond-1}\delta\sum_{t\in\mathbf{T}} 2^{q^\diamond+1}\mathbb{E}[|H_{t-\delta}|^{2q^\diamond}|Y_{t-\delta}|^{4q^\diamond}\mathbf{1}_{\sup_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]\leqslant\epsilon^{-qu}}]\\
\leqslant&\delta^{q^\diamond}\epsilon^{-q^\diamond(2+v^\diamond-4u)}2^{3q^\diamond+1}|\delta|\mathbf{T}||^{5q^\diamond-1}(\delta\sum_{t\in\mathbf{T}} \mathbb{E}[|Y_{t-\delta}|^{4q^\diamond}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^{4q^\diamond}|\mathcal{F}^Y_t]\mathbf{1}_{\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]\leqslant\epsilon^{-qu}}]\\
&+\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^{4q^\diamond}|\tilde{\Delta}^Y_{t-\delta}|^{4q^\diamond}\mathbf{1}_{\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]\leqslant\epsilon^{-qu}}])
\end{aligned}$$

with, as soon as $q^\diamond \leqslant q/4$,

$$\begin{aligned}
\mathbb{E}[|Y_{t-\delta}|^{4q^\diamond}|\tilde{\Delta}^Y_t|^{4q^\diamond}\mathbf{1}_{\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]\leqslant\epsilon^{-qu}}] =&\mathbb{E}[|Y_{t-\delta}|^{4q^\diamond}\mathbb{E}[|\tilde{\Delta}^Y_t|^{4q^\diamond}|\mathcal{F}^Y_t]\mathbf{1}_{\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^{4q^\diamond}|\mathcal{F}^Y_{t-\delta}]\leqslant\epsilon^{-4q^\diamond u}}]\\
\leqslant&\epsilon^{-4q^\diamond u}\mathbb{E}[|Y_{t-\delta}|^{4q^\diamond}].
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{P}(\delta^6 \sum_{t\in\mathbf{T}} |\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^4(|H_{t-\delta}|^2 + \mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}])|Y_{t-\delta}|^4 \geqslant \epsilon^{2+v^\diamond-4u}, \mathcal{A}_{Y,u,q})\\
&\qquad\leqslant\delta^{q^\diamond}\epsilon^{-(2+v^\diamond)q^\diamond}2^{3q^\diamond+2}|\delta|\mathbf{T}||^{5q^\diamond-1}\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^{4q^\diamond}].
\end{aligned}$$

Notice, that from the same approach we obtain

$$\begin{aligned}
&\mathbb{P}(|\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\tilde{\Delta}^Y_{w-\delta}|^2 - \mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}]| \geqslant \frac{\epsilon^s}{8}, \mathcal{A}_{Y,u,q}) = \mathbb{P}(\delta^2|\sum_{t\in\mathbf{T}}\mathfrak{n}_{\mathbf{T},\delta,t-\delta}H_{t-\delta}| \geqslant \frac{\epsilon^s}{8}, \mathcal{A}_{Y,u,q})\\
\leqslant&\mathbb{P}(\delta^2\sum_{t\in\mathbf{T}}\mathfrak{n}_{\mathbf{T},\delta,t-\delta}H_{t-\delta} \geqslant \frac{\epsilon^s}{8}, \delta^4\sum_{t\in\mathbf{T}}|\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2(|H_{t-\delta}|^2 + \mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]) < \frac{\epsilon^{4s}}{8}, \mathcal{A}_{Y,u,q})\\
&+\mathbb{P}(\delta^4\sum_{t\in\mathbf{T}}|\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2(|H_{t-\delta}|^2 + \mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]) \geqslant \frac{\epsilon^{4s}}{8}, \mathcal{A}_{Y,u,q}),
\end{aligned}$$

where the first term is bounded, using (4.31), by $2\exp(-\frac{1}{16}\epsilon^{-4s})$. Moreover, it follows from the Hölder inequality that, for every $q^\diamond \in [1, \frac{q}{4}]$

$$\begin{aligned}
\mathbb{P}(\delta^4\sum_{t\in\mathbf{T}}|\mathfrak{n}_{\mathbf{T},\delta,t-\delta}|^2(|H_{t-\delta}|^2 + \mathbb{E}[|H_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]) \geqslant \frac{\epsilon^{4s}}{8}, \mathcal{A}_{Y,u,q})&\\
\leqslant\delta^{q^\diamond}\epsilon^{-4(s+u)q^\diamond}2^{6q^\diamond+2}|\delta|\mathbf{T}||^{3q^\diamond-1}&.
\end{aligned}$$

We also remark that, since $\sup_{t\in\mathbf{T}}|\bar{\Delta}^Y_{t-\delta}|\mathbf{1}_{\mathcal{A}_{Y,u,q}} \leqslant \epsilon^{-u}$, it follows from the Cauchy-Schwarz inequality that

$$|\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\bar{\Delta}^Y_{w-\delta}Y_{w-\delta}|\mathbf{1}_{\mathcal{A}_{Y,u,q}}\mathbf{1}_{|Y_0|<\epsilon^{-v}} < |\mathbf{T}|^{\frac{3}{2}}\epsilon^{-u}(\epsilon^{-2v} + \sum_{t\in\mathbf{T}}|Y_t|^2)^{\frac{1}{2}},$$

and, for $v > 0$, as soon as $\epsilon \in [|32|\delta|\mathbf{T}||^{\frac{3}{2}}\delta^{\frac{1}{2}}|^{\frac{1}{u+s+v}}, |32|\delta|\mathbf{T}||^{\frac{3}{2}}|^{-\frac{1}{\frac{1}{2}-s-u}}]$,

$$\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2 < \epsilon, |\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}2\bar{\Delta}^Y_{w-\delta}Y_{w-\delta}| \geqslant \frac{\epsilon^s}{8}, \mathcal{A}_{Y,u,q}, |Y_0| < \epsilon^{-v}) = 0$$

Moreover, from Markov inequality, for every $q^\diamond \geqslant \frac{p}{v}$, $\mathbb{P}(|Y_0| \geqslant \epsilon^{-v}) \leqslant \epsilon^p\mathbb{E}[|Y_0|^{q^\diamond}]$

Now for every $\epsilon \geqslant |16\delta^3|\mathbf{T}|^2|^{\frac{1}{s+2u}}$, using the Markov and Hölder inequalities yields

$$\begin{aligned}
\mathbb{P}(\delta^2 \sum_{\substack{w\in\mathbf{T}\\ w\leqslant t}} |\delta^{\frac{1}{2}} 2\tilde{\Delta}^Y_{w-\delta}\bar{\Delta}^Y_{w-\delta} + \delta|\bar{\Delta}^Y_{w-\delta}|^2| \geqslant \frac{\epsilon^s}{8}, \mathcal{A}_{Y,u,q})&\\
\leqslant \mathbb{P}(\delta^{5/2} \sum_{\substack{w\in\mathbf{T}\\ w\leqslant t}} |2\tilde{\Delta}^Y_{w-\delta}\bar{\Delta}^Y_{w-\delta}| \geqslant \frac{\epsilon^s}{8} - \delta^3|\mathbf{T}|^2\epsilon^{-2u}, \mathcal{A}_{Y,u,q})&\\
\leqslant \mathbb{E}[32^q|\mathbf{T}|^{2q-2}\delta^{5q/2} \sum_{\substack{w\in\mathbf{T}\\ w\leqslant t}} |\tilde{\Delta}^Y_{w-\delta}|^q \epsilon^{-q(s+u)} \mathbf{1}_{\mathcal{A}_{Y,u,q}}]&\\
\leqslant 32^q|\mathbf{T}|^{2q-2}\epsilon^{-q(s+u)}\delta^{5q/2} \sum_{\substack{w\in\mathbf{T}\\ w\leqslant t}} \mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^q \mathbf{1}_{\mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^q|\mathcal{F}^Y_{w-\delta}]\leqslant\epsilon^{-qu}}]&\\
\leqslant 32^q\delta^{\frac{q}{2}}\epsilon^{-q(s+2u)}|\delta|\mathbf{T}||^{2q}.&
\end{aligned}$$

In particular, taking $q^\diamond = 4q$, we have proved that for every $\epsilon \in [\underline{\epsilon}_1(\delta), |32|\delta|\mathbf{T}||^{\frac{3}{2}}|^{-\frac{1}{\frac{1}{2}-s-u}}]$,

$$\begin{aligned}
\mathbb{P}(\delta \sum_{t\in\mathbf{T}} |Y_t|^2 <\epsilon, \mathfrak{A}_2^c, \mathcal{A}_{Y,u,q}) \leqslant{}& \epsilon^p\mathbb{E}[|Y_0|^{\frac{p}{v}}]) + \mathbb{P}(\delta|Y_0|^2 \geqslant \epsilon) + \delta^{\frac{q}{4}}\epsilon^{-q(s+u)}2^{\frac{3q}{2}+2}|\delta|\mathbf{T}||^{\frac{3q}{4}-1}\\
&+ 32^q\delta^{\frac{q}{2}}\epsilon^{-q(s+2u)}|\delta|\mathbf{T}||^{2q} + \delta^{\frac{q}{4}}\epsilon^{-\frac{(2+v^\diamond)q}{4}}2^{\frac{3q}{4}+2}|\delta|\mathbf{T}||^{\frac{5q}{4}-1}\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^q]\\
&+ 2\exp(-\frac{\epsilon^{-4s}}{16}) + 2\exp(-\frac{\epsilon^{-v^\diamond}}{2}) + 2\exp(-\frac{\epsilon^{2(s+u)-1}}{2^{11}|\delta|\mathbf{T}||^2}).
\end{aligned}$$

At this point, we remark that

$$\begin{aligned}
\mathfrak{A}_2 \subset&\{\delta\sum_{t\in\mathbf{T}}|Y_0|^2 + \delta^2 \sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}} |\tilde{\Delta}^Y_{w-\delta}|^2 < \delta\sum_{t\in\mathbf{T}}|Y_t|^2 + \frac{\epsilon^s}{2}\}\\
&\cap \{\delta\sum_{t\in\mathbf{T}}|Y_0|^2 + \delta^2 \sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}} \mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}] < \delta\sum_{t\in\mathbf{T}}|Y_t|^2 + \frac{\epsilon^s}{2}\}
\end{aligned}$$

It follows that, for every $\epsilon \leqslant 2^{-\frac{1}{1-s}}$,

$$\{\delta\sum_{t\in\mathbf{T}}|Y_t|^2 < \epsilon\} \cap \mathfrak{A}_2 \subset \{\delta\sum_{t\in\mathbf{T}}|Y_t|^2 < \epsilon\} \cap \mathfrak{A}_1 \cap \{|Y_0|^2 < \frac{\epsilon^s}{\delta|\mathbf{T}|}\},$$

and the proof of **Step 1** is completed.
Step 2. We show that for every $\epsilon \in (0, \overline{\epsilon}_2(\delta)]$ and $u \in (0, \frac{s}{4} - \frac{3r}{4})$,

$$\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2 < \epsilon, \delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}] \geqslant \frac{\epsilon^r}{2}, \mathfrak{A}_1, \mathcal{A}_{Y,u,q}) = 0.$$

with $\overline{\epsilon}_2(\delta) = |2^7\delta|\mathbf{T}||^{-\frac{1}{s-3r-4u}}$. First, we notice that,
on the set $\{\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}] \geqslant \frac{\epsilon^r}{2}\} \cap \mathcal{A}_{Y,u,q}$, we have

$$\delta\sum_{t\in\mathbf{T}}\mathbf{1}_{\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]\geqslant\frac{\epsilon^r}{4\delta|\mathbf{T}|}} \geqslant \frac{\epsilon^{r+2u}}{4}$$

and it follows that

$$\begin{aligned}
\delta^2 \sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}} \mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}] \geqslant{}& \delta^2 \sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}} \mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}]\mathbf{1}_{\mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}]\geqslant\frac{\epsilon^r}{4\delta|\mathbf{T}|}}\\
\geqslant{}& \frac{\epsilon^r}{4\delta|\mathbf{T}|}\delta^2 \sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}} \mathbf{1}_{\mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}]\geqslant\frac{\epsilon^r}{4\delta|\mathbf{T}|}}\\
\geqslant{}& \frac{\epsilon^r}{4\delta|\mathbf{T}|}\frac{1}{2}\frac{\epsilon^{r+2u}}{4}(\frac{\epsilon^{r+2u}}{4}+1) \geqslant \frac{\epsilon^{3r+4u}}{2^7\delta|\mathbf{T}|}.
\end{aligned}$$

In particular for every $\epsilon \in (0, |2^7\delta|\mathbf{T}||^{-\frac{1}{s-3r-4u}}]$

$$\{\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]\geqslant\frac{\epsilon^r}{2}\}\cap\{\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}]<\epsilon^s\}\cap\mathcal{A}_{Y,u,q}=\emptyset.$$

and the proof of **Step 2** is completed.

Step 3. In this part we show that for every $\epsilon\in(\underline{\epsilon}_3(\delta),\overline{\epsilon}_3(\delta))$, every $h,s\in(3r,\frac{1}{2})$ with $2h<s$, $u\in(0,\min(\frac{s}{2}-h,\frac{h}{4}-\frac{3r}{4}))$,

$$\begin{aligned}\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2<\epsilon,&\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_t]<\frac{\epsilon^r}{2},\delta\sum_{t\in\mathbf{T}}|\bar{\Delta}^Y_{t-\delta}|^2\geqslant\frac{\epsilon^r}{2},\mathbf{A}_1,|Y_0|^2<\frac{\epsilon^s}{\delta|\mathbf{T}|},\mathcal{A}_{Y,u,q})\\ \leqslant&\delta^{\frac{q}{4}}(\delta^{\frac{q}{4}}\epsilon^{-q(h+2u)}+\epsilon^{-\frac{(2+v^\diamond)q}{4}})2^{5q+1}(1+T^{2q})(1+\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^q])\\ &+\epsilon^{\frac{q(s-2h-2u)}{2}}2^{3q}T^q+\mathbb{P}(\delta|Y_0|^2\geqslant\epsilon)\\ &+2\exp(-\frac{\epsilon^{2(h+u)-1}}{2^9T^2})+2\exp(-\frac{\epsilon^{-v^\diamond}}{2})+2\exp(-\frac{\epsilon^{2h+2u-s}}{2^7T})\end{aligned}$$

with

$$\begin{aligned}\underline{\epsilon}_3(\delta)=&|16\delta T^2|^{\frac{1}{h+2u}},\\ \overline{\epsilon}_3(\delta)=&\min(|2^8\delta|\mathbf{T}||^{-\frac{1}{h-3r-4u}},(4\delta|\mathbf{T}|)^{-\frac{1}{\frac{1}{2}-h-u}},|4\delta|\mathbf{T}||^{-\frac{1}{s-2h-2u}},1).\end{aligned}$$

We begin by writing for every $t\in\mathbf{T}$

$$\begin{aligned}Y_t\bar{\Delta}^Y_t=&Y_0\bar{\Delta}^Y_0+\sum_{\substack{w\in\mathbf{T}\\ w\leqslant t}}\delta^{\frac{1}{2}}(\tilde{\Delta}^Y_{w-\delta}\bar{\Delta}^Y_{w-\delta}+\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}Y_{w-\delta})\\ &+\delta(|\bar{\Delta}^Y_{w-\delta}|^2+\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\tilde{\Delta}^Y_{w-\delta})\\ &+\delta^{\frac{3}{2}}(\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\bar{\Delta}^Y_{w-\delta}+\tilde{\Delta}^Y_{w-\delta}\bar{\Delta}^{\bar{\Delta}^Y}_{w-\delta})+\delta^2\bar{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\bar{\Delta}^Y_{w-\delta}\end{aligned}$$

and we define for $h\in(3r,\frac{s}{2})$

$$\begin{aligned}\mathfrak{A}_3:=&\{|\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\delta^{\frac{3}{2}}\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}Y_{w-\delta}|<\frac{\epsilon^h}{8}\}\cap\{\delta^{\frac{3}{2}}|\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\tilde{\Delta}^Y_{w-\delta}\bar{\Delta}^Y_{w-\delta}|<\frac{\epsilon^h}{8}\}\\ &\cap\{|\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\tilde{\Delta}^Y_{w-\delta}|<\frac{\epsilon^h}{8}\}\\ &\cap\{\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\delta^{\frac{1}{2}}(\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\bar{\Delta}^Y_{w-\delta}+\tilde{\Delta}^Y_{w-\delta}\bar{\Delta}^{\bar{\Delta}^Y}_{w-\delta})+\delta\bar{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\bar{\Delta}^Y_{w-\delta}|<\frac{\epsilon^h}{8}\}.\end{aligned}$$

We take $u\in(0,\frac{s}{2}-h)$. Using the exact same approach as in **Step 1**, (4.31) together with the Markov and Hölder inequalities imply that, for every $v^\diamond>0$,

$$\begin{aligned}\mathbb{P}(|\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\delta^{\frac{3}{2}}\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}Y_{w-\delta}|\geqslant\frac{\epsilon^h}{8},\delta\sum_{t\in\mathbf{T}}|Y_t|^2<\epsilon,\mathcal{A}_{Y,u,q})\\ \leqslant\delta^{\frac{q}{4}}\epsilon^{-\frac{(2+v^\diamond)q}{4}}2^{\frac{3q}{4}+2}|\delta|\mathbf{T}||^{\frac{5q}{4}-1}\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^q]\\ +\mathbb{P}(\delta|Y_0|^2\geqslant\epsilon)+2\exp(-\frac{\epsilon^{2(h+u)-1}}{2^9|\delta|\mathbf{T}||^2})+2\exp(-\frac{\epsilon^{-v^\diamond}}{2}).\end{aligned}$$

In the same way, the inequality (4.31) yields

$$\mathbb{P}(\delta^{\frac{3}{2}}|\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\tilde{\Delta}^Y_{w-\delta}\bar{\Delta}^Y_{w-\delta}|\geqslant\frac{\epsilon^h}{8},\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\mathbb{E}[|\tilde{\Delta}^Y_{w-\delta}|^2|\mathcal{F}^Y_{w-\delta}]<\epsilon^s,\mathcal{A}_{Y,u,q})\leqslant\quad 2\exp(-\frac{\epsilon^{2h+2u-s}}{2^7\delta|\mathbf{T}|}).$$

Moreover,

$$\mathbb{P}(|\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\tilde{\Delta}^Y_{w-\delta}|\geqslant\frac{\epsilon^h}{8},\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\tilde{\Delta}^Y_{w-\delta}|^2<\epsilon^s,\mathcal{A}_{Y,u,q})$$
$$\leqslant\mathbb{P}(\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}|^2\geqslant\frac{\epsilon^{2h-s}}{64},\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\tilde{\Delta}^Y_{w-\delta}|^2<\epsilon^s,\mathcal{A}_{Y,u,q}).$$

From Markov and Hölder inequalities, we have

$$\mathbb{P}(\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}|^2\geqslant\frac{\epsilon^{2h-s}}{64},\mathcal{A}_{Y,u,q})\leqslant\epsilon^{\frac{q(s-2h)}{2}}2^{3q}\mathbb{E}[|\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}|^2|^{\frac{q}{2}}\mathbf{1}_{\mathcal{A}_{Y,u,q}}]$$
$$\leqslant\epsilon^{\frac{q(s-2h)}{2}}2^{3q}\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}\mathbb{E}[|\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}|^q\mathbf{1}_{\mathcal{A}_{Y,u,q}}]|\delta^2|\mathbf{T}|^2|^{q/2-1}$$
$$\leqslant\epsilon^{\frac{q(s-2h-2u)}{2}}2^{3q}|\delta|\mathbf{T}||^q.$$

Besides, for every $\epsilon\geqslant|16\delta^3|\mathbf{T}|^2|^{\frac{1}{h+2u}}$, using Markov and Hölder inequalities yields

$$\mathbb{P}(\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\delta^{\frac{1}{2}}(\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\bar{\Delta}^Y_{w-\delta}+\tilde{\Delta}^Y_{w-\delta}\bar{\Delta}^{\bar{\Delta}^Y}_{w-\delta})+\delta\bar{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\bar{\Delta}^Y_{w-\delta}|\geqslant\frac{\epsilon^h}{8},\mathcal{A}_{Y,u,q})$$
$$\leqslant\mathbb{P}(\delta^{5/2}\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\tilde{\Delta}^{\bar{\Delta}^Y}_{w-\delta}\bar{\Delta}^Y_{w-\delta}+\tilde{\Delta}^Y_{w-\delta}\bar{\Delta}^{\bar{\Delta}^Y}_{w-\delta}|\geqslant\frac{\epsilon^h}{16},\mathcal{A}_{Y,u,q})$$
$$\leqslant2^{5q+1}\delta^{\frac{q}{2}}\epsilon^{-q(h+2u)}|\delta|\mathbf{T}||^{2q}.$$

In particular, for every $\epsilon\geqslant\underline{\epsilon}_3(\delta)$,

$$\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2<\epsilon,\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]<\frac{\epsilon^r}{2},\mathfrak{A}_1,\mathfrak{A}_3^c,\mathcal{A}_{Y,u,q})\leqslant$$
$$\delta^{\frac{q}{4}}(2^{5q+1}\delta^{\frac{q}{4}}\epsilon^{-q(h+2u)}T^{2q}+\epsilon^{-\frac{(2+v^\diamond)q}{4}}2^{\frac{3q}{4}+2}T^{\frac{5q}{4}-1}\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^q])$$
$$+\epsilon^{\frac{q(s-2h-2u)}{2}}2^{3q}T^q+\mathbb{P}(\delta|Y_0|^2\geqslant\epsilon)$$
$$+2\exp(-\frac{\epsilon^{2(h+u)-1}}{2^9|\delta|\mathbf{T}||^2})+2\exp(-\frac{\epsilon^{-v^\diamond}}{2})+2\exp(-\frac{\epsilon^{2h+2u-s}}{2^7\delta|\mathbf{T}|}).$$

We notice that, similarly as in **Step 1**,

$$\mathfrak{A}_3\subset\{\delta\sum_{t\in\mathbf{T}}Y_0\bar{\Delta}^Y_0+\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\bar{\Delta}^Y_{w-\delta}|^2<\delta|\sum_{t\in\mathbf{T}}Y_t\bar{\Delta}^Y_t|+\frac{\epsilon^h}{2}\}.$$

It follows from the Cauchy-Schwarz inequality, that for every $\epsilon\leqslant|4\delta|\mathbf{T}||^{\frac{1}{1-2u-2h}}$,

$$\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2<\epsilon,\delta\sum_{t\in\mathbf{T}}|\bar{\Delta}^Y_{t-\delta}|^2\geqslant\frac{\epsilon^r}{2},\mathfrak{A}_1,\mathfrak{A}_3,|Y_0|^2<\frac{\epsilon^s}{\delta|\mathbf{T}|},\mathcal{A}_{Y,u,q})$$
$$\leqslant\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|\bar{\Delta}^Y_{t-\delta}|^2>\frac{\epsilon^r}{2},\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\bar{\Delta}^Y_{w-\delta}|^2<\epsilon^h+|\delta|\mathbf{T}||^{\frac{1}{2}}\epsilon^{-u}(\epsilon^{\frac{s}{2}}+\epsilon^{\frac{1}{2}}),\mathcal{A}_{Y,u,q}).$$

In particular, for $\epsilon\leqslant1\wedge|4\delta|\mathbf{T}||^{-\frac{1}{s-2h-2u}}$, the *r.h.s.* of the above inequality is bounded by

$$\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|\bar{\Delta}^Y_{t-\delta}|^2>\frac{\epsilon^r}{2},\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\bar{\Delta}^Y_{w-\delta}|^2<2\epsilon^h,\mathcal{A}_{Y,u,q}).$$

Similarly as in **Step 2**, we notice that, on the set $\{\delta\sum_{t\in\mathbf{T}}|\bar{\Delta}^Y_{t-\delta}|^2>\frac{\epsilon^r}{2}\}\cap\mathcal{A}_{Y,u,q}$ then

$$\delta\sum_{t\in\mathbf{T}}\mathbf{1}_{|\bar{\Delta}^Y_t|^2\geqslant\frac{\epsilon^r}{4\delta|\mathbf{T}|}}\geqslant\frac{\epsilon^{r+2u}}{4}\quad\text{and}\quad\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\bar{\Delta}^Y_{w-\delta}|^2\geqslant\frac{\epsilon^{3r+4u}}{2^7\delta|\mathbf{T}|}.$$

In particular for every $\epsilon\leqslant|2^8\delta|\mathbf{T}||^{-\frac{1}{h-3r-4u}}$,

$$\{\delta\sum_{t\in\mathbf{T}}|\bar{\Delta}^Y_{t-\delta}|^2\geqslant\frac{\epsilon^r}{2}\}\cap\{\delta^2\sum_{\substack{w,t\in\mathbf{T}\\ w\leqslant t}}|\bar{\Delta}^Y_{w-\delta}|^2<2\epsilon^h\}=\emptyset.$$

and the proof of **Step 3** is completed.

Step 4. We now show (4.37). In the first three **Steps**, we have proved that for , every $h,s\in(3r,\frac{1}{2})$ with $2h<s$, $u\in(0,\min(\frac{1}{2}-s,\frac{s}{2}-h,\frac{h}{4}-\frac{3r}{4}))$, every $v,v^\diamond>0$, every $q\geqslant4$ and every $\epsilon\in[\max(\underline{\epsilon}_1(\delta),\underline{\epsilon}_3(\delta)),\min(1,\bar{\epsilon}_1(\delta),\bar{\epsilon}_2(\delta),\bar{\epsilon}_3(\delta))]$,

$$\begin{aligned}&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2<\epsilon,\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]+|\bar{\Delta}^Y_{t-\delta}|^2\geqslant\epsilon^r,\mathcal{A}_{Y,u,q})\\ &\leqslant\epsilon^p\mathbb{E}[|Y_0|^{\frac{p}{v}}])+2\mathbb{P}(\delta|Y_0|^2\geqslant\epsilon)+\epsilon^{\frac{q(s-2h-2u)}{2}}2^{3q}T^q\\ &\quad+\delta^{\frac{q}{4}}(2\delta^{\frac{q}{4}}\epsilon^{-q(s+2u)}+3\epsilon^{-q\frac{(2+v^\diamond)}{4}})2^{5q+1}(1+T^{2q})(1+\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^q])\\ &\quad+2\exp(-\frac{\epsilon^{-4s}}{16})+4\exp(-\frac{\epsilon^{-v^\diamond}}{2})+6\exp(-\frac{\epsilon^{2s+2u-1}}{2^{11}(1+T^2)})\end{aligned}$$

We first observe that

$$\begin{aligned}\mathbb{P}(\mathcal{A}^c_{Y,u,q})\leqslant&\epsilon^p(\mathbb{E}[\sup_{t\in\mathbf{T}}|\bar{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^{\frac{p}{u}}]+\mathbb{E}[\sup_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]^{\frac{p}{qu}}]\\ &+\mathbb{E}[\sup_{t\in\mathbf{T}}|\bar{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^{\frac{p}{u}}]+\mathbb{E}[\sup_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]^{\frac{p}{qu}}]).\end{aligned}$$

At this point, we assume that $q\geqslant\frac{2p}{s-2h-2u}$. Since $\epsilon\geqslant\delta^{\frac{1}{2+v^\diamond+\frac{p}{q}}}$, then

$$\begin{aligned}&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2<\epsilon,\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]+|\bar{\Delta}^Y_{t-\delta}|^2\geqslant\epsilon^r,\mathcal{A}_{Y,u,q})\\ &\leqslant\epsilon^p\mathbb{E}[|Y_0|^{\frac{p}{v}}])+2\mathbb{P}(\delta|Y_0|^2\geqslant\epsilon)\\ &\quad+\epsilon^p2^{5q+4}(1+T^{2q})(1+\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^q])\\ &\quad+2\exp(-\frac{\epsilon^{-4s}}{16})+4\exp(-\frac{\epsilon^{-v^\diamond}}{2})+6\exp(-\frac{\epsilon^{2s+2u-1}}{2^{11}(1+T^2)})\end{aligned}$$

Moreover, for every $q^\diamond>0$ such that $\epsilon\geqslant\delta^{\frac{q^\diamond}{q^\diamond+2p}}$, $\mathbb{P}(\delta|Y_0|^2\geqslant\epsilon)\leqslant\epsilon^p\mathbb{E}[|Y_0|^{q^\diamond}]$. In particular, we take $q^\diamond=\frac{2pq}{q(1+v^\diamond)+p}$ so that this inequality is satisfied for $\epsilon\geqslant\delta^{\frac{1}{2+v^\diamond+\frac{p}{q}}}$.

Now we fix $s=s(r):=\frac{5}{11}+\frac{6}{11}r$, $h=h(r):=\frac{2}{11}+\frac{9}{11}r$ and take $u<\frac{1}{22}-\frac{6}{11}r$. Since $r\in(0,\frac{1}{12})$, $s(r)\in(6r,\frac{1}{2})$, $h(r)\in(3r,\frac{s(r)}{2})$ and $\min(\frac{1}{2}-s(r),\frac{s(r)}{2}-h,\frac{h(r)}{4}-\frac{3r}{4}))>0$. Moreover, taking $v=\frac{6}{11}-\frac{6}{11}r-u+\frac{v^\diamond}{2}+\frac{p}{2q}$, and $q\geqslant\max(4,\frac{2p}{\frac{1}{11}-\frac{12}{11}r-2u})$, we have, for every $\epsilon\in[|2^{10}(1+T^3)\delta|^{\frac{1}{2+v^\diamond+\frac{p}{q}}},\min(|2^8T|^{-\frac{1}{\frac{2}{11}-\frac{24}{11}r-4u}},2^{-\frac{1}{\frac{6}{11}-\frac{6}{11}r}})]$,

$$
\begin{aligned}
&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2<\epsilon,\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]+|\bar{\Delta}^Y_{t-\delta}|^2\geqslant\epsilon^r,\\
&\leqslant\epsilon^p\mathbb{E}[|Y_0|^{\frac{p}{\frac{6}{11}-\frac{6}{11}r-u+\frac{v^\diamond}{2}+\frac{p}{2q}}}])+2\epsilon^p\mathbb{E}[|Y_0|^{\frac{2pq}{q(1+v^\diamond)+p}}]\\
&\quad+\epsilon^p2^{5q+4}(1+T^{2q})(1+\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^q])\\
&\quad+\epsilon^p(\mathbb{E}[\sup_{t\in\mathbf{T}}|\bar{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^{\frac{p}{u}}]+\mathbb{E}[\sup_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]^{\frac{p}{qu}}]\\
&\quad+\mathbb{E}[\sup_{t\in\mathbf{T}}|\bar{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^{\frac{p}{u}}]+\mathbb{E}[\sup_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^q|\mathcal{F}^Y_{t-\delta}]^{\frac{p}{qu}}])\\
&\quad+4\exp(-\frac{\epsilon^{-v^\diamond}}{2})+8\exp(-\frac{\epsilon^{-\frac{1}{11}+\frac{12}{11}r+2u}}{2^{11}(1+T^2)}).
\end{aligned}
$$

Now we take $u=\frac{1}{44}-\frac{3}{11}r$ and $q=q(r,p)=\max(4,\frac{2p}{\frac{1}{11}-\frac{12}{11}r-2u},\frac{p}{u})=\max(4,\frac{44p}{1-12r})$ (in particular $q(r,p)\geqslant\frac{2p}{s-2h-2u}$). It follows that, for every $\epsilon\in[|2^{10}(1+T^3)\delta|^{\frac{1}{2+v^\diamond+\frac{p}{q(r,p)}}},|2^8(1+T)|^{-\frac{11}{1-12r}}]$,

$$
\begin{aligned}
&\mathbb{P}(\delta\sum_{t\in\mathbf{T}}|Y_t|^2<\epsilon,\delta\sum_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^2|\mathcal{F}^Y_{t-\delta}]+|\bar{\Delta}^Y_{t-\delta}|^2\geqslant\epsilon^r,\\
&\leqslant 3\epsilon^p\mathbb{E}[|Y_0|^{\frac{2pq(r,p)}{q(r,p)(1+v^\diamond)+p}}]\\
&\quad+\epsilon^p2^{5q(r,p)+4}(1+T^{2q(r,p)})(1+\sup_{t\in\mathbf{T}}\mathbb{E}[|Y_{t-\delta}|^{q(r,p)}])\\
&\quad+\epsilon^p(2+\mathbb{E}[\sup_{t\in\mathbf{T}}|\bar{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^{q(r,p)}]+\mathbb{E}[\sup_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^Y_{t-\delta}|^{q(r,p)}|\mathcal{F}^Y_{t-\delta}]]\\
&\quad+\mathbb{E}[\sup_{t\in\mathbf{T}}|\bar{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^{q(r,p)}]+\mathbb{E}[\sup_{t\in\mathbf{T}}\mathbb{E}[|\tilde{\Delta}^{\bar{\Delta}^Y}_{t-\delta}|^{q(r,p)}|\mathcal{F}^Y_{t-\delta}]])\\
&\quad+4\exp(-\frac{\epsilon^{-v^\diamond}}{2})+8\exp(-\frac{\epsilon^{-\frac{1}{22}+\frac{6}{11}r}}{2^{11}(1+T^2)}).
\end{aligned}
$$

Since $q(r,p)>p$ and $v^\diamond>0$, $\mathbb{E}[|Y_0|^{\frac{2pq(r,p)}{q(r,p)(1+v^\diamond)+p}}]\leqslant 1+\mathbb{E}[|Y_0|^{q(r,p)}]$. We fix $v^\diamond=\frac{1}{22}-\frac{6}{11}r$ and the proof of (4.37) is completed.

□

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