THERMO- ACOUSTIC TOMOGRAPHY

AN OPTIMAL CONTROL APPROACH

M. Bergounioux

University of Orléans, Laboratory MAPMO, CNRS, UMR 7349,
Bâtiment de Mathématiques, BP 6759,
45067 Orléans cedex 2, France

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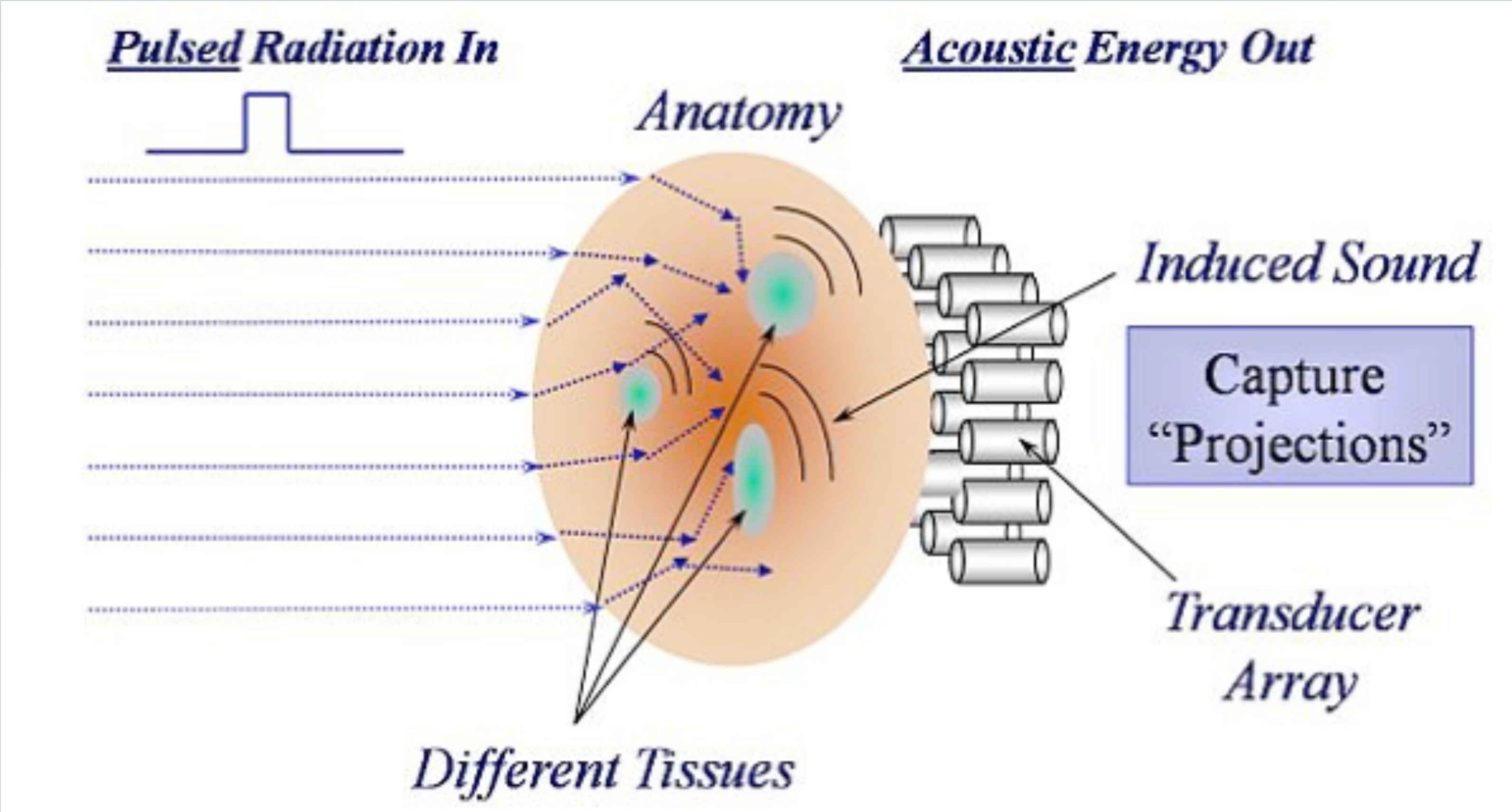
Outline

1. The direct problem
2. The inverse problem
3. Numerical results

1. The PAT direct problem

➡ Biologic tissue is irradiated by an energy source that is absorbed by the body. The source of energy is non-specific, but typically consists of visible light, near infrared, radio waves or microwaves.

➡ The absorbed energy is converted to heat, which raises the temperature of the tissue, typically by less than 0.001 degree Celsius.



➡ The increase in the temperature of the tissue causes the tissue to expand in volume, however slightly.

➡ This mechanical expansion produces an acoustic wave that propagates outward in all directions from the sight of energy absorption at the velocity of sound in biologic tissue, approximately 1.5 mm per microsecond.

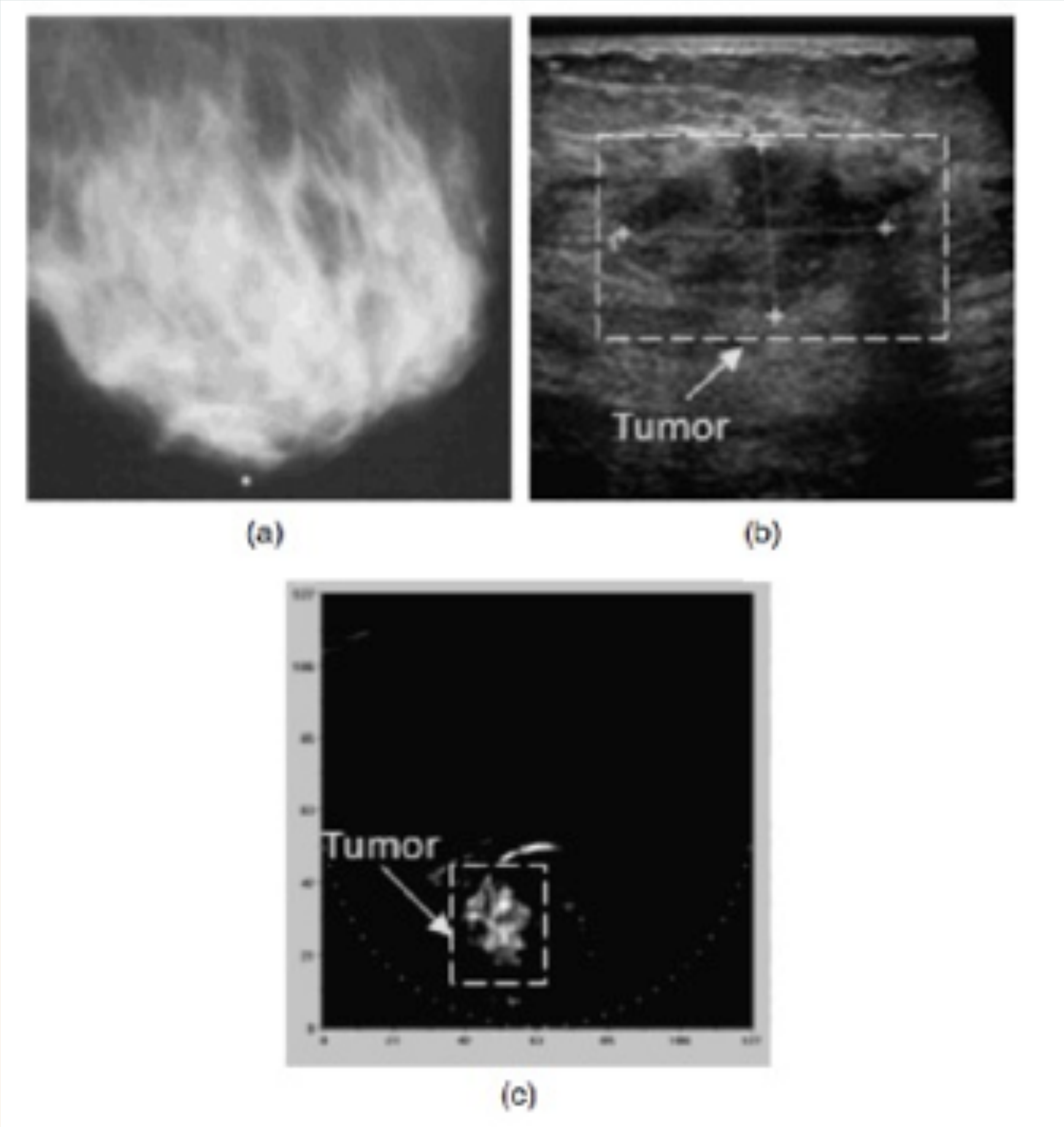
The term "photoacoustic" applies to this phenomenon when the stimulating radiation is optical, while "thermoacoustic" is the more general term and refers to all radiating sources, including optical.

Combine the advantages of non invasive techniques

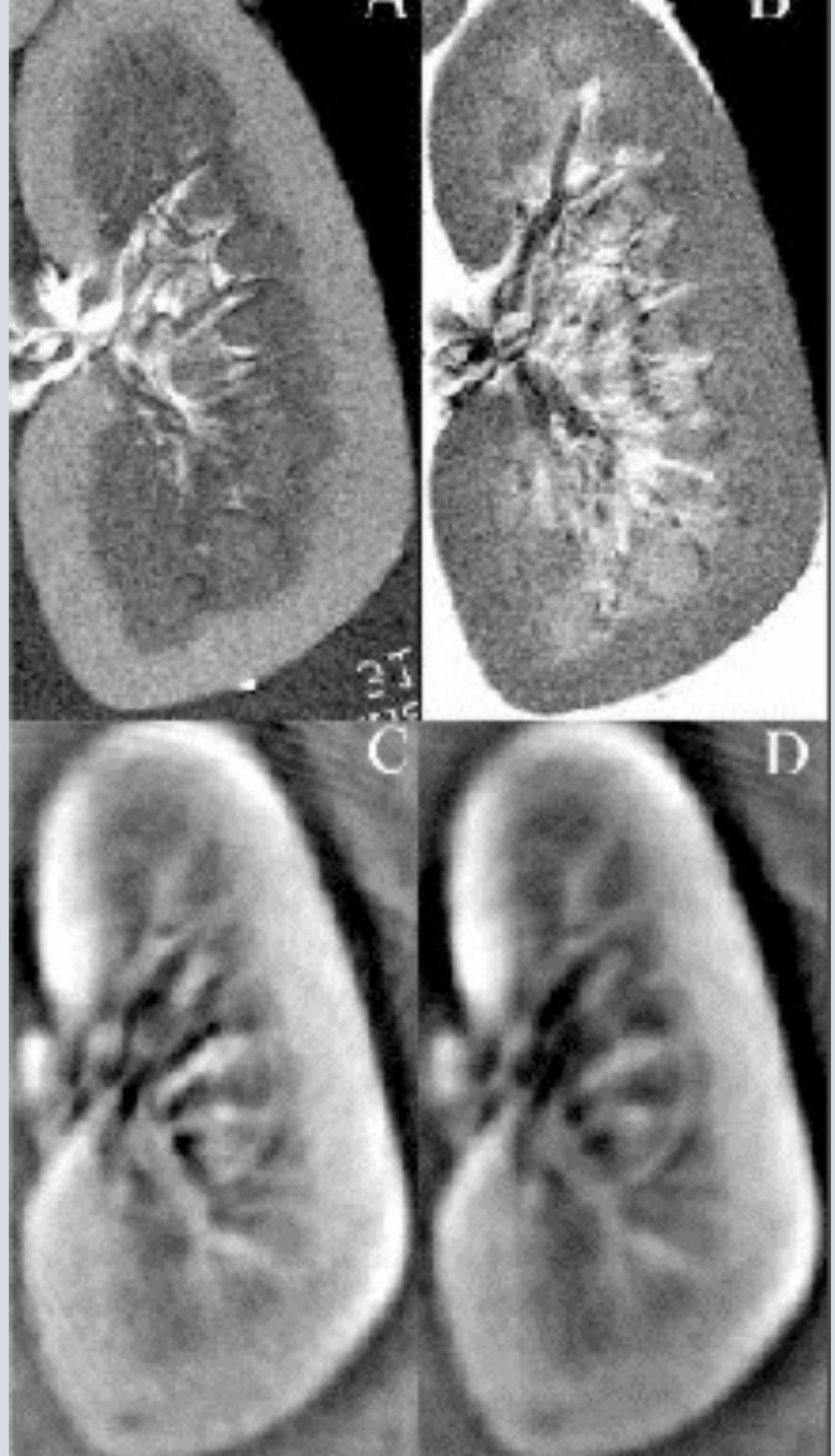
➡ Optics = contrast, sensitivity : functional and quantitative information

➡ Acoustics = high resolution thanks to the small acoustic wave diffusion

Morphological and functional information with the same device

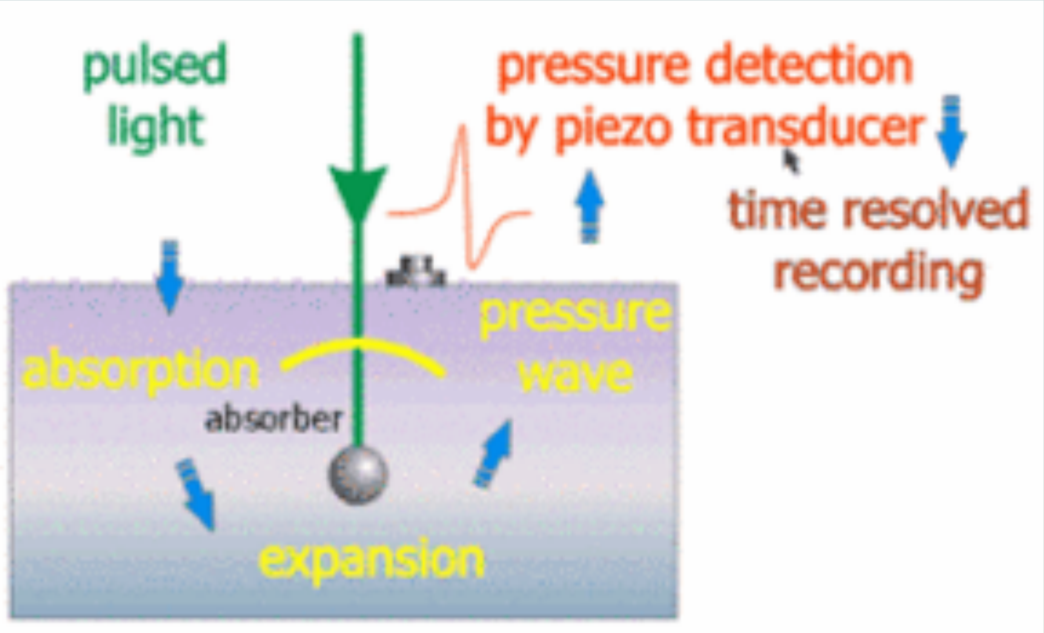


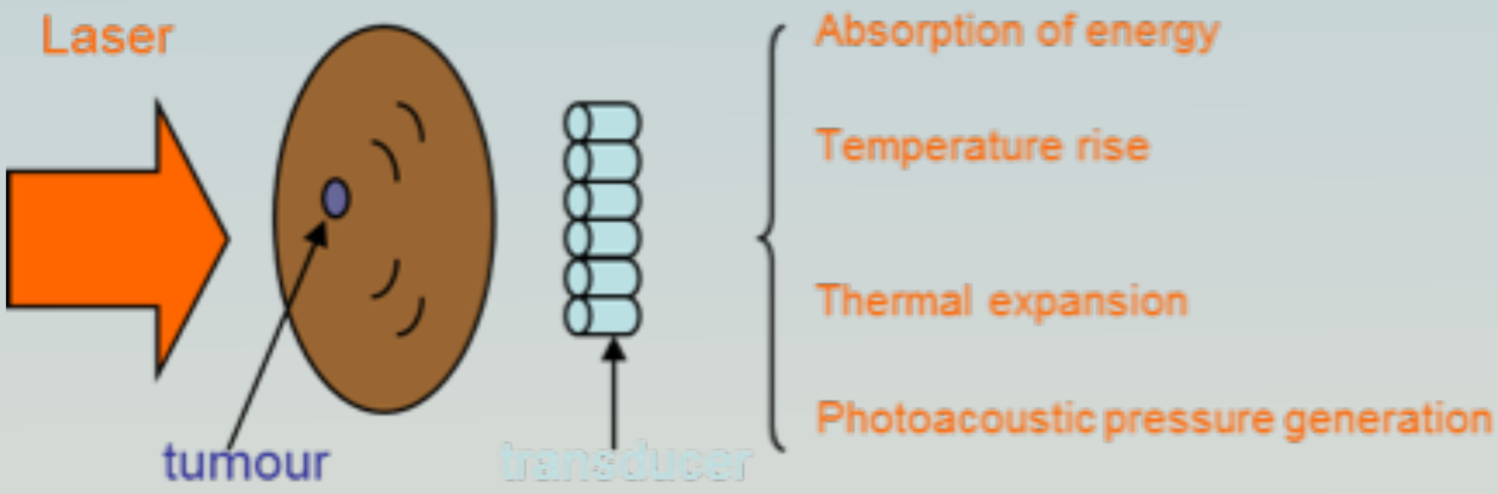
Clinical image showing breast tumor: (a) mammography, (b) ultrasonic and (c) photoacoustic images. High contrast of the PA image implies advanced angiogenesis indicative of a malignant tumor

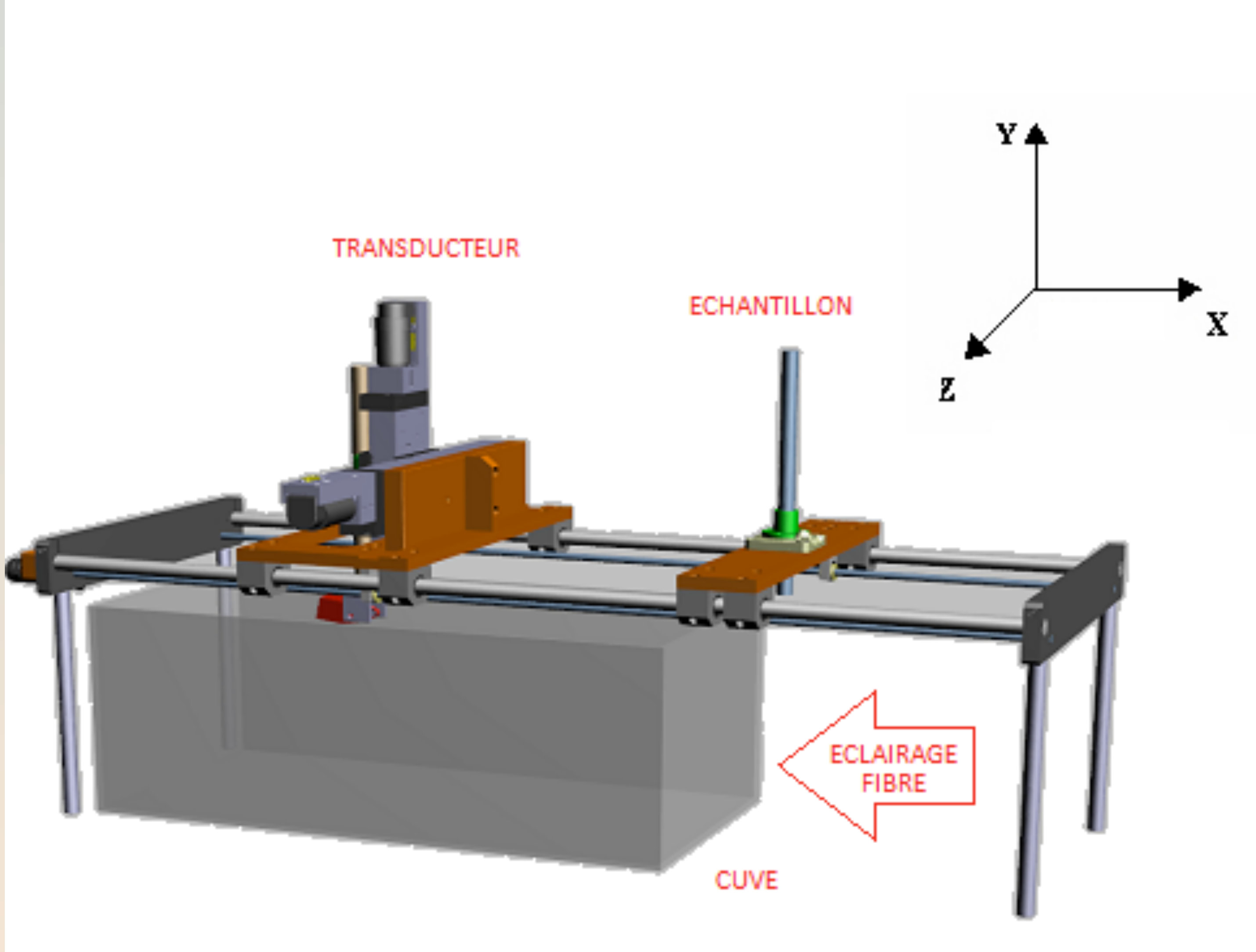


Thermoacoustic images of biologic tissue (lamb kidney)(C,D) in comparison to MRI images of the same kidney (A,B).

1.2. Experimental device (LMA- Fresnel Institute Marseille)

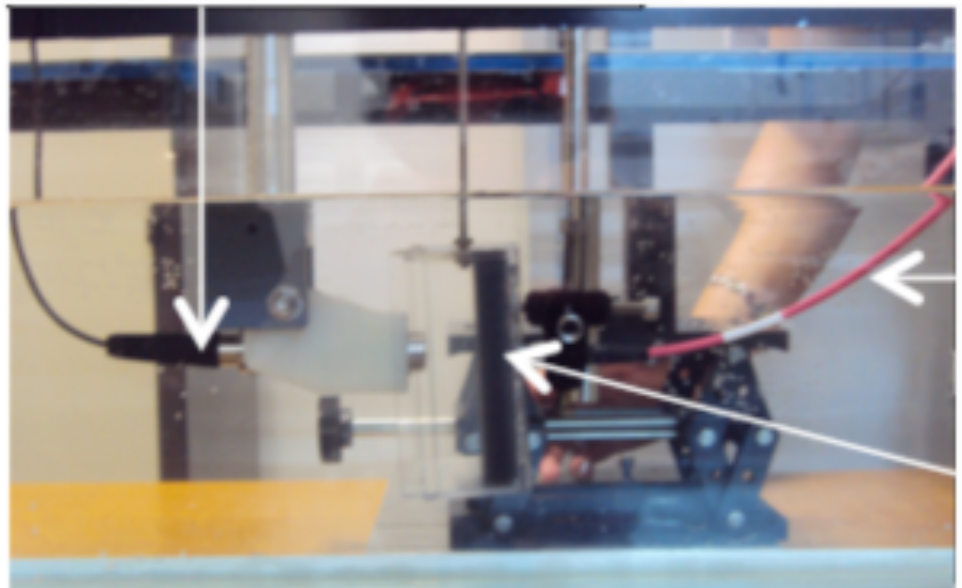






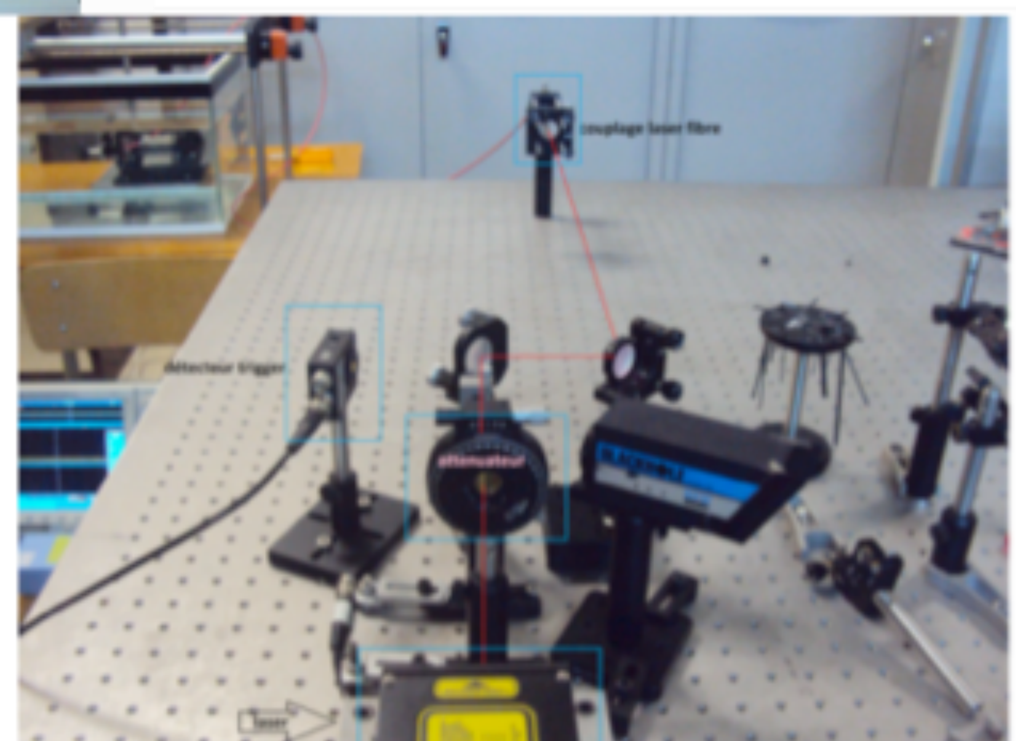
Source : Anabela Da Silva
(Fresnel Institute)

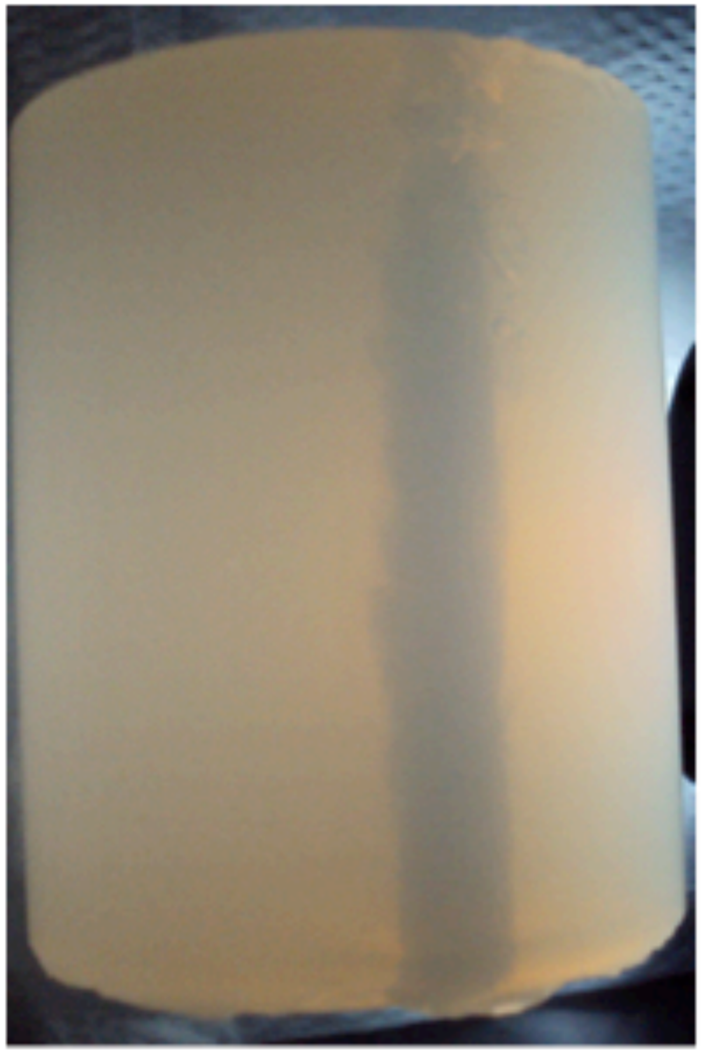
Acoustic detector : transducer (focused at 2cm and 3.5 MHz)

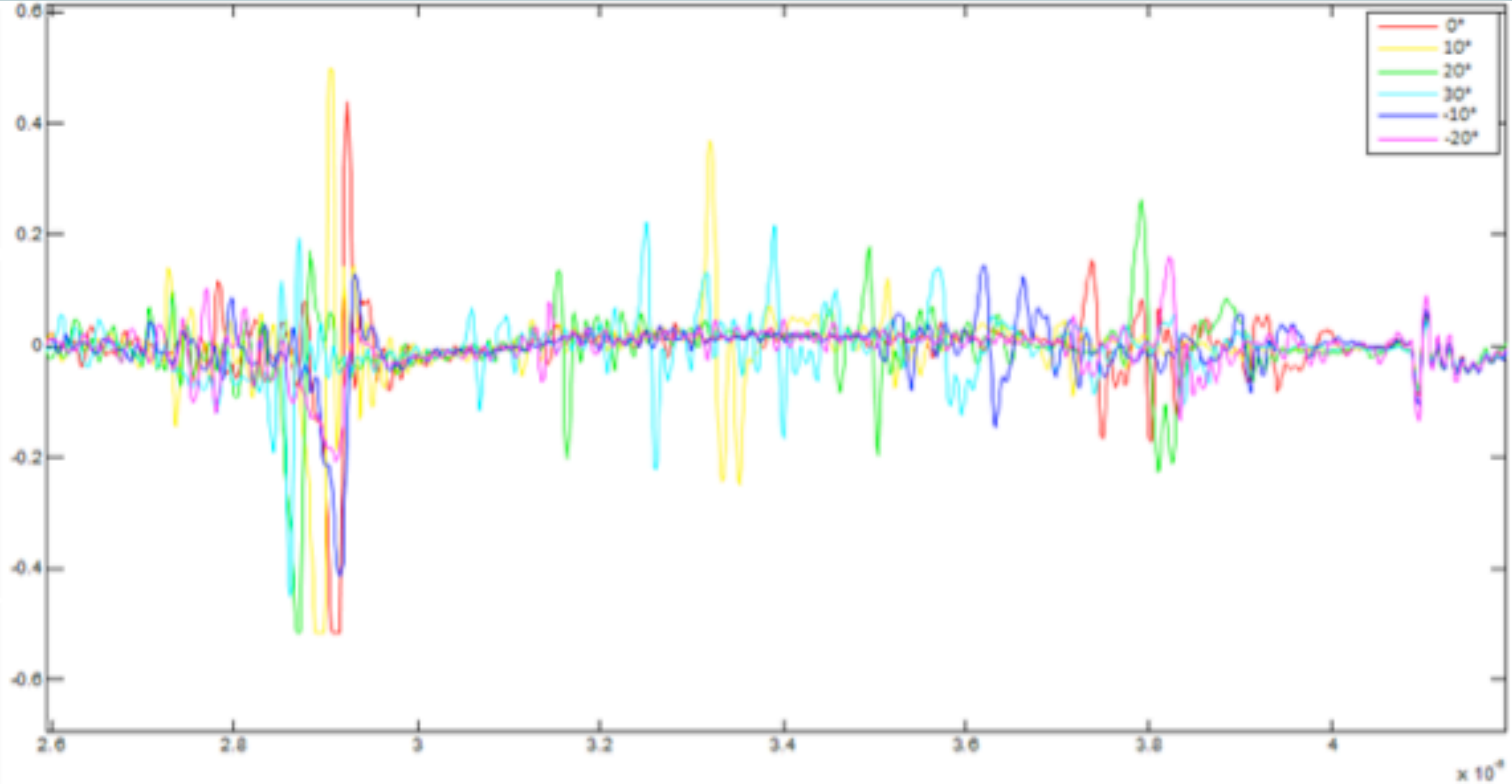


Optical source via optical fiber

Phantom : Agar-agar+ water+ black ink
to simulate optical properties of biological tissues







Rotations of the sample are performed : the red curve corresponds to 0°and so on. At 41 μs we get the optical fiber signal since all signals are superposed. At 28.7 μs we observe a more important signal due to the fictitious tumor in the phantom. We may estimate the distance between the tumor and the lightened edge (red curve) : (4.09-2.87). 10^{-5} . 1540 = 0.018788 m (sound speed 1540 ms^{-1}). The measured value is 18.27 mm. Others signals are due to interfaces.

1.1 Modeling optical effect

EM wave propagation through biological tissues: Visible wavelength range (PAT):

Biological tissues are not transparent to visible light:

- There is a strong absorption, except in the red and near infrared regions;
- Even though one chooses to study the tissue in the latter wavelength range, the light undergo strong scattering. **The physical problem then reduces to the modelling of light propagation in absorbing and scattering media.**

The Radiation Transfer Equation (RTE) allows for modelling in a general setting (mesoscopic scale) the propagation of the luminance L(r,t,s) [expressed in $W.m^{-2}.sr^{-1}$], representing the power measured at a position r, at a time t, in the direction of observation s, in a diffusive medium.

✓ μ_a absorption coefficient : this coefficient shows a good contrast between different kinds of soft tissues (healthy or not, etc.). Note that μ_a=0 outside Ω.

$\mu_a \in [\mu_a^{min}, \mu_a^{max}]$, $\mu_a^{min}>0$.

✓ D : diffusion coefficient $D = [3(\mu_a + \mu_s)^{-1}]$ where μ_s is the reduced scattering coefficient with $\mu_s \in [\mu_s^{min}, \mu_s^{max}]$, $\mu_s^{min}>0$.

μ_s:=(1-g) μ_s', with μ_s' the scattering coefficient and g is the anisotropy factor (g=0 outside Ω.) In general, $\mu_a \ll \mu_s$, so that $D \approx (3\mu_s)^{-1}$

Under the assumptions that the studied media

- o are far more scattering than absorbing (μ_s(r) >>μ_a(r)), and
- o thick compared to the mean free path,

assumptions which are quite reasonable in many cases, the RTE reduces to:

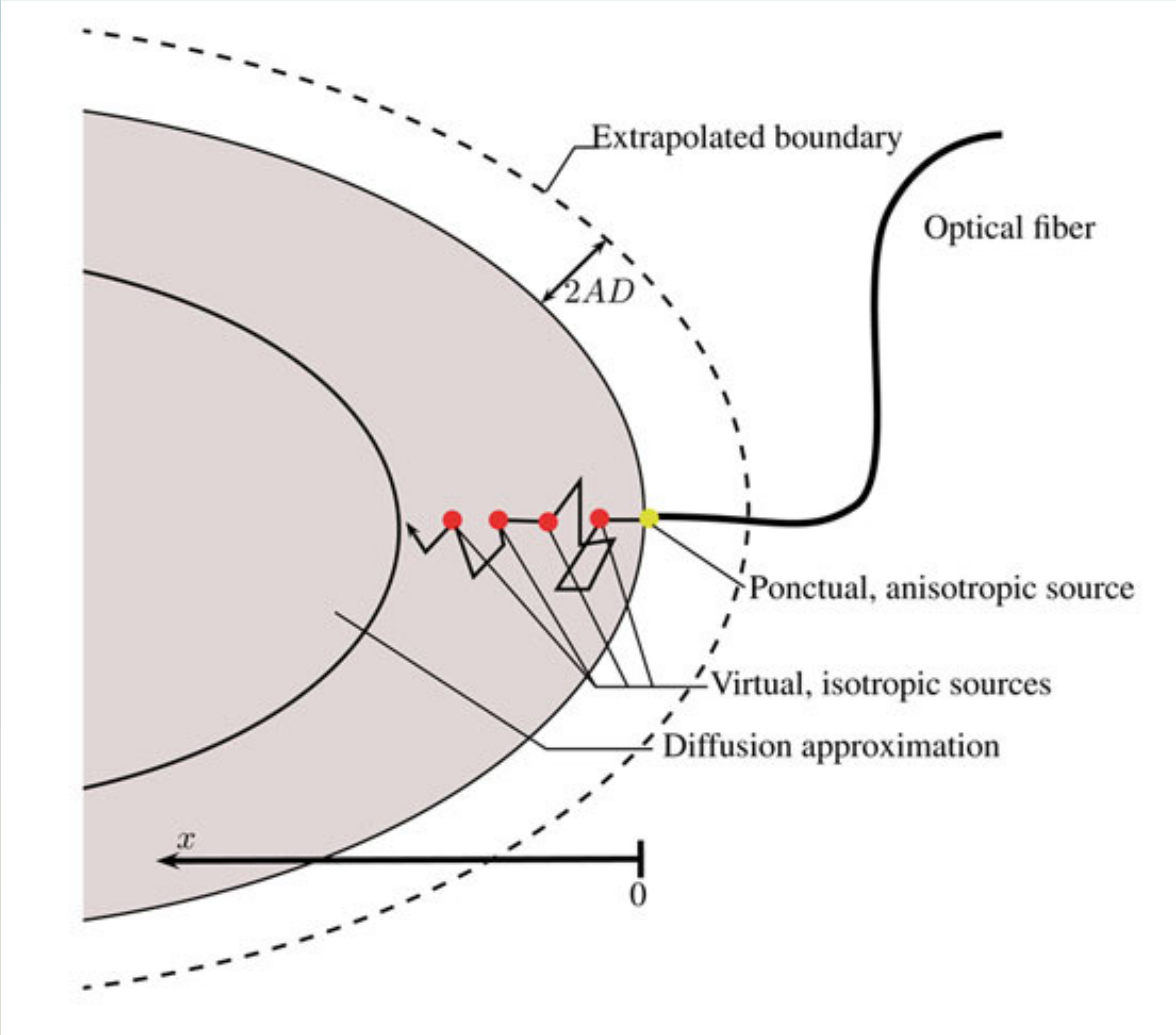
$$\begin{cases} \frac{1}{\nu}\frac{\partial I}{\partial t}(t,x) + \mu_a(x)I(t,x) - \mathrm{div}(D\nabla I)(t,x) = S(t,x), & (t,x) \in [0,T] \times \Omega \\ I(0,x) = 0, & x \in \Omega, \end{cases}$$

where the fluence rate I is the mean value of the luminance intensity L

$$I = \int_{B(0,1)} L(r,t,s)\, ds$$

- ν is the light speed
- S is the source
- D is the diffusion coefficient
- μ_a is the absorption coefficient

Boundary conditions



Ω is the part of the object where the diffusion approximation is relevant. Usually, the tissues close to the surface are not included. However, we may use the approximation in the whole oject if the source is chosen in a appropriate way. Then the usual Robin condition is replaced by a Dirichlet boundary condition exending the domain Ω

1.2 Modeling thermal and acoustic effects

$$\frac{\partial}{\partial t}\left(T - \frac{\gamma - 1}{\gamma\alpha} p_0\right) - \frac{K}{\rho C_P}\Delta T = \frac{H}{\rho C_P}$$

$$H = \mu_a I$$

$$\frac{\partial^2 p_0}{\partial t^2} - \mathrm{div}\left(\frac{v_s^2}{\gamma}\nabla p_0\right) = \alpha \frac{\partial^2 T}{\partial t^2}$$

- T is the temperature
- p_0 is the pressure
- α pressure coefficient expansion
- K thermal conductivity
- ρ density
- γ specific heat ratio
- v_s is the sound speed

Time scale for launching a sound wave may be far shorter than that for thermal conduction : Zero thermal conduction assumption

Thermal conductivity K set to 0 $\Rightarrow \gamma = 1$

$$\frac{\partial}{\partial t}\left(T - \frac{\gamma - 1}{\gamma\alpha} p_0\right) - \frac{K}{\rho C_P}\Delta T = \frac{H}{\rho C_P}$$

$$\frac{\partial T}{\partial t} = \frac{\mu_a I}{\rho C_P}$$

$$\frac{\partial^2 p_0}{\partial t^2} - \mathrm{div}\ \left(v_s^2 \nabla p_0\right) = \frac{\mu_a}{\rho C_P}\frac{\partial I}{\partial t}$$

Final pression equation (acoustic wave)

$$\begin{cases} \dfrac{\partial^2 p^0}{\partial t^2}(t,x) - \mathrm{div}(v_s^2 \nabla p^0)(t,x) = \mathbb{1}_\Omega(x)\Gamma(x)\mu_a(x)\dfrac{\partial I}{\partial t}(t,x), \ (t,x) \in [0,T] \times \mathcal{B}, \\ p^0(0,x) = \dfrac{\partial p^0}{\partial t}(0,x) = 0, \qquad x \in \mathcal{B} \end{cases}$$

- v_s is the sound speed
- $\mathbb{1}_\Omega$ is the indicatrix function of $\mathbf{\Omega}$
- Γ is the Grüneisen coefficient
- μ_a is the absorption coefficient
- $\mathcal{B}$ is a large ball

→ $$p(t,x) = \int_0^t p^0(s,x)\mathrm{d}s.$$

1.3 The full direct problem

$$
\begin{cases}
\dfrac{\partial^2 p}{\partial t^2}(t,x) - \mathrm{div}(v_s^2 \nabla)(t,x) = \mathbb{1}_\Omega(x)\Gamma(x)\mu_a(x)I(t,x), & (t,x) \in (0,T)\times\mathcal{B}, \\
p(t,x) = 0, & (t,x) \in (0,T)\times\partial\mathcal{B}, \\
p(0,x) = \dfrac{\partial p}{\partial t}(0,x) = 0, & x \in \mathcal{B},
\end{cases}
$$

$$
\begin{cases}
\dfrac{1}{\nu}\dfrac{\partial I}{\partial t}(t,x) + \mu_a(x)I(t,x) - \mathrm{div}(D\nabla I)(t,x) = S(t,x), & (t,x) \in (0,T)\times\Omega, \\
I(0,x) = 0, & x \in \Omega, \\
I(t,x) = 0 & (t,x) \in \mathcal{B}\backslash\Omega \\
I(t,x) = 0, & (t,x) \in \Sigma.
\end{cases}
$$

$$
\mu := (\mu_a, D),\ \ \mu_a \in [\mu_a^{\min}, \mu_a^{\max}],\ \ D \in [D^{\min}, D^{\max}],\ \ S \in L^\infty(\mathcal{Q})
$$

Case of many sources S_k

$$
\begin{cases}
\dfrac{\partial^2 p_k}{\partial t^2}(t,x) - \mathrm{div}(v_s^2 \nabla p_k)(t,x) = \mathbb{1}_\Omega(x)\Gamma(x)\mu_a(x) I_k(t,x), & (t,x) \in (0,T)\times\mathcal{B}, \\
p_k(t,x) = 0, & (t,x) \in (0,T)\times\partial\mathcal{B}, \\
p_k(0,x) = \dfrac{\partial p_k}{\partial t}(0,x) = 0, & x \in \mathcal{B},
\end{cases}
\tag{2.3}
$$

and

$$
\begin{cases}
\dfrac{1}{\nu}\dfrac{\partial I_k}{\partial t}(t,x) + \mu_a(x) I_k(t,x) - \mathrm{div}(D\nabla I_k)(t,x) = S_k(t,x), & (t,x) \in (0,T)\times\Omega, \\
I_k(0,x) = 0, & x \in \Omega, \\
I_k(t,x) = 0 & x \in \mathcal{B}\backslash\Omega \\
I_k(t,x) = 0, & (t,x) \in \Sigma.
\end{cases}
$$

$$
\boldsymbol{\mu} := (\mu_a, D) \qquad \mu_a \in [\mu_a^{\min}, \mu_a^{\max}] \qquad D \in [D^{\min}, D^{\max}]
$$

$(S_k)_{1\le k\le s}$ with $s \ge 2$ and each S_k in $L^\infty(\mathcal{Q})$.

1.4 Stationary approximation

$$\begin{cases} \dfrac{1}{\nu}\dfrac{\partial I}{\partial t}(t,x) + \mu_a(x)I(t,x) - \operatorname{div}(D\nabla I)(t,x) = S(t,x), & (t,x) \in (0,T)\times\Omega, \\ I(0,x) = 0, & x \in \Omega, \\ I(t,x) = 0 & (t,x) \in \mathcal{B}\backslash\Omega \\ I(t,x) = 0, & (t,x) \in \Sigma. \end{cases}$$

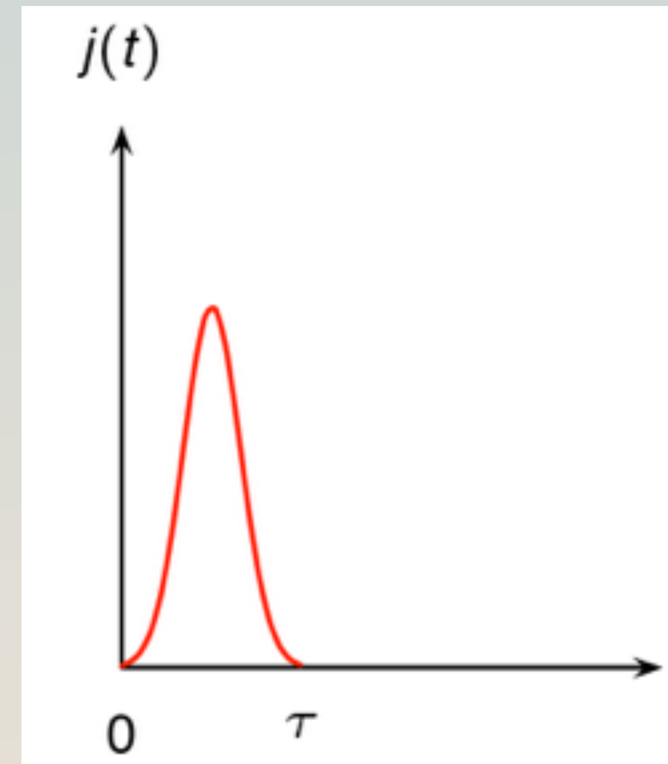
The speed of light is very large so that the light effect may be considered as instantaneous. So, assume

$$I = I^0(x)\delta_0(t)$$

$$\begin{cases} \mu_a(x)I^0(x) - \operatorname{div}(D\nabla I^0)(x) = S(x), & x \in \Omega, \\ I^0((x) = 0 & x \in \mathcal{B}\backslash\Omega \\ I^0((x) = 0, & \text{on } \partial\Omega \end{cases}$$

$$
\begin{cases}
\dfrac{\partial^2 p}{\partial t^2} - div(v^2 \nabla p) = C v^2 \mu_a 1_\Omega \dfrac{\partial I}{\partial t} & \text{in } (0,T)\times\mathcal{B} \\
p(0,\cdot) = \dfrac{\partial p}{\partial t}(0,\cdot) = 0 & \text{in } \mathcal{B} \\
p = 0 & \text{in } (0,T)\times\partial\mathcal{B},
\end{cases}
$$

$$
\begin{cases}
\dfrac{\partial^2 p}{\partial t^2} - div(v^2 \nabla p) = u_0(x) \dfrac{\partial j}{\partial t} & \text{in } (0,T)\times\mathcal{B} \\
p(0,\cdot) = \dfrac{\partial p}{\partial t}(0,\cdot) = 0 & \text{in } \mathcal{B} \\
p = 0 & \text{in } (0,T)\times\partial\mathcal{B},
\end{cases}
$$



u_0 (= H) is the energy deposition function

This is the usual simplified model for TAT (RF source)

Usual simplified model for TAT (RF source)

$$(TAT)\begin{cases} \dfrac{\partial^2}{\partial t^2}(x,t) - c^2(x)\Delta p(x,t) & = & f(x)\dfrac{\partial j}{\partial t}(t), \\ p(x,0) & = & 0, \\ \dfrac{\partial p}{\partial t}(x,0) & = & 0. \end{cases}$$

$$p(x,t) = \left(\frac{\mathrm{d}j}{\mathrm{d}t} \circledast \big(tRf\big)\right)(x,t). \qquad (5$$

Here, the *convolution* operation $\circledast$ is defined by

$$\big(g \circledast h\big)(x,t) = \int_0^t g(t-s)h(x,s)\,\mathrm{d}s$$

and the operator R, referred to as the *spherical Radon transform*, is defined by

$$(Rf)(x,t) := \frac{1}{4\pi}\int_{S^2} f(x+t\omega)\,\mathrm{d}\omega.$$

In TAT, one is rather confronted to the *inverse problem*:

> *Recover the energy deposition function $f(x)$ from measurements of $p(x,t)$ for x over a surface S outside the illuminated fluid.*

$$\frac{\mathrm{d}}{\mathrm{d}t}(j \circledast h)(x,t) = \left(\frac{\mathrm{d}j}{\mathrm{d}t} \circledast h\right)(x,t).$$

If $j \approx \delta$, then $\frac{\mathrm{d}j}{\mathrm{d}t} \circledast h \approx \frac{\mathrm{d}h}{\mathrm{d}t}$, so that $\left(\frac{\mathrm{d}j}{\mathrm{d}t} \circledast \big(tRf\big)\right)(x,t) \approx \frac{\mathrm{d}}{\mathrm{d}t}(tRf)$. Consequently, Equation (5) can be approximated by

$$p(x,t) = \frac{\mathrm{d}}{\mathrm{d}t}(tRf). \tag{6}$$

However, this integral formulation is purely *linear* and cannot be genelarized to non-linear models. From the numerical point of view, the integral formulation leads to the so-called filtered back-projection method.

Usual methods

- *The filtered backprojection* approach is the most popular[17,22,23,24,29,41]. However, it is not clear that backprojection-type formulae could be written for any closed observation surface $\mathcal{S}$. In [23], inversion formulae are provided assuming odd dimensions and constant sound speed. Indeed, in this case the Huygens' principle holds. Roughly speaking, it asserts that for any initial source with a compact support, the wave leaves any bounded domain in a finite time. This is no longer true if the spatial dimension is even and/or the sound speed is not constant. All known formulae of filtered backprojection type assume constant sound speed and thus are not available for acoustically inhomogeneous media. In addition, the only closed bounded surface $\mathcal{S}$ for which such formulae are known is a sphere. Let us also mention [34] where a reconstruction algorithm in this vein (using the Radon transform) is proposed.

OK if S is closed and infinite (great) number of measures

- *Expansion series* are useful in the case where the Huyghens principle is valid. This approach was extended to the constant speed and arbitrary closed observation surface and modified by the use of the eigenfunctions of the Laplacian with Dirichlet conditions on $\mathcal{S}$[10]. It theoretically works for any closed surface and for variable sound speeds[38]. One can also refer to [28,36].
- *The time reversal method* (see for example [25,26]) can be used to approximate the initial pressure when the sound speed inside the object is variable. It works for arbitrary geometries of the closed observation surface $\mathcal{S}$. Ammari *et al.*[3,5] have performed sharp analysis of these problems both from the modeling and numerical point of view.

1.5 Sensitivity analysis

The fluence and the pressure equations have unique solutions

$$
\begin{aligned}
I\colon\ \mathcal{U}_{ad} &\longrightarrow \mathcal{C}^0(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega)) \\
\boldsymbol{\mu} &\longmapsto I[\boldsymbol{\mu}]
\end{aligned}
$$

$$
\begin{aligned}
p\colon\ \mathcal{U}_{ad} &\longrightarrow \mathcal{C}^0(0,T;H^1_0(\mathcal{B})) \\
\boldsymbol{\mu} &\longmapsto p[\boldsymbol{\mu}]
\end{aligned}
$$

$$\boldsymbol{\mu} = (\mu_a, D)$$

$$\mathcal{U}_{ad} = \left\{ (\mu_a, D) \in [L^\infty(\mathcal{B})]^2 \mid \mu_a \in [\mu_a^{min}, \mu_a^{max}] \text{ and } D \in [D^{\min}, D^{\max}] \text{ a.e. in } \mathcal{B} \right\}$$

The operator p is continuous from $\mathcal{U}_{ad}$ endowed with the weak (for μ_a) - strong (for D) L^2 topology to $L^2(\mathcal{B})$ (with the strong topology).

2. The inverse problem

Recover μ_a and D from measurements of p(x, t) for x over a surface ω outside the illuminated fluid.

This is an ill posed problem

⇨ Formulation as an optimal control problem

$$\min_{\boldsymbol{\mu} \in \mathcal{U}_{ad}} J(\boldsymbol{\mu})$$

$$\mathcal{U}_{ad} = \{\boldsymbol{\mu} = (\mu_a, D) \in [L^\infty(\mathcal{B})]^2 \mid \mu_a \in [\mu_a^{min}, \mu_a^{max}] \text{ and } D \in [D^{\min}, D^{\max}] \text{ a.e. in } \mathcal{B}\}$$

2.1 The cost functional

$$J(\boldsymbol{\mu}) = \mathcal{F}(\boldsymbol{\mu}) + f(\boldsymbol{\mu})$$

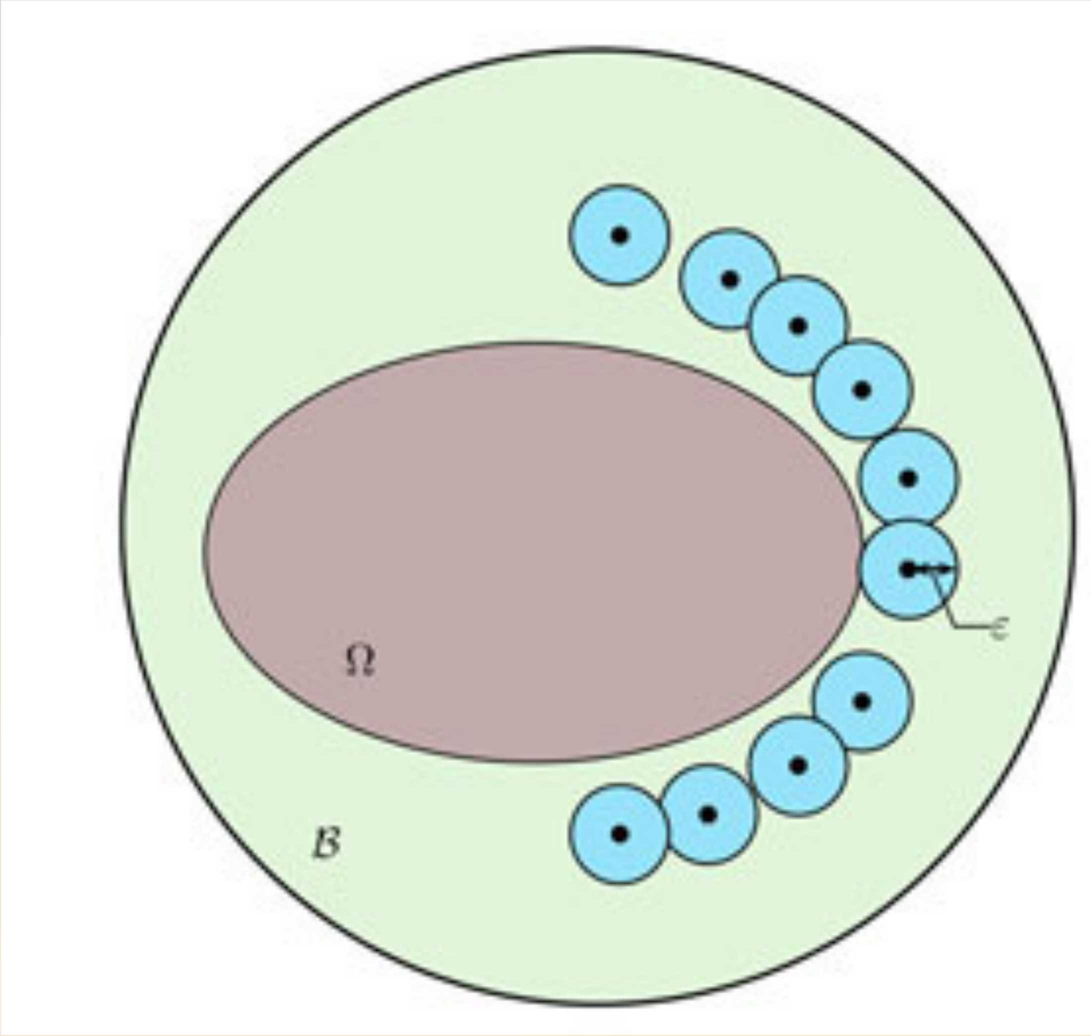
- Fitting data term

$$\mathcal{F}(\boldsymbol{\mu}) = \frac{1}{2}\int_{[0,T]\times\omega} (p[\boldsymbol{\mu}](t,x) - p^{obs}(t,x))^2 \mathrm{d}x\, \mathrm{d}t$$

ω is not closed, and codimension 1 or 2

$$\omega = \bigcup_{i=1}^{N}\{x_i\} \rightarrow \omega_\varepsilon = \bigcup_{x\in\omega} B(x,\varepsilon)$$

$$\mathcal{S} \rightarrow \mathcal{S}_\varepsilon = \bigcup_{x\in\mathcal{S}} B(x,\varepsilon)$$



Theorem 2.1. *Assume that* $\alpha \geq 0$ *and* $\beta > 0$.
Then, Problem $(\mathcal{P})$ *has at least a solution* $\bar{\boldsymbol{\mu}} = (\bar{\mu}_a, \bar{D})$.

Idea of the proof

- choose a minimizing sequence $(\mu_a^n, D^n)_{n\in\mathbb{N}}$
- a priori estimates and convergences

  μ_a^n weakly converges to $\bar{\mu}_a$ in $L^2(\Omega)$

  D^n strongly converges to $\bar{D}$ in $L^2(\Omega)$
- use the lower semi-continuity of the functional J
- $\mathcal{U}_{ad}$ is closed

Uniqueness is an open problem (at least 2 measurements)

$$J(\boldsymbol{\mu}) = \mathcal{F}(\boldsymbol{\mu}) + f(\boldsymbol{\mu})$$

- Regularization term

$$f(\boldsymbol{\mu}) = \begin{cases} \alpha \int_\Omega (B\mu_a)^2(x)\mathrm{d}x + \beta TV(D) \text{ if } D \in BV(\Omega) \\ +\infty \qquad \text{otherwise.} \end{cases}$$

B may a pass band filter or mollifier operator, or the identity.

The TV term seems to be the weakest one that provides an existence result while respecting the physical requirements since discontinuities (and contours) are preserved.

2.3 « Basic » example : the classical TAT case

$$\begin{cases} \dfrac{\partial^2 p^0}{\partial t^2}(t,x) - \mathrm{div}(v_s^2 \nabla p^0)(t,x) = \mathbb{1}_\Omega(x)\Gamma(x)\dfrac{\partial H}{\partial t}(t,x), \; (t,x) \in [0,T] \times \mathcal{B}, \\ p^0(0,x) = \dfrac{\partial p^0}{\partial t}(0,x) = 0, \qquad\qquad x \in \mathcal{B} \end{cases}$$

$$H = \mu_a I$$

Set $u(t,x) = \mathbb{1}_\Omega(x)\Gamma(x)H(t,x)$ **and** $p(t,x) = \int_0^t p^0(s,x)\mathrm{d}s$

$$\left(\frac{\partial^2 p}{\partial t^2} - \mathrm{div}(v_s^2 \nabla p)\right)(t,x) = u(t,x), \quad (t,x) \in [0,T] \times \mathcal{B},$$

$$p(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{B},$$

$$p(0,x) = 0, \quad \frac{\partial p}{\partial t}(0,x) = 0, \quad x \in \mathcal{B}$$

→ $p = p[u]$

$$J(u) = \frac{1}{2}\left\|p[u] - p^{obs}\right\|^2_{L^2([0,T]\times\omega)} + \frac{\alpha}{2}\left\|u\right\|^2_{L^2([0,T]\times\Omega)}$$

$$\min_{u\in\mathcal{U}_{ad}} J(u)$$

In this case u $\longrightarrow$ p[u] is linear and J is strictly convex.

There exists a unique solution $\bar{u}$ characterized by

$$\forall u \in \mathcal{U}_{ad} \qquad (J'(\bar{u}), u - \bar{u})_{L^2} \geq 0$$

$\mathcal{U}_{ad}$ is a set of bounded functions with support in Ω

Computation of (J'(u),v)

$$J'(u) \cdot v = \left(p[u] - p^{obs}, p'[u] \cdot v\right)_{L^2((0,T)\times\omega)} + \alpha\,(u, v)_{L^2((0,T)\times\Omega)}$$

$\dot{p} := p'[u] \cdot v$ is the solution of the (linearized) equation

$$\begin{cases} \dfrac{\partial^2 \dot{p}}{\partial t^2} - \mathrm{div}(v_s^2 \nabla \dot{p}) = v \text{ in } [0,T] \times \mathcal{B} \\ \dot{p}(0,\cdot) = \dfrac{\partial \dot{p}}{\partial t}(0,\cdot) = 0 \quad \text{in } \mathcal{B} \\ \dot{p} = 0 \quad \text{on } [0,T] \times \partial\mathcal{B} \end{cases}$$

Optimal control (duality) technique : use of the adjoint state q

$$\begin{cases} \dfrac{\partial^2 q}{\partial t^2} - \mathrm{div}(v_s^2 \nabla q) = (p[\bar{u}] - p^{\mathrm{obs}})\mathbb{1}_\omega & \text{in } [0,T] \times \mathcal{B} \\ q(T,\cdot) = \dfrac{\partial q}{\partial t}(T,\cdot) = 0 & \text{in } \mathcal{B} \\ q = 0 & \text{on } [0,T] \times \partial\mathcal{B} \end{cases}$$

$$\left(p[u] - p^{obs}, p'[u]\cdot v\right)_{L^2((0,T)\times\omega)} = \int_{(0,T)\times\mathcal{B}} (p[\bar{u}] - p^{obs}) 1_\omega \dot{p}\, dt\ dx$$

$$= \int_{(0,T)\times\mathcal{B}} \left(\frac{\partial^2 q}{\partial t^2} - \mathrm{div}(v_s^2 \nabla q)\right) \dot{p}\, dt\ dx$$

$$= \int_{(0,T)\times\mathcal{B}} \left(\frac{\partial^2 \dot{p}}{\partial t^2} - \mathrm{div}(v_s^2 \nabla \dot{p})\right) q\, dt\ dx$$

$$= \int_{(0,T)\times\Omega} q\, v\, dt\ dx$$

$$J'(\bar{u}) \cdot (u - \bar{u}) = \int_{(0,T)\times\Omega} (q + \alpha\bar{u})(u - \bar{u})\, dt\ dx$$

Finally $\quad \forall u \in \mathcal{U}_{ad} \qquad (q + \alpha\bar{u}, u - \bar{u}) \geq 0$

$$\forall u \in \mathcal{U}_{ad} \qquad (\bar{u} - q - \alpha\bar{u} - \bar{u}, u - \bar{u}) \leq 0$$

That is

$$\bar{u} = \Pi_{U_{ad}}(\bar{u} - q - \alpha\bar{u})$$

Finally, the solution $\bar{u}$ is characterized by

State equation

$$\begin{cases} \left(\dfrac{\partial^2 \bar{p}}{\partial t^2} - \mathrm{div}(v_s^2 \nabla \bar{p})\right)(t,x) = \bar{u}(t,x), \quad (t,x) \in [0,T] \times \mathcal{B}, \\ \bar{p}(t,x) = 0, \quad (t,x) \in [0,T] \times \partial\mathcal{B}, \\ \bar{p}(0,x) = 0, \quad \dfrac{\partial \bar{p}}{\partial t}(0,x) = 0, \quad x \in \mathcal{B} \end{cases}$$

Adjoint equation

$$\begin{cases} \dfrac{\partial^2 \bar{q}}{\partial t^2} - \mathrm{div}(v_s^2 \nabla \bar{q}) = (\bar{p} - p^{\mathrm{obs}})\mathbb{1}_\omega & \text{in } [0,T] \times \mathcal{B} \\ \bar{q}(T,\cdot) = \dfrac{\partial \bar{q}}{\partial t}(T,\cdot) = 0 & \text{in } \mathcal{B} \\ \bar{q} = 0 & \text{on } [0,T] \times \partial\mathcal{B} \end{cases}$$

Projection equation

$$\bar{u} = \Pi_{U_{ad}}(\bar{u} - \bar{q} - \alpha \bar{u})$$

2.4 The general case

State function : (p, I)

Control function (parameters to identify) $\boldsymbol{\mu} = (\mu_a, D)$

Admissible control set

$$\mathcal{U}_{ad} = \left\{ \boldsymbol{\mu} = (\mu_a, D) \in [L^\infty(\mathcal{B})]^2 \mid \mu_a \in [\mu_a^{min}, \mu_a^{max}] \text{ and } D \in [D^{\min}, D^{\max}] \text{ a.e. in } \mathcal{B} \right\}$$

State equations

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2}(t,x) - \operatorname{div}(v_s^2 \nabla)(t,x) = \mathbb{1}_\Omega(x)\Gamma(x)\mu_a(x)I(t,x), & (t,x) \in (0,T) \times \mathcal{B}, \\ p(t,x) = 0, & (t,x) \in (0,T) \times \partial\mathcal{B}, \\ p(0,x) = \dfrac{\partial p}{\partial t}(0,x) = 0, & x \in \mathcal{B}, \end{cases}$$

$$\begin{cases} \dfrac{1}{\nu}\dfrac{\partial I}{\partial t}(t,x) + \mu_a(x)I(t,x) - \operatorname{div}(D\nabla I)(t,x) = S(t,x), & (t,x) \in (0,T) \times \Omega, \\ I(0,x) = 0, & x \in \Omega, \\ I(t,x) = 0 & (t,x) \in \mathcal{B}\backslash\Omega \\ I(t,x) = 0, & (t,x) \in \Sigma. \end{cases}$$

Cost functional and optimization problem

$$J(\boldsymbol{\mu}) = \mathcal{F}(\boldsymbol{\mu}) + f(\boldsymbol{\mu})$$

$$\mathcal{F}(\boldsymbol{\mu}) = \frac{1}{2}\int_{[0,T]\times\omega} (p[\boldsymbol{\mu}](t,x) - p^{obs}(t,x))^2 \mathrm{d}x\,\mathrm{d}t$$

$$f(\boldsymbol{\mu}) = \begin{cases} \alpha \int_{\Omega} (B\mu_a)^2(x)\mathrm{d}x + \beta TV(D) & \text{if } D \in BV(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

$$\min_{\boldsymbol{\mu}\in\mathcal{U}_{ad}} J(\boldsymbol{\mu})$$

$\mu \mapsto p[\mu]$ is no longer linear (because of the term $\mu_a I[\mu]$)

J is no longer convex

Existence OK but no uniqueness

The optimality system is no longer a necessary and sufficient condition

J is not Gâteaux differentiable because of the TV term

$$\min_{\boldsymbol{\mu} \in \mathcal{U}_{ad}} J(\boldsymbol{\mu})$$

$$\Updownarrow$$

$$\min_{\boldsymbol{\mu} \in [L^\infty(\Omega)]^2} \mathcal{F}(\boldsymbol{\mu}) + f(\boldsymbol{\mu}) + \iota_{\mathcal{U}_{ad}}(\boldsymbol{\mu})$$

$$\iota_{\mathcal{U}_{ad}}(x) = \begin{cases} 0 & \text{if } x \in \mathcal{U}_{ad} \\ +\infty & \text{otherwise.} \end{cases}$$

$$\Downarrow$$

$$0 \in \partial(\mathcal{F}(\bar{\boldsymbol{\mu}}) + f(\bar{\boldsymbol{\mu}}) + \iota_{\mathcal{U}_{ad}}(\bar{\boldsymbol{\mu}}))$$

Moreover

$$\partial(\mathcal{F}(\bar{\boldsymbol{\mu}}) + f(\bar{\boldsymbol{\mu}}) + \iota_{\mathcal{U}_{ad}}(\bar{\boldsymbol{\mu}})) \subset D\mathcal{F}(\bar{\boldsymbol{\mu}}) + \partial f(\bar{\boldsymbol{\mu}}) + \partial\iota_{\mathcal{U}_{ad}}(\bar{\boldsymbol{\mu}}))$$

Computation of $d\mathcal{F}$

Proposition 2.1. *For every* $\xi = (\xi_a, \xi_D) \in L^2(\Omega) \times L^2(\Omega)$*, the functional* $\mathcal{F}$ *is Gâteaux-differentiable at* $\boldsymbol{\mu} = (\mu_a, D)$ *in the direction* ξ *and*

$$\begin{aligned}\langle d\mathcal{F}(\boldsymbol{\mu}), \xi \rangle_{L^2(\Omega)} &= \int_\Omega \nabla \mathcal{F}(\boldsymbol{\mu})(x)\xi(x)\mathrm{d}x \\ &= \int_\Omega \left(\frac{\partial \mathcal{F}}{\partial \mu_a}(\mu_a, D)(x)\xi_{\mu_a} + \frac{\partial \mathcal{F}}{\partial D}(\mu_a, D)(x)\xi_D(x) \right) \mathrm{d}x \qquad (2.7)\end{aligned}$$

where

$$\nabla \mathcal{F}(\boldsymbol{\mu}) = \left(\frac{\partial \mathcal{F}}{\partial \mu_a}(\boldsymbol{\mu}), \frac{\partial \mathcal{F}}{\partial D}(\boldsymbol{\mu}) \right) = \left(\int_0^T (\mathbb{1}_\Omega \Gamma q_1 - q_2) I, -\int_0^T \nabla q_2 \cdot \nabla I \right).$$

Adjoint state equations

$$\begin{cases} \dfrac{\partial^2 q_1}{\partial t^2} - \operatorname{div}(v_s^2 \nabla q_1) = (p - p^{\text{obs}})\mathbb{1}_{\omega_\varepsilon} & \text{in } [0,T] \times \mathcal{B} \\ q_1(T,\cdot) = \dfrac{\partial q_1}{\partial t}(T,\cdot) = 0 & \text{in } \mathcal{B} \\ q_1 = 0 & \text{on } [0,T] \times \partial\mathcal{B} \end{cases}$$

$$\begin{cases} -\dfrac{1}{\nu}\dfrac{\partial q_2}{\partial t} + \mu_a q_2 - \operatorname{div}(D \nabla q_2) = \Gamma \mu_a q_1 & \text{in } \mathcal{Q} \\ q_2(T,\cdot) = 0 & \text{on } \Omega \\ q_2 = 0 & \text{on } \Sigma. \end{cases}$$

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Projection equations (1)

Equation on μ_a. For every $\mu_a \in L^\infty(\Omega)$ such that $\mu_a \in [\mu_a^{\min}, \mu_a^{\max}]$,

$$\left\langle \frac{\partial \mathcal{F}}{\partial \mu_a}(\bar{\mu}_a, \bar{D}) + 2\alpha B^* B \bar{\mu}_a \, , \, \mu_a - \bar{\mu}_a \right\rangle_{L^2(\Omega)} \geq 0 \; ,$$

Equation on D.

$$- \frac{\partial \mathcal{F}}{\partial D}(\bar{\mu}_a, \bar{D}) \in \partial TV(\bar{D}) + \partial \iota_{[D^{\min}, D^{\max}]},$$

where B^* is the L^2-adjoint operator of B.

Projection equations (2)

$$\forall \mu_a \in L^\infty(\Omega), \text{ s.t. } \mu_a \in [\mu_a^{\min}, \mu_a^{\max}],$$

$$\left(\int_0^T \mathbb{1}_\Omega \Gamma(q_1 - q_2 I)\, dt + 2\alpha B^* B \bar{\mu}_a, \mu_a - \bar{\mu}_a \right)_{L^2(\Omega)} \geq 0$$

$$\exists \delta^* \in \partial TV(\bar{D}), \quad \forall D \in L^\infty(\Omega) \text{ s.t. } D \in [D^{\min}, D^{\max}]$$

$$\left(\int_0^T (\nabla I \cdot \nabla q_2)\, dt - \delta^*, D - \bar{D} \right)_{L^2(\Omega)} \geq 0.$$

Optimality system **If** $\bar{\boldsymbol{\mu}} = (\bar{\mu}_a, \bar{D})$ **is a solution, then**

State equations

Adjoint equations

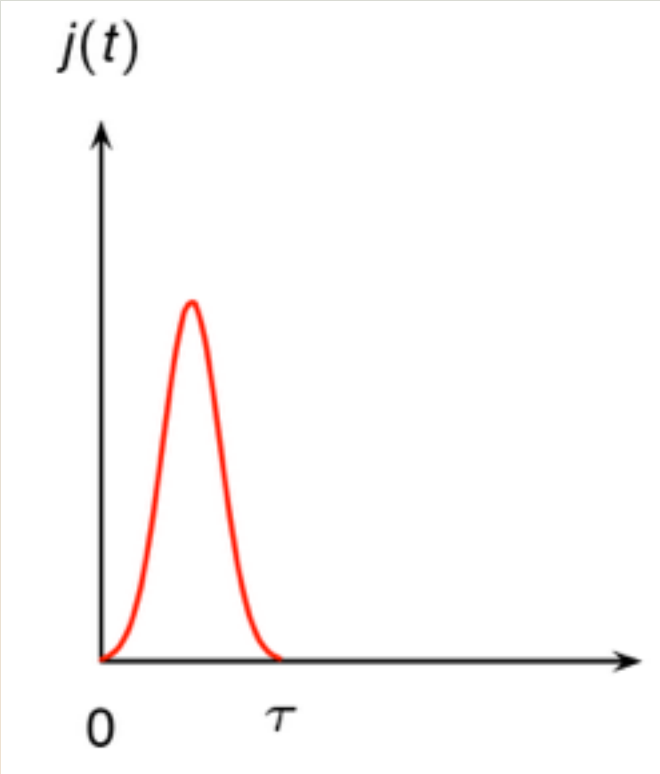
Projection equations

3. Numerical computation (simplified case)

The «simplified model» for TAT (no optical effect)

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2} - div(v^2 \nabla p) = C v^2 \mu_a 1_\Omega \dfrac{\partial I}{\partial t} & \text{in } (0,T) \times \mathcal{B} \\ p(0,\cdot) = \dfrac{\partial p}{\partial t}(0,\cdot) = 0 & \text{in } \mathcal{B} \\ p = 0 & \text{in } (0,T) \times \partial\mathcal{B}, \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2} - div(v^2 \nabla p) = u_0(x) \dfrac{\partial j}{\partial t} & \text{in } (0,T) \times \mathcal{B} \\ p(0,\cdot) = \dfrac{\partial p}{\partial t}(0,\cdot) = 0 & \text{in } \mathcal{B} \\ p = 0 & \text{in } (0,T) \times \partial\mathcal{B}, \end{cases}$$



u_0 is the energy deposition function

$$\begin{cases} \dfrac{\partial^2 p}{\partial t^2}(t,x) - \operatorname{div}(v_s^2 \nabla p)(t,x) = 0 & (t,x) \in [0,T] \times \mathcal{B}, \\ p(t,x) = 0, & (t,x) \in [0,T] \times \partial\mathcal{B}, \\ p(0,x) = \mathbb{1}_\Omega(x)\Gamma(x)\mu_a(x)I(x) & x \in \mathcal{B}, \\ \dfrac{\partial p}{\partial t}(0,x) = 0, & x \in \mathcal{B}. \end{cases}$$

$$\begin{cases} \mu_a(x)I(x) - \operatorname{div}(D(x)\nabla I(x)) = S(x) & x \in \Omega, \\ I(x) = 0, & x \in \partial\Omega. \end{cases}$$

$$\min_{(\mu_a, D) \in U_{ad}} \mathcal{F}(\mu_a, D) := \int_{\omega \times (0,T)} (p(\mu_a, D) - p^{obs})^2 + \frac{\alpha}{2} \int_\Omega \mu_a^2 + \beta \sqrt{\|\nabla D\|^2 + \eta^2}$$

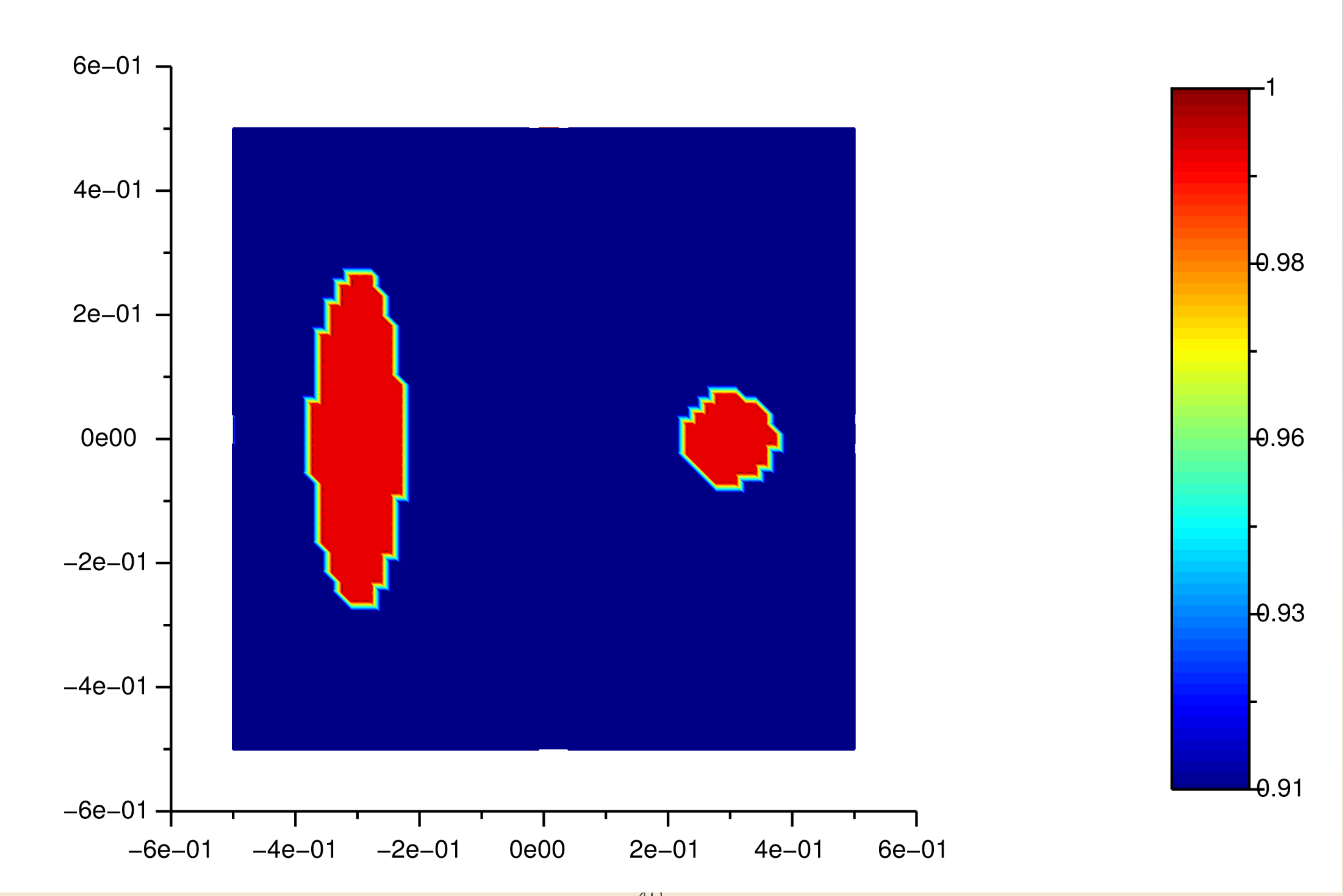
Optimality system

$$\begin{cases} \dfrac{\partial^2 q_1}{\partial t^2} - \mathrm{div}(v_s^2 \nabla q_1) = (p - p^{\mathrm{obs}}) \mathbb{1}_{\omega_\varepsilon} & \text{in } [0,T] \times \mathcal{B} \\ q_1(T,\cdot) = \dfrac{\partial q_1}{\partial t}(T,\cdot) = 0 & \text{in } \mathcal{B} \\ q_1 = 0 & \text{on } [0,T] \times \partial\mathcal{B} \end{cases}$$

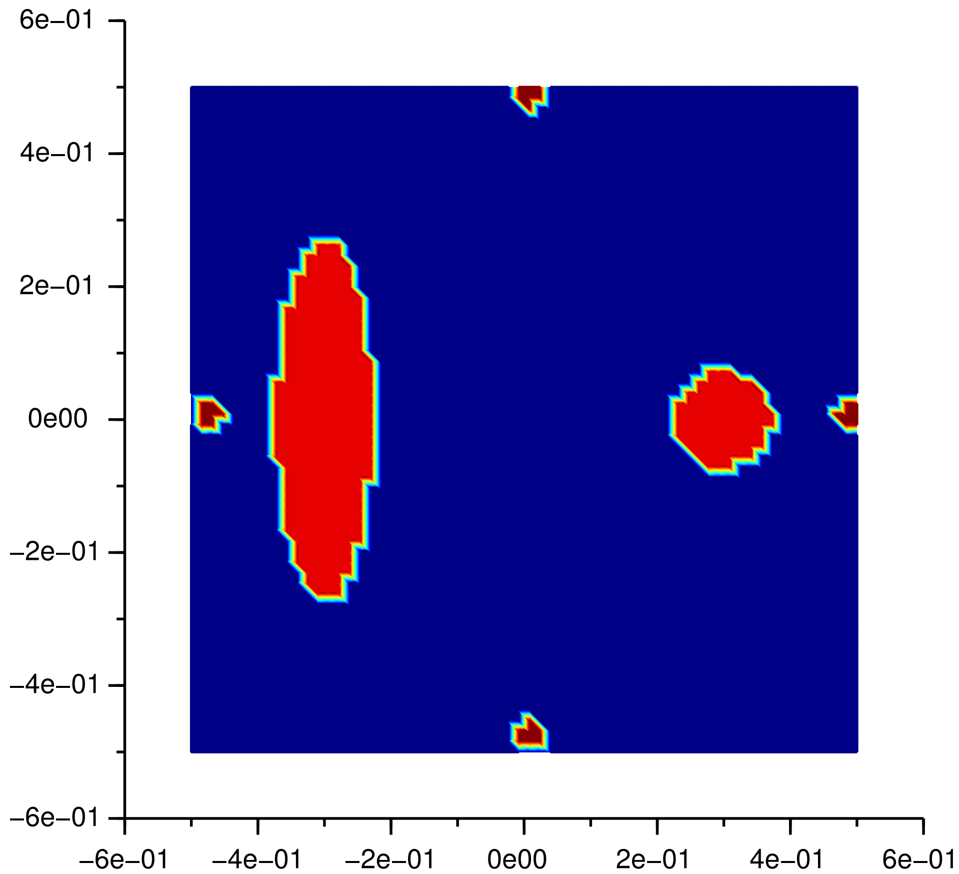
$$\begin{cases} \mu_a q_2 - \mathrm{div}(D \nabla q_2) = -\Gamma \mu_a \dfrac{\partial q_1}{\partial t}(0) & \text{in } \Omega \\ q_2 = 0 & \text{on } \partial\Omega \end{cases}$$

$$\begin{aligned} \langle d\mathcal{F}(\mu_a, D), (\xi_{\mu_a}, \xi_D) \rangle &= \int_\Omega -(\partial q_1(0,\cdot)\Gamma + q_2) I \xi_{\mu_a} - (\nabla q_2 \cdot \nabla I) \xi_D \\ &+ \alpha \int_\Omega \mu_a \xi_{\mu_a} + \beta_{\mathrm{TV}} \int_\Omega d\left(\sqrt{\|\nabla D\|^2 + \epsilon_{\mathrm{TV}}^2}\right) \\ &= 0 \ (\leq 0 \text{ if we add constraints on } (\mu_a, D)) \end{aligned}$$

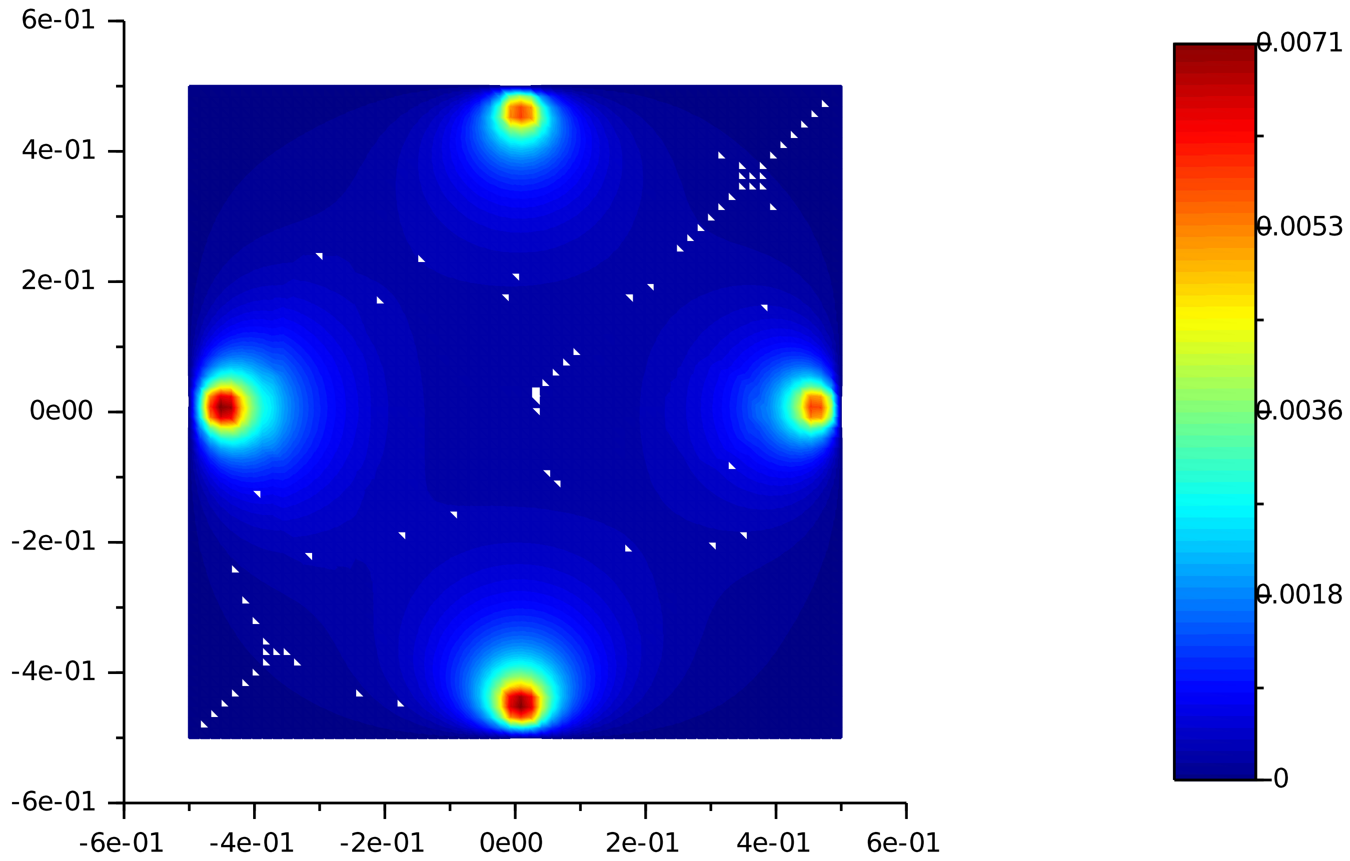
We chose to solve the optimality system by means of the gradient algorithm. The forward and backward problems are solved by means of a leapfrog discretization scheme on a staggered grid. In order to avoid handling large grids (due to the size of B), we use an appropriate PML (Perfectly Matched Layer) technique



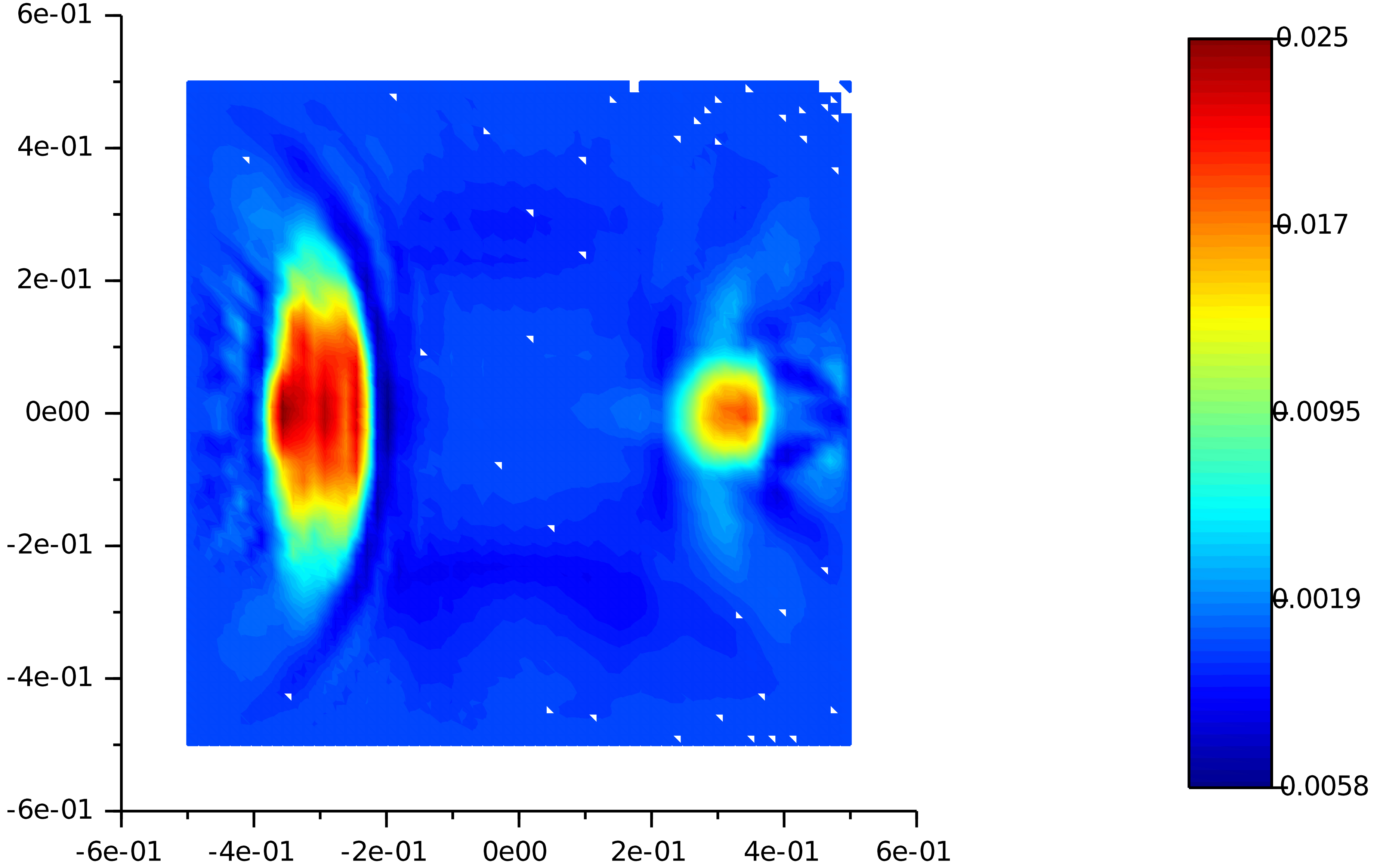
4 sources (lightning)



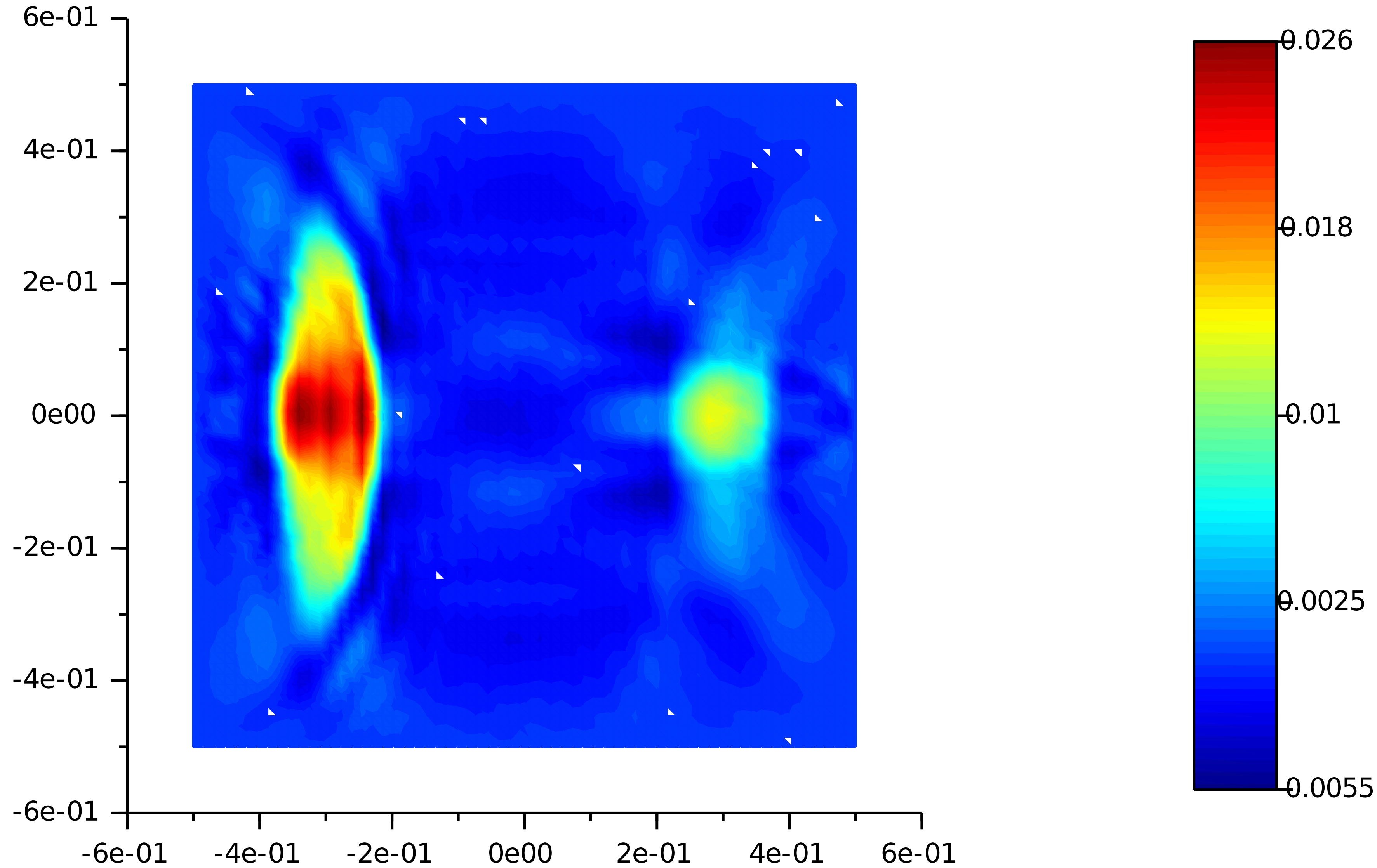
Summation ($\Gamma \mu_a I$)



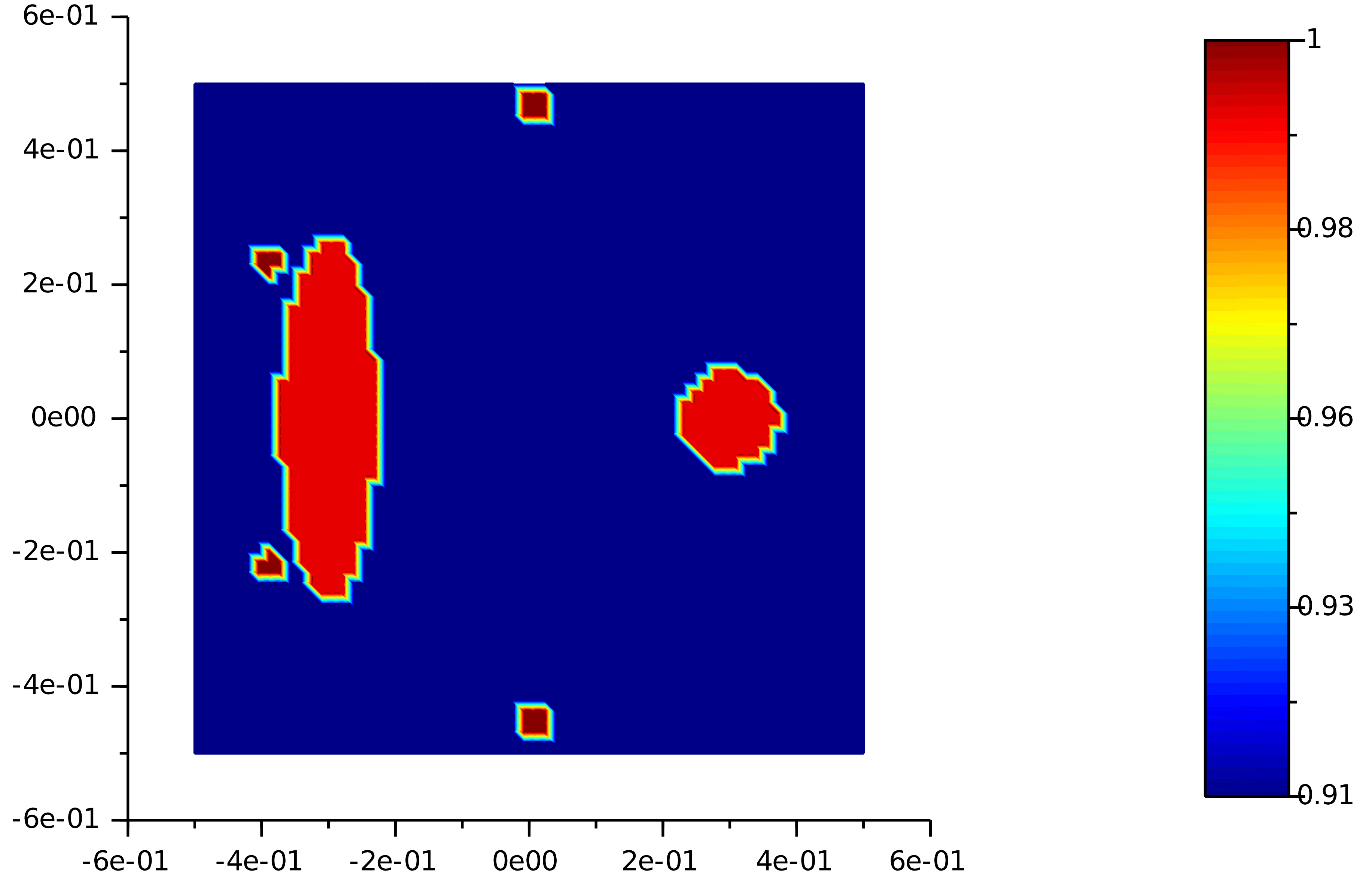
Reconstruction without D (=1) - 4 sensors - 64 pixels - α = 0.1



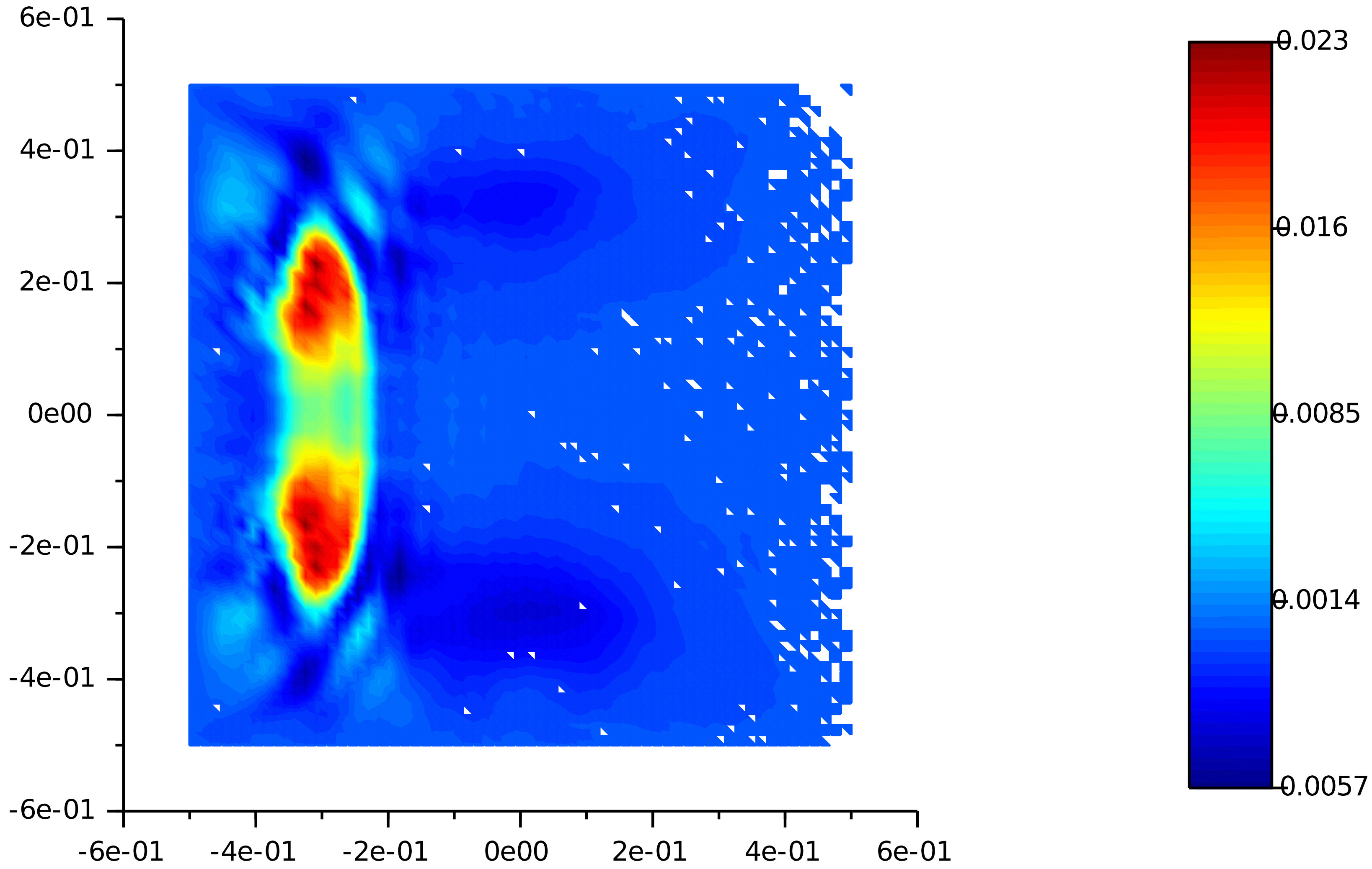
Reconstruction without D (=1) - 2 sensors (left/right) - 64 pixels - α = 0.1



4 sources (lightning)



Reconstruction without D (=1) - 4 sensors - 64 pixels - α = 0.1



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