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Inverse Scattering and Interior Transmission Eigenvalue Problems for Isotropic Media

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Plan

(A) A Scattering Problem
(B) The Linear Sampling Method and the Factorization Method
(C) The Corresponding Interior Transmission Eigenvalue Problem

Literature:

- F. Cakoni, D. Colton: A Qualitative Approach to Inverse Scattering Theory. Springer, 2013.
- D. Colton. R. Kress: Inverse Acoustic and Electromagnetic Scattering Theory. 3rd Edition. Springer, 2013.
- A. Kirsch, F. Hettlich: Maxwell's Equations. Springer, 2014.
- A. Kirsch, N. Grinberg: The Factorization Method for Inverse Problems. Oxford University Press, 2008.
- F. Cakoni, H. Haddar (guest editors): Special Issue by *Inverse Problems* on transmission eigenvalue problems (2013)

(A) A Scattering Problem

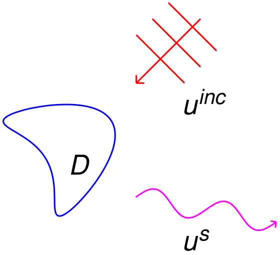
Wave propagationn in frequency domain: incident wave u^{inc} satisfies the reduced wave equation; that is, Helmholtz equation

$$\Delta u^{inc} + k^2 u^{inc} = 0 \text{ in } \mathbb{R}^2 .$$

u^{inc} is scattered by
a (bounded) medium and
generates scattered field u^s.

Here, $k = \frac{\omega}{c} > 0$ wave number.

Total field: $u = u^{inc} + u^s$



Requirement: u^s is radiating; that is, satisfies radiation condition

$$\frac{\partial u^s(r\hat{x})}{\partial r} - iku^s(r\hat{x}) = \mathcal{O}(r^{-3/2}), \quad r = |x| \to \infty ,$$

uniformly wrt $\hat{x} = x/|x| \in S^1 = \{y \in \mathbb{R}^2 : |y| = 1\}$.

A Scattering Problem, cont.

Radiation condition implies asymptotic form

$$u^s(x) \;=\; \frac{\exp(ikr)}{\sqrt{8\pi kr}}\, u^\infty(\hat{x}) \;+\; \mathcal{O}\left(\frac{1}{r^{3/2}}\right), \quad |x| \to \infty\,,$$

uniformly wrt $\hat{x} \in S^1$.

$u^\infty(\hat{x}) \in \mathbb{C}$ is called far field or scattering amplitude of u^s.

Important for uniqueness:

Lemma of Rellich: If $u^\infty = 0$ on S^1 then $u^s = 0$ in exterior of D.

Still missing: type of scattering medium. Examples:

(A) Sound soft obstacle (or perfect conductor in E-mode):

$$\Delta u + k^2 u \;=\; 0 \text{ in } \mathbb{R}^2 \setminus \overline{D}\,, \quad u = 0 \text{ auf } \partial D \quad \text{or}$$

(B) Inhomogeneous medium with contrast q $\quad (= 0 \text{ in } \mathbb{R}^2 \setminus D)$

$$\Delta u \;+\; k^2(1+q)u \;=\; 0 \text{ in } \mathbb{R}^2\,, \quad u, \nabla u \text{ continuous in } \mathbb{R}^2\,.$$

A Scattering Problem, cont.

Direct scattering problem:

Given: Incident plane wave $u^{inc}(x) = e^{ik\hat{\theta}\cdot x}$ with $\hat{\theta} \in S^1$ and open bounded set $D \subset \mathbb{R}^2$ (Lipschitz boundary, exterior connected) and contrast q (in example (B)).

Determine: Scattered field u^s and far field u^∞ with:

(A) $\Delta u + k^2 u = 0$ in $\mathbb{R}^2 \setminus \overline{D}$ and $u = 0$ auf ∂D or

(B) $\Delta u + k^2(1+q)u = 0$ in $\mathbb{R}^2$, respectively,

for total field $u = u^{inc} + u^s$, and u^s satisfies radiation condition.

For $q \in L^\infty(D)$ these direct problems are uniquely solvable in $H^1_{loc}(\mathbb{R}^2 \setminus \overline{D})$ or $H^1_{loc}(\mathbb{R}^2)$, respectively.

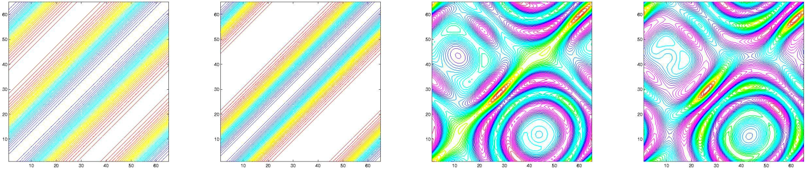
Inverse scattering problem:

Given: Far field $u^\infty(\hat{x}, \hat{\theta})$ for all directions of observation $\hat{x} \in S^1$ and some or all directions $\hat{\theta} \in S^1$ of incident plane waves u^{inc}.

Determine: Form of D!

Two Examples

Which scattering media $D \subset \mathbb{R}^2$ belong to these far field patterns? $u^\infty(\phi, \theta)$, $\phi, \theta \in [0, 2\pi]$?



Re u^∞ Im u^∞ Re u^∞ Im u^∞

Left example simple:

Theorem of Karp: If $u^\infty(\phi, \theta) = f(\phi - \theta)$ for some function f then D is a disk.

Proof is consequence of uniqueness result:

Theorem: Let u_j^∞ far field corresponding to D_j for $j = 1, 2$. If $u_1^\infty(\hat{x}, \hat{\theta}) = u_2^\infty(\hat{x}, \hat{\theta})$ for all $\hat{x}, \hat{\theta} \in S^1$, then $D_1 = D_2$.

The Far Field Operator

Define $F : L^2(S^1) \to L^2(S^1)$ by

$$(Fg)(\hat{x}) = \int_{S^1} u^\infty(\hat{x}, \hat{\theta})\, g(\hat{\theta})\, ds(\hat{\theta})\,, \quad \hat{x} \in S^1\,,$$

for the scattering under Dirichlet boundary conditions or by an inhomogeneous medium. This is far field corresponding to incident field

$$v_g^{inc}(x) = \int_{S^1} e^{ik\hat{\theta}\cdot x}\, g(\hat{\theta})\, ds(\hat{\theta})\,, \quad x \in \mathbb{R}^2\,.$$

Properties of F:

- F is compact and normal, $\mathcal{S} = I + \frac{ik}{8\pi^2} F$ is unitary.
- F is one-to-one if k^2 is no eigenvalue of "corresponding" evp.

Idea of proof for Dirichlet b.c.: If $Fg = 0$ then far field corresponding to scattered wave v_g^s and incident wave v_g^{inc} vanishes. Lemma of Rellich implies that $v_g^s = 0$ in $\mathbb{R}^2 \setminus D$; thus $0 = v_g^{inc} + v_g^s = v_g^{inc}$ on ∂D. Therefore, $\Delta v_g^{inc} + k^2 v_g^{inc} = 0$ in D, $v_g^{inc} = 0$ on ∂D; i.e. k^2 is Dirichlet eigenvalue of $-\Delta$. By assumption $v_g^{inc} = 0$ in D and thus $g = 0$. ∎

The Interior Transmission Eigenvalue Problem

Analogously for scattering by an inhomogeneous medium:

Theorem: F one-to-one if $(u, w) = (0, 0)$ is the only solution of

$$\Delta u + k^2(1+q)u = 0 \text{ in } D\,, \quad \Delta w + k^2 w = 0 \text{ in } D\,,$$

$$u = w \text{ on } \partial D\,, \quad \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} \text{ on } \partial D\,.$$

This is the corresponding interior transmission eigenvalue problem.

Proof: If $Fg = 0$ then far field corresponding to scattered wave v_g^s and incident wave $v_g^{inc}(x) = \int_{S^1} \exp(ik\hat{\theta}\cdot x)\, g(\hat{\theta})\, ds(\hat{\theta})$ vanishes. Lemma of Rellich implies that $v_g^s = 0$ in $\mathbb{R}^2 \setminus D$, thus $u := v_g^{inc} + v_g^s$ and $w := v_g^{inc}$ satisfy interior transmission eigenvalue problem. By assumption v_g^{inc} vanishes in D and thus also g. ■

Note: The converse does not hold, i.e. F can be one-to-one even if k^2 is eigenvalue (e.g. if D has corners, see Blasten/Päivärinta/Sylvester 2013)

(B) The Linear Sampling Method and the Factorization Method

Consider simultanously: Scattering by sound soft obstacle or inhomogeneous medium.

Inverse Problem: Given far field operator F, determine D!

Remember: $(Fg)(\hat{x}) = \int_{S^1} u^\infty(\hat{x}, \hat{\theta})\, g(\hat{\theta})\, ds(\hat{\theta})$, $\hat{x} \in S^1$, is far field corresponding to v_g^s outside of D.

For $z \in \mathbb{R}^2$ define $\phi \in L^2(S^1)$ by $\boxed{\phi_z(\hat{x}) = \exp(-ikz \cdot \hat{x}),\ \hat{x} \in S^1}$.

ϕ_z is far field pattern of $x \mapsto \Phi(x,z) = \frac{i}{4} H_0^{(1)}(k|x-z|)$, $x \neq z$.

1. case: z $\in$ D. Lemma of Rellich implies:

$$Fg = \phi_z \iff v_g^s = \Phi(\cdot, z) \text{ outside of } D \iff$$

Dirichlet boundary conditions: $v_g^{inc} = -\Phi(\cdot, z)$ on ∂D

Transmission conditions: Let $u = v_g^s + v_g^{inc}$ total field. Then $u - v_g^{inc} = \Phi(\cdot, z)$ on ∂D and $\frac{\partial}{\partial \nu}(u - v_g^{inc}) = \frac{\partial}{\partial \nu}\Phi(\cdot, z)$ on ∂D .

The Linear Sampling Method, cont.

Recall Herglotz function $\quad v_g^{inc}(x) = \int_{S^1} \exp(ik\hat{\theta}\cdot x) g(\hat{\theta})\, ds(\hat{\theta})\,,\ x \in \mathbb{R}^2$.
Therefore, $Fg = \phi_z$ is equivalent to $v := v_g^{inc}$ and u satisfying

(1) $\quad \Delta v + k^2 v = 0$ in $D\,,\quad v = -\Phi(\cdot, z)$ on $\partial D\,,\qquad$ or, resp.,

(2) $$\left\{ \begin{array}{cc} \Delta v + k^2 v = 0 \text{ in } D\,, & \Delta u + k^2(1+q)u = 0 \text{ in } D\,, \\ u - v = \Phi(\cdot, z) \text{ on } \partial D\,, & \frac{\partial}{\partial \nu}(u-v) = \frac{\partial}{\partial \nu}\Phi(\cdot, z) \text{ on } \partial D\,, \end{array} \right\}$$

These problems are uniquely solvable if k^2 is not an eigenvalue (clear?).

2. case: $z \notin D$. $Fg = \phi_z \;\Leftrightarrow\; v_g^s = \Phi(\cdot, z)$ outside of $D \cup \{z\}$. But v_g^s smooth at $z \notin D$ and $\Phi(\cdot, z)$ singular at z. Therefore, $Fg = \phi_z$ not solvable for $z \notin D$.

Drawback: Even if $z \in D$ the solution v of (1) or (2) is almost never of the form $v = v_g^{inc}$ with some $g \in L^2(S^1)$! However:

Theorem: The space $\{v_g^{inc}|_D : g \in L^2(S^1)\}$ is always dense in $\{v \in H^1(D) : \Delta v + k^2 v = 0 \text{ in } D\}$.

The Linear Sampling Method, cont.

$\boxed{Fg = \phi_z}$ is improperly posed because $F : L^2(S^1) \to L^2(S^1)$ is compact.

Recall from previous page: For $z \notin D$ equation never solvable, for $z \in D$ "sometimes". In any case use Tikhonov regularization; that is, solve

$$\varepsilon\, g_{z,\varepsilon} \;+\; F^* F\, g_{z,\varepsilon} \;=\; F^* \phi_z$$

which is uniquelly solvable for all $\varepsilon > 0$ and $z \in \mathbb{R}^2$. With the help of the Factorization Method one can prove (later!):

Theorem: Let k^2 be not an eigenvalue (either Dirichlet eigenvalue in case of Dirichlet boundary conditions or interior transmission eigenvalue in case of transmission conditions). Then $z \in D$ if, and only if, the mapping

$$\varepsilon \;\mapsto\; (g_{z,\varepsilon}, \phi_z)_{L^2(S^1)} = v^{inc}_{g_{z,\varepsilon}}(z)$$

is bounded from $\mathbb{R}_{>0}$ to $\mathbb{C}$.

This gives method to visualize D by plotting contour lines of $z \mapsto v^{inc}_{g_{z,\varepsilon}}(z)$ for small values of ε.

The Factorization Method

Theorem 1: Let k^2 be no eigenvalue of the corresponding eigenvalue problem. Then F can be factorized in the form $F = H^* T H$.

$$
\begin{array}{ccc}
L^2(S^1) & \xrightarrow{F} & L^2(S^1) \\
H \downarrow & & \uparrow H^* \\
H^{1/2}(\partial D) & \xrightarrow{T} & H^{-1/2}(\partial D)
\end{array}
\qquad\qquad
\begin{array}{ccc}
L^2(S^1) & \xrightarrow{F} & L^2(S^1) \\
H \downarrow & & \uparrow H^* \\
L^2(D) & \xrightarrow{T} & L^2(D)
\end{array}
$$

Furthermore, T is a compact perturbation of a coercive operator and H is compact, more precicely:

The Factorization Method, cont.

$$(Hg)(x) = \int_{S^1} e^{ik\hat{\theta}\cdot x} g(\hat{\theta})\, ds(\hat{\theta}), \quad x \in \partial D \quad \text{bzw.} \quad x \in D.$$

Theorem 2: For $z \in \mathbb{R}^2$ define $\phi_z \in L^2(S^1)$ by

$$\phi_z(\hat{x}) := e^{-ikz\cdot\hat{x}}, \quad \hat{x} \in S^1.$$

Then: $$z \in D \iff \phi_z \in \mathcal{R}(H^*)$$

Remember: $$F = H^* T H$$

Theorem 3: (Range identity) Let k^2 be no correponding eigenvalue. Then:

$$\mathcal{R}(H^*) = \mathcal{R}\big((F^*F)^{1/4}\big)$$

Combined: $$z \in D \iff \phi_z \in \mathcal{R}\big((F^*F)^{1/4}\big)$$

FM studies solvability of $(F^*F)^{1/4}g = \phi_z$, LSM of: $Fg = \phi_z$!

The Factorization Method, cont.

$$\phi_z(\hat{x}) = e^{-ikz\cdot\hat{x}}\,,\ \hat{x}\in S^1\,,\quad z\in D \iff \phi_z\in\mathcal{R}\big((F^*F)^{1/4}\big)$$

$F: L^2(S^1)\to L^2(S^1)$ compact, normal and one-to-one. Therefore, there exists complete ONS $\{\psi_j\}$ of eigenfunctions of F with corresponding eigenvalues $\lambda_j\in\mathbb{C}$.

Condition $\phi_z\in\mathcal{R}\big((F^*F)^{1/4}\big)$ is equivalent to

$$z\in D \iff \sum_{j\in\mathbb{N}}\frac{\big|\langle\phi_z,\psi_j\rangle\big|^2}{|\lambda_j|} < \infty$$

$$\iff w(z) = \left[\sum_{j\in\mathbb{N}}\frac{\big|\langle\phi_z,\psi_j\rangle\big|^2}{|\lambda_j|}\right]^{-1} > 0$$

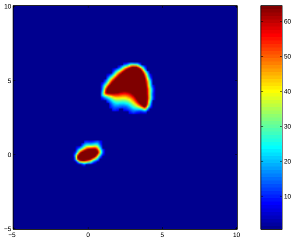
Therefore, sign w is characteristic function of D.

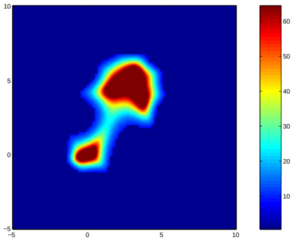
Simulations

The following numerical simulations show contour plots of

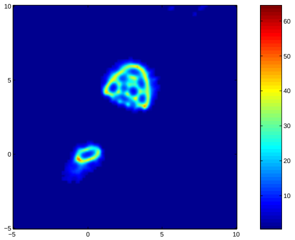
$$w(z) = \left[\sum_{j=1}^{32} \frac{\left| \langle \phi_z, \psi_j \rangle \right|^2}{|\lambda_j|} \right]^{-1}, \quad z \in \mathbb{R}^2,$$

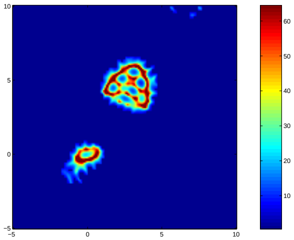
for 32 incident directions and 32 directions of observations (replace operator F by matrix $F \in \mathbb{C}^{32 \times 32}$ and ϕ_z by vector $\phi_z \in \mathbb{C}^{32}$).



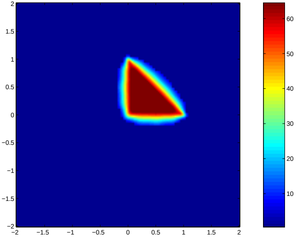


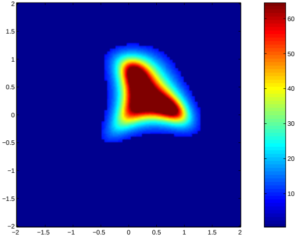
Simulations



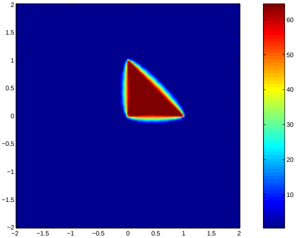


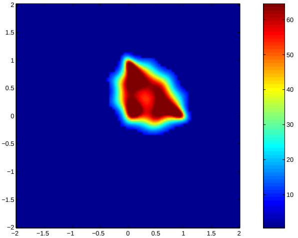
Simulations



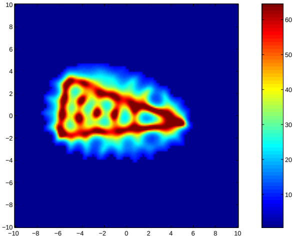


Simulations





Simulations



A Link between Linear Sampling and Factorization Method

General functional analytic situation:
Let $F : X \to Y$ compact and injective operator between Hilbert spaces X and Y. Let $\{\sigma_n, \psi_n, \varphi_n\}$ be a singular system of F; that is, $\{\psi_n : n \in \mathbb{N}\}$ and $\{\varphi_n : n \in \mathbb{N}\}$ are complete ONS in X and Y, resp., and $F\psi_n = \sigma_n \varphi_n$ and $F^* \varphi_n = \sigma_n \psi_n$ for all n.
Define $J : Y \to X$ by $\sum_n \alpha_n \varphi_n \mapsto \sum_n \alpha_n \psi_n$. Then $J : Y \to X$ is an isometry and:

Theorem: Let $\phi \in Y$ and $g_\varepsilon \in X$ be the Tikhonov regularization of $Fg = \phi$; that is, $\varepsilon g_\varepsilon + F^* F g_\varepsilon = F^* \phi$ for $\varepsilon > 0$.
Then $\phi \in \mathcal{R}\big((F^*F)^{1/4}\big)$ if, and only if, the mapping $\varepsilon \mapsto (g_\varepsilon, J\phi)_X$ is bounded.

Theorem: Let k^2 be not an eigenvalue and $\varepsilon g_{z,\varepsilon} + F^* F g_{z,\varepsilon} = F^* \phi_z$. Then $z \in D$ if, and only if, the mapping

$$\varepsilon \;\mapsto\; (g_{z,\varepsilon}, \phi_z)_{L^2(S^1)} = v^{inc}_{g_{z,\varepsilon}}(z)$$

is bounded. Proof on blackboard!

(C) The Interior Transm. Eigenvalue Problem

Classical formulation (setting $\lambda = k^2$):

$$\Delta u + \lambda(1+q)u = 0 \text{ in } D\,, \quad \Delta w + \lambda w = 0 \text{ in } D\,,$$

$$u = w \text{ on } \partial D\,, \quad \frac{\partial u}{\partial \nu} = \frac{\partial w}{\partial \nu} \text{ auf } \partial D\,.$$

Variational formulation? Solution space?

(1) Ultra Weak Formulation: Let $u, w \in L^2(D)$ and $(\psi, \phi) \in V$ where

$$V = \big\{(\psi,\phi) \in H^2(D) \times H^2(D) : \psi = \phi \text{ on } \partial D \text{ and } \partial\psi/\partial\nu = \partial\phi/\partial\nu \text{ on } \partial D\big\}.$$

Green's second theorem:

$$\begin{aligned} 0 &= \int_D [\Delta u + \lambda(1+q)u]\psi\,dx \\ &= \int_D [\Delta\psi + \lambda(1+q)\psi]u\,dx \;+\; \int_{\partial D}\left[\psi\,\frac{\partial u}{\partial\nu} - u\,\frac{\partial\psi}{\partial\nu}\right] ds \end{aligned}$$

$$\int_D [\Delta\psi + \lambda(1+q)\psi]u\,dx \;=\; \int_D [\Delta\phi + \lambda\phi]w\,dx \quad \forall\,(\psi,\phi) \in V\,.$$

The Interior TEVP, cont.

(2) Semi Weak Formulation: Set $v = w - u$. Then $v \in H_0^2(D)$ and $w \in L^2(D)$ and $\Delta w + \lambda w = 0$ in D and

$$(*) \qquad \Delta v + \lambda(1+q)v \;=\; \lambda q\, w \quad \text{in } D\,.$$

Variational formulation (replace λw by w):

$$\int_D [\Delta v + \lambda(1+q)v - q\,w]\phi\, dx = 0 \quad \text{for all } \phi \in L^2(D)\,,$$

$$\int_D [\Delta \psi + \lambda \psi]\, w\, dx = 0 \quad \text{for all } \psi \in H_0^2(D)\,.$$

Here, $(v, w) \in H_0^2(D) \times L^2(D)$ and $(\psi, \phi) \in H_0^2(D) \times L^2(D)$.

(3) $H_0^2(D)-$Formulation: Assume: $q \in L^\infty(D)$ and $q(x) \geq q_0 > 0$ on D.

From $(*)$: $[\Delta + \lambda]\frac{1}{q}[\Delta v + \lambda(1+q)v] \;=\; 0$; that is, $v \in H_0^2(D)$ and

$$\int_D [\Delta v + \lambda(1+q)v]\,[\Delta\psi + \lambda\psi]\,\frac{dx}{q} \;=\; 0 \quad \text{for all } \psi \in H_0^2(D)\,.$$

Discreteness

Assumption: $q \in L^\infty(D)$ and $q(x) \geq q_0 > 0$ on R where $R \subset D$ is some open subdomain with $\partial D \subset \overline{R}$.

Semi Weak Formulation: Determine $\lambda > 0$ and non-trivial $(v, w) \in X := H_0^2(D) \times L^2(D)$ with

$$a_\lambda(v, w; \psi, \phi) := \int_D [\Delta\psi + \lambda\psi] w \, dx + \int_D [\Delta v + \lambda(1+q)v - qw] \, \phi \, dx = 0$$

for all $(\psi, \phi) \in X$. Define also the symmetric form

$$\hat{a}_\lambda(v, w; \psi, \phi) := \int_D [\Delta\psi + \lambda\psi] w \, dx + \int_D \{ [\Delta v + \lambda v] \, \phi - qw\phi \} \, dx$$

Lemma: There exist $\hat{c} > 0$ and $\alpha > 0$ such that for all $\lambda < 0$:

$$\int_{D \setminus R} |w|^2 dx \ \leq \ \hat{c} \, e^{-2\alpha\sqrt{|\lambda|}} \int_R |w|^2 dx$$

for all solutions $w \in L^2(D)$ of $\Delta w + \lambda w = 0$ in D.

Proof of Lemma (in $\mathbb{R}^3$ for simplicity)

Let $\lambda = -k^2$ and $R' \subset R$ with $\partial D \subset \overline{R'}$ and $\alpha = \text{dist}(D \setminus R, R') > 0$ and $\rho \in C^\infty(D)$ with compact support in D and $\rho = 1$ in $D \setminus R'$. Green's representation theorem to ρw in D where $\Delta w - k^2 w = 0$ in D:

$$\begin{aligned} \rho(x)\, w(x) &= -\int_D \big[\Delta(\rho w)(y) - k^2 (\rho w)(y)\big] \frac{\exp(-k|x-y|)}{4\pi|x-y|}\, dy \\ &= 2\int_{R'} w(y)\, \text{div}_y \left(\nabla\rho(y)\, \frac{\exp(-k|x-y|)}{4\pi|x-y|} \right) dy \\ &\quad - \int_{R'} \Delta\rho(y)\, \frac{\exp(-k|x-y|)}{4\pi|x-y|} w(y)\, dy\,. \end{aligned}$$

For $x \in D \setminus R$:

$$\big|w(x)\big| \le c_1\, e^{-\alpha k} \int_{R'} \big|w(y)\big|\, dy \le c_1\, e^{-\alpha k} \int_R \big|w(y)\big|\, dy\,, \quad \text{thus}$$

$$\big|w(x)\big|^2 \le c_1^2\, e^{-2\alpha k}\, |R| \int_R \big|w(y)\big|^2\, dy\,.$$

Integration with respect to x over $D \setminus R$ yields the assertion. ■

Theorem (inf-sup condition)

$$\hat{a}_\lambda(v, w; \psi, \phi) := \int_D [\Delta\psi + \lambda\psi] w \, dx + \int_D \{[\Delta v + \lambda v]\phi - q \, w \, \phi\} \, dx$$

There exists $\lambda_0 < 0$ and $c > 0$ such that for all $\lambda \leq \lambda_0$

$$\sup_{(\psi,\phi)\neq 0} \frac{|\hat{a}_\lambda(v, w; \psi, \phi)|}{\|(\psi, \phi)\|_X} \geq c \|(v, w)\|_X \quad \text{for all } (v, w) \in X \, .$$

Proof (sketch): Fix $\lambda_0 < 0$ with (by Lemma!)

$$\int_D q|w|^2 \, dx = \int_{D\setminus R} q|w|^2 \, dx + \int_R q|w|^2 \, dx \geq \frac{q_0}{2} \int_R |w|^2 \, dx$$

for all solutions w of $\Delta w + \lambda w = 0$ in D and all $\lambda \leq \lambda_0$.
Proof by contradiction: Otherwise there exist $(v_j, w_j) \in X$ with $\|(v_j, w_j)\|_X = 1$ and $\sup_{(\psi,\phi)\neq 0} |\hat{a}_\lambda(v_j, w_j; \psi, \phi)| \to 0$ as $j \to \infty$. There exist weakly convergence subsequences $(v_j, w_j) \rightharpoonup (v, w)$ in X. Then $\hat{a}_\lambda(v, w; \psi, \phi) = 0 \; \forall (\psi, \phi) \in X$, thus $\Delta w + \lambda w = 0$ in D.

Theorem, Proof cont.

$$0 = \hat{a}_\lambda(v, w; \psi, \phi) = \int_D [\Delta\psi + \lambda\psi] w \, dx + \int_D \{[\Delta v + \lambda v]\phi - qw\phi\} \, dx$$

Set $\psi = -v$ and $\phi = w$, then $\int_D q \, w^2 dx = 0$, thus $w = 0$ in R. From this: $w = 0$ in D. With $\psi = 0$ and $\phi = v$ also $v = 0$ follows.

Choose $\rho \in C^\infty(D)$ with $\rho = 1$ in neighborhood R' of ∂D and $\rho = 0$ in $D \setminus R$. Set $\psi = \rho v_j$ and $\phi = -\rho w_j$. Then

$$\int_R [\Delta(\rho v_j) - \lambda\rho v_j] \, w_j \, dx - \int_R (\Delta v_j - \lambda v_j) \, \rho w_j - q \, \rho w_j^2 \, dx$$

tends to zero. Because $v_j \to 0$ in $H^1(D)$ we have $\int_R q \, \rho |w_j|^2 \, dx \to 0$, thus $\int_{R'} |w_j|^2 dx \to 0$. Similar arguments yield $w_j \to 0$ in $L^2(D)$ and $v_j \to 0$ in $H^2(D)$. Contradition to $\|(v_j, w_j)\|_X = 1$! ■

Theorem on Discreteness

Karlsruhe Institute of Technology

$$a_\lambda(v,w;\psi,\phi) = \int_D [\Delta\psi + \lambda\psi]\, w\, dx + \int_D [\Delta v + \lambda(1+q)v - qw]\,\phi\, dx$$

$$\hat{a}_\lambda(v,w;\psi,\phi) = \int_D [\Delta\psi + \lambda\psi]\, w\, dx + \int_D [\Delta v + \lambda v - qw]\,\phi\, dx$$

Theorem of Riesz yields existence of bounded $A_\lambda, \hat{A}_\lambda : X \to X$ with $\big(\hat{A}_\lambda(v,w),(\psi,\phi)\big)_X = \hat{a}_\lambda(v,w;\psi,\phi)$, $\big(A_\lambda(v,w),(\psi,\phi)\big)_X = a_\lambda(v,w;\psi,\phi)$ for all $(v,w),(\psi,\phi) \in X = H_0^2(D) \times L^2(D)$. Generalized Theorem of Lax-Milgram yields that $\hat{A}_\lambda$ is isomorphism from X onto itself.

Furthermore, $\hat{A}_\lambda - A_\mu$ and $A_\lambda - A_\mu$ are compact (simple) and A_λ one-to-one for $\lambda \le \lambda_1$ for some $\lambda_1 \le \lambda_0$ (complicated).

Therefore, also A_λ is isomorphism from X onto itself for $\lambda \le \lambda_1$.

Rewrite $A_\lambda(v,w) = 0$ into $(v,w) + A_{\lambda_1}^{-1}(A_\lambda - A_{\lambda_1})(v,w) = 0$. Also $A_\lambda - A_{\lambda_1} = (\lambda - \lambda_1)K$ and K compact. Thus:

Theorem: The set of transmission eigenvalues is discrete with infinity as the only accumulation point.

Existence of Eigenvalues

Assumption: $q \in L^\infty(D)$ and $q(x) \geq q_0 > 0$ on D.

$H_0^2(D)-$Formulation: Determine $\lambda > 0$ and non-trivial $v \in H_0^2(D)$ with

$$h_\lambda(v,\psi) \;:=\; \int_D [\Delta v + \lambda(1+q)v]\,[\Delta\psi + \lambda\psi]\,\frac{dx}{q} \;=\; 0 \quad \text{for all } \psi \in H_0^2(D)\,.$$

$$h_\lambda(v,\psi) \;=\; a(v,\psi) + \lambda\, b(v,\psi) + \lambda^2\, c(v,\psi) \quad \text{for all } \psi \in H_0^2(D), \text{ where}$$

$$a(v,\psi) \;=\; \int_D \Delta v\,\Delta\psi\,\frac{dx}{q}\,, \qquad c(v,\psi) \;=\; \int_D \frac{1+q}{q}\,v\,\psi\,dx\,,$$

$$b(v,\psi) \;=\; \int_D \big[(1+q)\,v\,\Delta\psi + \psi\,\Delta v\big]\,\frac{dx}{q}\,.$$

This is a quadratic eigenvalue problem (not self–adjoint!)

Theorem: (Colton/K./Päivärinta 1989, Päivärinta/Sylvester 2009, Cakoni/Gintides/Haddar 2009)

The set of eigenvalues λ forms a discrete set, and there is a sequence of real eigenvalues which converge to infinity.

Sketch of Proof, Part 1

First: Let D be a disk centered at 0 with radius $R > 0$, and q constant. Let again $\lambda = k^2$ and $\rho = \sqrt{1+q}$. For fixed $n \in \mathbb{N}$ the functions

$$u(r,\phi) = a\, J_n(k\rho r)\, e^{in\phi}\,, \quad w(r,\phi) = b\, J_n(kr)\, e^{in\phi}\,,$$

$r \in [0,R]$, $\phi \in [0,2\pi]$, are solutions of the differential equations. The constants a, b are determined from boundary conditions:

$$a\, J_n(k\rho R) - b\, J_n(kR) = 0\,, \quad a\,\rho\, J_n'(k\rho R) - b\, J_n'(kR) = 0\,.$$

Nontrivial solutions exist if

$$\det \begin{bmatrix} J_n(k\rho R) & -J_n(kR) \\ \rho\, J_n'(k\rho R) & -J_n'(kR) \end{bmatrix} = J_n(kR)\,\rho\, J_n'(k\rho R) - J_n(k\rho R)\, J_n'(kR) = 0\,.$$

Asymptotic behaviour of $J_n(t)$ for $t \to \infty$ yields assertion.

Sketch of Proof, **Part 2**

Treat λ as parameter and consider the (real valued) function

$$f(\lambda) \;=\; \inf_{\|u\|_{H_0^2(D)}=1} h_\lambda(u,u)\,. \quad \text{It is } f(0)>0 \text{ because } a \text{ is coercive.}$$

Lemma: f is continuous on $\mathbb{R}_{\geq 0}$. Every zero $\lambda > 0$ of f is an interior transmission eigenvalue.

Goal: Determine $\hat{\lambda} > 0$ and $\hat{u} \in H_0^2(D)$ with $\hat{u} \neq 0$ and $h_{\hat{\lambda}}(\hat{u},\hat{u}) \leq 0$.

Choose disk B in D and interior TEV $\hat{\lambda}$ with eigenfunction $\hat{u} \in H_0^2(B)$ in B corresponding to constant q_0 (possible by part 1!). Extend $\hat{u}$ by 0 into D, then $\hat{u} \in H_0^2(D)$.

$$\begin{aligned} h_{\hat{\lambda}}(\hat{u},\hat{u}) &= \int_D \left[\Delta\hat{u} + \hat{\lambda}(1+q)\hat{u}\right]\left[\Delta\hat{u} + \hat{\lambda}\hat{u}\right]\frac{dx}{q} \\ &= \int_D |\Delta\hat{u} + \hat{\lambda}\hat{u}|^2 \frac{dx}{q} \;+\; \hat{\lambda}\int_D \hat{u}(\Delta\hat{u} + \hat{\lambda}\hat{u})\,dx \\ &\leq \int_B |\Delta\hat{u} + \hat{\lambda}\hat{u}|^2 \frac{dx}{q_0} + \hat{\lambda}\int_B \hat{u}(\Delta\hat{u} + \hat{\lambda}\hat{u})\,dx \;=\; 0\,. \end{aligned}$$

■

This ends the lecture! Fioralba will continue with more advanced and recent results.

Thank you for your attention!