Passive correlation based imaging

Chrysoula Tsogka

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University of Crete and IACM-FORTH

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is the solution of the wave equation when one sensor acts as source and the other as receiver

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  - Seismic fault monitoring

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Model problem

- We consider a domain Ω containing a reflector $\mathcal{O}$
- and $u(t, \vec{\mathbf{x}})$ solution of the acoustic wave equation on $\Omega \setminus \mathcal{O}$:

$$\frac{1}{c(\vec{\mathbf{x}})^2} \frac{\partial^2 u(t, \vec{\mathbf{x}})}{\partial t^2} - \Delta u(t, \vec{\mathbf{x}}) = n(t, \vec{\mathbf{x}}).$$

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- Propagation speed $c(\vec{\mathbf{x}})$ is given

$$c(\vec{\mathbf{x}})^2 = c_0^2 \qquad \text{(homogeneous case)},$$

$$\frac{1}{c(\vec{\mathbf{x}})^2} = \frac{1}{c_0^2}\left[1 + \sigma\mu(\frac{\vec{\mathbf{x}}}{\ell})\right] \qquad \text{(inhomogeneous case)},$$

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- c_0 is known speed of sound in the background medium,
- μ is the normalized fluctuations,
- ℓ is the correlation length,
- σ controls the strength of the fluctuations,
- $n(t, \vec{\mathbf{x}})$ models noise sources.

The Green's function
Time domain

- The Green's function $G(t, \vec{\mathbf{x}}, \vec{\mathbf{y}})$ is the solution of

$$\frac{1}{c(\vec{\mathbf{x}})^2}\frac{\partial^2 G(t, \vec{\mathbf{x}}, \vec{\mathbf{y}})}{\partial t^2} - \Delta G(t, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = \delta(t)\delta(\vec{\mathbf{x}} - \vec{\mathbf{y}})$$

- Assuming $G(t, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = 0$ for $t < 0$ we obtain in a homogeneous medium $c(\vec{\mathbf{x}}) = c_0$, the well-known expression

$$G(t, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = \frac{1}{4\pi|\vec{\mathbf{x}} - \vec{\mathbf{y}}|}\delta(t - \frac{|\vec{\mathbf{x}} - \vec{\mathbf{y}}|}{c_0})$$

The Green's function
Frequency domain

- In the frequency domain, the time-harmonic Green's function

$$\widehat{G}(\omega, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = \int G(t, \vec{\mathbf{x}}, \vec{\mathbf{y}}) e^{i\omega t} dt$$

  is the solution of

$$\frac{\omega^2}{c(\vec{\mathbf{x}})^2} \widehat{G}(\omega, \vec{\mathbf{x}}, \vec{\mathbf{y}}) + \Delta \widehat{G}(\omega, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = -\delta(\vec{\mathbf{x}} - \vec{\mathbf{y}})$$

  with the Sommerfeld radiation condition (assuming $c(\vec{\mathbf{x}}) = c_0$ at infinity)

$$\lim_{|\vec{\mathbf{x}}| \to \infty} |\vec{\mathbf{x}}| \left(\frac{\vec{\mathbf{x}}}{|\vec{\mathbf{x}}|} \nabla_{\vec{\mathbf{x}}} - \frac{i\omega}{c_0} \right) \widehat{G}(\omega, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = 0$$

- In a homogeneous medium we obtain

$$\widehat{G}(\omega, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = \frac{1}{4\pi |\vec{\mathbf{x}} - \vec{\mathbf{y}}|} e^{i \frac{\omega}{c_0} |\vec{\mathbf{x}} - \vec{\mathbf{y}}|}$$

The Green's function

wave equation solution

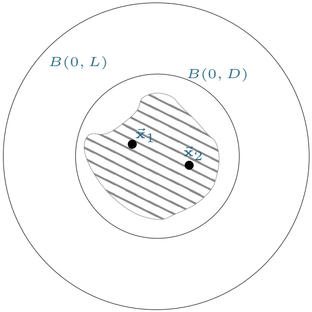
- The solution of the wave equation for a source $n(t, \vec{\mathbf{x}})$ is

$$u(t, \vec{\mathbf{x}}) = \int \int G(t - s, \vec{\mathbf{x}}, \vec{\mathbf{y}}) n(s, \vec{\mathbf{y}}) d\vec{\mathbf{y}} ds$$

and in the frequency domain

$$\widehat{u}(\omega, \vec{\mathbf{x}}) = \int \widehat{G}(\omega, \vec{\mathbf{x}}, \vec{\mathbf{y}}) \widehat{n}(\omega, \vec{\mathbf{y}}) d\vec{\mathbf{y}}$$

The Kirchhoff Helmholtz identity



Assume that the medium is homogeneous outside $B(0, D)$, then

$$\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) - \overline{\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2)} = \frac{2i\omega}{c_0} \int_{\partial B(0,L)} \overline{\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{y}})} \widehat{G}(\omega, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) ds(\vec{\mathbf{y}})$$

for $L \gg D$ and $\forall \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_1 \in B(0, D)$ (the medium can be heterogeneous in $B(0, D)$).

The Kirchhoff Helmholtz identity

Proof

- Let $G(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)$ and $G(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2)$ solutions of

$$\frac{\omega^2}{c(\vec{\mathbf{x}})^2}\overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)} + \Delta\overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)} = -\delta(\vec{\mathbf{y}} - \vec{\mathbf{x}}_1) \quad (1)$$

$$\frac{\omega^2}{c(\vec{\mathbf{x}})^2}\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2) + \Delta\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2) = -\delta(\vec{\mathbf{y}} - \vec{\mathbf{x}}_2) \quad (2)$$

- Multiply (2) by $\overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)}$ and (1) by $\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2)$ and subtract

$\overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)} \times (2) - \widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2) \times (1)$:

$$\nabla \cdot \left(\overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)}\nabla\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2) - \widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2)\nabla\overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)}\right) =$$

$$= -\delta(\vec{\mathbf{y}} - \vec{\mathbf{x}}_2)\overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)} + \delta(\vec{\mathbf{y}} - \vec{\mathbf{x}}_1)\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2)$$

$$= -\delta(\vec{\mathbf{y}} - \vec{\mathbf{x}}_2)\overline{\widehat{G}(\omega, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_1)} + \delta(\vec{\mathbf{y}} - \vec{\mathbf{x}}_1)\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2)$$

(reciprocity) $$= -\delta(\vec{\mathbf{y}} - \vec{\mathbf{x}}_2)\overline{\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2)} + \delta(\vec{\mathbf{y}} - \vec{\mathbf{x}}_1)\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2)$$

The Kirchhoff Helmholtz identity

Proof continues

- By integrating the previous expression over the ball $B(0,L)$ and using the divergence theorem, we obtain

$$\int_{\partial B(0,L)} \frac{\vec{\mathbf{y}}}{|\vec{\mathbf{y}}|} \cdot \left(\overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)} \nabla \widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2) - \widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2) \nabla \overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)} \right) ds(\vec{\mathbf{y}})$$
$$= -\overline{\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2)} + \widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2)$$

- Using the Sommerfeld radiation condition (letting $L \to \infty$) we replace $\frac{\vec{\mathbf{y}}}{|\vec{\mathbf{y}}|} \cdot \nabla \widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2)$ by $\frac{i\omega}{c_0} \widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_2)$ and $\frac{\vec{\mathbf{y}}}{|\vec{\mathbf{y}}|} \cdot \nabla \overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)}$ by $\frac{-i\omega}{c_0} \overline{\widehat{G}(\omega, \vec{\mathbf{y}}, \vec{\mathbf{x}}_1)}$ to obtain

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$$\boxed{\frac{2i\omega}{c_0}\int_{\partial B(0,L)} \overline{\widehat{G}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{y}})}\widehat{G}(\omega,\vec{\mathbf{x}}_2,\vec{\mathbf{y}}) ds(\vec{\mathbf{y}}) = \widehat{G}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) - \overline{\widehat{G}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2)}}$$

Noise sources

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Noise sources

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- its correlation function is

$$E\left\{n(\vec{\mathbf{x}}_1, t_1)n(\vec{\mathbf{x}}_2, t_2)\right\} = F(t_1 - t_2)K(\vec{\mathbf{x}}_1)\delta(\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2)$$

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- assume Gaussian statistics for the noise sources
- assume that the coherence time of the sources is small compared to the typical travel times, i.e., $F(t) = F_\varepsilon(t) = F\left(\frac{t}{\varepsilon}\right)$
- $K(\vec{\mathbf{x}})$ characterizes the spatial support of the sources (if $K(\vec{\mathbf{x}}) \equiv 1$ there are noise sources everywhere)

Computation of cross-correlations

- Let $u(t, \vec{\mathbf{x}}_1)$ and $u(t, \vec{\mathbf{x}}_2)$ be the wave fields recorded at $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$. Their empirical cross correlation is

$$C_T(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) = \frac{1}{T} \int_0^T u(t, \vec{\mathbf{x}}_1) u(t + \tau, \vec{\mathbf{x}}_2) dt$$

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Theorem

The expectation of the empirical cross correlation C_T (with respect to the distribution of the sources) is independent of T :

$$\langle C_T(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) \rangle = \frac{1}{2\pi} \int \int \overline{\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{y}})} \widehat{G}(\omega, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) \widehat{F_\varepsilon}(\omega) e^{-i\omega\tau} d\vec{\mathbf{y}} d\omega,$$

Proof

The solution of the wave equation can be written as (change of variable $\tilde{s} \to t - s$)

$$u(t, \vec{\mathbf{x}}) = \int_{-\infty}^{t} \int G(t - s, \vec{\mathbf{x}}, \vec{\mathbf{y}}) n_\varepsilon(s, \vec{\mathbf{y}}) d\vec{\mathbf{y}} ds = \int_{0}^{\infty} \int G(\tilde{s}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) n_\varepsilon(t - \tilde{s}, \vec{\mathbf{y}}) d\vec{\mathbf{y}} d\tilde{s}$$

Extending $G(t, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = 0$ for $t \leq 0$, we get

$$u(t, \vec{\mathbf{x}}) = \int_{-\infty}^{\infty} \int G(s, \vec{\mathbf{x}}, \vec{\mathbf{y}}) n_\varepsilon(t - s, \vec{\mathbf{y}}) d\vec{\mathbf{y}} ds$$

The stationarity of n_ε implies stationarity of $u(t, \vec{\mathbf{x}})$, hence the mean of C_T does not depend on T and is given by

$$C^{(1)}(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) := \langle C_T(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) \rangle = \langle u(0, \vec{\mathbf{x}}_1) u(\tau, \vec{\mathbf{x}}_2) \rangle$$

$$= \int d\vec{\mathbf{y}}_1 \int d\vec{\mathbf{y}}_2 \int ds \int ds' G(s, \vec{\mathbf{x}}_1, \vec{\mathbf{y}}_1) G(s', \vec{\mathbf{x}}_2, \vec{\mathbf{y}}_2) \langle n_\varepsilon(-s, \vec{\mathbf{y}}_1) n_\varepsilon(\tau - s', \vec{\mathbf{y}}_2) \rangle$$

$$= \int d\vec{\mathbf{y}} \int ds \int ds' G(s, \vec{\mathbf{x}}_1, \vec{\mathbf{y}}) G(s', \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) F_\varepsilon(-s - \tau + s')$$

Proof

doing a change of variables $s_1 = s$, $s_2 = -s - \tau + s'$ we get

$$C^{(1)}(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) = \int d\vec{\mathbf{y}} \int ds_1 \int ds_2 G(s_1, \vec{\mathbf{x}}_1, \vec{\mathbf{y}}) G(\tau + s_1 + s_2, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) F_\varepsilon(s_2)$$

using Fourier transform and going in the frequency domain we obtain the result,

$$C^{(1)}(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) =$$

$$\int e^{-i\omega_1 s_1 - i\omega_2(\tau + s_1 + s_2)} \widehat{G}(\omega_1, \vec{\mathbf{x}}_1, \vec{\mathbf{y}}) \widehat{G}(\omega_2, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) F_\varepsilon(s_2) d\vec{\mathbf{y}} ds_1 ds_2 \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi}$$

$$\left(\int ds_1 \Rightarrow \omega_1 = -\omega_2\right)$$

$$\int e^{-i\omega_2(\tau + s_2)} \widehat{G}(-\omega_2, \vec{\mathbf{x}}_1, \vec{\mathbf{y}}) \widehat{G}(\omega_2, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) F_\varepsilon(s_2) d\vec{\mathbf{y}} ds_2 \frac{d\omega_2}{2\pi}$$

$\left(\widehat{F}_\varepsilon(-\omega_2) = \left(\int e^{-i\omega_2 s_2} F_\varepsilon(s_2) ds_2\right)\right.$ and $\widehat{F}_\varepsilon$ real valued and even

$$\int e^{-i\omega_2 \tau} \widehat{G}(-\omega_2, \vec{\mathbf{x}}_1, \vec{\mathbf{y}}) \widehat{G}(\omega_2, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) \widehat{F}_\varepsilon(\omega_2) d\vec{\mathbf{y}} \frac{d\omega_2}{2\pi}$$

$(f(t)$ real $\Rightarrow \widehat{f}(-\omega) = \overline{\widehat{f}(\omega)})$

$$\int e^{-i\omega\tau} \overline{\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{y}})} \widehat{G}(\omega, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) \widehat{F}_\varepsilon(\omega) d\vec{\mathbf{y}} \frac{d\omega}{2\pi}$$

Expectation of cross-correlation

We showed that the expectation of the empirical cross correlation C_T is independent of T and given by :

$$C^{(1)}(\tau,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = \frac{1}{2\pi}\int\int \overline{\widehat{G}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{y}})}\widehat{G}(\omega,\vec{\mathbf{x}}_2,\vec{\mathbf{y}})K(\vec{\mathbf{y}})\widehat{F_\varepsilon}(\omega)e^{-i\omega\tau}d\vec{\mathbf{y}}d\omega,$$

We can re-write $C^{(1)}(\tau,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2)$ as

$$C^{(1)}(\tau,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = \int e^{-i\omega\tau}\widehat{D}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2)\widehat{F}_\varepsilon(\omega)\frac{d\omega}{2\pi}$$

with

$$\widehat{D}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = \int \overline{\widehat{G}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{y}})}\widehat{G}(\omega,\vec{\mathbf{x}}_2,\vec{\mathbf{y}})K(\vec{\mathbf{y}})d\vec{\mathbf{y}}$$

Self averaging property

Theorem

The empirical cross correlation C_T is a self-averaging quantity

$$C_T(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) \stackrel{T\to\infty}{\longrightarrow} C^{(1)}(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2)$$

in probability with respect to the distribution of the sources.

- C_T is a statistically stable quantity : for large T, C_T is independent of the realization of the noise sources !
- This implies, in particular, that the SNR of the cross-correlations is proportional to $\sqrt{T}$!

$$SNR(X) = \frac{< X >}{\sqrt{Var(X)}}$$

Proof : show that the variance of C_T tends to zero as $T \to \infty$ (the rate is $O(1/T)$).
The covariance of the empirical cross correlation C_T is :

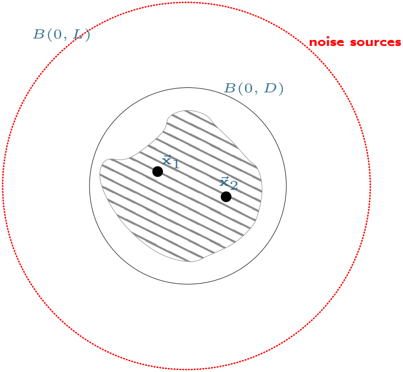
$$\mathrm{Cov}\big(C_T(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2), C_T(\tau', \vec{\mathbf{x}}_3, \vec{\mathbf{x}}_4)\big)$$
$$= \frac{1}{2\pi T}\int \widehat{D}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_3)\overline{\widehat{D}(\omega, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_4)}(\widehat{F_\varepsilon}(\omega))^2 e^{-i\omega(\tau'-\tau)}d\omega$$
$$+\frac{1}{2\pi T}\int \widehat{D}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_4)\overline{\widehat{D}(\omega, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3)}(\widehat{F_\varepsilon}(\omega))^2 e^{-i\omega(\tau'+\tau)}d\omega$$

For details see

J. Garnier and G. Papanicolaou, "Passive sensor imaging using cross correlations of noisy signals in a scattering medium", SIAM J. Imaging Sciences, 2 :396–437, 2009.

Emergence of the Green's function from noise cross-correlations

- Assume that the medium is homogeneous outside $B(0, D)$ and that the noise sources are distributed with uniform density on the surface of a sphere $B(0, L)$ with $L \gg D$



Emergence of the Green's function from noise cross-correlations

- Assume that the medium is homogeneous outside $B(0,D)$ and that the noise sources are distributed with uniform density on the surface of a sphere $B(0,L)$ with $L \gg D$
- Then for any $\vec{\mathbf{x}}_1,\ \vec{\mathbf{x}}_2 \in B(0,D)$ we have

$$C^{(1)}(\tau,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = \int_{\partial B(0,L)} e^{-i\omega\tau}\overline{\widehat{G}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{y}})}\widehat{G}(\omega,\vec{\mathbf{x}}_2,\vec{\mathbf{y}})\widehat{F}_\varepsilon(\omega)d\vec{\mathbf{y}}\frac{d\omega}{2\pi}$$

- Using the Kirchhoff-Helmholtz identity

$$\frac{2i\omega}{c_0}\int_{\partial B(0,L)}\overline{\widehat{G}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{y}})}\widehat{G}(\omega,\vec{\mathbf{x}}_2,\vec{\mathbf{y}})ds(\vec{\mathbf{y}}) = \widehat{G}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) - \overline{\widehat{G}(\omega,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2)}$$

  we obtain (up to a multiplicative constant),

$$\frac{\partial}{\partial\tau}C^{(1)}(\tau,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = -F_\varepsilon \star G(\tau,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) + F_\varepsilon \star G(-\tau,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2)$$

  where $\star$ denotes convolution in τ.

Emergence of the Green's function from noise cross-correlations

- If $\varepsilon \ll 1$ then F_ε behaves like a delta-function and we get

$$\boxed{\frac{\partial}{\partial \tau} C^{(1)}(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) \approx G(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) - G(-\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2)}$$

Emergence of the Green's function from noise cross-correlations

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- In the more general case, for spatially localised noise sources, the cross correlation between $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ is expected to have a singular component at the travel time between the two points only if the ray going through $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ reaches into the source region, that is, into the support of the function $K(\vec{\mathbf{y}})$. This is shown using stationary phase.

- We assume a slowly varrying $c(\vec{\mathbf{x}})$ and use the WKB (Wentzell-Kramers-Brillouin) asymptotics for the Green's function

$$\widehat{G}(\frac{\omega}{\varepsilon}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) \approx a(\vec{\mathbf{x}}, \vec{\mathbf{y}}) e^{i\omega \frac{\tau(\vec{\mathbf{x}}, \vec{\mathbf{y}})}{\varepsilon}}$$

  The amplitude $a(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ and travel time $\tau(\vec{\mathbf{x}}, \vec{\mathbf{y}})$ are smooth except at $\vec{\mathbf{x}} = \vec{\mathbf{y}}$.
- To obtain this we seek for an expansion of $\widehat{G}(\frac{\omega}{\varepsilon}, \vec{\mathbf{x}}, \vec{\mathbf{y}})$ as $\varepsilon \to 0$ of the form

$$\widehat{G}(\frac{\omega}{\varepsilon}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = e^{i\frac{\omega}{\varepsilon} T(\vec{\mathbf{x}}, \vec{\mathbf{y}})} \sum_{j=0}^{\infty} A_j(\vec{\mathbf{x}}, \vec{\mathbf{y}}) \frac{\varepsilon^j}{\omega^j}$$

  substituting in

$$\frac{\omega^2}{\varepsilon^2 c(\vec{\mathbf{x}})^2} \widehat{G}(\frac{\omega}{\varepsilon}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) + \Delta \widehat{G}(\frac{\omega}{\varepsilon}, \vec{\mathbf{x}}, \vec{\mathbf{y}}) = -\delta(\vec{\mathbf{x}} - \vec{\mathbf{y}})$$

- we get

Terms of $\mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$: $\left|\nabla T(\vec{\mathbf{x}},\vec{\mathbf{y}})\right|^2 - \dfrac{1}{c(\vec{\mathbf{x}})^2} = 0,$ eikonal for the travel time

Terms of $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$: $2\nabla T(\vec{\mathbf{x}},\vec{\mathbf{y}})\cdot\nabla A_0(\vec{\mathbf{x}},\vec{\mathbf{y}}) + A_0(\vec{\mathbf{x}},\vec{\mathbf{y}})\Delta T(\vec{\mathbf{x}},\vec{\mathbf{y}}) = 0,$

transport for the amplitude

that we can solve with the method of characteristics (rays)

- We have $a(\vec{\mathbf{x}},\vec{\mathbf{y}}) = A_0(\vec{\mathbf{x}},\vec{\mathbf{y}})$ and $\tau(\vec{\mathbf{x}},\vec{\mathbf{y}}) = T(\vec{\mathbf{x}},\vec{\mathbf{y}})$

In the homogeneous case : $a(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \dfrac{1}{4\pi\left|\vec{\mathbf{x}}-\vec{\mathbf{y}}\right|}$ and $\tau(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \dfrac{\left|\vec{\mathbf{x}}-\vec{\mathbf{y}}\right|}{c_0}$

- The travel time can be obtained from Fermat's principle :

$$\tau(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \inf\left\{T s.t. \exists (X_t)_{t\in[0,T]} \in \mathcal{C}^1, X_0 = \vec{\mathbf{x}}, X_T = \vec{\mathbf{y}}, \left|\frac{dX_t}{dt}\right| = c(X_t)\right\}$$

- The minimising curve X_t is a ray and we assume that $c(\vec{\mathbf{x}})$ is such that there is a unique ray joining any $\vec{\mathbf{x}}, \vec{\mathbf{y}}$. We need the following Lemma :

Lemma

If $\nabla_{\vec{\mathbf{y}}}\tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}) = \nabla_{\vec{\mathbf{y}}}\tau(\vec{\mathbf{x}}_2,\vec{\mathbf{y}})$ then $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ lie on the same ray issuing from $\vec{\mathbf{y}}$ and

$$\tau(\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = \left|\tau(\vec{\mathbf{x}}_2,\vec{\mathbf{y}}) - \tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}})\right|.$$

While if $\nabla_{\vec{\mathbf{y}}}\tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}) = -\nabla_{\vec{\mathbf{y}}}\tau(\vec{\mathbf{x}}_2,\vec{\mathbf{y}})$ then $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ lie on the opposite sides of the same ray issuing from $\vec{\mathbf{y}}$ and

$$\tau(\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = \tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}) + \tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}).$$

Spatially localised noise sources

back to cross-correlations

- We have

$$C^{(1)}(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) = \int e^{-i\omega\tau} \overline{\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{y}})} \widehat{G}(\omega, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) \widehat{F}_\varepsilon(\omega) d\vec{\mathbf{y}} \frac{d\omega}{2\pi}$$

- Now use that $\widehat{F}_\varepsilon(\omega) = \varepsilon \widehat{F}(\varepsilon\omega)$ (since $F_\varepsilon(t) = F(\frac{t}{\varepsilon})$) to get

$$C^{(1)}(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) = \varepsilon \int e^{-i\omega\tau} \overline{\widehat{G}(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{y}})} \widehat{G}(\omega, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) \widehat{F}(\varepsilon\omega) d\vec{\mathbf{y}} \frac{d\omega}{2\pi}$$

- With the change of variables $\tilde{\omega} = \varepsilon\omega$,

$$C^{(1)}(\tau, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) = \int e^{-i\frac{\tilde{\omega}}{\varepsilon}\tau} \overline{\widehat{G}(\frac{\tilde{\omega}}{\varepsilon}, \vec{\mathbf{x}}_1, \vec{\mathbf{y}})} \widehat{G}(\frac{\tilde{\omega}}{\varepsilon}, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) K(\vec{\mathbf{y}}) \widehat{F}(\tilde{\omega}) d\vec{\mathbf{y}} \frac{d\tilde{\omega}}{2\pi}$$

Spatially localised noise sources

stationary phase

- Using the WKB approximation for $\widehat{G}$ we get

$$C^{(1)}(\tau,\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = \int \overline{a(\vec{\mathbf{x}}_1,\vec{\mathbf{y}})}a(\vec{\mathbf{x}}_2,\vec{\mathbf{y}})e^{i\frac{\omega}{\varepsilon}T(\vec{\mathbf{y}})}K(\vec{\mathbf{y}})d\vec{\mathbf{y}}\frac{d\omega}{2\pi}$$

  with phase

$$\omega T(\vec{\mathbf{y}}) = \omega\left(\tau(\vec{\mathbf{x}}_2,\vec{\mathbf{y}}) - \tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}) - \tau\right)$$

- Stationary phase implies that the main contributions come from the critical points where

$$\frac{\partial}{\partial\omega}\left(\omega T(\vec{\mathbf{y}})\right) = 0,\ \nabla_{\vec{\mathbf{y}}}\left(\omega T(\vec{\mathbf{y}})\right) = 0$$

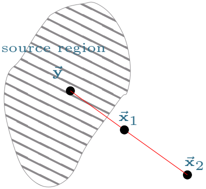
  from where it follows that

$$\tau(\vec{\mathbf{x}}_2,\vec{\mathbf{y}}) - \tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}) = \tau,\ \nabla_{\vec{\mathbf{y}}}\tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}) = \nabla_{\vec{\mathbf{y}}}\tau(\vec{\mathbf{x}}_2,\vec{\mathbf{y}})$$

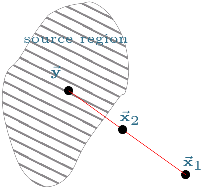
  The previous Lemma implies that $\vec{\mathbf{x}}_1$ and $\vec{\mathbf{x}}_2$ lie on the same ray issuing from $\vec{\mathbf{y}}$! In order for a stationary point to contribute we need $K(\vec{\mathbf{y}}) \neq 0$which means that $\vec{\mathbf{y}}$ has to be in the source region !

Spatially localised noise sources

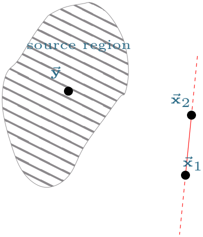
Possible configurations when $\nabla_{\vec{\mathbf{y}}}\tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}) = \nabla_{\vec{\mathbf{y}}}\tau(\vec{\mathbf{x}}_2,\vec{\mathbf{y}})$



$$\tau(\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = \tau(\vec{\mathbf{x}}_2,\vec{\mathbf{y}}) - \tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}})$$



$$\tau(\vec{\mathbf{x}}_1,\vec{\mathbf{x}}_2) = -(\tau(\vec{\mathbf{x}}_2,\vec{\mathbf{y}}) - \tau(\vec{\mathbf{x}}_1,\vec{\mathbf{y}}))$$



no contribution !

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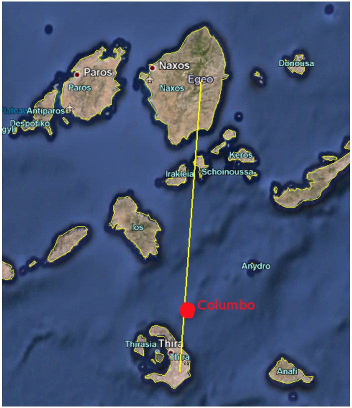
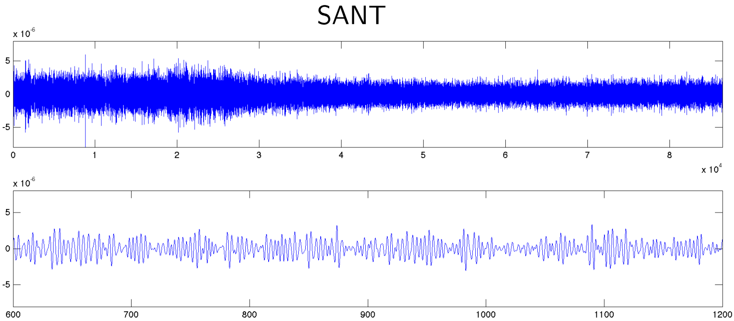
Seismic noise : An example

Cross-correlations

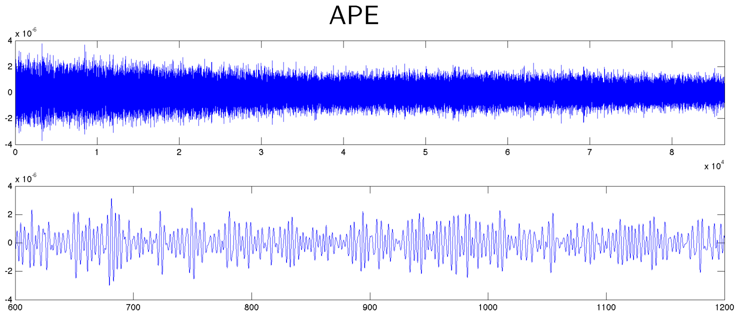


Two seismic stations (Santorini and Naxos)

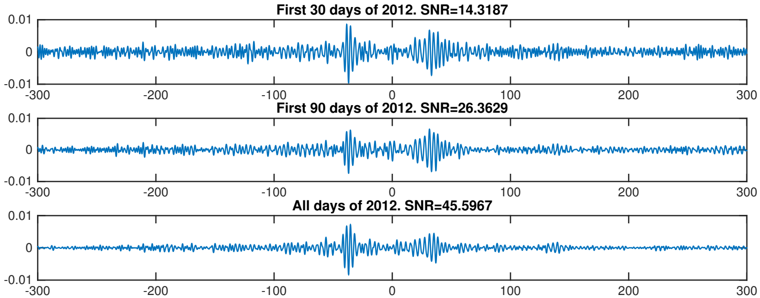


The data recorded on Santorini filtered in frequency range $[0.2, 0.5]$Hz

Seismic noise : An example
Cross-correlations



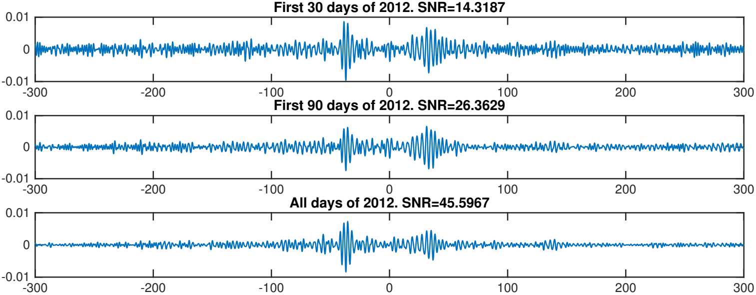
The data recorded on Naxos filtered in frequency range $[0.2, 0.5]$Hz



The cross-correlations between the two stations as a function of recording time.
Distance : 78Km , Mean Speed : 2.1 Km/s $\Rightarrow$ expected peak at ± 37s
Recall SNR analysis ($\sqrt{T}$)

Seismic noise : An example

Cross-correlations



The cross-correlations between the two stations as a function of recording time.

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Use these cross-correlations to : Estimate the velocity structure in the crust
Volcano monitoring (Santorini/seismic activity)
Seismic fault monitoring

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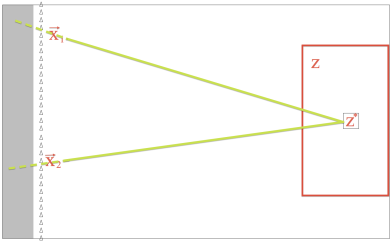
The imaging problem

- noise sources are spatially localized,
- sensors $(\vec{x}_q)_{1 \leqslant q \leqslant N_q}$ are located between the sources and the reflector



- This is the daylight configuration (Josselin's talk) : there are rays that emanating from the source region meet first a sensor and then the reflector !
- Stationary phase analysis shows that the cross correlation between two sensors $\vec{x}_1$ and $\vec{x}_2$ has a singular component at $+/-$ the sum of the travel times between the sensors and the reflector

$$\tau(z^*, \vec{x}_1) + \tau(z^*, \vec{x}_2)$$

Imaging functional

- To image the reflector it seems a good idea to compute

$$\mathcal{I}^{\mathrm{D}}(\mathbf{z}) = \sum_{q,q'=1}^{N_{\mathrm{q}}} C_T(\tau(\mathbf{z},\vec{\mathbf{x}}_{q'}) + \tau(\mathbf{z},\vec{\mathbf{x}}_q), \vec{\mathbf{x}}_q, \vec{\mathbf{x}}_{q'}),$$

  for points $\mathbf{z}$ in a search domain.
- we could also use the $-(\tau(\mathbf{z},\vec{\mathbf{x}}_{q'}) + \tau(\mathbf{z},\vec{\mathbf{x}}_q)))$, but it should not make a difference since we expect $C_T(-\tau,\vec{\mathbf{x}}_q,\vec{\mathbf{x}}_{q'}) = C_T(\tau,\vec{\mathbf{x}}_{q'},\vec{\mathbf{x}}_q)$. In practice we may want to average the cross-correlation over the positive and negative times.
- $\mathcal{I}^{\mathrm{D}}(\mathbf{z})$ is Kirchhoff migration (or travel time migration) and has been analyzed in detail (Beylkin, Bleistein, Symes, ...) in the case of an active array, i.e., when we send pulses from sources at the array and record the echoes at the receivers.
- Here the difference is that the array is passive and records noisy signals. By forming the cross-correlations of the recorded signals we transform the "passive array" into an active one.

Simulation setup

- wave equation on the rectangle $[0, 50\lambda] \times [-15\lambda, 15\lambda]$, with a reflector located on $[44\lambda, 46\lambda] \times [-\lambda, \lambda]$,
- random distribution of sources has support on the rectangle $[0, 4\lambda] \times [-15\lambda, 15\lambda]$,
- we record the solution u of the wave equation at N_q receivers located at $\vec{\mathbf{x}}_q = (5\lambda, (q - (N_\mathrm{q} + 1)/2)\lambda/2)$, for $1 \leqslant q \leqslant N_\mathrm{q}$,
- $\lambda = 6\text{mm}$ and $c_0 = 3\text{km/s}$,
- the reflector is modeled as a soft acoustic scatterer, *i.e.* $u = 0$ on the boundary of the reflector
- we surround domain by PML

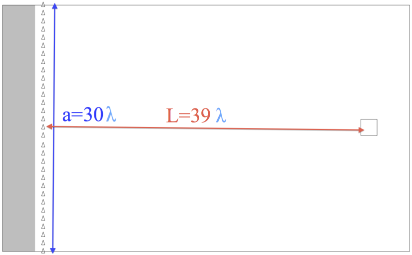


Image domain : square of size 20λ centered on the reflector

Numerical solution of wave equation

- We solve the wave equation with the code Montjoie (http://montjoie.gforge.inria.fr/).

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- For the numerical examples considered here we use 7th order finite elements in space and 4th order finite differences in time.
- We added the computation of cross-correlations and imaging functionals in Montjoie.

Results (homogeneous test case)

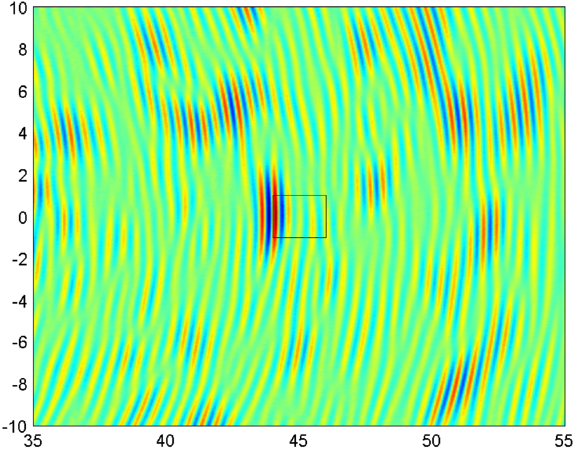

Figure : Imaging functional for the homogeneous medium. $N_{\mathrm{q}} = 21$

Results (homogeneous test case)

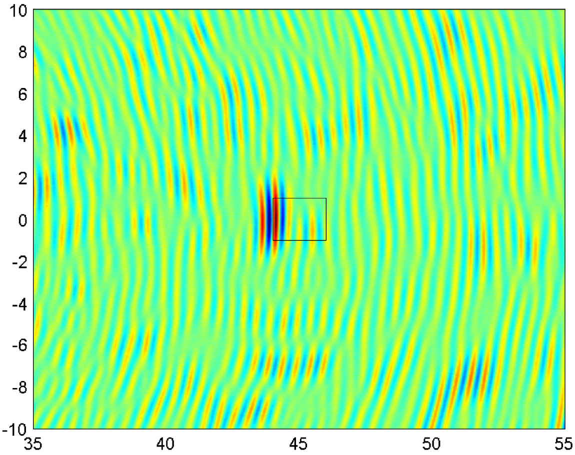

Figure : Imaging functional for the homogeneous medium. $N_{\text{q}} = 31$

Results (homogeneous test case)

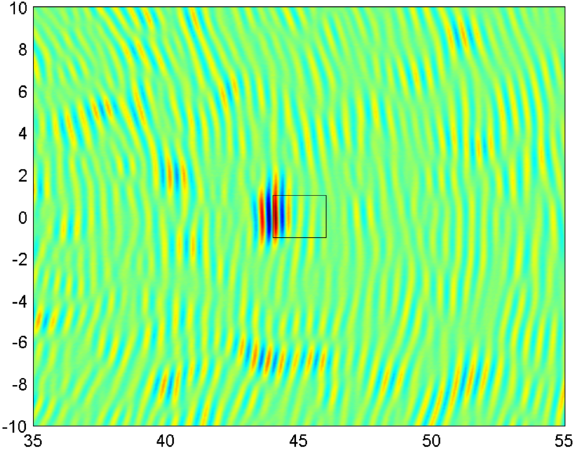

Figure : Imaging functional for the homogeneous medium. $N_{\mathrm{q}} = 41$

Results (homogeneous test case)

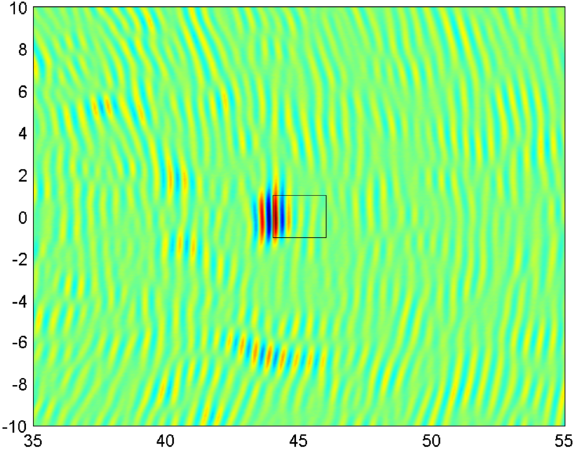

Figure : Imaging functional for the homogeneous medium. $N_{\mathrm{q}} = 51$

Image quality

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- In other words, "What are the parameters that control the quality of the image, and how"?
- Resolution analysis (same as for the active case)
  - Cross range resolution : $\lambda L/a$.
  - Range resolution : c_0/B

J. Garnier and G. Papanicolaou, "Resolution analysis for imaging with noise", Inverse Problems, Vol. 26, pp. 074001, 2010.

Signal to Noise Ratio analysis

- SNR analysis : Assuming that the sources surround the region of interest, and that the receiver array is sampled at half-a-wavelength apart (or larger), we show that

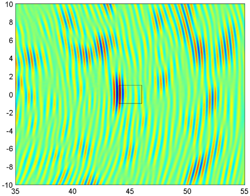
$$\mathrm{SNR}^{\mathrm{D}} = \frac{\langle \mathcal{I}^{\mathrm{D}}(\mathbf{z}_{\mathrm{r}}) \rangle}{\mathrm{Var}\big(\mathcal{I}^{\mathrm{D}}(\mathbf{z})\big)^{1/2}} \sim \frac{N_{\mathrm{q}}^2 B}{\sqrt{N_{\mathrm{q}}^2 B/T}} = N_{\mathrm{q}}\sqrt{BT}$$

- Numerically we compute

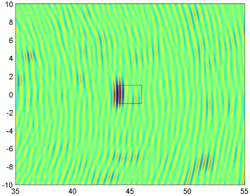
$$\mathsf{SNR} = \frac{|\mathcal{I}^{\mathsf{D}}|(\mathbf{z}^*)}{\max_{\mathbf{z}\neq\mathbf{z}^*}|\mathcal{I}^{\mathsf{D}}|(\mathbf{z})}$$

where $\mathbf{z}^*$ is the point where the image admits its maximal value and $\mathbf{z} \neq \mathbf{z}^*$ means that squares of size $2\lambda \times 2\lambda$ centered at $\mathbf{z}$ and $\mathbf{z}^*$ do not intersect.

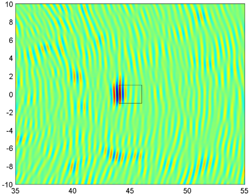
Array size



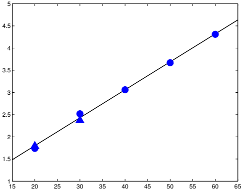
$N_{\mathrm{q}} = 21, a = 10\lambda$



$N_{\mathrm{q}} = 31, a = 15\lambda$



$N_{\mathrm{q}} = 41, a = 20\lambda$

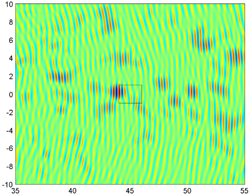


SNR vs $N_{\mathrm{q}}{}^*$

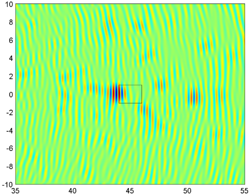
* disks $\Delta X_{\mathrm{q}} = \lambda/2$, triangles $\Delta X_{\mathrm{q}} = \lambda$

Resolution improves with array size and SNR increases linearly with number of receivers

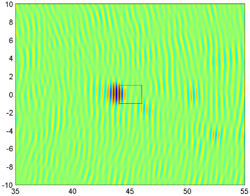
Recording time



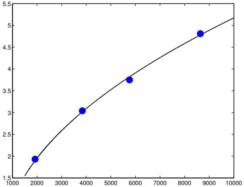
$T = 1920\mu s$



$T = 3840\mu s$

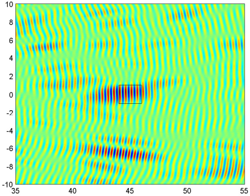


$T = 8640\mu s$

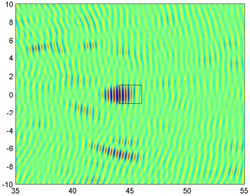


SNR vs T

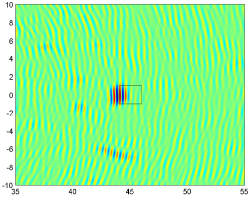
SNR is proportional to $\sqrt{T}$



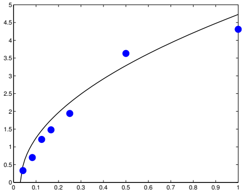
$B = 56\,\text{kHz}$



$B = 125\,\text{kHz}$



$B = 250\,\text{kHz}$



SNR vs B

Range resolution improves with bandwidth and SNR is proportional to $\sqrt{B}$

To resume

Our analysis shows that the important parameters for imaging are :

1. *The number of sensors* N_q. Both cross-range resolution and SNR linearly improve with N_q.
2. *The bandwidth of the noise sources* B. Range resolution improves linearly with B, while the SNR is proportional to $\sqrt{B}$.
3. *The recording time* T. The SNR of the cross-correlations, and therefore the SNR of the image as well, is proportional to $\sqrt{T}$.

Numerical results are in very good agreement with the theory.

J. Garnier, G. Papanicolaou, A. Semin and C.T., "Signal to Noise Ratio estimation in passive correlation based imaging", SIAM J. Imaging Sci. 6-2 (2013), pp. 1092-1110.

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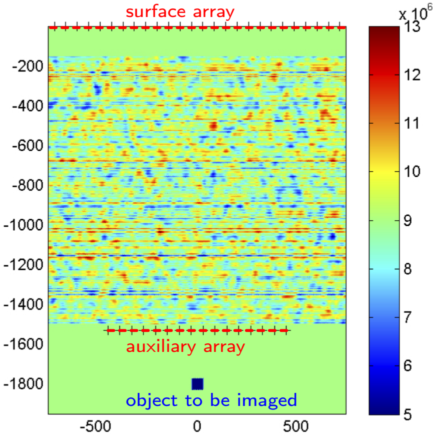
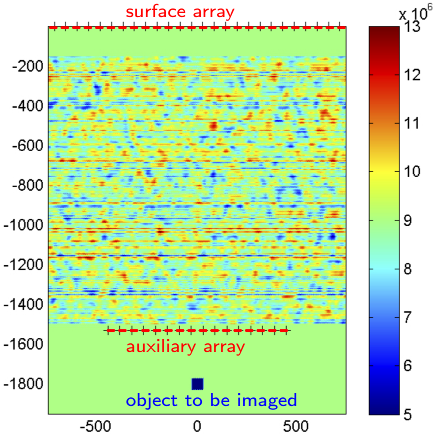
Imaging with waves in complex media

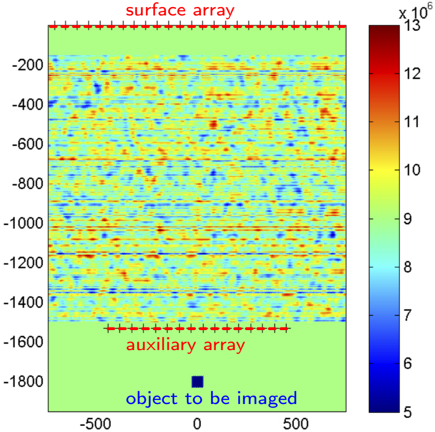


- motivation : exploration geophysics

Imaging with waves in complex media

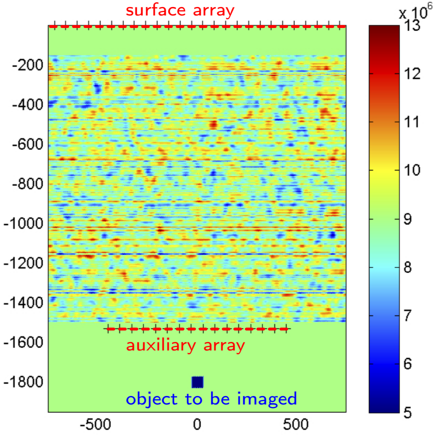


- motivation : exploration geophysics
- traditional approach : surface array imaging



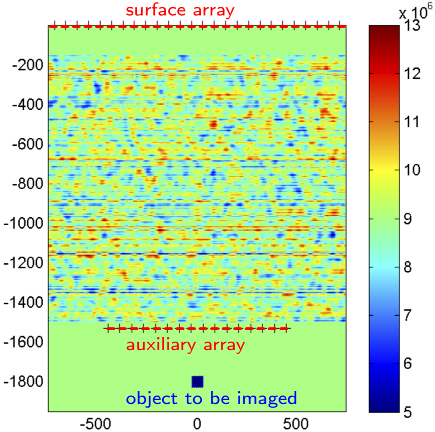
- motivation : exploration geophysics
- traditional approach : surface array imaging
- complex medium impedes the imaging process

problem : small signal to noise ratio, i.e., scatterer echoes are overwhelmed by reflections from the background medium.



- motivation : exploration geophysics
- traditional approach : surface array imaging
- complex medium impedes the imaging process
- auxiliary receiver array does it helps and how ?

How to use the recorded signals on the auxiliary array so as to make a good image ?



- motivation : exploration geophysics
- traditional approach : surface array imaging
- complex medium impedes the imaging process
- auxiliary receiver array does it helps and how ?

Mathematical Model : study wave propagation in inhomogeneous random media with a velocity that fluctuates arround a known mean

- Here there are no random sources, the sources are deterministic
- the randomness comes form the complex medium. Here μ, is obtained by combining an isotropic and a layered random variable,

$$\mu(\vec{\mathbf{x}}) = \frac{1}{\sqrt{2}} \left(\mu_i(\vec{\mathbf{x}}) + \mu_l(\vec{\mathbf{x}})\right),$$

with standard deviation $\sigma = 0.08$. The isotropic part $\mu_i(\vec{\mathbf{x}}) = \mu\left(\frac{x}{\ell}, \frac{z}{\ell}\right)$, has a Gaussian correlation function

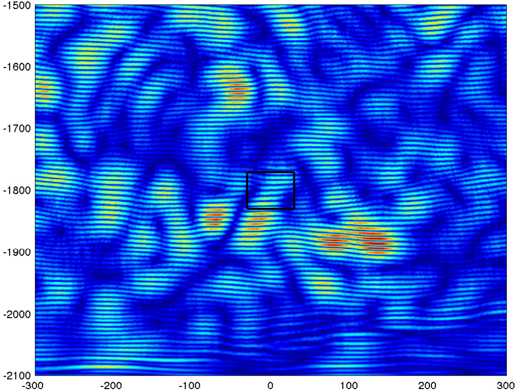
$$E\{\mu_i(\vec{\mathbf{x}}_1)\mu_i(\vec{\mathbf{x}}_2)\} = R(\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) = e^{-\frac{|\vec{\mathbf{x}}_1 - \vec{\mathbf{x}}_2|^2}{2\ell^2}}, \qquad \ell = \lambda/2$$

and the layered random variable, $\mu_l(\vec{\mathbf{x}}) = \mu\left(\frac{z}{\ell_z}\right)$, satisfies

$$E\{\mu_l(z_1)\mu_l(z_2)\} = \left(1 + \frac{|z_1 - z_2|}{\ell_z}\right) e^{-\frac{|z_1 - z_2|}{\ell_z}}, \qquad \ell_z = \lambda/30.$$

Traditional Imaging

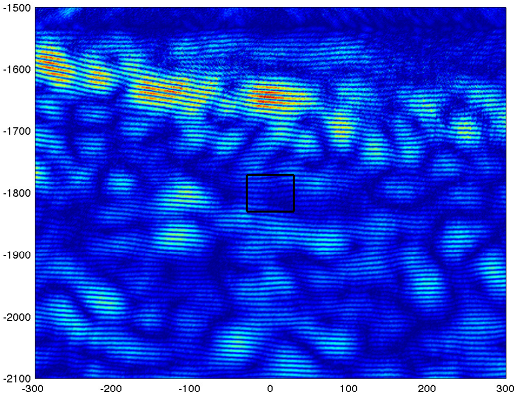
Image obtained using the array at the top



Statistically unstable and with very low signal to noise ratio : good image only for very low fluctuations

Traditional Imaging

Image obtained using the bottom array



Statistically unstable and with very low signal to noise ratio : good image only for very low fluctuations

Correlation based Imaging

- We compute cross-correlations at the auxiliary array by

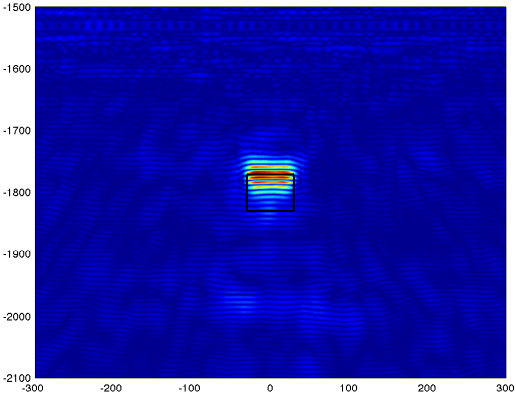
$$C(\tau, \vec{\mathbf{x}}_l, \vec{\mathbf{x}}_m) = \frac{1}{T} \sum_{s=1}^{N_s} \int u(t, \vec{\mathbf{x}}_l; \vec{\mathbf{x}}_s) u(t+\tau, \vec{\mathbf{x}}_m; \vec{\mathbf{x}}_s) dt$$

- Note that this is coherent averaging over the sources which is not what we do in the ambient noise problem where

$$C(\tau, \vec{\mathbf{x}}_l, \vec{\mathbf{x}}_m) = \frac{1}{T} \int \left(\sum_{s=1}^{N_s} u(t, \vec{\mathbf{x}}_l; \vec{\mathbf{x}}_s) \right) \left(\sum_{s=1}^{N_s} u(t+\tau, \vec{\mathbf{x}}_m; \vec{\mathbf{x}}_s) \right) dt$$

Correlation based Imaging

Correlation based imaging with the auxiliary array



Statistically stable : high SNR even for strong medium fluctuations
(depends on bandwidth, complex medium characteristics, array characteristics)

Wave propagation through a strongly scattering medium

- *The field transmitted through the random strongly scattering medium has Gaussian statistics, mean zero, and cross correlations of the form*

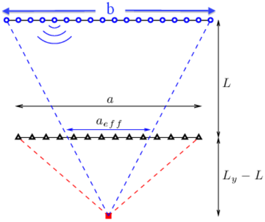
$$
\begin{aligned}
\mathbb{E}\left[\widehat{u}(\omega,\vec{\mathbf{x}}_q;\vec{\mathbf{x}}_s)\widehat{u}(\omega',\vec{\mathbf{x}}_{q'};\vec{\mathbf{x}}_{s'})\right] &= 0 \\
\mathbb{E}\left[\widehat{u}(\omega,\vec{\mathbf{x}}_q;\vec{\mathbf{x}}_s)\overline{\widehat{u}(\omega',\vec{\mathbf{x}}_{q'};\vec{\mathbf{x}}_{s'})}\right] &= \widehat{F}(\omega)\overline{\widehat{F}(\omega')}\exp\left(-\frac{(\omega-\omega')^2}{\omega_c^2}\right) \\
&\times \exp\left(-\frac{|\mathbf{x}_q-\mathbf{x}_{q'}|^2}{X_{cq}^2}-\frac{|\mathbf{x}_s-\mathbf{x}_{s'}|^2}{X_{cs}^2}\right),
\end{aligned}
$$

- ω_c : correlation frequency of the field in the plane $z=-L$ of the auxiliary receiver array,
- X_{cq} is the correlation radius of the field at the auxiliary receiver array (when emitted from a point source at the source array),
- X_{cs} is the correlation radius of the field at the source array (when emitted from a point source at the auxiliary receiver array).

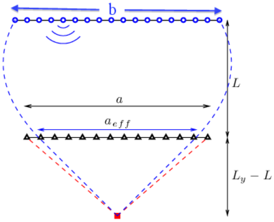
Cross range resolution : $\lambda(L_y - L)/a_{\text{eff}}$.

$$\text{with } a_{\text{eff}}^{\text{hom}} = b\frac{L_y - L}{L} \text{ and } a_{\text{eff}}^{r} = \frac{\lambda(L_y - L)}{X_{\text{cq}}}$$

and is improved in the random paraxial case.



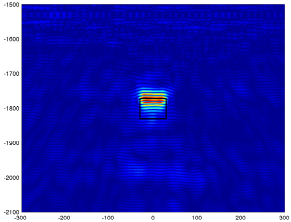
(a) Homogeneous medium



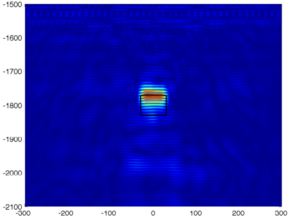
(b) Random medium

J. Garnier and G. Papanicolaou, Role of scattering in virtual source imaging, SIAM Journal of Imaging Science 7 (2014), pp. 1210-1236.

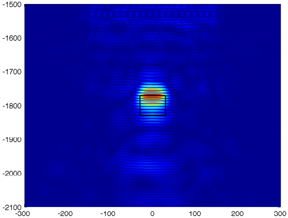
Cross-range resolution : $\lambda(L_y - L)/a_{\text{eff}}$



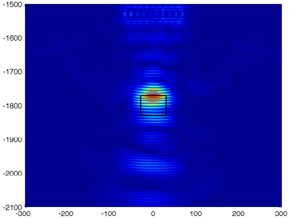
$a = 30\lambda$



$a = 20\lambda$



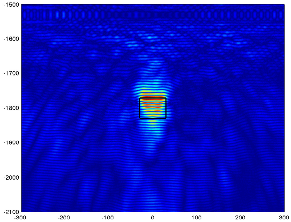
$a = 10\lambda$



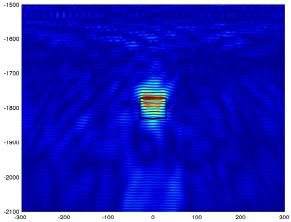
$a = 5\lambda$

The array size does not seem to affect the resolution ! (here $a_{\text{eff}} = 9\lambda$)

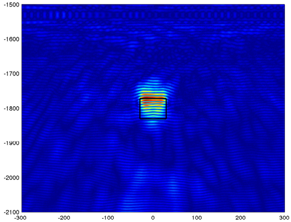
Range resolution : c_0/B



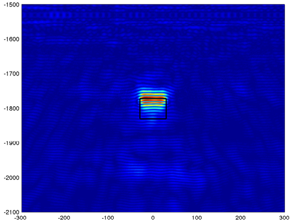
$B = 7.5$Hz



$B = 10$Hz



$B = 15$Hz



$B = 40$Hz

Range resolution improves with B

Image quality : SNR analysis

We show that, the SNR defined by,

$$\mathrm{SNR}_{\mathrm{CC}} = \frac{|\mathbb{E}[\mathcal{I}_{\mathrm{CC}}(\vec{\mathbf{y}})]|}{\mathrm{Var}(\mathcal{I}_{\mathrm{CC}}(\vec{\mathbf{y}}^S))^{1/2}}, \qquad (1)$$

satisfies (for $X_{\mathrm{cs}} \ll b$ and $\omega_c \ll B$)

$$\mathrm{SNR}_{\mathrm{CC}} \approx \frac{\sigma_{\mathrm{ref}} X_{\mathrm{cq}}}{\lambda^2 (L_y - L)} \Big(\frac{b}{\max\{\Delta X_{\mathrm{s}}, X_{\mathrm{cs}}\}} \Big)^{1/2} \Big(\frac{a}{\max\{\Delta X_{\mathrm{q}}, X_{\mathrm{cq}}\}} \Big) \Big(\frac{B}{\omega_{\mathrm{c}}} \Big)^{1/2}.$$

When the correlation radius X_{cq} is small, it is relevant to have a dense auxiliary receiver array for a given aperture in order to get good stability.

For a given aperture a, the SNR increases when the inter-distance ΔX_{q} decreases, until the inter-distance becomes of the order of the correlation radius X_{cq} of the illumination field, and then the SNR reaches a value determined by X_{cq}.

SNR vs bandwidth

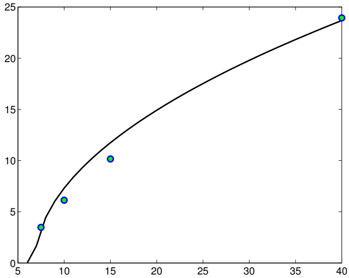


Figure : SNR vs bandwidth

- SNR is proportional to $\sqrt{B}$

SNR vs number of sources

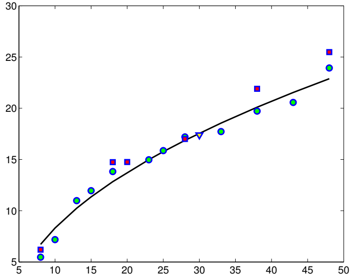
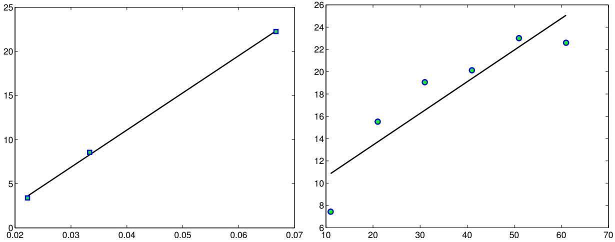


Figure : SNR as a function of b (in λ). Blue circles correspond to $\Delta X_{\rm s} = \lambda/2$, red squares to $\Delta X_{\rm s} = \lambda$ and yellow triangles to $\Delta X_{\rm s} = 3\lambda/2$.

- SNR is proportional to $\sqrt{b}$ ($X_{\rm cs} > \Delta X_{\rm s}$)

SNR vs number of receivers



(a) SNR as a function of $1/\Delta X_q$ (measured in $\lambda/2$) for a fixed array aperture $a = 21\lambda$.

(b) SNR as a function of N_q

- SNR is proportional to N_q $(X_{cq} < \Delta X_q)$

Conclusions

Virtual source array imaging : Imaging by migrating the cross-correlation matrix of the auxiliary array data gives much better images then migrating the data !

In fact the random medium effects are eliminated and it is "even better" then if an active array near the object was used for imaging (only the down-going part of the Green's function is reconstructed)
Differences w.r.t the homogeneous case :

1. SNR depends on the inter-element (auxiliary) array distance which should be of the order of X_{cq} for optimal SNR.
2. SNR is proportional to $\sqrt{N_{\mathrm{s}}}$. SNR also depends on the inter-element (source) array distance which should be of the order of X_{cs} for optimal SNR.

Numerical results are in very good agreement with the theory. Resolution analysis in agreement with results obtained in the random paraxial regime.

J. Garnier, G. Papanicolaou, A. Semin and C.T., Signal to Noise Ratio estimation in virtual source array imaging, SIAM Journal on Imaging Science 8 (2015), pp. 248-279.

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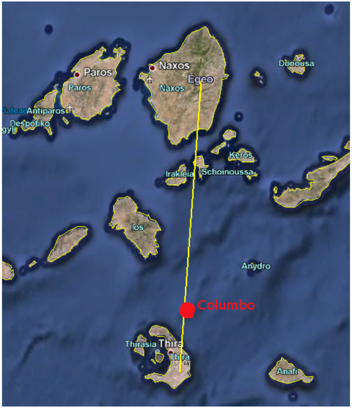
Seismic noise : An example

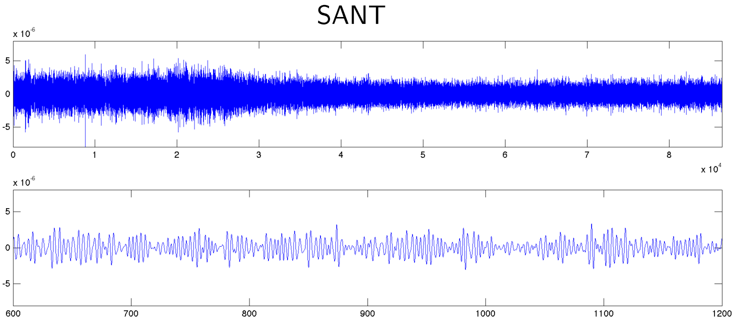
The seismic stations



Two seismic stations (Santorini and Naxos)

Seismic noise : An example

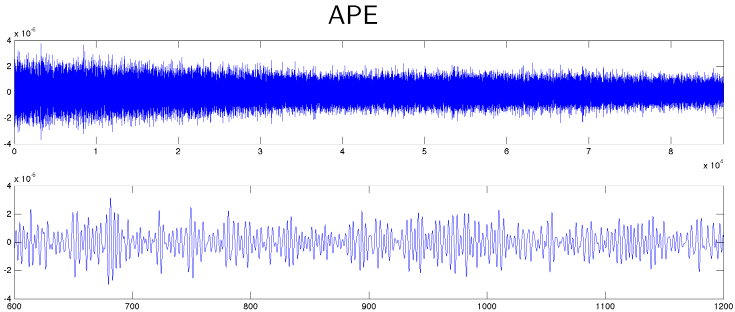
Recordings on Santorini station



The data recorded on Santorini filtered in frequency range $[0.2, 0.5]$Hz

Seismic noise : An example

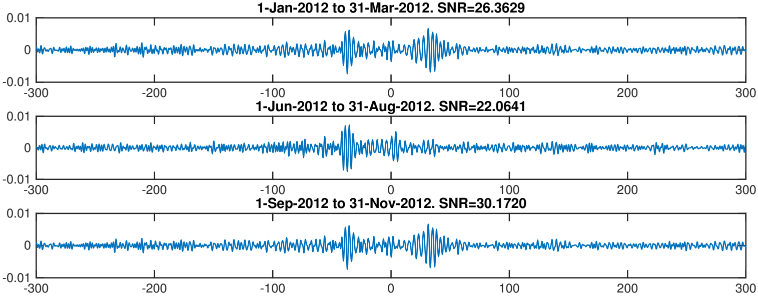
Recordings on Naxos station



The data recorded on Naxos filtered in frequency range $[0.2, 0.5]$Hz

Seismic noise : An example

Cross-correlations (3 months)



Observe variability w.r.t to the season of the year !

Due to seasonal variations of the medium or of the noise sources ? (non-stationarity)

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Two methods to estimate dc/c

We want to use the cross-correlations to estimate relative velocity changes in the propagating medium.

Two methods are mainly used to estimate dc/c. They both compare :

- $\mathbb{CC}_{\mathrm{ref}}$: the average $\mathbb{CC}$ over a long period (reference)
- $\mathbb{CC}_{\mathrm{cur}}$: average of $\mathbb{CC}$ over a small period around the current day

Two methods to estimate dc/c

We want to use the cross-correlations to estimate relative velocity changes in the propagating medium.
Two methods are mainly used to estimate dc/c. They both compare :

$\mathbb{CC}_{\mathrm{ref}}$: the average $\mathbb{CC}$ over a long period (reference)

$\mathbb{CC}_{\mathrm{cur}}$: average of $\mathbb{CC}$ over a small period around the current day

- MWCS method operates in the frequency domain
- SM operates in the time domain

A small change in the velocity $c \to c(1 + dc/c)$ induces a change in travel time $t \to t(1 + dt/t)$ (with $dc/c = -dt/t$). We want to measure the stretching coefficient $\varepsilon = dt/t$ by comparing $\mathbb{C}_{\text{ref}}$ and $\mathbb{C}_{\text{cur}}$.

We seek ε that maximizes :

$$C(\varepsilon) = \frac{\displaystyle\int \mathbb{C}_{\text{cur}}(t(1+\varepsilon))\mathbb{C}_{\text{ref}}(t)dt}{\sqrt{(\int \mathbb{C}^2_{\text{cur}}(t(1+\varepsilon))dt)(\int \mathbb{C}^2_{\text{ref}}(t)dt)}},$$

where the integration is over a specific time window.

Two methods to estimate dc/c

Stretching Method

A small change in the velocity $c \to c(1 + dc/c)$ induces a change in travel time $t \to t(1 + dt/t)$ (with $dc/c = -dt/t$). We want to measure the stretching coefficient $\varepsilon = dt/t$ by comparing $\mathbb{CC}_{\text{ref}}$ and $\mathbb{CC}_{\text{cur}}$.

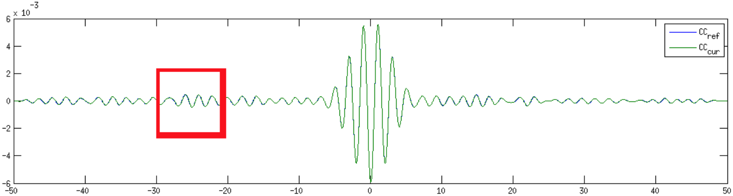
We seek ε that maximizes :

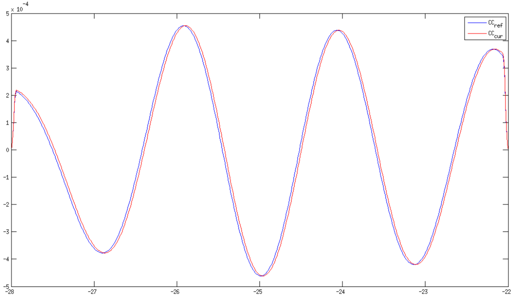
$$C(\varepsilon) = \frac{\displaystyle\int \mathbb{CC}_{\text{cur}}(t(1+\varepsilon))\mathbb{CC}_{\text{ref}}(t)dt}{\sqrt{(\int \mathbb{CC}^2_{\text{cur}}(t(1+\varepsilon))dt)(\int \mathbb{CC}^2_{\text{ref}}(t)dt)}},$$

where the integration is over a specific time window.
We seek for the parameter ε in a search interval $[-\varepsilon_{max} : \varepsilon_{max}]$, with an accuracy $d\varepsilon = 10^{-6}$ (using adaptive refinement).

The MWCS method detects a phase shift in the frequency domain.

We divide each cross-correlation function into N_w windows, with each window centered around time $t_i, \;\; i = 1, ..., N_w$. For each central time t_i we get a measurement dt_i using the corresponding windowed segments of $\mathbb{CC}_{\text{ref}}$ and $\mathbb{CC}_{\text{cur}}$.





Those segments after being window tapered, they are being fourier transformed and called $F_{\mathrm{ref}}(\nu) = \mathcal{F}\left(\mathbb{C}\mathrm{C}_{\mathrm{ref}}\right)$ and $F_{\mathrm{cur}}(\nu) = \mathcal{F}\left(\mathbb{C}\mathrm{C}_{\mathrm{ref}}\right)$ respectively.
Then the cross-spectrum is calculated as

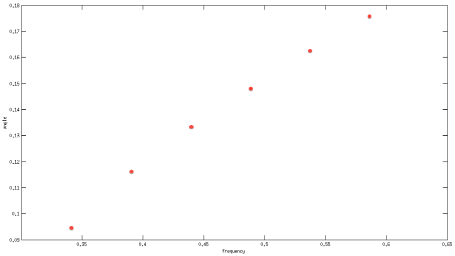
$$X(\nu) = F_{\mathrm{ref}}(\nu)\ F^{*}_{\mathrm{cur}}(\nu),$$

Two methods to estimate dc/c

MWCS Method

We estimate the phase of the cross-spectrum $\varphi_i(\nu_j)$ as a function of frequency ν_j,

$$\varphi_i(\nu_j) = 2\pi \; dt_i \; \nu_j,$$



and then we estimate dt_i the time shift corresponding to the central time t_i. From this we get $dc/c = -dt/t$ using weighted least squares.

D. Clarke and L. Zaccarelli and N.M. Shapiro and F. Brenguier "Assessment of resolution and accuracy of the Moving Window Cross Spectral technique for monitoring crustal temporal variations using ambient seismic noise", Geophys. J. Int. **186** (2011) 867-882.

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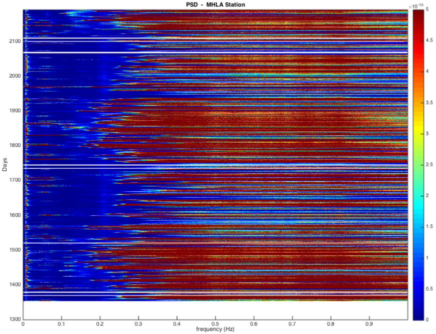
What are seasonal variations ?

It has been observed that noisy recordings have seasonal variations



Power spectral density of recordings on Milos island. Vertical axis : time in days (850 days ≈ 2.3 years). Horizontal axis : frequency between 0 and 1 Hz.

What are seasonal variations ?

Seasonal variations in the seismic velocity are observed in

Ueli Meier, Nikolai M. Shapiro and Florent Brenguier, "Detecting seasonal variations in seismic velocities within Los Angeles basin from correlations of ambient seismic noise", Geophys. J. Int. (2010) 181, 985–996

where it is suggested that they are due to hydrological and/or thermoelastic variations of the medium.

On the other hand

Zhongwen Zhan, Victor C. Tsai and Robert W. Clayton, "Spurious velocity changes caused by temporal variations in ambient noise frequency content", Geophys. J. Int. (2013) 194, 1574–1581

suggest that the observed variations are spurious and are due to seasonal variations in the amplitude spectrum of the noise sources.

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suggest that the observed variations are spurious and are due to seasonal variations in the amplitude spectrum of the noise sources.

To study the effect of seasonal variations of the ambient noise sources on the estimated changes of the seismic velocity we design simple but realistic numerical experiments

Modelling the seasonal variations : the wave equation

We consider the acoustic wave equation :

$$\frac{1}{c(\vec{\mathbf{x}})^2}\frac{\partial^2 u}{\partial t^2}(t,\vec{\mathbf{x}}) - \Delta u(t,\vec{\mathbf{x}}) = n(t,\vec{\mathbf{x}}),$$

where $n(t,\vec{\mathbf{x}})$ models the noise sources.

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where $n(t,\vec{\mathbf{x}})$ models the noise sources.The solution at a given point $\vec{\mathbf{x}}$ can be written as,

$$u(t,\vec{\mathbf{x}}) = \int\int G(t-s,\vec{\mathbf{x}},\vec{\mathbf{y}})n(s,\vec{\mathbf{y}})d\vec{\mathbf{y}}ds,$$

or equivalently in the frequency domain,

$$\hat{u}(\omega,\vec{\mathbf{x}}) = \int \hat{G}(\omega,\vec{\mathbf{x}},\vec{\mathbf{y}})\hat{n}(\omega,\vec{\mathbf{y}})d\vec{\mathbf{y}}.$$

Here hat denotes the Fourier transform and $\hat{G}(\omega,\vec{\mathbf{x}},\vec{\mathbf{y}})$ is the Green's function.

We assume that $n(t, \vec{\mathbf{x}})$ is a zero-mean stationary in time random process with a covariance function

$$\langle n(t_1, \vec{\mathbf{y}}_1), n(t_2, \vec{\mathbf{y}}_2)\rangle = \Gamma(t_2 - t_1, \vec{\mathbf{y}}_1)\delta(\vec{\mathbf{y}}_2 - \vec{\mathbf{y}}_1).$$

Here $\langle\cdot\rangle$ stands for statistical averaging. The function $t \to \Gamma(t, \vec{\mathbf{y}})$ is the time correlation function of the noise signals emitted by the noise sources at location $\vec{\mathbf{y}}$.

Modelling the seasonal variations : The sources

We assume that $n(t, \vec{\mathbf{x}})$ is a zero-mean stationary in time random process with a covariance function

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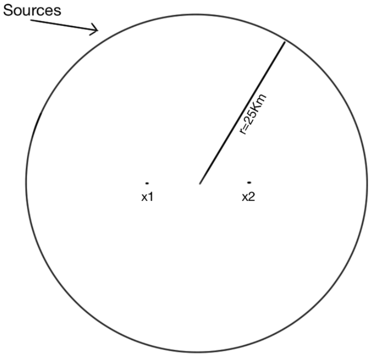
The function $\vec{\mathbf{y}} \to \Gamma(0, \vec{\mathbf{y}})$ characterizes the spatial support of the sources. In our case we assume that the sources are uniformly distributed on a circle $\mathcal{C}$ of radius of $R_{\mathcal{C}} = 25$km :

$$\Gamma(t, \vec{\mathbf{y}}) = \frac{1}{2\pi R_{\mathcal{C}}} \Gamma_0(t, \vec{\mathbf{y}})\delta_{\mathcal{C}}(\mathbf{y}).$$

Modelling the seasonal variations : setup

The noise sources are located on a circle of radius 25km and the wave field is recorded at two receivers $\vec{\mathbf{x}}_1 = (-5, 0)$km and $\vec{\mathbf{x}}_2 = (5, 0)$km.



Modelling the seasonal variations :

To model seasonal variations we assume that the statistics of the noise sources change from one day to another and denote $\Gamma_0^j(t, \vec{\mathbf{y}})$ the covariance function at day j. The wave field at $\vec{\mathbf{x}}$ is computed by ($i = 1, N_s$ the noise sources)

$$\hat{u}^j(\omega, \vec{\mathbf{x}}) = \frac{1}{N_s} \sum_{i=1}^{N_s} \hat{G}^j(\omega, \vec{\mathbf{x}}, \vec{\mathbf{y}}_i) \hat{n}_i^j(\omega), \quad (2)$$

where $\hat{n}_i^j(\omega)$ is the frequency content of the noise sources at $\vec{\mathbf{y}}_i$ during day j, which is random such that $\left\langle \hat{n}_i^j(\omega) \right\rangle = 0$ and

$$\left\langle \hat{n}_i^j(\omega) \overline{\hat{n}_i^j}(\omega') \right\rangle = 2\pi \hat{\Gamma}_0^j(\omega, \vec{\mathbf{y}}_i) \delta(\omega - \omega').$$

Modelling the seasonal variations : Variation models

Our model for the power spectral density of the noise sources is

$$\hat{\Gamma}_0^j(\omega, \vec{\mathbf{y}}) = \hat{F}(\omega)\hat{s}^j(\omega, \vec{\mathbf{y}}),$$

Here the unperturbed noise source distribution is uniform over the circle $\mathcal{C}$ and has power spectral density $\hat{F}(\omega)$, and $\hat{s}^j(\omega, \vec{\mathbf{y}})$ is the daily perturbation of the power spectral density at location $\vec{\mathbf{y}}$.

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Modelling the seasonal variations : Variation models

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For uniform daily perturbations, we have $\hat{s}^j(\omega, \vec{\mathbf{y}}) = \hat{f}^j(\omega)l(\vec{\mathbf{y}})$,

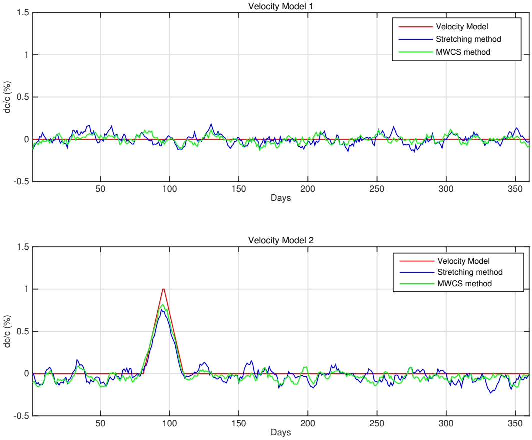
$$\left\langle \hat{\mathrm{CC}}^j(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2) \right\rangle = \hat{F}(\omega)\hat{f}^j(\omega) \int_{\mathcal{C}} d\sigma(\vec{\mathbf{y}}) \, \overline{\hat{G}^j(\omega, \vec{\mathbf{x}}_1, \vec{\mathbf{y}})} \hat{G}^j(\omega, \vec{\mathbf{x}}_2, \vec{\mathbf{y}}) l(\vec{\mathbf{y}}),$$

and the daily perturbation factors out of the integral ...

Results

Reference $\mathbb{C}$-function = average 360 daily CC-functions
Current $\mathbb{C}$-function for day j = average 7 days around day j.

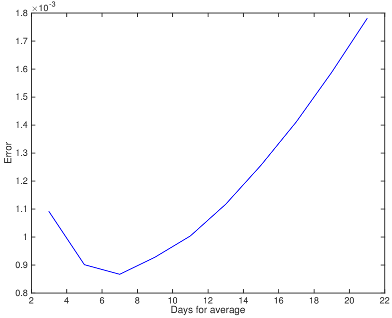
Without seasonal variations



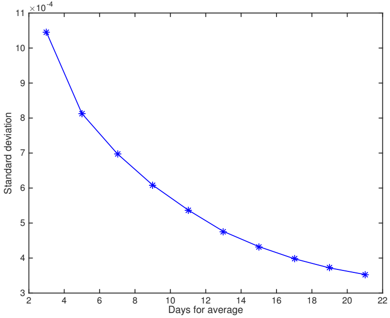
Results

Accuracy vs Stability

L^2 error for days where $dc/c \neq 0$

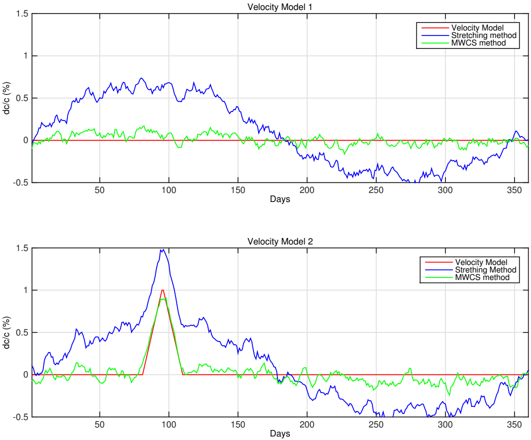


std for days where $dc/c = 0$



Results

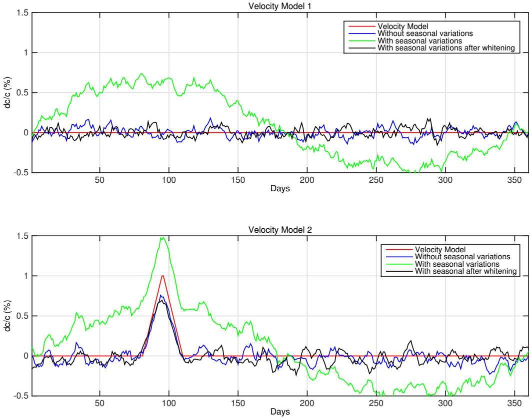
With seasonal variations



We clearly see that only the SM estimation is affected by the seasonal variations in the amplitude spectra of the noise sources.

Spectral Whitening

A simple way to remove them is to use spectral whitening (normalize the amplitude spectra) of the $\mathbb{CC}$-functions.



Spectral whitening works for perturbations of separable form, i.e., uniform seasonal variations (a reasonable assumption for receivers that are not very far apart)

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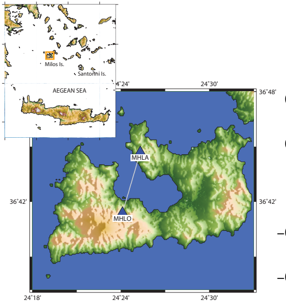
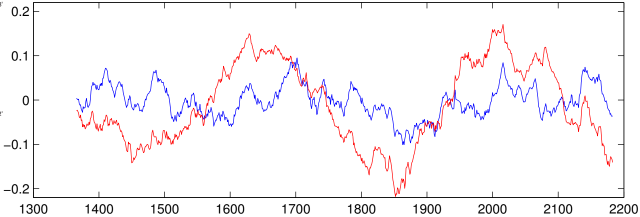
SNR increase by spectral whitening

We consider Milos, an island in Aegean sea and two stations 6 Km apart.



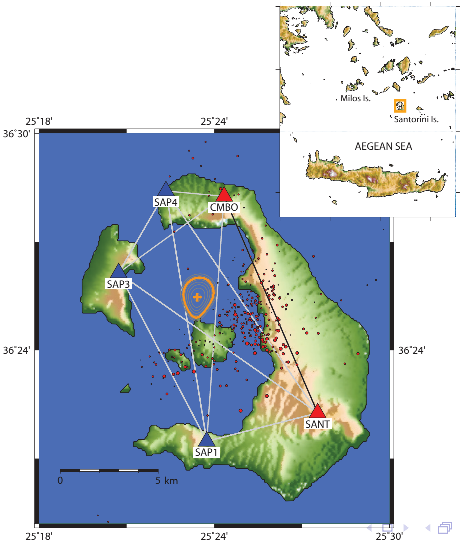


Red : dc/c estimation without using Spectral Whitening. Blue : dc/c estimation using Spectral Whitening

SNR improvement of an order of 3 (std).

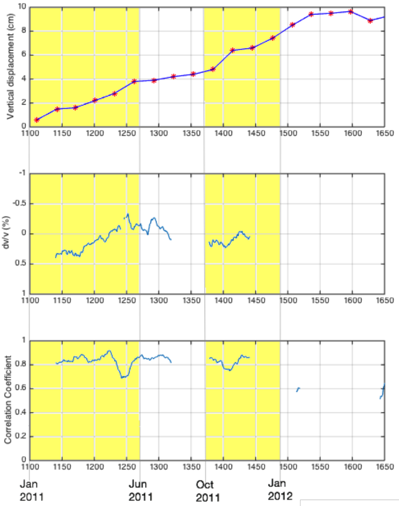
Santorini 2011-2012 unrest

The Santorini 2011-2012 seismic unrest begins on January 2011 and ends on February 2012 measuring a total of 10 cm uplift of the caldera of Santorini.



Slow event,
difficult to follow without removing seasonal variations !

We observe
decrease in the velocity of seismic waves in the caldera of Santorini which is correlated with the accumulated elevation measured with GPS.



Conclusions/Future work

- Potential of developing monitoring tools which provide accurate results even with sparse seismic networks

  E. Daskalakis, C. Evangelidis, J. Garnier, N. Melis, G. Papanicolaou and C. T., "Robust seismic velocity change estimation using ambient noise recordings", preprint 2015.

- Study how errors propagate into the estimation (UQ)
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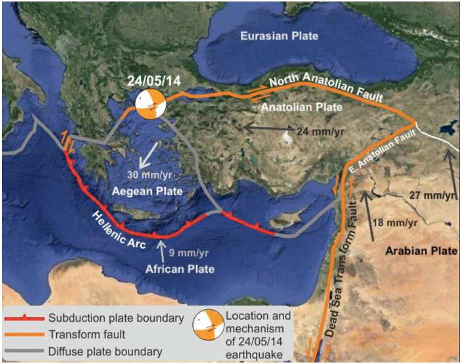
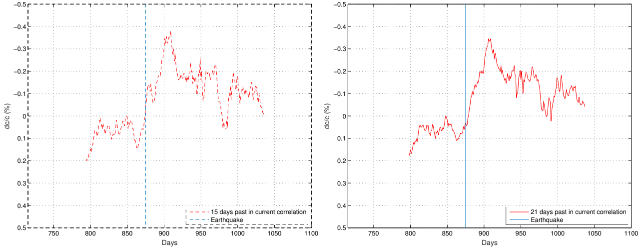


Results obtained with the SM method when in the computation of $\mathbb{C}_{cur}$ we only use days in the past.



On the left we use a window of 15 days and on the right 21 days. The day of the earthquake (M6.9) is shown with the blue vertical curve. We clearly observe a decrease in the relative velocity of about 0.4% starting a few days before and continuing for several days after the earthquake.
Similar results have been obtained for the Tohuku-Oki earthquake in Japan

N. Nakata and R. Snieder, "Near-surface weakening in Japan after the 2011 Tohuku-Oki earthquake", Geophys. Res. Lett. (2011) 38, L17302.

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