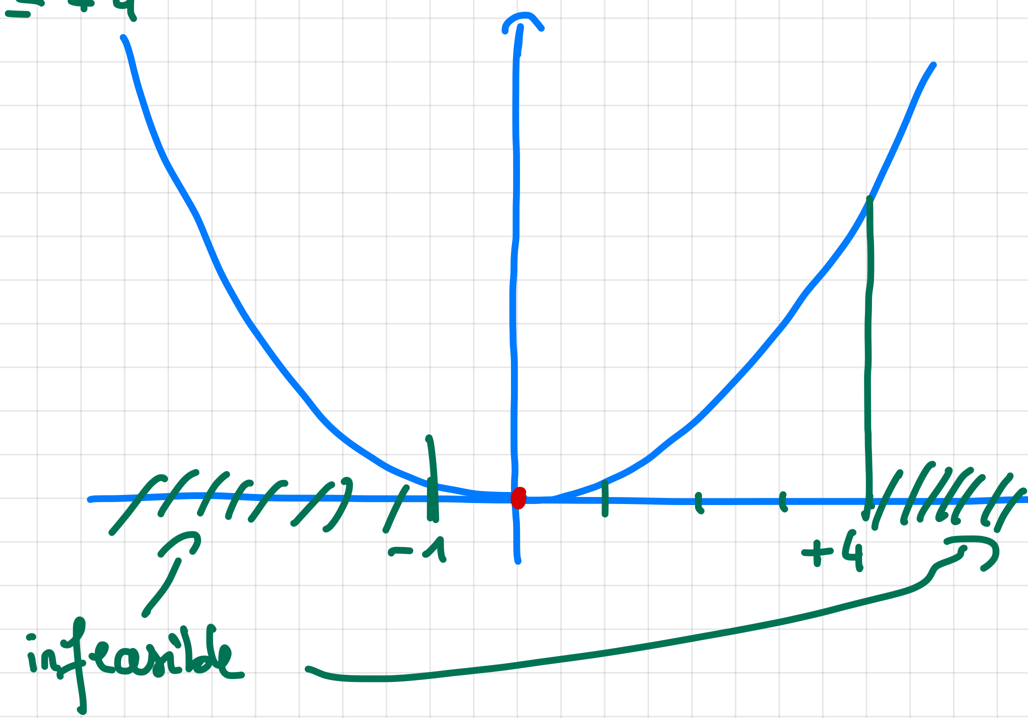
CONSTRAINED OPTIMIZATION

$\min_{x \in \mathbb{R}} f(x) \wedge$ st $-1 \leq x \leq +4$

say $f(x) = x^2$

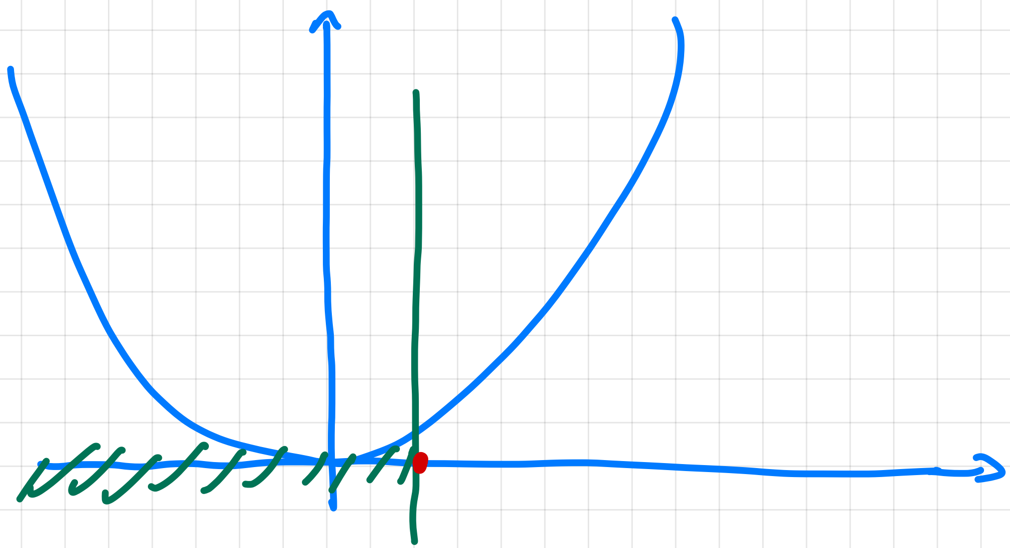


$\hookrightarrow x^* = 0$

$-1 < x^* < 4$

constraints are not active

$\min_{x \in \mathbb{R}} f(x) = x^2$, $x \geq 1$



$\hookrightarrow x^* = 1$

it is on the boundary of the constraint. The constraint is active

Definition : Consider inequality constraint, say $g : \mathbb{R}^n \to \mathbb{R}$ and the constraint $g(x) \leq 0$. It is active at the optimum x^* if $g(x^*) = 0$

If we have an equality constraint $g(x) = 0$, it is active if $g(x^*) = 0$, i.e if the constraint is satisfied.

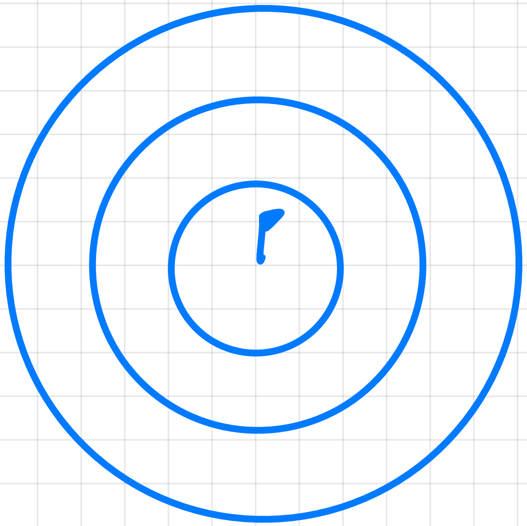
MORE INTUITIONS ON CONSTRAINT OPTIMIZATION

$$\min_{x \in \mathbb{R}^2} f(x) = \|x\|^2$$

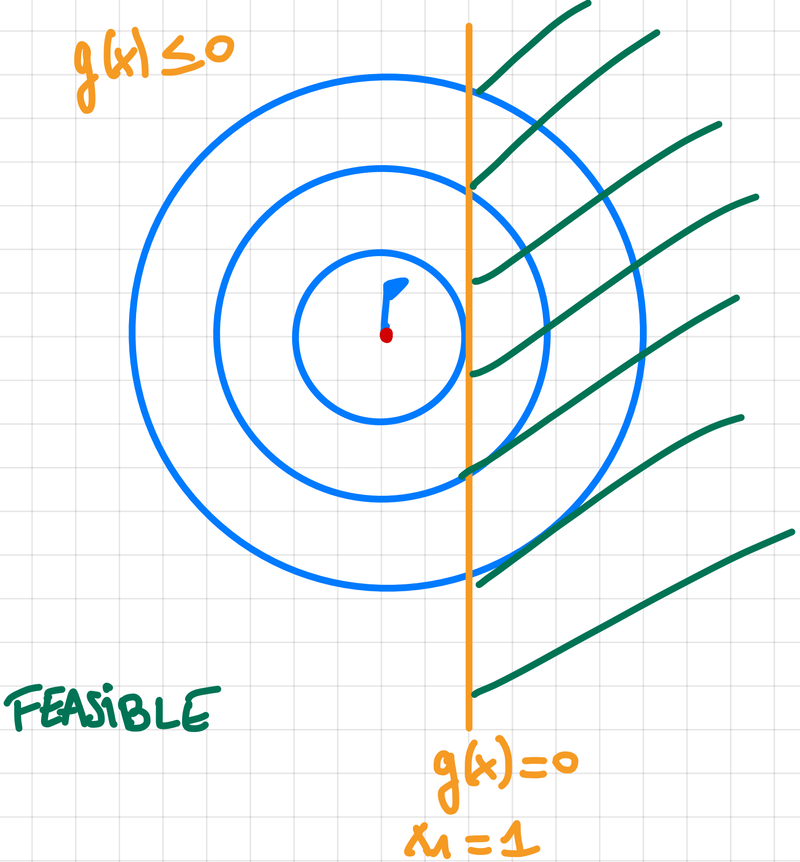
$g(x) = x_1 - 1$

$$\min_{x \in \mathbb{R}^2} f(x) = \|x\|^2$$

- s.t. $g(x) \leq 0$ (1)
- st $g(x) \geq 0$ (2)
- st $g(x) = 0$ (3)



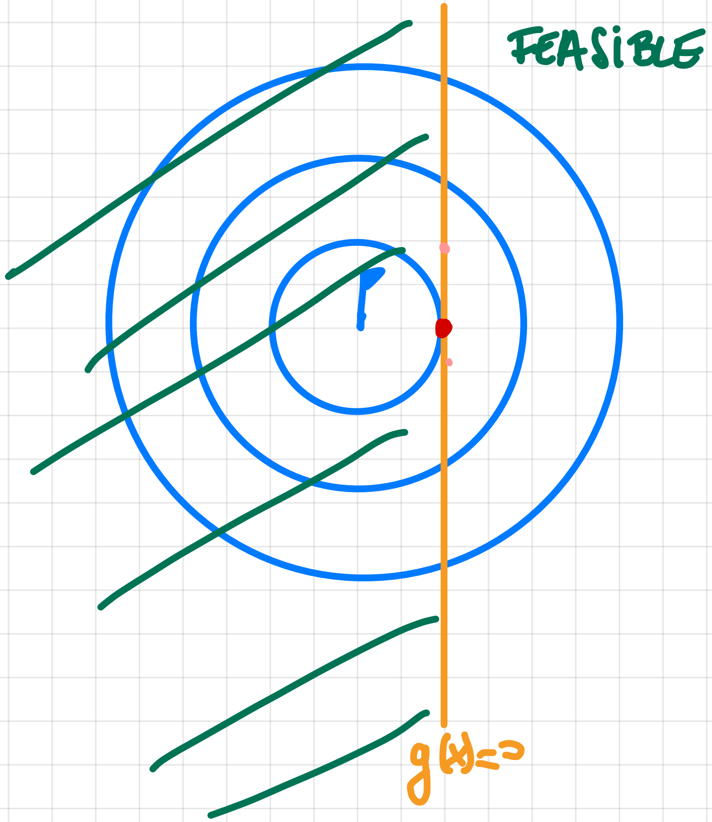
DRAW - constraint and find optimum of problem



$x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

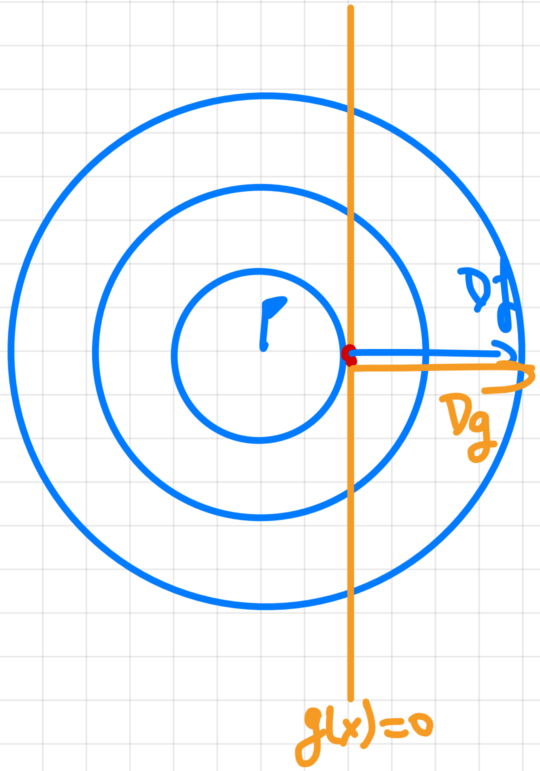
(constraint not active)

1



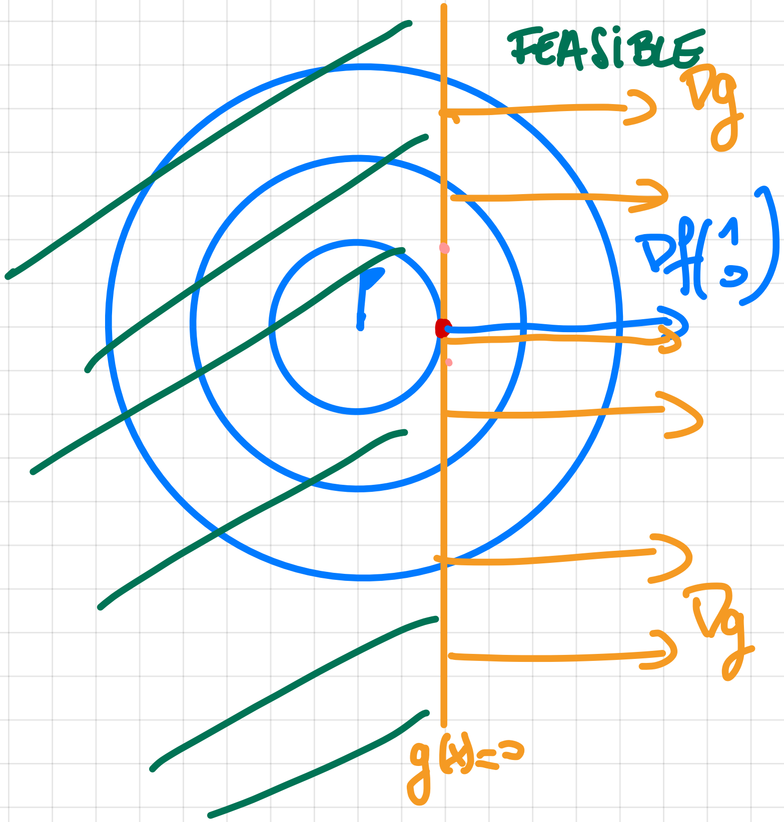
$x^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2



$x^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

3



$\nabla f(x^*)$ and $\nabla g(x^*)$ are colinear

<u>ie</u> $\exists \lambda \in \mathbb{R}$ such that

$$\nabla f(x^*) = -\lambda \nabla g(x^*)$$

$$\Leftrightarrow \quad \nabla f(x^*) + \lambda \nabla g(x^*) = 0$$

RELAXATION

Idea : transform a constrained problem into unconstrained one

Example (from M. Bilicire class)

Deal: offer 1 € per meter of altitude.

Constraint : Alpinst must stay in France.

$x \in \mathbb{R}^2$ position (latitude, longitude)

$$\max_x f(x) \qquad f : \text{altitude}$$

$$\text{st} \quad x \in \text{France}.$$

$x^* =$ top Mont Blanc

$f(x^*) = 4807$ €

New deal: → Also climb outside France
→ Pay a fee (fine)

$a(x)$: fine $\qquad a(x) = 0 \quad x \in \text{France}$.

$$(B) \left[\max_{x \in \mathbb{R}^2} \underbrace{f(x) - a(x)}_{\mathcal{L}(x)} \right.$$

→ No constraint anymore

Relaxed problem.

BILIONAIRE: Design $a(x)$, penalization such that solution of (B) is still solut° of original problem.

$f(x_{Everest}) = 8848$ €

$a(x) = 4041$€ if $x \notin$ France

$\mathcal{L}(x_{Everest}) = 8848 - 4041 = 4807$

$f(x_{mont\text{-}blanc}) = 4807$

This example introduced the idea of relaxing the constraint.
We need to introduce a penalty such that the optimum of the relaxed problem equals the optimum of the constrained pb.

LAGRANGIAN:

$$\begin{aligned} \min\ & f(x) & f: \mathbb{R}^n \to \mathbb{R} \\ \text{s.t}\ & h(x) = 0 & h: \mathbb{R}^n \to \mathbb{R}^m \\ & \underset{\mathbb{R}^p}{g(x)} \le 0 & g: \mathbb{R}^n \to \mathbb{R}^p \end{aligned}$$

PRIMAL PROBLEM

$\le$ true component-wise

Lagrangian: $\mathcal{L}(x, \lambda, \mu) = f(x) + \underset{\mathbb{R}^m}{\lambda}^T h(x) + \underset{\mathbb{R}^p}{\mu}^T g(x)$

DUAL FUNCTION: $q: \mathbb{R}^{m+p} \to \mathbb{R}$

LAGRANGIAN:

$$\min f(x) \qquad f: \mathbb{R}^n \to \mathbb{R}$$

$$\text{s.t.} \quad h(x) = 0 \qquad h: \mathbb{R}^n \to \mathbb{R}^m$$

$$\underset{\mathbb{R}^p}{g(x)} \leq 0 \qquad g: \mathbb{R}^n \to \mathbb{R}^p$$

true component-wise

PRIMAL PROBLEM

Lagrangian:

$$L(x,\lambda,\mu) = f(x) + \underbrace{\underset{\mathbb{R}^m}{\lambda}^T h(x)}_{\sum_{k=1}^{m} \lambda_k h_k(x)} + \underbrace{\underset{\mathbb{R}^p}{\mu}^T g(x)}_{\sum_{k=1}^{p} \mu_k g_k(x)}$$

DUAL FUNCTION:

$$q: \begin{cases} \mathbb{R}^{m+p} \to \mathbb{R} \\ q(\lambda,\mu) = \min\limits_{x \in \mathbb{R}^n} L(x,\lambda,\mu) \end{cases}$$

if $g(x) > 0$ (constraint violated) $\quad \mu^T g(x) \geq 0 \quad \mu \geq 0$

if constraint violated $L(x,\lambda,\mu) > f(x)$

DUAL BOUND

If x^* is an optimum of the constrained problem
If $\lambda \in \mathbb{R}^m$, $\mu \in \mathbb{R}^p$, $\mu \geq 0$ then

$$q(\lambda,\mu) \leq f(x^*) \leq f(x) \quad \forall x \text{ feasible}$$

↳ optimizing the relaxed problemed gives a lower bound on the constrained problem.

PROOF:

$$q(\lambda,\mu) = \min_x f(x) + \lambda^T h(x) + \mu^T g(x)$$
$$\leq f(x^*) + \lambda^T \underbrace{h(x^*)}_{=0} + \underbrace{\mu^T}_{\geq 0} \underbrace{g(x^*)}_{\leq 0} \leq f(x^*)$$

DUAL PROBLEM :

Find best lower-bound : $\max\limits_{(\lambda,\mu)} \ q(\lambda,\mu)$

$\underline{\text{s.t}} \ \mu \geq 0$

WEAK DUALITY THEOREM :

x^* opt constrained pb

(λ^*, μ^*) is the optimum of DUAL PROBLEM

$$q(\lambda^*, \mu^*) \leq f(x^*)$$

OPTIMALITY OF PRIMAL DUAL:

$\exists\, x^*, \lambda^*, \mu^*$ st $q(\lambda^*, \mu^*) = f(x^*)$

Then x^* is the optimal for constrained pb.

(λ^*, μ^*) — DUAL PROBLEM.