Rational computation in dioid algebra and its application to performance evaluation of Discrete Event Systems

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Abstract

Modeling and analysis of a specific class of Discrete Event Systems lead to introduce an exotic algebra of formal series (*dioid* algebra). In particular, the behavior of these systems is characterized by computing *transfer matrices*. In this paper, we study the algebraic problems which arise when considering rational computations in this particular dioid. The main theorem states that rational elements are periodic, in the sense they represent the eventual periodic behavior of Timed Event Graphs. Then the algebra of periodicities is investigated. Some formulae and algorithms are presented. In particular, we show how the computation of the periodic behavior is related to the Frobenius problem for linear diophantine equations. These algorithms have been implemented in MAPLE. An application to a simple flowshop is presented.

Introduction

Our approach follows the Linear System Theory for Discrete Event Systems developed by Cohen, Dubois, Moller, Quadrat, Viot (see [2, 3]). This theory extends to a restricted class of Discrete Event Systems the main concepts of Control Theory such as transfer function and state-space representation. The first part of the paper reviews the algebraic tools needed to deal with Timed Event Graphs, summarizing the presentation given in [3]. We first explain how Timed Event Graphs can be modeled using dater functions, shift operators in dating and shift operators in counting. Then we focus on the algebraic properties of shifts. This leads to the introduction of a specific algebra of formal series called $\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle$. Solving the state equation for Timed Event Graphs results in making rational computations in this algebra, that is computing the transfer matrix of the system, $H = CA^*B$ (A^* plays a role analogous to $(sI - A)^{-1}$ in classical theory). We study the properties of rationals. The important theorem 4.11 characterizes rational series as periodic series, i.e. series corresponding to a periodic behavior of the system. Then we investigate the algebra of periodicities, i.e. the way periodicities are transformed when systems are put in parallel (proposition 5.3), in cascade (proposition 5.5) and in feedback. The results can be summarized by saying that the slowest periodicity is absorbing, which is very natural when dealing with Timed Event Graphs. This is the simple case. When the systems which are put in cascade have the same periodic slope (the same periodic throughput), some more complex "arithmetical" features appear. This corresponds to the case when the *critical subgraph* of the Timed Event Graphs is not reduced to a single circuit. In this case, the computation of the transient behavior is closely related with the "Frobenius problem" for Linear Diophantine equations. We conclude by shortly presenting an application to a simple manufacturing system (flowshop), obtained with our current implementation in MAPLE.

1 Modeling Timed Event Graphs

1.1 Dater equations

We refer the reader to [3] where it is explained in detail how Timed Event Graphs can be mod-

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eled from the dioid point of view. We just recall here the few basic facts needed for our purpose. Whith each transition i of an event graph, we associate a *dater function*, which is a map $\mathbb{Z} \to \mathbb{Z} \cup \{\pm\infty\}, n \mapsto x_i(n)$, defined by:

$x_i(n) = t \quad \Leftrightarrow$ "the firing numbered n of the transition i occurs at date t"

Dater functions are increasing. The restriction $n \in \mathbb{Z}$ is related to the discrete nature of the events, and we assume that the time only takes integer or infinite values ($x_i(n) \in \mathbb{Z} \cup \{\pm\infty\}$). This means there is a clock giving absolute times, and all the firings occur only at these times. Allowing $x_i(n) = -\infty$ or $x_i(n) = +\infty$ allows modeling situations when all the firings up to n have already occurred before we consider the system ($x_i(n) = -\infty$), or the firing numbered n never occurs ($x_i(n) = +\infty$). For

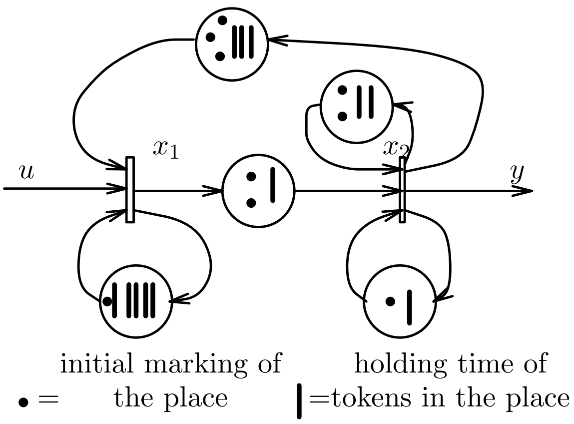

Figure 1: A simple Timed Event Graph

instance, for the event graph of Figure 1, it is immediate to obtain the following inequalities:

$$\begin{aligned} x_1(n) &\geq \max[5+x_1(n-1), 3+x_2(n-3), u(n)] \\ x_2(n) &\geq \max[1+x_1(n-2), 2+x_2(n-2), 1+x_2(n-1)] \\ y(n) &\geq x_2(n) \end{aligned} \tag{1}$$

Of course, the solution of (1) is not unique. We have to make the additional assumption that the transitions are fired as soon as possible (*earliest behavior*), which is equivalent to selecting the least solution of (1).

1.2 Operator representation

Let us denote by $\mathcal{T}$ the set of dater functions ("signals"). The elementary shift in dating $\delta : \mathcal{T} \to \mathcal{T}$ and the elementary shift in counting $\gamma : \mathcal{T} \to \mathcal{T}$ are defined by $\gamma x(n) = x(n-1)$ and $\delta x(n) = x(n) + 1$. The set of operators $\mathcal{T}^{\mathcal{T}}$ is naturally endowed with two laws: -the addition (denoted by $\oplus$) corresponding to the max of signals, -the composition product, denoted as usual by . or concatenation. With these notations, we write for instance $x_2(n-2) = \gamma^2 x_2(n)$ and $\max[2+x_2(n-2), 1+x_2(n-1)] = (\gamma^2\delta^2 \oplus \gamma\delta)x_2(n)$. Thus, (1) rewrites as follows:

$$\begin{cases} x_1 \geq \gamma\delta^5 x_1 \oplus \gamma^3\delta^3 x_2 \oplus u \\ x_2 \geq \gamma^2\delta x_1 \oplus (\gamma^2\delta^2 \oplus \gamma\delta)x_2 \\ y \geq x_2 \end{cases} \tag{2}$$

This system can be written in a standard state-space form:

$$x \geq Ax \oplus Bu, \quad y \geq Cx \tag{3}$$

where A, B, C are matrices the entries of which are sum of shift operators. The main concern of this paper is to find the minimal solution of (3) in the general case by means of effective algorithms. The solution (x, y) (i.e. all the dater functions associated with transitions and outputs) will provide a complete knowledge of the earliest behavior of the Timed Event Graph.

Two examples of this approach, applied to this simple Timed Event Graph and to a more complex one corresponding to a flexible workshop can be found in Section 7.

2 Algebraic model

2.1 Absorption properties of shift operators

Since γ and δ commute, it should be clear that the only operators we need to model Timed Event Graphs can be written as finite sums of operators $\gamma^n\delta^t$ (with $n, t \in \mathbb{N}$). Conversely, given an operator $L = \bigoplus_{i=1}^p \gamma^{n_i}\delta^{t_i}$, we may ask whether such a decomposition with respect to γ and δ is unique. This is *not* the case because we have important absorption properties. First, if $t \geq t'$, then $(\delta^t \oplus \delta^{t'})x(k) = \max(t + x(k), t' + x(k)) = \max(t, t') + x(k) = t + x(k) = \delta^t x(k)$. Similarly, let $n \geq n'$. We have: $(\gamma^n \oplus \gamma^{n'})x(k) = \max(x(k-n), x(k-$

$n')) = x(k-n') = \gamma^{n'}x(k)$, for the dater function $k \mapsto x(k)$ is increasing and $k-n \le k-n'$. This implies the following fundamental rules:

$$\gamma^n \oplus \gamma^{n'} = \gamma^{\min(n,n')} \tag{4}$$
$$\delta^t \oplus \delta^{t'} = \delta^{\max(t,t')} \tag{5}$$

2.2 The MinMax$\langle\langle\gamma,\delta\rangle\rangle$ algebra

We now describe the algebraic structure which reflects the properties of shift operators. Let us denote by $\mathbb{B} = \{\varepsilon, e\}$ the set of Booleans, with addition $\oplus$ and product $\otimes$, ε as zero and e as the unit. $\mathbb{B}\langle\langle\gamma,\delta\rangle\rangle$ denotes the set of formal series with indeterminates γ, δ, boolean coefficients, and exponents in $\mathbb{Z}$, endowed with the usual addition $\oplus$ and product $\otimes$. Indeed, a formal series is a mapping $\mathbb{Z}^2 \to \mathbb{B}$, $(n,t) \mapsto s(n,t)$. s admits an unique expression

$$s = \bigoplus_{n,t\in\mathbb{Z}} s(n,t) \otimes \gamma^n \otimes \delta^t. \tag{6}$$

Clearly, $\oplus$ is *idempotent* ($(\forall s) \quad s \oplus s = s$). $\oplus$ and $\otimes$ both are commutative monoid laws over $\mathbb{B}\langle\langle\gamma,\delta\rangle\rangle$, $\otimes$ is distributive with respect to $\oplus$ (i.e. $(s\oplus s')\otimes r = s\otimes r \oplus s'\otimes r$) and ε is absorbing. Such a structure is known as a commutative-idempotent-semiring with absorbing zero, or simply as a commutative *dioid* (see [3]). As usual, we will omit the $\otimes$ sign and simply write $s(n,t)\gamma^n\delta^t$ instead of $s(n,t)\otimes\gamma^n\otimes\delta^t$. It is important to notice that there is a natural order associated with $\oplus$ and which is compatible with the product, namely

$$s \le r \iff s \oplus r = r.$$

The ordered set $(\mathbb{B}\langle\langle\gamma,\delta\rangle\rangle, \le)$ is *complete* (i.e. every subset X admits a least upper bound, naturally denoted by $\bigoplus_{x\in X} x$). Moreover, the infinite distributivity law holds:

$$(\bigoplus_{x\in X} x)y = (\bigoplus_{x\in X} xy). \tag{7}$$

Thus, $((\mathbb{B}\langle\langle\gamma,\delta\rangle\rangle, \oplus, \otimes)$ is a *complete* dioid [3].

The *support* of a series s is defined by $\mathrm{Supp}(s) = \{(n,t) \in \mathbb{Z}^2; \quad s(n,t) \ne \varepsilon\}$. Since s can be written as $s = \bigoplus_{(n,t)\in \mathrm{Supp}(s)} \gamma^n\delta^t$, it is clear Boolean formal series are characterized by their support. When $\mathrm{Supp}(s)$ is finite, s is defined to be a *polynomial*. The subdioid of polynomials will be denoted by $\mathbb{B}\langle\gamma,\delta\rangle$.

We define the (total) *degree* of a series s as the upper bound of $\mathrm{Supp}(s)$: $\deg(s) = (\deg_\gamma(s), \deg_\delta(s)) = \sup \quad \mathrm{Supp}(s)$. The *valuation* is defined dually.

Examples 2.1 $\varepsilon \oplus \gamma^n\delta^t = \gamma^n\delta^t$, $(e \oplus \gamma^3\delta^2)(\delta^{-4} \oplus \gamma^5\delta^{-4}) = \delta^{-4} \oplus \gamma^5\delta^{-4} \oplus \gamma^3\delta^{-2} \oplus \gamma^8\delta^{-2}$. Because negative exponents are allowed, γ^{-1} is a polynomial. $\deg(\gamma \oplus \delta^2) = \sup[(1,0),(0,2)] = (1,2)$.

Proposition 2.2 *The only invertible elements in* $\mathbb{B}\langle\langle\gamma,\delta\rangle\rangle$ *are monomials.*

Proof $\gamma^{-n}\delta^{-t}$ is the inverse of $\gamma^n\delta^t$. Conversely, noticing that $\deg(ss') = \deg(s) + \deg(s')$, val (ss') = val (s) + val (s') and $\deg(s) \ge$ val (s), $ss' = e$ implies $0 = \deg(e) -$ val $(e) = (\deg(s) -$ val $(s)) + (\deg(s') -$ val $(s'))$, which is possible only if $\deg(s) -$ val $(s) = 0$ and $\deg(s') -$ val $(s') = 0$. This implies s is a monomial. ■

Before carrying on the study of formal series, we have to introduce a new operation: the star. Roughly speaking, the star plays a role analogous to the inverse in classical algebra. This is why the star is called a rational operation.

Definition 2.3 (Star operation) $a^* = e \oplus a \oplus a^2 \oplus a^3 \oplus \ldots$

Since $\mathbb{B}\langle\langle\gamma,\delta\rangle\rangle$ is a complete dioid, this infinite sum is well defined. The following properties are straightforward:

$$\begin{array}{ll} \text{(i)} & (a^*)^* = a^* \\ \text{(ii)} & (a\oplus b)^* = a^*b^* \\ \text{(iii)} & a^* = a^*a^* \end{array} \tag{8}$$

In the $\mathbb{B}\langle\langle\gamma,\delta\rangle\rangle$ algebra, we have not yet translated the absorption rules (4),(5). This can be done by taking the quotient of $\mathbb{B}\langle\langle\gamma,\delta\rangle\rangle$ by an appropriate congruence. Indeed, the first absorption rule implies $e \oplus \gamma = e$ (this is the case $n' = 0$ and $n = 1$). Writing $e\oplus\gamma = e\oplus e\gamma = e\oplus(e\oplus\gamma)\gamma$, we obtain after an immediate induction that:

$$e = e \oplus \gamma \oplus \gamma^2 \oplus \gamma^3 \oplus \ldots = \gamma^* \quad \text{(as operators)} \tag{9}$$

This suggests that e and γ^* should be identified. For the same reason, the second absorption rule requires e be identified with (δ^{-1}). This identification is done by introducing the following quotient:

Definition 2.4 (MinMax$\langle\!\langle\gamma,\delta\rangle\!\rangle$ **algebra)** *The map* $\varphi : \mathbb{B}\langle\!\langle\gamma,\delta\rangle\!\rangle \to \mathbb{B}\langle\!\langle\gamma,\delta\rangle\!\rangle$, $s \mapsto s\gamma^*(\delta^{-1})^*$ *is a congruence. The quotient dioid* $\mathbb{B}\langle\!\langle\gamma,\delta\rangle\!\rangle/\varphi$ *is called the* MinMax$\langle\!\langle\gamma,\delta\rangle\!\rangle$ *dioid.*

Since $\gamma^*\gamma^* = \gamma^*$, we have $\varphi(e) = \gamma^*\delta^* = \gamma^*(\gamma^*\delta^*) = \varphi(\gamma^*)$. This shows that e and γ^* belong to the same equivalence class. Of course the same property holds for $(\delta^{-1})^*$, which shows that φ realizes the desired identification: rules (4),(5) are valid in MinMax$\langle\!\langle\gamma,\delta\rangle\!\rangle$. In the following, we will deal with objects in MinMax$\langle\!\langle\gamma,\delta\rangle\!\rangle$ without precising they are equivalence classes modulo φ. The equivalence classes of polynomials will be again called *polynomials*, and the subdioid of polynomials will be denoted by MinMax$\langle\gamma,\delta\rangle$. MinMax$\langle\!\langle\gamma,\delta\rangle\!\rangle$ is complete, and the lower bound will be denoted by $\wedge$.

We now present a very useful graphic representation of elements of MinMax$\langle\!\langle\gamma,\delta\rangle\!\rangle$, which makes it obvious how the simplification rules work.

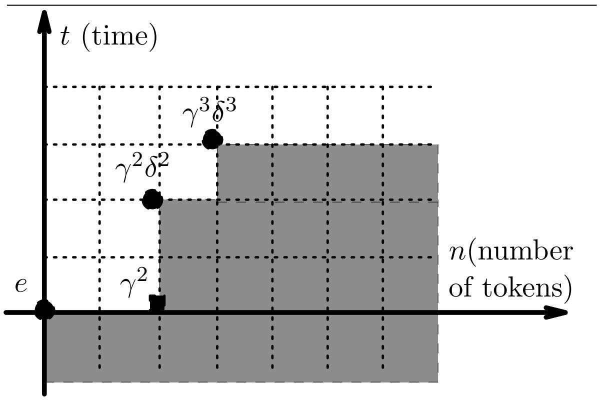

Figure 2: Graphic representation of $p = e \oplus \gamma^2\delta^2 \oplus \gamma^2 \oplus \gamma^3\delta^3$

Graphic representation It is immediate to check that given $s \in \mathbb{B}\langle\!\langle\gamma,\delta\rangle\!\rangle$, there is a *maximal representative* of the equivalence class of s modulo φ, namely $\varphi(s) = s\gamma^*(\delta^{-1})^*$. Then, a monomial $\gamma^\nu\delta^\tau \in \mathbb{B}\langle\!\langle\gamma,\delta\rangle\!\rangle$ can be represented by a point of the $\mathbb{Z}^2$-plane with coordinates (ν,τ), and the class of s by the collection of points corresponding to $\varphi(s)$, i.e. $\operatorname{Supp}\varphi(s)$. For instance, to the equivalence class of $\gamma^2\delta^2$ corresponds the "south-east" cone of the $\mathbb{Z}^2$-plane $\{(2+n', 2-t');\ (n',t') \in \mathbb{N}^2\}$. More generaly, an arbitrary element of MinMax$\langle\!\langle\gamma,\delta\rangle\!\rangle$ is simply represented by an union of south-east elementary cones of $\mathbb{Z}^2$. Addition corresponds to union and product to the usual addition of subsets. For instance, Figure 2 represents the polynomial $p = e \oplus \gamma^2 \oplus \gamma^2\delta^2 \oplus \gamma^3\delta^3$. The monomial γ^2 can be droped since it lies in the "shadow" of $\gamma^2\delta^2$.

2.3 Canonical form of polynomials

In the case of polynomials, there exists a canonical form which is simpler than the maximal representative already mentioned.

Proposition 2.5 *Let* $S \in$ MinMax$\langle\gamma,\delta\rangle$. *There exists a unique minimal* $s \in \mathbb{B}\langle\gamma,\delta\rangle$ *among the representatives of* S.

Proof (i) existence: let $s \in \mathbb{B}\langle\gamma,\delta\rangle$ be a representative of S. We write s as a sum of n monomials: $s = \bigoplus_{i=1}^n a_i$. We say the monomial a_i is redundant if $\varphi(\bigoplus_{j\neq i} a_i) = \varphi(s)$. If s is not minimal, we can eliminate a redundant monomial, and so on. Hence we obviously get a minimal representative.

(ii) uniqueness: let us now assume that $p = \bigoplus_{i\in I} a_i$ and $q = \bigoplus_{j\in J} b_j$ both are minimal representatives (a_i, b_j being monomials). Let $c = \gamma^*(\delta^{-1})^*$. $\varphi(p) = \varphi(q)$ is equivalent to $(\bigoplus_{i\in I} a_i)c = (\bigoplus_{j\in J} b_j)c$, i.e. $\bigoplus_{i\in I} a_i c = \bigoplus_{j\in J} b_j c$ Since $e \leq c$, we have $a_i \leq a_i c \leq \bigoplus_{j\in J} b_j c$, and there exists $j \in J$ such that $a_i \leq b_j c$. Thus $a_i c \leq b_j c^2 = b_j c$. For the same reason, $b_j c \leq a_k c$ for some k. Indeed, we must have $k = i$ (otherwise, a_i should be redundant), hence $\varphi(a_i) = a_i c = b_j c = \varphi(b_j)$. This implies $a_i = b_j$ since the restriction of φ to the set of monomials is clearly injective (cf. the graphic interpretation of φ). This shows that $p = q$ and ends the proof. ■

The geometric meaning of the minimal representative is obvious. In Figure 2, the minimal representative of p corresponds to the set of "north-west" corners of its maximal representative, namely $p' = e \oplus \gamma^2\delta^2 \oplus \gamma^3\delta^3$ (corresponding to the black points). The minimal representative has been obtained by eliminating the redundant monomial γ^2 (black square).

Proposition 2.5 allows extending the definition of support, degree and valuation to $\mathrm{MinMax}\langle\gamma,\delta\rangle$. These concepts are now defined in terms of canonical representatives. The number of monomials of the canonical representative of p is called the *complexity* of p (denoted by $\mathrm{compl}(p)$).

Remark 2.6 In $\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle$, there is in general no canonical representative. Consider $\delta^* = e \oplus \delta \oplus \delta^2 \oplus \ldots$. We also have $\delta^* = \delta^k \oplus \delta^{k+1} \oplus \ldots$ for all k. This suggests that the minimal representative of δ^* should be something like $\delta^{+\infty}$ which does not exist in $\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle$.

3 System Theory

We have now provided enough material to solve at least "formally" our main problem (3). The star notation which has been defined for scalars of $\mathbb{B}\langle\langle\gamma,\delta\rangle\rangle$ (and hence of $\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle$) obviously extends to $\mathcal{M}_{n,n}(\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle)$, the algebra of $n \times n$ square matrices with entries in $\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle$. The star is related to solving equations by the following well known result [3]:

Proposition 3.1
Let $A \in \mathcal{M}_{n,n}(\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle)$ and $b \in (\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle)^n$. The least solution of

$$x \geq Ax \oplus b \tag{10}$$

is given by A^*b.

Thus, the least solution of (3) is given by $x = A^*Bu$, $y = CA^*Bu$. $H = CA^*B$ is called the *transfer matrix* of the system.

Usually, the transfer function is characterized as the "impulse response" of the system. This still holds in the $\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle$ algebra, and has important consequencies. First, we associate a formal series with a dater function $n \mapsto x(n)$ in a natural way:

$$\mathcal{S}_x = \bigoplus_{n\in\mathbb{Z}} \gamma^n \delta^{x(n)} \tag{11}$$

with the convention $\delta^{-\infty} = \varepsilon$ and $\delta^{+\infty} = \delta^*$. Conversely, given a series s, the only dater function $\mathcal{D}_s$ such that $s = \mathcal{S}_x$ is given by:

$$\mathcal{D}_s = \sup\{t, \gamma^n\delta^t \leq s_x\} \tag{12}$$

This correspondence allows identifying signals and formal series. All the notions already defined for series (support,valuation,...) are obviously extended to signals.

Let us now consider a system $u \mapsto y$ with transfer series: $H = \bigoplus_{p\in\mathbb{Z}} \gamma^p \delta^{\mathcal{D}_H(p)}$. Applying the definition of shift operators, we have $Hu(n) = \max_{p\in\mathbb{Z}}[\mathcal{D}_H(p) + u(n-p)]$. Then, the input output-relation for dater function is given by the following *max-convolution* of daters:

$$y(n) \stackrel{\text{def}}{=} (\mathcal{D}_H \star u)(n) = \max_{p\in\mathbb{Z}}[\mathcal{D}_H(p) + u(n-p)] \tag{13}$$

The translation of (13) using formal series is simply a product:

$$\mathcal{S}_y = \bigoplus_{n\in\mathbb{Z}} \gamma^n \delta^{y(n)} =$$
$$= (\bigoplus_{n\in\mathbb{Z}} \gamma^n \delta^{\mathcal{D}_H(n)})(\bigoplus_{n\in\mathbb{Z}} \gamma^n \delta^{u(n)}) = H\mathcal{S}_u$$

Because $e = \bigoplus_{n\geq 0} \gamma^n \delta^0$, the dater function associated with e is given by $\mathcal{D}_e(n) = -\infty$ if $n < 0$ and $\mathcal{D}_e(n) = 0$ if $n \geq 0$. This means that the events numbered $0, 1, 2, \ldots$ occur at date 0. For this reason, e is called the *impulse*, it models the case when an infinite "quantity" of inputs is available at date 0. Since $\mathcal{S}_y = He = H$, the dater function associated with H determines the output y corresponding to an impulse (*impulse response*).

4 Rationality and periodicity

The *rational closure* of a subset $\mathcal{E}$ of a dioid is by definition the smallest subdioid $\mathcal{F}$ such that $\mathcal{E} \subset \mathcal{F}$ and $\mathcal{F}$ is rationally stable (i.e. stable for the operations $\oplus$,$\otimes$,and $*$). The notation $\mathcal{E}^*$ for $\mathcal{F}$ is standard. When modeling systems, only *causal polynomials* occur, that is polynomials whith nonnegative exponents in γ in δ. Let us denote by $\mathrm{MinMax}^+\langle\gamma,\delta\rangle$ the subdioid of causal polynomials. Then, transfer matrices are objects of the form CA^*B

with $A^* \in [\mathcal{M}_{n,n}(\text{MinMax}^+\langle\gamma,\delta\rangle)]^*$, $C \in \mathcal{M}_{p,n}(\text{MinMax}^+\langle\gamma,\delta\rangle)$ and $B \in \mathcal{M}_{n,q}(\text{MinMax}^+\langle\gamma,\delta\rangle)$. Since

$$[\mathcal{M}_{n,n}(\text{MinMax}^+\langle\gamma,\delta\rangle)]^* = \mathcal{M}_{n,n}[(\text{MinMax}^+\langle\gamma,\delta\rangle)^*]$$

(this is simply a part of the Kleene-Schützenberger Theorem, see [3] for a proof specific to this context), we are reduced to computing the stars of scalars. Indeed, the computation of A^* provided the stars of scalars are known is no more than the Gauss elimination algorithm applied to the matrix equation $X = AX \oplus \text{Id}$. The simplest algorithm is perhaps the Jacobi-like variant developed in [6, 5]. There is no difficulty here and we shall only consider the scalar case.

Definition 4.1 (Rational series) *We call* rational series*, or simply* rationals *the series belonging to the rational closure $\mathcal{R}$ of the subdioid of causal polynomials.*

Definition 4.2 (Periodicity) *Let $(\nu,\tau) \in (\mathbb{N}\setminus\{0\} \times \mathbb{N}\setminus\{0\}) \cup \{(1,0),(0,+\infty)\}$. A dater function $n \mapsto d(n)$ is (ν,τ)-*periodic *if there exists $N \in \mathbb{N}$ such that:*

$$(\forall n \geq N) \quad d(n+\nu) = d(n) + \tau \tag{14}$$

Periodicity has a straightforward interpretation for Timed Event Graphs. (14) means that after a transient behavior of length N, *every τ units of time, ν firings occur.* The *periodic throughput* or the *periodic slope* of d is the ratio $\lambda(d) \overset{\text{def}}{=} \frac{\tau}{\nu}$. The minimal value of N is called the *length of the transient*, and the minimal value of (ν,τ) is called *the* periodicity of d. The *degenerate* cases $(\nu,\tau) = (1,0)$ and $(0,+\infty)$ represent respectively the situation when an infinite number of event occur in a finite time and when no events occur after the $N-1$ th.

Remark 4.3 Indeed, definition 4.2 is not the usual one for periodic functions. Since dater functions are increasing, they cannot be periodic in the classical sense and this is not confusing. An equivalent formulation is to introduce $\tilde{d}(n) = d(n+1) - d(n)$ and to observe that (14) implies $(\forall n \geq N) \quad \tilde{d}(n+\nu) = \tilde{d}(n)$ which means that the *increases* of d are eventually periodic in the usual sense.

Definition 4.4 (Periodic series) *A series $s \in \text{MinMax}\langle\langle\gamma,\delta\rangle\rangle$ is (ν,τ)-periodic if the dater function $\mathcal{D}_s$ associated with s is (ν,τ)-periodic.*

The following provides an algebraic characterization of periodic series:

Proposition 4.5 *s is (ν,τ)-periodic iff there exists two polynomials p,q such that*

$$s = p \oplus q(\gamma^\nu\delta^\tau)^* \tag{15}$$

This is called a periodic representation *of s, with* periodicity $\gamma^\nu\delta^\tau$.

Proof The degenerate cases being straightforward, we assume $\tau \neq 0$ and $\tau \neq +\infty$. Let $s = \bigoplus_{n\in\mathbb{N}} \gamma^n\delta^{\mathcal{D}_s(n)}$ be (ν,τ)-periodic with transient of lenght N. We have:

$$\begin{aligned}\bigoplus_{n\geq N} \gamma^n\delta^{\mathcal{D}_s(n)} &= (\bigoplus_{n=N}^{n=N+\nu-1} \gamma^n\delta^{\mathcal{D}_s(n)}) \otimes \\ &\otimes(e \oplus \gamma^\nu\delta^\tau \oplus \gamma^{2\nu}\delta^{2\tau} \oplus \ldots) = \\ &= (\bigoplus_{n=N}^{n=N+\nu-1} \gamma^n\delta^{\mathcal{D}_s(n)})(\gamma^\nu\delta^\tau)^*\end{aligned}$$

Taking

$$p = \bigoplus_{n=0}^{N-1} \gamma^n\delta^{\mathcal{D}_s(n)} \quad \text{and} \quad q = \bigoplus_{n=N}^{n=N+\nu-1} \gamma^n\delta^{\mathcal{D}_s(n)}$$

We obviously get a representation like (15).

Conversely, assume we have $\mathcal{D}_s = \max(\mathcal{D}_p, \mathcal{D}_{q(\gamma^n\delta^t)^*})$ and $\mathcal{D}_{q(\gamma^n\delta^t)^*} = \mathcal{D}_q \star \mathcal{D}_{(\gamma^n\delta^t)^*}$. The conclusion results from observing that:

Lemma 4.6
-(i) *The* max *of two (ν,τ)-periodic functions is (ν,τ)-periodic.*
-(ii) *If d has finite support and* d' *is (ν,τ)-periodic, then the max-convolution $d \star d'$ is (ν,τ)-periodic.*

Proof of the Lemma: (i) is obvious. Since

$$d \star d'(n) = \max_{u\in\text{Supp}(d)} [d(u) + d'(n-u)] \tag{16}$$

and d has finite support, (16) expresses $d \star d'$ as the max of a finite number of functions $n \mapsto d(u) + d'(n-u)$ which have periodicity (ν,τ).

By applying several times the point (i) of the Lemma, we get that $d \star d'$ is (ν,τ)-periodic. This concludes the proof of lemma 4.6 and of proposition 4.5. ■

Considering $(\gamma\delta)^* = (e \oplus \gamma\delta)(\gamma^2\delta^2)^*$, we see that periodic representations are not unique. However, $(\gamma\delta)^*$ obviously seems to be simpler than $(e \oplus \gamma\delta)(\gamma^2\delta^2)^*$. We explain now which minimality is involved here, in order to obtain a canonical form of periodic series.

Definition 4.7 (Proper representations) $s = p \oplus q(\gamma^\nu\delta^\tau)^*$ *is* proper *if: (i)* $\deg p < \operatorname{val} q$ *and (ii)* $\deg q - \operatorname{val} q < (n,t)$.

The meaning of these two conditions is clear: (i) guarantees there is no overlapping between the monomials of p and q, similarly, (ii) guarantees the monomials of q,$q\gamma^\nu\delta^\tau$,$q\gamma^{2\nu}\delta^{2\tau}, \ldots$ are all different. In the case of proper representations, p can be interpreted as a *transient*, and q as a *pattern* which is moved by the successive translations: e, $\gamma^\nu\delta^\tau$, $\gamma^{2\nu}\delta^{2\tau}$,etc.

Definition 4.8 (Reduced representation) *Given two proper periodic representations* $\mathfrak{P}$: $s = p \oplus q(\gamma^\nu\delta^\tau)^*$ *and* $\mathfrak{P}'$: $s = p' \oplus q'(\gamma^{\nu'}\delta^{\tau'})^*$ *of series* s*, we say that* $\mathfrak{P}$ *is simpler than* $\mathfrak{P}'$ *(denoted by* $\mathfrak{P} \preceq \mathfrak{P}'$*) if* $(\nu,\tau) \leq (\nu',\tau')$ *and* $\deg(p) \leq \deg(p')$.

The $\preceq$ relation is obviously reflexive and transitive. It is also antisymmetric, for if $\deg p$ and (ν,τ) are known, p and q are necessarily equal to:

$$p = \bigoplus_{k=0}^{\deg_\gamma(p)} \gamma^k\delta^{\mathcal{D}_s(k)} \qquad q := \bigoplus_{k=\deg_\gamma(p)+1}^{\deg_\gamma(p)+\nu} \gamma^k\delta^{\mathcal{D}_s(k)}$$

Indeed, a proper representation is well determined by (N,ν,τ) with $N = \deg_\gamma(p)$ and we shall write

$$\mathfrak{P} = (N,\nu,\tau) \tag{17}$$

Theorem 4.9 *A periodic series* s *admits a* simplest *(i.e.* $\preceq$*-minimal) periodic proper representation, called the canonical form of* s.

Sketch of proof If $s = p \oplus q(\gamma^\nu\delta^\tau)^*$ and $s = p' \oplus q'(\gamma^{\nu'}\delta^{\tau'})^*$ are two periodic representations, we have to show that there exists a proper representation $s = (p' \wedge p) \oplus q''(\gamma^{\gcd(\nu,\nu')}\delta^{\gcd(\tau,\tau')})^*$, i.e. that greatest lower bounds exist for the $\preceq$ relation. This can be easily seen by reasoning in terms of dater functions, and proves the uniqueness. Since the set of periodicities, together with $\preceq$ is artinian (by (17), we can identify it to a sub-ordered set of $((\mathbb{N}\cup\{+\infty\})^3, \leq)$, there exists a simplest proper representation. ■

Remark 4.10 The Theorem 4.9 admits an algorithmic translation, which is a bit involved but presents no conceptual difficulties.

Theorem 4.11 (Main Theorem) *The rational series are precisely the periodic series.*

Proof of theorem 4.11: In proposition 4.5, we have shown that polynomials are periodic , i.e.:

$$\mathrm{MinMax}^+\langle\gamma,\delta\rangle \subset \mathcal{P} \tag{18}$$

Since $\mathcal{R}$ is the smallest rationally closed subdioid satisfying (18), we only have two check that $\mathcal{P}$ is rationally closed to get $\mathcal{R} \subset \mathcal{P}$. As the other inclusion is obvious, this will conclude the proof.

The next section is devoted to showing that the set of the periodic series $\mathcal{P}$ is stable for $\oplus$, $\otimes$, and $*$, by means of formulae and algorithms. This is the material needed for the proof of theorem 4.11, and it is fundamental in order to make effective rational computations in $\mathrm{MinMax}\langle\langle\gamma,\delta\rangle\rangle$.

5 Rational properties of periodic series

5.1 Sum of periodic series

Let us introduce a new operation over the set of periodicities. The $\sqcup$ of periodicities if the commutative operation defined by

$$\begin{array}{ll} \gamma^\nu\delta^\tau \sqcup \gamma^{\nu'}\delta^{\tau'} = \gamma^\nu\delta^\tau & \text{if } \frac{\tau'}{\nu'} < \frac{\tau}{\nu} \\ \gamma^\nu\delta^\tau \sqcup \gamma^{\nu'}\delta^{\tau'} = \gamma^{\operatorname{lcm}(\nu,\nu')}\delta^{\operatorname{lcm}(\tau,\tau')} & \text{if } \frac{\tau}{\nu} = \frac{\tau'}{\nu'} \\ \gamma^\nu\delta^\tau \sqcup \delta^{+\infty} = \delta^{+\infty} & \\ \gamma^\nu\delta^\tau \sqcup \gamma = \gamma^\nu\delta^\tau & \end{array}$$

$\sqcup$ is clearly associative and idempotent.

We first consider the sum of two simple periodic series:

Proposition 5.1 *Let* $s = \gamma^n\delta^t(\gamma^\nu\delta^\tau)^*$ *and* $s' = \gamma^{n'}\delta^{t'}(\gamma^{\nu'}\delta^{\tau'})^*$, *then* $s \oplus s'$ *is* $\gamma^\nu\delta^\tau \sqcup \gamma^{\nu'}\delta^{\tau'}$-*periodic. Moreover, if* $\frac{\tau}{\nu} \neq \frac{\tau'}{\nu'}$, *then* $\gamma^\nu\delta^\tau \sqcup \gamma^{\nu'}\delta^{\tau'}$ *is the minimal periodicity.*

Proof -If $\frac{\tau}{\nu} = \frac{\tau'}{\nu'}$: we write $\text{lcm}(\nu,\nu') = k\nu = k'\nu'$, and develop:

$$(\gamma^\nu\delta^\tau)^* = (e \oplus \gamma^\nu\delta^\tau \oplus \ldots \oplus \gamma^{(k-1)\nu}\delta^{(k-1)\tau})(\gamma^{k\nu}\delta^{k\tau})^*. \quad (19)$$

Because $k\tau = \lambda(s)k\nu = \lambda(s)k'\nu' = k'\tau'$ and $\gcd(k,k') = 1$, we get $k\tau = \text{lcm}(\tau,\tau')$. Thus (19) is a $\gamma^{\text{lcm}(\nu,\nu')}\delta^{\text{lcm}(\tau,\tau')}$-periodic representation of s. Since a similar representation also holds for s', we obviously obtain the required form for $s \oplus s'$.

-case $\frac{\tau}{\nu} > \frac{\tau'}{\nu'}$. First, we need a technical result:

Lemma 5.2 *Assume* $\frac{\tau}{\nu} > \frac{\tau'}{\nu'}$, *and* n, n', t, t' *are arbitrary integers. Then there exists* $K \geq 0$ *such that*

$$\gamma^{n'}\delta^{t'}\gamma^{K\nu'}\delta^{K\tau'}(\gamma^{\nu'}\delta^{\tau'})^* \leq \gamma^n\delta^t(\gamma^\nu\delta^\tau)^* \quad (20)$$

Applying the Lemma to $s \oplus s'$, we obtain the formula:

$$\gamma^n\delta^t(\gamma^\nu\delta^\tau)^* \oplus \gamma^{n'}\delta^{t'}(\gamma^{\nu'}\delta^{\tau'})^* = \gamma^n\delta^t(\gamma^\nu\delta^\tau)^* \oplus \oplus\gamma^{n'}\delta^{t'} \oplus \ldots \gamma^{n'+(K-1)\nu'}\delta^{t'+(K-1)\tau'}$$

which shows that $s \oplus s'$ has periodicity $\gamma^\nu\delta^\tau$.

Sketch of proof of Lemma 5.2. The simplest proof consists in observing that $\lim_{n\to+\infty} \mathcal{D}_s(n) - \mathcal{D}_{s'}(n) = +\infty$ (for the slope of $\mathcal{D}_s$ is greater than the one of $\mathcal{D}_{s'}$). Let p_0 such that for all $n \geq p_0$, $\mathcal{D}_{s'}(n) \leq \mathcal{D}_s(n)$: it should be clear that the monomials of s' of valuation in γ greater than p_0 are dominated by s, which yields (20). ■

Proposition 5.3 *The sum of a two periodic series* $s = p \oplus q\varpi^*$ *and* $s' = p' \oplus q'\varpi'^*$ *is periodic with periodicity* $\varpi \sqcup \varpi'$.

Proof We show that $s \oplus s'$ has a periodic representation of the form:

$$p'' \oplus q''(\varpi \sqcup \varpi')^* \quad (21)$$

Proposition 5.1 gives the result when q and q' are monomials. Applying again 5.1 and using the associativity and idempotence of $\sqcup$, it should be clear that if we add terms like $\gamma^{n_i}\delta^{t_i}\varpi^*$ or $\gamma^{n_j}\delta^{t_j}(\varpi')^*$ to (21), we still obtain an expression of the same form. ■

5.2 Product of periodic series

We now introduce another law which will play for product the role $\sqcup$ plays for sum. The $\sqcap$ of periodicities is the commutative operation defined by:

$$\begin{array}{ll} \gamma^\nu\delta^\tau \sqcap \gamma^{\nu'}\delta^{\tau'} = \gamma^\nu\delta^\tau & \text{if } \frac{\tau'}{\nu'} < \frac{\tau}{\nu} \\ \gamma^\nu\delta^\tau \sqcap \gamma^{\nu'}\delta^{\tau'} = \gamma^{\gcd(\nu,\nu')}\delta^{\gcd(\tau,\tau')} & \text{if } \frac{\tau}{\nu} = \frac{\tau'}{\nu'} \\ \gamma^\nu\delta^\tau \sqcap \delta^{+\infty} = \delta^{+\infty} & \\ \gamma^\nu\delta^\tau \sqcap \gamma = \gamma^\nu\delta^\tau & \end{array}$$

The $\sqcap$ is also associative and idempotent. The set of periodicities equipped with $\sqcup$ and $\sqcap$ is *not* a lattice, because the absorption property $\varpi \sqcup (\varpi \sqcap \varpi') = \varpi$ does not hold (consider $\varpi = \gamma\delta$ and $\varpi' = \gamma\delta^2$). However, we do have a weaker property which will be sufficient for our purpose:

$$(\varpi \sqcup \varpi') \sqcup (\varpi \sqcap \varpi') = (\varpi \sqcup \varpi') \quad (22)$$

Proposition 5.4 *Let* $s = (\gamma^\nu\delta^\tau)^*$ *and* $s' = (\gamma^{\nu'}\delta^{\tau'})^*$. *Then* ss' *is* $\gamma^\nu\delta^\tau \sqcap \gamma^{\nu'}\delta^{\tau'}$-*periodic, and* $\gamma^\nu\delta^\tau \sqcap \gamma^{\nu'}\delta^{\tau'}$ *is the minimal periodicity.*

Proof
(i) Case $\frac{\tau}{\nu} = \frac{\tau'}{\nu'}$. We have

$$(\gamma^\nu\delta^\tau)^*(\gamma^{\nu'}\delta^{\tau'})^* = \bigoplus_{i,j\geq 0} \gamma^{i\nu+j\nu'}\delta^{i\tau+j\tau'} \quad (23)$$

obviously, the only possible values for $i\nu + j\nu'$ are multiples of $\gcd(\nu,\nu')$. A well known result of the theory of Linear Diophantine equations states there exists a least $n = k\gcd(\nu,\nu')$ such that all the multiples of $\gcd(\nu,\nu')$ greater or equal to n can be expressed as $i\nu + j\nu'$ for some $i, j \geq 0$ (this is explained with more details in the next section). This n is called the *conductor* of (ν,ν'), denoted by $\text{cond}(\nu,\nu')$. Let us show that:

$$\begin{cases} i_0\nu + j_0\nu' = \text{cond}(\nu,\nu') \\ i_1\nu + j_1\nu' = \text{cond}(\nu,\nu') + \gcd(\nu,\nu') \\ i_2\nu + j_2\nu' = \text{cond}(\nu,\nu') + 2\gcd(\nu,\nu') \ldots \end{cases} \quad (24)$$

implies

$$\begin{cases} i_0\tau + j_0\tau' = \mathrm{cond}(\tau,\tau') \\ i_1\tau + j_1\tau' = \mathrm{cond}(\tau,\tau') + \gcd(\tau,\tau') \\ i_2\tau + j_2\tau' = \mathrm{cond}(\tau,\tau') + 2\gcd(\tau,\tau') \dots \end{cases} \quad (25)$$

In fact, multiplying (24) by $\lambda = \frac{\tau}{\nu}$, we have $i_0\tau + j_0\tau' = \lambda\mathrm{cond}(\nu,\nu')$, $i_1\tau + j_1\tau' = \lambda\mathrm{cond}(\nu,\nu') + \lambda\gcd(\nu,\nu') = \lambda\mathrm{cond}(\nu,\nu') + \gcd(\tau,\tau')$, and so on. This proves that $\mathrm{cond}(\tau,\tau') \le \lambda\mathrm{cond}(\nu,\nu')$. Applying this result to $\nu = \lambda^{-1}(\lambda\nu)$, $\nu' = \lambda^{-1}(\lambda\nu')$, we obtain the second inequality: $\mathrm{cond}(\nu,\nu') \le \lambda^{-1}\mathrm{cond}(\lambda\nu,\lambda\nu')$. This proves $\lambda\mathrm{cond}(\nu,\nu') = \mathrm{cond}(\tau,\tau')$ and shows that (24) implies (25). Then (23) obvioulsy rewrites:

$$ss' = (\bigoplus_{i\nu+j\nu' \le \mathrm{cond}(\nu,\nu')-1} \gamma^{i\nu+j\nu'}\delta^{i\tau+j\tau'}) \oplus$$
$$\oplus\gamma^{\mathrm{cond}(\nu,\nu')}\delta^{\mathrm{cond}(\tau,\tau')}(\gamma^{\gcd(\nu,\nu')}\delta^{\gcd(\tau,\tau')})^*$$

which yields a $\gamma^{\nu}\delta^{\tau} \sqcap \gamma^{\nu'}\delta^{\tau'}$-periodic representation of ss' an concludes Case (i).

(ii) Case $\frac{\tau}{\nu} > \frac{\tau'}{\nu'}$. We use again the technical lemma 5.2: there exists $k \ge 0$ such that

$$\gamma^{k\nu'}\delta^{k\tau'}(\gamma^{\nu'}\delta^{\tau'})^* \le \gamma^{n}\delta^{t}(\gamma^{\nu}\delta^{\tau})^*. \quad (26)$$

and multiplying this identity by $(\gamma^{\nu}\delta^{\tau})^*$, we obtain $\gamma^{k\nu'}\delta^{k\tau'}(\gamma^{\nu'}\delta^{\tau'})^*(\gamma^{\nu}\delta^{\tau})^* \le (\gamma^{\nu}\delta^{\tau})^*(\gamma^{\nu}\delta^{\tau})^* = (\gamma^{\nu}\delta^{\tau})^*$. Then the identity

$$(\gamma^{\nu}\delta^{\tau})^*(\gamma^{\nu'}\delta^{\tau'})^* =$$
$$= (\gamma^{\nu}\delta^{\tau})^*[e \oplus \gamma^{\nu'}\delta^{\tau'} \oplus \gamma^{2\nu'}\delta^{2\tau'} \oplus \dots$$
$$\oplus\gamma^{(k-1)\nu'}\delta^{(k-1)\tau'} \oplus \gamma^{k\nu'}\delta^{k\tau'}(\gamma^{\nu'}\delta^{\tau'})^*.] \quad (27)$$

rewrites

$$ss' = (\gamma^{\nu}\delta^{\tau})^*[e \oplus \gamma^{\nu'}\delta^{\tau'} \oplus \gamma^{2\nu'}\delta^{2\tau'} \oplus \dots$$
$$\oplus\gamma^{(k-1)\nu'}\delta^{(k-1)\tau'}] \quad (28)$$

This concludes the proof of the proposition. ■

Proposition 5.5 *The product of two periodic series $s = p \oplus q\varpi^*$ and $s' = p' \oplus q'\varpi'^*$ is periodic with periodicity $\varpi \sqcup \varpi'$.*

Proof $ss' = pp' \oplus pq'\varpi'^* \oplus p'q\varpi^* \oplus qq'\varpi^*\varpi'^*$. Applying proposition 5.4, we obtain $qq'\varpi^*\varpi'^* = p'' \oplus q''(\varpi \sqcap \varpi')^*$. Then the proposition 5.3 shows that ss' has periodicity $(\varpi \sqcup \varpi') \sqcup (\varpi \sqcap \varpi') = (\varpi \sqcup \varpi')$. This concludes the proof. ■

5.3 Star of periodic series

Let $s = p \oplus q\varpi^*$. The following formula reduces computing the star of s to computing the star of polynomials:

$$(p \oplus q\varpi^*)^* = p^*(e \oplus q(q \oplus \varpi)^*) \quad (29)$$

We obtain the star of a polynomial p of complexity n (p is the sum of n monomials $\bigoplus_{i=1}^n m_i$) by a recursive application of the rule

$$p^* = m_n^*(\bigoplus_{i=1}^{n-1} m_i)^* \quad (30)$$

which reduces the problem to a polynomial of complexity $n-1$. This provides periodic forms for p^* and s^*, and shows that $\mathcal{P}$ is stable for the $*$ operation. This concludes the proof of the main Theorem 4.11. ■

Remark 5.6 If the time takes non integer values, the relation between rationality and periodicity vanishes (consider the sum $(\gamma\delta)^* \oplus (\gamma^{\sqrt{2}}\delta^{\sqrt{2}})^*$). This difficulty is far from being an artificial problem. In fact, it is the limit of some pathology which already occurs with integer time. When the slopes of series are very close, the size of the transient becomes too important and the periodic representation is practically impossible to use. In these cases, a truncation has to be made. It should also be possible to use more conventional representations of rationals, like $s = \bigoplus_i a_i b_i^*$, where the a_i, b_i are monomials. The following example illustrates these difficulties:

Example 5.7 $(\gamma^{20}\delta)^* \oplus \delta(\gamma^{21}\delta)^* = \delta \oplus \gamma^{21}\delta^2 \oplus \gamma^{42}\delta^3 \oplus \gamma^{63}\delta^4 \oplus \gamma^{84}\delta^5 \oplus \gamma^{105}\delta^6 \oplus \gamma^{126}\delta^7 \oplus \gamma^{147}\delta^8 \oplus \gamma^{168}\delta^9 \oplus \gamma^{189}\delta^{10} \oplus \gamma^{210}\delta^{11} \oplus \gamma^{231}\delta^{12} \oplus \gamma^{252}\delta^{13} \oplus \gamma^{273}\delta^{14} \oplus \gamma^{294}\delta^{15} \oplus \gamma^{315}\delta^{16} \oplus \gamma^{336}\delta^{17} \oplus \gamma^{357}\delta^{18} \oplus \gamma^{378}\delta^{19} \oplus \gamma^{399}\delta^{20} \oplus \gamma^{420}\delta^{21}(\gamma^{20}\delta)^*$.

6 Star operation and diophantine linear equations

We have already seen that the product of periodic series reduces to computing the conductor of a simple Diophantine Linear equation. We now consider the problem of computing the star of a polynomial a, when all the monomials have the *same* slope. This is a generalization of

Proposition 5.4, which corresponds to a polynomial with complexity 2: $a = \gamma^{\nu}\delta^{\tau} \oplus \gamma^{\nu'}\delta^{\tau'}$. In a similar way, we shall show that the distribution of the monomials of a^* is related with the distribution of solutions of a more general linear diophantine equation, the theory of which affords a more efficient algorithm than the one given in Section5.3.

Let

$$a = \gamma^{\nu_1}\delta^{\tau_1} \oplus \gamma^{\nu_2}\delta^{\tau_2} \oplus \cdots \oplus \gamma^{\nu_l}\delta^{\tau_l}$$

with $\lambda = \frac{\tau_1}{\nu_1} = \ldots = \frac{\tau_l}{\nu_l}$. For all $i > 0$, each term in a^i is of the form

$$\gamma^{\nu_1 x_1 + \nu_2 x_2 + \cdots + \nu_l x_l} \, \delta^{\tau_1 x_1 + \tau_2 x_2 + \cdots + \tau_l x_l}$$

with $\sum_k x_k = i$. This leads us to consider the following problem:

$$\begin{aligned} &\text{find} \quad (x_1, \ldots, x_l) \in \mathbb{N}^l \quad \text{such that} \\ &\qquad \nu_1 x_1 + \nu_2 x_2 + \cdots + \nu_l x_l = c \end{aligned} \tag{31}$$

It can be shown [1] that if $\gcd(\nu_1, \nu_2, \ldots, \nu_l) = 1$, there exists $K \in \mathbb{N}$ such that for all $c \geq K$, (31) has a solution. The minimal value c_o of K is called the conductor of $(\nu_1, \ldots, \nu_l)$. The question of the determination of c_o is not solved in the general case. It is known as the "Frobenius problem" for linear diophantine equations.

In the case $l = 2$, the conductor for the equation $\nu_1 x_1 + \nu_2 x_2 = c$ is exactly known

$$\text{cond}(\nu_1, \nu_2) = (\nu_1 - 1)(\nu_2 - 1) \tag{32}$$

We now assume that $l > 2$ and $\nu_1 < \nu_2 < \cdots < \nu_l$. A simple upper bound is given by

$$s = (\nu_1 - 1)(\nu_l - 1) \tag{33}$$

A better bound is furnished by

$$t = \nu_2 \frac{d_1}{d_2} + \cdots + \nu_l \frac{d_{l-1}}{d_l} - \sum_i \nu_i \tag{34}$$

where d_i denotes the gcd of $\nu_1, \ldots, \nu_i$.

The existence of these bounds leads to a simple method for computing a^*, which consists in generating all the terms of a^* up to the bound. Then, since the periodicity of a^* is obvioulsy equal to $\gamma^{\nu_1}\delta^{\tau_1} \sqcap \gamma^{\nu_2}\delta^{\tau_2} \sqcap \ldots \sqcap \gamma^{\nu_l}\delta^{\tau_l}$, we obtain a periodic proper representation of a^*, which can be reduced to minimal form.

Example 6.1 Let us take $a = \gamma^6\delta^6 \oplus \gamma^{10}\delta^{10} \oplus \gamma^{15}\delta^{15}$. Since 6,10,15 are coprime the periodicity of a^* must be $(1, 1)$. Indeed, we have: $a^* = e \oplus \gamma^6\delta^6 \oplus \gamma^{10}\delta^{10} \oplus \gamma^{12}\delta^{12} \oplus \gamma^{16}\delta^{16} \oplus \gamma^{18}\delta^{18} \oplus \gamma^{20}\delta^{20} \oplus \gamma^{22}\delta^{22} \oplus \gamma^{24}\delta^{24} \oplus \gamma^{26}\delta^{26} \oplus \gamma^{28}\delta^{28} \oplus \gamma^{30}\delta^{30}(\gamma\delta)^*$.

Remark 6.2 The formal series point of view should be compared with that presented in [2]. Instead of using formal series, one can introduce the matrix equation in the $(\max, +)$ algebra $x_n = Ax_{n-1} \oplus Bu_n$, where A and B are matrices whith entries in $\mathbb{R} \cup \{-\infty\}$. Then, the study of the periodic behaviour of the *autonomous* system $x_n = Ax_{n-1}$ leads to consider the sequence $\{A^n\}_n$. The approach developed in [2] extends some of the Frobenius' results on the cyclicity of nonnegative matrices to the $(\max, +)$ algebra. Indeed, to the authors' knowledge, the "Frobenius problem" for Linear Diophantine Equations comes from this context, and it is not surprising that this problem still plays a central role in the study of formal series.

7 Illustrating examples

A preliminary implementation has been realized as a set of MAPLE macros. In a further stage, it is intended to support symbolic computation in dioids, which can be used to evaluate and optimize the periodic slope (this is the resource optimization problem, see [4]). Another version devoted to the non symbolic problem is being developed in FORTRAN. The following examples have been dealt with using the MAPLE implementation.

A simple application Let us consider again the Timed Event Graph of Figure 1. It is modeled by the following system:

$$\begin{aligned} x = \begin{bmatrix} \gamma\delta^5 & \gamma^3\delta^3 \\ \gamma^2\delta & \gamma\delta \oplus \gamma^2\delta^2 \end{bmatrix} x \oplus \begin{bmatrix} e \\ \varepsilon \end{bmatrix} u = \\ = Ax \oplus Bu, \\ y = \begin{bmatrix} \varepsilon & e \end{bmatrix} x = Cx \end{aligned}$$

We obtain the transfer matrix:

$$H = CA^*B = \gamma^2\delta(\gamma\delta^5)^*$$

which indeed is very simple! We can interpret H in terms of impulse response: if an infinite quantity of tokens becomes available at date 0, then the output is simply the dater function associated with H, i.e. two tokens exit at date 1, then, the periodic behavior is reached, and 1 token exits every 5 unit of times. In particular, from the input-output point of view, the Timed Event Graph of Figure 1 is equivalent to the Timed Event Graph of Figure 3.

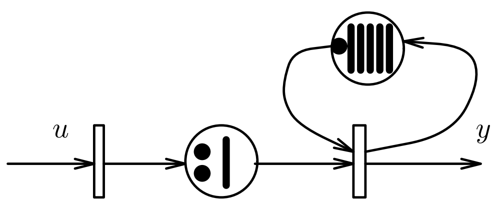

Figure 3: Reduced Timed Event Graph

Interpretation in terms of Timed Event Graphs We classicaly define the weight of a path as the product of the operators associated with the elementary arcs. Then a well known result states that the entry (i,j) of A^* represents the sum of the weights of all paths $j \mapsto i$ (see [5]). From the absorption properties of slopes, it should be clear that the periodicity (ν_i, τ_i) of $(A^*)_{i,i}$ has maximal ratio $\frac{\nu_i}{\tau_i}$ among all the weights of circuits $i \mapsto i$. When the event graph is strongly connected, the maximal ratio $\frac{\nu_i}{\tau_i}$ is the same for all the transitions. In this case, the set of the arcs of circuits which realize this maximal ratio (*critical circuits*) is called the *critical subgraph*, and the periodic throughput associated with any transition i is simply characterized as the slope of the critical circuits, while the gcd and lcm which appear in the computations are related to the structure of the critical subgraph.

Example 7.1 In the case of the event graph of Figure 1, the critical subgraph is reduced to the arc with weigth with $\gamma\delta^5$. We have:

$$A^* = \begin{bmatrix} (\gamma\delta^5)^* & \gamma^3\delta^3(\gamma\delta^5)^* \\ \gamma^2\delta(\gamma\delta^5)^* & \% \end{bmatrix}$$

with $\% = e \oplus \gamma\delta \oplus \gamma^2\delta^2 \oplus \gamma^3\delta^3 \oplus \gamma^4\delta^4 \oplus \gamma^5\delta^5 \oplus \gamma^6\delta^9(\gamma\delta^5)^*$. Since the graph is strongly connected, all the entries have the same periodic slope.

Application to a flowshop We consider the simple flowshop with 3 parts P_1, P_2, P_3 and 3 machines M1, M2, M3, shown in Figure 4. We neglect the transportation times between machines. The schedule is the same for all parts : M1 $\to$ M2 $\to$ M3 $\to$ M1. The processing times are given by the following table:

	P1	P2	P3
M1	9	2	1
M2	2	7	2
M3	1	10	1

This means part P1 must be processed at least 9 units of time on machine M1, etc. Then part P1 becomes *instantaneously* available for the next machine M2, and machine M1 is also available for the next job. This leads to draw the Timed Event Graph of Figure 4. The vertical circuits correspond to the circulations of parts, and the the horizontal circuits correspond to machines. The position of tokens in the places is related to a specific initial state of the system. A more detailed account of modeling flowshop and jobshops using dioids can be found in [2].

We have introduced 3 inputs (u_1 in x_1, u_2 in x_2, u_3 in x_3). The output transitions are x_7, x_8, x_9. Then, the following matrix representation can be writen:

$$A = \begin{bmatrix} \varepsilon & \varepsilon & \gamma\delta & \varepsilon & \varepsilon & \varepsilon & \gamma\delta & \varepsilon & \varepsilon \\ \delta^9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^{10} & \varepsilon \\ \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma \\ \delta^9 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta^2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \delta^2 & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \delta & \varepsilon & \delta^7 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \gamma\delta \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^7 & \varepsilon & \delta & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^2 & \varepsilon & \delta^{10} & \varepsilon \end{bmatrix}$$

$$B = \begin{bmatrix} e & \varepsilon & \varepsilon \\ \varepsilon & e & \varepsilon \\ \varepsilon & \varepsilon & e \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon \end{bmatrix}$$

$$C = \begin{bmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta^{10} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \delta \end{bmatrix}$$

We obtain the transfer matrix:

$$H = \begin{bmatrix} \delta^{12} \oplus \gamma\delta^{30}(\gamma\delta^{19})^* & \gamma\delta^{21}(\gamma\delta^{19})^* & \gamma\delta^{13} \oplus \gamma^2\delta^{31}(\gamma\delta^{19})^* \\ \delta^{28}(\gamma\delta^{19})^* & \delta^{19}(\gamma\delta^{19})^* & \gamma\delta^{29}(\gamma\delta^{19})^* \\ \delta^{29}(\gamma\delta^{19})^* & \delta^{20}(\gamma\delta^{19})^* & \delta^{4} \oplus \gamma\delta^{30}(\gamma\delta^{19})^* \end{bmatrix}$$

Because the event graph is strongly connected, the periodic slope is the same for all the entries of the transfer matrix. This periodicity corresponds to the weigth $\gamma\delta^{19}$ of the *unique* critical circuit (the vertical circuit of part P2).

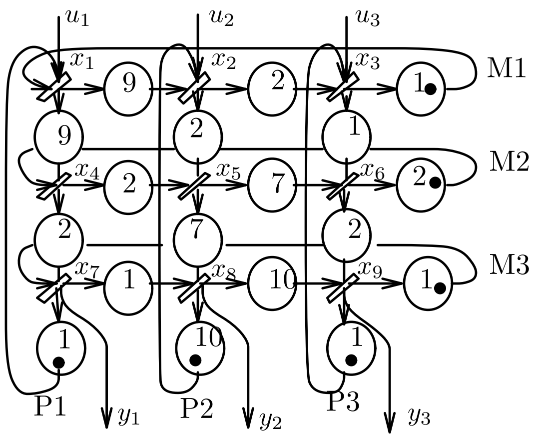

Figure 4: 3-machines,3-parts flowshop

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