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THE LOCAL CAUCHY PROBLEM FOR IONIZED MAGNETIZED REACTIVE GAS MIXTURES

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Abstract

We investigate a system of partial differential equations modeling ionized magnetized reactive gas mixtures. In this model, transport fluxes are anisotropic linear combinations of fluid macroscopic variable gradients and also include the effect of macroscopic electromagnetic forces. By using entropic variables, we first recast the system of conservation laws into a partially symmetric conservative form and next into a partial normal form, that is, in the form of a quasilinear partially symmetric hyperbolic-parabolic system. Using a result of Vol'Pert and Hudjaev, we prove local existence and uniqueness of a bounded smooth solution to the Cauchy problem.

1 Introduction

Ionized magnetized reactive gas mixtures—or reactive plasmas—have many practical applications such laboratory plasmas, high-speed gas flows or atmospheric phenomena. In this paper, we investigate the structure and properties of the corresponding systems of partial differential equations.

The kinetic theory of ionized gas mixtures can be used to obtain the equations governing high density low temperature plasmas. The resulting systems are different according to the various characteristic lengths and times of the phenomena under investigation. Assuming that there is a single temperature in the mixture—this is the case for various practical applications—the corresponding governing equations are derived in Ferziger and Kaper [1] and Giovangigli and Graille [2] for general reactive polyatomic gas mixtures.

The corresponding equations—governing ionized magnetized reactive gas mixtures—can be split into conservation equations, transport fluxes, thermochemistry and Maxwell's equations. A remarkable aspect is that the magnetic field yields anisotropic diffusion mass fluxes, heat flux and viscous tensor. In particular, diffusion fluxes involve anisotropic linear combinations of macroscopic fluid variable gradients as well as zeroth order terms arising from the action of macroscopic electromagnetic forces. The corresponding structural mathematical assumptions concerning thermoelectrochemistry and transport coefficients are also derived from the kinetic theory of gases [2] and generalize the situation of neutral species [3, 4].

The governing equations for reactive ionized magnetized gas mixtures constitute a second-order quasilinear system of conservation laws. By using entropic variables, we first recast the system into a partially symmetric conservative form and next into a partially normal form, that is, in the form of a quasilinear partially symmetric hyperbolic-parabolic system. We use the term partially symmetric in contrast with the neutral regime. The system of neutral gas mixtures is indeed completely symmetric in the sense that the convective matrices which couple together the hyperbolic and parabolic subsystems are fully symmetric [5, 3].

For the resulting partially symmetric hyperbolic-parabolic system, we prove existence of a unique solution to the Cauchy problem with smooth initial conditions, locally in time, in the Vol'Pert space $V_l(\mathbb{R}^3)$. Our method of proof relies on the results of Vol'pert and Hudjaev concerning the Cauchy problem for symmetric quasilinear hyperbolic-parabolic composite systems of partial differential equations [6].

The governing equations for ionized magnetized reactive gas mixtures are presented in section 2 and the quasilinear form is obtained in section 3. In section 4, we investigate partial symmetrizability, normal form and existence of solutions for an abstract system. Finally, in section 5, we apply these results to the system of partial differential equations modeling multicomponent ionized magnetized reactive gas mixtures.

2 Equations for Ionized Magnetized Reactive Gas Mixtures

The equations governing dissipative plasmas can be split between conservation equations, transport fluxes, thermochemistry, and Maxwell's equations. These equations can be derived in the framework of the kinetic theory of gases by using a first order Enskog expansion [1, 2].

2.1 Conservation's Equations

We denote by $\mathfrak{S}$ the species indexing set $\mathfrak{S} = \{1, \dots, n^s\}$, n^s the number of species, n_k, ρ_k and q_k the number of moles, the mass and the charge per unit volume of the k^{th} species and m_k the molar mass of the k^{th} species.

The species mass conservation equations read

$$\partial_t \rho_k + \boldsymbol{\partial_x}\cdot(\rho_k \boldsymbol{v}) + \boldsymbol{\partial_x}\cdot(\boldsymbol{\mathcal{F}}_k) = m_k \omega_k, \quad k \in \mathfrak{S}, \tag{2.1}$$

where $\boldsymbol{v}$ is the macroscopic velocity of the mixture, $\boldsymbol{\mathcal{F}}_k$ the diffusion flux and ω_k the chemical source term of the k^{th} species.

The following equation expresses momentum conservation

$$\partial_t(\rho \boldsymbol{v}) + \boldsymbol{\partial_x}\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v} + p\mathbb{I}) + \boldsymbol{\partial_x}\cdot\boldsymbol{\Pi} = \rho\boldsymbol{g} + q\boldsymbol{E} + \boldsymbol{J}\wedge\boldsymbol{B}, \tag{2.2}$$

with ρ the total mass per unit volume, p the pressure, $\mathbb{I}$ the 3 by 3 unit tensor, $\boldsymbol{\Pi}$ the viscous stress tensor, q the total charge per unit volume, $\boldsymbol{E}$ the electric field, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{g}$ a species independent external force and $\boldsymbol{J}$ the total electric current density defined by $\boldsymbol{J} = \boldsymbol{j} + q\boldsymbol{v}$ with $\boldsymbol{j}$ the conduction current density

We introduce the fluid energy per unit mass e^{f} which is defined by $e^{\mathrm{f}} = e+\boldsymbol{v}\cdot\boldsymbol{v}/2$, where e is the internal energy per unit mass. The energy conservation equation written in terms of e^{f} reads

$$\partial_t(\rho e^{\mathrm{f}}) + \boldsymbol{\partial_x}\cdot\left[\left(\rho e^{\mathrm{f}} + p\right)\boldsymbol{v}\right] + \boldsymbol{\partial_x}\cdot(\boldsymbol{Q} + \boldsymbol{\Pi}\cdot\boldsymbol{v}) = \rho\boldsymbol{g}\cdot\boldsymbol{v} + \boldsymbol{J}\cdot\boldsymbol{E}, \tag{2.3}$$

where $\boldsymbol{Q}$ is the heat flux.

2.2 Transport Fluxes

A remarkable aspect of dissipative plasmas is that transport fluxes in strong magnetic fields are anisotropic [7, 1, 2]. In order to express these anisotropic transport fluxes, we define the unitary vector $\boldsymbol{\mathcal{B}} = \boldsymbol{B}/B$, where B is the norm of the magnetic field $\boldsymbol{B}$, and for any vector $\boldsymbol{X}$, we introduce the three associated vectors

$$\boldsymbol{X}^{\parallel} = (\boldsymbol{\mathcal{B}}\cdot\boldsymbol{X})\boldsymbol{\mathcal{B}}, \quad \boldsymbol{X}^{\perp} = \boldsymbol{X} - \boldsymbol{X}^{\parallel} \quad \text{and} \quad \boldsymbol{X}^{\odot} = \boldsymbol{\mathcal{B}}\wedge\boldsymbol{X},$$

which are mutually orthogonal $\boldsymbol{X}^{\perp}\cdot\boldsymbol{X}^{\parallel} = 0$, $\boldsymbol{X}^{\odot}\cdot\boldsymbol{X}^{\parallel} = 0$, $\boldsymbol{X}^{\perp}\cdot\boldsymbol{X}^{\odot} = 0$. In addition, for any vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, we have the relations $\boldsymbol{X}^{\perp}\cdot\boldsymbol{Y}^{\odot} + \boldsymbol{X}^{\odot}\cdot\boldsymbol{Y}^{\perp} = 0$ and $\boldsymbol{X}^{\perp}\cdot\boldsymbol{Y}^{\perp} = \boldsymbol{X}^{\odot}\cdot\boldsymbol{Y}^{\odot}$.

The diffusion flux $\boldsymbol{\mathcal{F}}_k$, $k \in \mathfrak{S}$, is given by

$$\boldsymbol{\mathcal{F}}_k = \rho_k \boldsymbol{\mathcal{V}}_k, \quad k \in \mathfrak{S}, \tag{2.4}$$

where the diffusion velocity $\boldsymbol{\mathcal{V}}_k$, $k \in \mathfrak{S}$, reads

$$\begin{aligned} \boldsymbol{\mathcal{V}}_k \ = \ & -\sum_{l\in\mathfrak{S}} D_{kl}^{\parallel}\left(\boldsymbol{d}_l^{\parallel} + \chi_l^{\parallel}(\boldsymbol{\partial_x}\log T)^{\parallel}\right) \\ & -\sum_{l\in\mathfrak{S}} D_{kl}^{\perp}\left(\boldsymbol{d}_l^{\perp} + \chi_l^{\perp}(\boldsymbol{\partial_x}\log T)^{\perp} + \chi_l^{\odot}(\boldsymbol{\partial_x}\log T)^{\odot}\right) \\ & -\sum_{l\in\mathfrak{S}} D_{kl}^{\odot}\left(\boldsymbol{d}_l^{\odot} + \chi_l^{\perp}(\boldsymbol{\partial_x}\log T)^{\odot} - \chi_l^{\odot}(\boldsymbol{\partial_x}\log T)^{\perp}\right). \end{aligned} \tag{2.5}$$

and where the species diffusion driving force $\boldsymbol{d}_k$, $k \in \mathfrak{S}$, is given by

$$\boldsymbol{d}_k = \frac{1}{p}\left[\boldsymbol{\partial_x} p_k - \rho_k\boldsymbol{g} - q_k(\boldsymbol{E} + \boldsymbol{v}\wedge\boldsymbol{B})\right]. \tag{2.6}$$

In these expressions, $D_{kl}^{\parallel}$, $D_{kl}^{\perp}$ and $D_{kl}^{\odot}$, $k, l \in \mathfrak{S}$, are the multicomponent diffusion coefficients, $\chi_k^{\parallel}$, $\chi_k^{\perp}$ and $\chi_k^{\odot}$, $k \in \mathfrak{S}$, the thermal diffusion ratios, T the absolute temperature and p_k, $k \in \mathfrak{S}$, the partial pressure of the $k^{\rm th}$ species. The corresponding conduction current density $\boldsymbol{j}$ reads

$$\boldsymbol{j} = \sum_{k\in\mathfrak{S}} q_k \boldsymbol{V}_k. \tag{2.7}$$

Note that for neutral gases, the charges q_k, $k \in \mathfrak{S}$, vanish and we recover the classical expressions since we then have $D^{\|}_{kl} = D^{\perp}_{kl}$, $D^{\odot}_{kl} = 0$, $k,l \in \mathfrak{S}$, $\chi^{\|}_k = \chi^{\perp}_k$, $\chi^{\odot}_k = 0$, $k \in \mathfrak{S}$, so that the diffusion velocities $\boldsymbol{V}_k$, $k \in \mathfrak{S}$, read $\boldsymbol{V}_k = -\sum_{l\in\mathfrak{S}} D_{kl}(\boldsymbol{d}_l + \chi_l \boldsymbol{\partial_x} \log T)$ [3]. For ionized gases, however the diffusion coefficients are different according to the three spatial directions as a consequence of anisotropy. We also observe that the species diffusion driving forces $\boldsymbol{d}_k$, $k \in \mathfrak{S}$, contain an additional term due to the macroscopic electromagnetic force $q_k(\boldsymbol{E} + \boldsymbol{v}\wedge\boldsymbol{B})$. Moreover, although our formalism uses the unitary vector $\boldsymbol{\mathcal{B}} = \boldsymbol{B}/B$, the fluxes behave smoothly as $\boldsymbol{B}$ goes to zero as shown in the following sections thanks to the properties of the transport coefficients.

The expression of the heat flux is

$$\begin{aligned} \boldsymbol{Q} \;=\; & -\lambda^{\|}(\boldsymbol{\partial_x} T)^{\|} - \lambda^{\perp}(\boldsymbol{\partial_x} T)^{\perp} - \lambda^{\odot}(\boldsymbol{\partial_x} T)^{\odot} \\ & + p\sum_{k\in\mathfrak{S}} \Big(\widehat{\chi}^{\|}_k \boldsymbol{V}^{\|}_k + \widehat{\chi}^{\perp}_k \boldsymbol{V}^{\perp}_k + \widehat{\chi}^{\odot}_k \boldsymbol{V}^{\odot}_k\Big) + \sum_{k\in\mathfrak{S}} \rho_k h_k \boldsymbol{V}_k, \end{aligned} \tag{2.8}$$

where h_k is the enthalpy per unit mass of the $k^{\rm th}$ species, $\lambda^{\|}$, $\lambda^{\perp}$ and $\lambda^{\odot}$ the thermal conductivities, and $\widehat{\chi}^{\|}_k$, $\widehat{\chi}^{\perp}_k$ and $\widehat{\chi}^{\odot}_k$ other thermal diffusion ratios, where we distinguish as well the coefficients according to the three spatial directions. We likewise note that expression (2.8) of the heat flux coincides with that for neutral gases since $\lambda^{\|} = \lambda^{\perp}$, $\lambda^{\odot} = 0$, $\widehat{\chi}^{\|}_k = \widehat{\chi}^{\perp}_k$, $\widehat{\chi}^{\odot}_k = 0$, $k \in \mathfrak{S}$, when the charges q_k, $k \in \mathfrak{S}$, vanish and the heat flux then reads $\boldsymbol{Q} = -\lambda \boldsymbol{\partial_x} T + \sum_{k\in\mathfrak{S}}(p\chi_k + \rho_k h_k)\boldsymbol{V}_k$ [3]. Moreover, the heat flux is smooth as $\boldsymbol{B}$ goes to zero as shown in the following thanks to the properties of the transport coefficients.

In order to simplify the writing of the transport properties, we define the real matrices $D^{\|}$, $D^{\perp}$, $D^{\odot}$, and the real vectors $\widehat{\chi}^{\|}$, $\chi^{\|}$, $\widehat{\chi}^{\perp}$, $\chi^{\perp}$, $\widehat{\chi}^{\odot}$,$\chi^{\odot}$, $\widehat{\theta}^{\|}$, $\theta^{\|}$, by

$$D^{\diamond} = (D^{\diamond}_{kl})_{k,l\in\mathfrak{S}}, \quad \widehat{\chi}^{\diamond} = (\widehat{\chi}^{\diamond}_k)_{k\in\mathfrak{S}}, \quad \chi^{\diamond} = (\chi^{\diamond}_k)_{k\in\mathfrak{S}},$$

where $\diamond$ denotes any symbol in $\{\|, \perp, \odot\}$, the real vectors $\widehat{\theta}^{\perp}$, $\theta^{\perp}$, $\widehat{\theta}^{\odot}$, $\theta^{\odot}$, by the following linear systems

$$\begin{aligned} \theta^{\|} = D^{\|}\chi^{\|}, &\quad \theta^{\perp} + \mathtt{i}\theta^{\odot} = (D^{\perp} + \mathtt{i}D^{\odot})(\chi^{\perp} + \mathtt{i}\chi^{\odot}), \\ \widehat{\theta}^{\|} = D^{\|\mathtt{T}}\widehat{\chi}^{\|}, &\quad \widehat{\theta}^{\perp} + \mathtt{i}\widehat{\theta}^{\odot} = (D^{\perp} + \mathtt{i}D^{\odot})^{\mathtt{T}}(\widehat{\chi}^{\perp} + \mathtt{i}\widehat{\chi}^{\odot}), \end{aligned}$$

and the real coefficients $\widehat{\lambda}^{\|}$, $\widehat{\lambda}^{\perp}$, $\widehat{\lambda}^{\odot}$ by

$$\begin{aligned} \widehat{\lambda}^{\|} &= \lambda^{\|} + \tfrac{p}{T}\widehat{\chi}^{\|\mathtt{T}} D^{\|}\chi^{\|}, \\ \widehat{\lambda}^{\perp} + \mathtt{i}\widehat{\lambda}^{\odot} &= \lambda^{\perp} + \mathtt{i}\lambda^{\odot} + \tfrac{p}{T}(\widehat{\chi}^{\perp} + \mathtt{i}\widehat{\chi}^{\odot})^{\mathtt{T}}(D^{\perp} + \mathtt{i}D^{\odot})(\chi^{\perp} + \mathtt{i}\chi^{\odot}). \end{aligned}$$

The viscous stress tensor can be written in the form

$$
\begin{aligned}
\boldsymbol{\Pi} \;=\; & -\kappa(\boldsymbol{\partial_x}\cdot\boldsymbol{v})\mathbb{I} - \eta_1\mathsf{S} - \eta_2(\mathsf{T}(\mathcal{B})\mathsf{S} - \mathsf{S}\mathsf{T}(\mathcal{B})) \\
& -\eta_3(-\mathsf{T}(\mathcal{B})\mathsf{S}\mathsf{T}(\mathcal{B}) + \mathcal{B}^{\mathsf{T}}\mathsf{S}\mathcal{B}\,\mathcal{B}{\otimes}\mathcal{B}) \\
& -\eta_4(\mathsf{S}\,\mathcal{B}{\otimes}\mathcal{B} + \mathcal{B}{\otimes}\mathcal{B}\,\mathsf{S} - 2\mathcal{B}^{\mathsf{T}}\mathsf{S}\mathcal{B}\,\mathcal{B}{\otimes}\mathcal{B}) \\
& -\eta_5(\mathcal{B}{\otimes}\mathcal{B}\,\mathsf{S}\,\mathsf{T}(\mathcal{B}) - \mathsf{T}(\mathcal{B})\,\mathsf{S}\,\mathcal{B}{\otimes}\mathcal{B}),
\end{aligned}
\tag{2.9}
$$

where $\mathcal{B} = \boldsymbol{B}/B$, κ is the volume viscosity, and η_1, η_2, η_3, η_4, η_5 are the shear viscosities. In this expression we have denoted by S the strain symmetric traceless two order tensor

$$\mathsf{S} = \boldsymbol{\partial_x}\boldsymbol{v} + \boldsymbol{\partial_x}\boldsymbol{v}^{\mathsf{T}} - \tfrac{2}{3}(\boldsymbol{\partial_x}\cdot\boldsymbol{v})\mathbb{I},$$

where $^{\mathsf{T}}$ denotes the transposition, and by $\mathsf{T}(\boldsymbol{a})$ the antisymmetric matrix defined for any vector $\boldsymbol{a} = (a_1, a_2, a_3)^{\mathsf{T}}$ by

$$\mathsf{T}(\boldsymbol{a}) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

We define for convenience the vectors $\mathsf{T}_i(\boldsymbol{a})$, $i \in \mathfrak{C}$, as the i^{th} column of the matrix $\mathsf{T}(\boldsymbol{a})$ so that

$$\mathsf{T}_1(\boldsymbol{a}) = (0, a_3, -a_2)^{\mathsf{T}}, \quad \mathsf{T}_2(\boldsymbol{a}) = (-a_3, 0, a_1)^{\mathsf{T}}, \quad \text{and} \quad \mathsf{T}_3(\boldsymbol{a}) = (a_2, -a_1, 0)^{\mathsf{T}}.$$

For neutral gases, the expression (2.9) of the viscous stress tensor becomes $\boldsymbol{\Pi} = -\kappa(\boldsymbol{\partial_x}\cdot\boldsymbol{v})\mathbb{I} - \eta_1\mathsf{S}$ as we have $\eta_2 = \eta_3 = \eta_4 = \eta_5 = 0$. The viscous stress tensor then reads as a linear combination of the identity matrix $\mathbb{I}$ and the strain tensor S, and, for ionized gases, as a linear combination of the identity matrix and all the symmetric traceless two order tensors built from the strain tensor S and the antisymmetric rotation tensor $\mathsf{T}(\mathcal{B})$ associated with the magnetic field $\boldsymbol{B}$. Furthermore, the viscous stress tensor behave smoothly as $\boldsymbol{B}$ goes to zero as shown in the following thanks to the properties of the transport coefficients.

2.3 Chemical source term Expression

We consider n^r elementary reversible reactions among the n^s species which can be formally written as

$$\sum_{k\in\mathfrak{S}} \nu_{kr}^{\mathrm{f}}\, \mathfrak{M}_k \;\rightleftarrows\; \sum_{k\in\mathfrak{S}} \nu_{kr}^{\mathrm{b}}\, \mathfrak{M}_k, \qquad r \in \mathfrak{R},$$

where $\mathfrak{M}_k$ is the chemical symbol of the k^{th} species, ν_{kr}^{f} and ν_{kr}^{b} are the forward and the backward stoichiometric coefficients of the k^{th} species in the r^{th} reaction, respectively, and $\mathfrak{R} = \{1, \dots, n^r\}$ is the set of reaction indexes.

The Maxwellian production rates given by the kinetic theory can be written

$$\omega_k = \sum_{r\in\mathfrak{R}} (\nu_{kr}^{\mathrm{b}} - \nu_{kr}^{\mathrm{f}})\tau_r, \quad k \in \mathfrak{S}, \tag{2.10}$$

where τ_r is the rate of progress of the r^{th} reaction. The rates of progress are given by the symmetric expression [3]

$$\tau_r = \mathcal{K}_r^{\text{s}} \left(\exp \langle \nu_r^{\text{f}}, \mathsf{M}\mu \rangle - \exp \langle \nu_r^{\text{b}}, \mathsf{M}\mu \rangle \right), \tag{2.11}$$

where $\nu_r^{\text{f}} = (\nu_{1r}^{\text{f}}, \dots, \nu_{n^s r}^{\text{f}})^{\mathsf{T}}$, $\nu_r^{\text{b}} = (\nu_{1r}^{\text{b}}, \dots, \nu_{n^s r}^{\text{b}})^{\mathsf{T}}$, $\mu = (\mu_1, \dots, \mu_{n^s})^{\mathsf{T}}$, with μ_k, $k \in \mathfrak{S}$, the species reduced chemical potential, M, the diagonal matrix defined by $\mathsf{M} = \text{diag}(m_1, \dots, m_{n^s})$ and $\mathcal{K}_r^{\text{s}}$ is the symmetric reaction constant. This symmetric formulation of the rates of progress is obtained by using the fundamental reciprocal relation between forward and backward reaction constants [8] that can be deduced from the kinetic theory [3].

2.4 Thermodynamics

Thermodynamics obtained in the framework of the kinetic theory of gases is valid out of equilibrium and has, therefore, a wider range of validity than classical thermodynamics introduced for stationary homogeneous equilibrium states. The formalism obtained from the kinetic theory still coincides with the Gibbs formalism applied to intensive variables [3].

The total mass per unit volume ρ, the total charge per unit volume q, and the total pressure p can be written in the form

$$\rho = \sum_{k \in \mathfrak{S}} \rho_k, \quad q = \sum_{k \in \mathfrak{S}} q_k, \quad p = \sum_{k \in \mathfrak{S}} p_k,$$

where the species partial pressure p_k, $k \in \mathfrak{S}$, is given by $p_k = r_k \rho_k T = n_k R T$, with $r_k = R/m_k$, R the perfect gas constant.

The internal energy e and the enthalpy h per unit mass can be decomposed into

$$\rho e = \sum_{k \in \mathfrak{S}} \rho_k e_k, \quad \rho h = \sum_{k \in \mathfrak{S}} \rho_k h_k = \sum_{k \in \mathfrak{S}} \rho_k (e_k + r_k T),$$

where e_k and h_k are the internal energy and the enthalpy per unit mass of the k^{th} species and T the temperature. Internal energy's expression is

$$e_k(T) = e_k^{\text{st}} + \int_{T^{\text{st}}}^{T} c_{v,k}(\tau) \, \mathrm{d}\tau,$$

where $e_k^{\text{st}} = e_k(T^{\text{st}})$ is the formation energy of the k^{th} species at the positive standard temperature T^{st} and $c_{v,k}$ is the constant-volume specific heat of the k^{th} species. The constant-volume and constant-pressure specific heats verify

$$\rho c_v = \sum_{k \in \mathfrak{S}} \rho_k c_{v,k}, \quad \rho c_p = \sum_{k \in \mathfrak{S}} \rho_k c_{p,k} = \sum_{k \in \mathfrak{S}} \rho_k (c_{v,k} + r_k).$$

The entropy s per unit mass can be expressed in terms of the species entropies s_k, $k \in \mathfrak{S}$, from the relation

$$\rho s = \sum_{k \in \mathfrak{S}} \rho_k s_k,$$

with

$$s_k(T,\rho_k) = s_k^{\rm st} + \int_{T^{\rm st}}^{T} \frac{c_{v,k}(\tau)}{\tau}\, {\rm d}\tau - r_k \log\left(\frac{\rho_k}{m_k \gamma^{\rm st}}\right),$$

where $\gamma^{\rm st} = p^{\rm st}/(RT^{\rm st})$ is the standard concentration, that is, the concentration at the standard state $T^{\rm st}$, $p^{\rm st}$. Similarly, we can express the Gibbs function g per unit mass in terms of the species Gibbs functions g_k, $k \in \mathfrak{S}$, from the relation

$$\rho g = \sum_{k\in\mathfrak{S}} \rho_k g_k,$$

where g_k is given by $g_k = h_k - T s_k$. We finally define the species reduced chemical potential μ_k by $\mu_k = g_k/(RT)$. The species Gibbs functions g_k and the species reduced chemical potential μ_k, $k \in \mathfrak{S}$, are functions of ρ_k and T. We can classically write

$$g_k(\rho_k, T) = g_k^u(T) + r_k T \log n_k, \quad \mu_k(\rho_k, T) = \mu_k^u(T) + \frac{1}{m_k} \log n_k,$$

with g_k^u, $k \in \mathfrak{S}$, the species unitary Gibbs functions and μ_k^u, $k \in \mathfrak{S}$, the species unitary reduced chemical potentials.

2.5 Maxwell's Equations

The electric and magnetic fields satisfy the four Maxwell's equations

$$\boldsymbol{\partial_x}\cdot\boldsymbol{E} = \frac{q}{\varepsilon_0}, \tag{2.12}$$

$$\boldsymbol{\partial_x}\wedge\boldsymbol{E} = -\partial_t \boldsymbol{B}, \tag{2.13}$$

$$\boldsymbol{\partial_x}\cdot\boldsymbol{B} = 0, \tag{2.14}$$

$$\boldsymbol{\partial_x}\wedge\boldsymbol{B} = \mu_0(\boldsymbol{J} + \varepsilon_0 \partial_t \boldsymbol{E}), \tag{2.15}$$

where ε_0 is the dielectric constant and μ_0 the magnetic permeability. It is well known that if the first and the third equations are verified at initial time $t = 0$, the two others insure that they hold at all time.

2.6 Mathematical assumptions

We describe in this subsection the mathematical assumptions concerning thermoelectrochemistry and transport coefficients. These assumptions are obtained from the kinetic theory [2] and are not enough intuitive to be guessed empirically.

The species of the mixture are assumed to be constituted by neutral atoms and electrons. We denote by $\mathfrak{A} = \{1, \dots, n^a\}$ the atoms indexing set, by n^a the number of atoms in the mixture, by $\widetilde{m}_l$, $l \in \mathfrak{A}$, the atom masses and by $\mathfrak{a}_{kl}$ the number of $l^{\rm th}$ atoms in the $k^{\rm th}$ species. We define $\mathfrak{a}_{k0}$ as the number of electrons in the $k^{\rm th}$ species and for notational convenience, $\overline{\mathfrak{A}} = \mathfrak{A} \cup \{0\} = \{0, \dots, n^a\}$. We also introduce the atomic vectors $\mathfrak{a}_l$, $l \in \mathfrak{A}$, defined by $\mathfrak{a}_l = (\mathfrak{a}_{1l}, \dots, \mathfrak{a}_{n^s l})^{\rm T}$,

$l \in \mathfrak{A}$, and the electron vector $\mathfrak{a}_0$, by $\mathfrak{a}_0 = (\mathfrak{a}_{10}, \dots, \mathfrak{a}_{n^s0})^{\mathsf{T}}$. Finally, we define $\varkappa$, as the absolute value of charge per unit mole for electrons.

We now define the reaction vectors by $\nu_r = (\nu_{1r}, \dots, \nu_{n^s r})^{\mathsf{T}}$, $r \in \mathfrak{R}$, where $\nu_{kr} = \nu_{kr}^{\mathrm{b}} - \nu_{kr}^{\mathrm{f}}$, $k \in \mathfrak{S}$, so that $\nu_r = \nu_r^{\mathrm{b}} - \nu_r^{\mathrm{f}}$, and we denote by $\mathcal{R}$ the linear space spanned by ν_r, $r \in \mathfrak{R}$. We also define the molar mass vector $\mathsf{m} = (m_1, \dots, m_{n^s})^{\mathsf{T}}$, the mass vector per unit volume $\varrho = (\rho_1, \dots, \rho_{n^s})^{\mathsf{T}}$, the charge vector per unit mass $\mathsf{z} = (z_1, \dots, z_{n^s})^{\mathsf{T}}$, the charge vector per unit volume $\mathsf{q} = (q_1, \dots, q_{n^s})^{\mathsf{T}}$, and the unit vector $\mathsf{u} = (1, \dots, 1)^{\mathsf{T}}$. We also have the relation $q_k = \rho_k z_k$, $k \in \mathfrak{S}$.

2.6.1 Assumption on thermoelectrochemistry

In all this paper, we assume that the following assumptions ($\mathsf{Th_1}$-$\mathsf{Th_4}$), which are derived from the kinetic theory, hold.

($\mathsf{Th_1}$) *The species molar masses m_k, $k \in \mathfrak{S}$, and the gas constant R are positive constants. The formation energies e_k^{st}, $k \in \mathfrak{S}$, and the formation entropies s_k^{st}, $k \in \mathfrak{S}$, are constants. The specific heats $c_{v,k}$, $k \in \mathfrak{S}$, are $\mathcal{C}^\infty$ functions of $T \geqslant 0$. Furthermore, there exist positive constants $\underline{c}_v$ and $\overline{c}_v$ with $0 < \underline{c}_v \leqslant c_{v,k}(T) \leqslant \overline{c}_v$, for $T \geqslant 0$ and $k \in \mathfrak{S}$.*

($\mathsf{Th_2}$) *The stoichiometric coefficients ν_{kr}^{f} and ν_{kr}^{b}, $k \in \mathfrak{S}$, $r \in \mathfrak{R}$, and the atomic coefficients $\mathfrak{a}_{kl}$, $k \in \mathfrak{S}$, $l \in \mathfrak{A}$, are nonnegative integers. The numbers of electrons $\mathfrak{a}_{k0}$, $k \in \mathfrak{S}$, are integers. The atomic vectors $\mathfrak{a}_l$, $l \in \overline{\mathfrak{A}}$, and the reaction vectors ν_r, $r \in \mathfrak{R}$, satisfy the conservation relations $\langle \nu_r, \mathfrak{a}_l \rangle = 0$, $r \in \mathfrak{R}$, $l \in \overline{\mathfrak{A}}$. This relation expresses atom conservation for $l \in \mathfrak{A}$ and charge conservation for $l = 0$.*

($\mathsf{Th_3}$) *The atom masses $\widetilde{m}_l$, $l \in \mathfrak{A}$, and the electron mass $\widetilde{m}_0$ are positive constants. Moreover, the species molar masses m_k, $k \in \mathfrak{S}$, are given by*

$$m_k = \sum_{l \in \mathfrak{A}} \widetilde{m}_l \mathfrak{a}_{kl} + \widetilde{m}_0 \mathfrak{a}_{k0}, \qquad k \in \mathfrak{S}.$$

These relations can be written in vector form

$$\mathsf{m} = \sum_{l \in \mathfrak{A}} \widetilde{m}_l \mathfrak{a}_l + \widetilde{m}_0 \mathfrak{a}_0.$$

We also have the proportionality relation between the species charge per unit volume q_k, $k \in \mathfrak{S}$, and the number of electrons in the k^{th} species, $q_k = -\varkappa \mathfrak{a}_{k0} n_k$, $k \in \mathfrak{S}$, where $\varkappa$ is a positive constant which represents the absolute value of charge per unit mole for electrons. This proportionality relation is equivalent to $z_k = -\varkappa \mathfrak{a}_{k0}/m_k$, $k \in \mathfrak{S}$.

($\mathsf{Th_4}$) *The rate constants $\mathcal{K}_r^{\mathrm{s}}$, $r \in \mathfrak{R}$, are $\mathcal{C}^\infty$ positive functions of $T > 0$.*

2.6.2 Assumptions on transport coefficients

In this subsection, we introduce a set of assumptions concerning the transport coefficients derived from the kinetic theory [2].

($\mathsf{Tr_1}$) *The flux diffusion coefficients* $D^{\parallel}_{kl}$, $D^{\perp}_{kl}$ *and* $BD^{\odot}_{kl}$, $k,l \in \mathfrak{S}$, *the thermal diffusion ratios* $\chi^{\parallel}_k$, $\chi^{\perp}_k$, $B\chi^{\odot}_k$, $\widehat{\chi}^{\parallel}_k$, $\widehat{\chi}^{\perp}_k$ *and* $B\widehat{\chi}^{\odot}_k$, $k \in \mathfrak{S}$, *the volume viscosity* κ, *the shear viscosities* $\eta_1, B\eta_2, \eta_3, \eta_4, B\eta_5$ *and the thermal conductivities* $\lambda^{\parallel}$, $\lambda^{\perp}$ *and* $B\lambda^{\odot}$ *are* $\mathcal{C}^\infty$ *functions of* $(T, \varrho, \boldsymbol{B})$ *for* $T > 0$, $\varrho > 0$ *and* $\boldsymbol{B} \in \mathbb{R}^3$, *where* B *is the norm of the magnetic field* $\boldsymbol{B}$. *Moreover, the coefficients* $D^{\parallel}_{kl}$, $k,l \in \mathfrak{S}$, *do not depend on the magnetic field* $\boldsymbol{B}$ *and we can write* $D^{\perp}_{kl} - D^{\parallel}_{kl} = B^2\phi^{\perp}_{kl}(B^2)$ *and* $D^{\odot}_{kl} = B\phi^{\odot}_{kl}(B^2)$, *where* $\phi^{\perp}_{kl}$ *and* $\phi^{\odot}_{kl}$, $k,l \in \mathfrak{S}$, *are* $\mathcal{C}^\infty([0,\infty),\mathbb{R})$ *functions. The coefficients* $\chi^{\parallel}_k$, $\widehat{\chi}^{\parallel}_k$, $k \in \mathfrak{S}$, *do not depend on the magnetic field* $\boldsymbol{B}$ *and we can write* $\chi^{\perp}_k - \chi^{\parallel}_k = B^2\psi^{\perp}_k(B^2)$, $\widehat{\chi}^{\perp}_k - \widehat{\chi}^{\parallel}_k = B^2\widehat{\psi}^{\perp}_k(B^2)$, $\chi^{\odot}_k = B\psi^{\odot}_k(B^2)$ *and* $\widehat{\chi}^{\odot}_k = B\widehat{\psi}^{\odot}_k(B^2)$, *where* $\psi^{\perp}_k$, $\widehat{\psi}^{\perp}_k$, $\psi^{\odot}_k$ *and* $\widehat{\psi}^{\odot}_k$ *are* $\mathcal{C}^\infty([0,\infty),\mathbb{R})$ *functions. The coefficient* $\lambda^{\parallel}$ *does not depend on the magnetic field* $\boldsymbol{B}$ *and we can write* $\lambda^{\perp} - \lambda^{\parallel} = B^2\varsigma^{\perp}(B^2)$ *and* $\lambda^{\odot} = B\varsigma^{\odot}(B^2)$, *where* $\varsigma^{\perp}$ *and* $\varsigma^{\odot}$ *are* $\mathcal{C}^\infty([0,\infty),\mathbb{R})$ *functions. Lastly, we have* $\eta_1 = \varphi_1(B^2)$, $\eta_2 = B\varphi_2(B^2)$, $\eta_3 = B^2\varphi_3(B^2)$, $\eta_4 = B^2\varphi_4(B^2)$, $\eta_5 = B^3\varphi_5(B^2)$ *and* $2\eta_4 - \eta_3 = B^4\varphi_6(B^2)$, *where* φ_α, $\alpha \in \{1,\dots,6\}$, *are* $\mathcal{C}^\infty([0,\infty),\mathbb{R})$ *functions.*

($\mathsf{Tr_2}$) *Thermal conductivities* $\lambda^{\parallel}$ *and* $\lambda^{\perp}$ *are positive functions. The volume viscosity* κ *is a nonnegative function and the shear viscosities* η_1, η_2, η_3, η_4, η_5 *verify* $\eta_1 + \eta_4 > 0$, $\eta_1 + \eta_3 > 0$, $\eta_1 - \eta_3 > 0$.

($\mathsf{Tr_3}$) *The matrices* $A^{\parallel}$, $A^{\perp}$ *and* $A^{\odot}$ *defined by*

$$A^{\diamond} = \begin{bmatrix} \frac{T}{p}\widehat{\lambda}^{\diamond} & \widehat{\theta}^{\diamond\mathsf{T}} \\ \theta^{\diamond} & D^{\diamond} \end{bmatrix}, \quad \diamond \in \{\parallel, \perp, \odot\},$$

verify $\langle A^{\parallel}\mathsf{x},\mathsf{x}\rangle \geqslant 0$ *and* $\langle A^{\perp}\mathsf{x},\mathsf{x}\rangle + \langle A^{\perp}\mathsf{y},\mathsf{y}\rangle + \langle A^{\odot}\mathsf{x},\mathsf{y}\rangle - \langle \mathsf{x}, A^{\odot}\mathsf{y}\rangle \geqslant 0$, *for* $\mathsf{x},\mathsf{y} \in \mathbb{R}^{n^s+1}$. *Moreover,* $\langle A^{\parallel}\mathsf{x},\mathsf{x}\rangle = 0$ *if, and only if, the vector* x *is proportional to the vector* $(0,\varrho^{\mathsf{T}})^{\mathsf{T}}$ *and* $\langle A^{\perp}\mathsf{x},\mathsf{x}\rangle + \langle A^{\perp}\mathsf{y},\mathsf{y}\rangle + \langle A^{\odot}\mathsf{x},\mathsf{y}\rangle - \langle \mathsf{x}, A^{\odot}\mathsf{y}\rangle = 0$ *if, and only if, both vectors* x *and* y *are proportional to the vector* $(0,\varrho^{\mathsf{T}})^{\mathsf{T}}$.

2.7 Entropy production

The entropy per unit mass s satisfies the following conservation equation

$$\partial_t(\rho s) + \boldsymbol{\partial_x}\cdot(\rho s \boldsymbol{v}) + \boldsymbol{\partial_x}\cdot\left(\frac{\boldsymbol{Q}}{T} - \sum_{k\in\mathfrak{S}} \frac{g_k}{T}\rho_k \boldsymbol{V}_k\right) = \Upsilon, \tag{2.16}$$

where Υ is the entropy production term given by

$$\Upsilon = -\sum_{k\in\mathfrak{S}} \frac{g_k m_k \omega_k}{T} - \frac{\boldsymbol{\Pi}:\boldsymbol{\partial_x v}}{T} - \Big(\boldsymbol{Q} - \sum_{k\in\mathfrak{S}} \rho_k h_k \boldsymbol{V}_k\Big)\cdot\frac{\boldsymbol{\partial_x} T}{T^2} - \sum_{k\in\mathfrak{S}} \frac{p}{T}\boldsymbol{V}_k\cdot\boldsymbol{d}_k. \tag{2.17}$$

The entropy production term Υ can be split into a sum of nonnegative terms and this property is important from several points of view. From a thermodynamical point of view, it shows that the macroscopic model satisfies the second principle, inherited from the kinetic model. From a mathematical point of view, entropy also plays a central role in establishing well posedness of the resulting system of partial differential equations. In order to establish that the entropy production term splits into a sum of nonnegative terms, we use the various assumptions on transport coefficients that have been given in the previous section.

Using Eqs (2.10), (2.11) the entropy production due to chemistry is easily rewritten in the form

$$-\sum_{k\in\mathfrak{S}} \frac{g_k m_k \omega_k}{T} = \sum_{r\in\mathfrak{R}} R\mathcal{K}_r^{\mathrm{s}}\left(\exp\langle\nu_r^{\mathrm{f}},\mathsf{M}\mu\rangle - \exp\langle\nu_r^{\mathrm{b}},\mathsf{M}\mu\rangle\right)\log\left[\frac{\exp\langle\nu_r^{\mathrm{f}},\mathsf{M}\mu\rangle}{\exp\langle\nu_r^{\mathrm{b}},\mathsf{M}\mu\rangle}\right]$$

Assumptions (Th_1) and (Th_4) on the positivity of constants R, $\mathcal{K}_r^{\mathrm{s}}$, $r\in\mathfrak{R}$, yield that the entropy production due to chemistry is nonnegative.

Furthermore, the entropy production due to viscous effects reads

$$\begin{aligned}-\tfrac{1}{T}\boldsymbol{\Pi}:\boldsymbol{\partial_x v} =& \tfrac{\kappa}{T}(\boldsymbol{\partial_x}\cdot\boldsymbol{v})^2 + 2(\eta_1+\eta_4)\,\mathrm{Tr}(p^{\|}\mathbf{S}p^{\perp}p^{\perp}\mathbf{S}p^{\|})\\ &+ (\eta_1+\eta_3)[(\mathrm{Tr}(p^{\|}\mathbf{S}p^{\|}))^2 + \tfrac{1}{2}(\mathrm{Tr}(p^{\perp}\mathbf{S}p^{\perp}))^2]\\ &+ (\eta_1-\eta_3)[\mathrm{Tr}(p^{\perp}\mathbf{S}p^{\perp}p^{\perp}\mathbf{S}p^{\perp}) - \tfrac{1}{2}(\mathrm{Tr}(p^{\perp}\mathbf{S}p^{\perp}))^2],\end{aligned}$$

where $\mathrm{Tr}(A)$ denotes the trace of a tensor A, $p^{\|}$ the orthogonal projection with range spanned by the vector $\boldsymbol{B}$ and $p^{\perp} = \mathbb{I} - p^{\|}$ the orthogonal projection with kernel spanned by the vector $\boldsymbol{B}$. According to assumption (Tr_2) on the volume viscosity κ and the shear viscosities η_1, η_2, η_3, η_4, η_5, we have to investigate the sign of the following quantities

$$\mathrm{Tr}(p^{\|}\mathbf{S}p^{\perp}p^{\perp}\mathbf{S}p^{\|}),\quad (\mathrm{Tr}(p^{\|}\mathbf{S}p^{\|}))^2 + \tfrac{1}{2}(\mathrm{Tr}(p^{\perp}\mathbf{S}p^{\perp}))^2,$$
$$\mathrm{Tr}(p^{\perp}\mathbf{S}p^{\perp}p^{\perp}\mathbf{S}p^{\perp}) - \tfrac{1}{2}(\mathrm{Tr}(p^{\perp}\mathbf{S}p^{\perp}))^2,$$

to prove the nonnegativity of the entropy production due to viscous effects. As these expressions are invariant under a change of coordinates, we choose an orthonormal basis $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ such as $\mathbf{e}_1$ is proportional to the vector $\boldsymbol{B}$. In this situation, we then obtain

$$\begin{aligned}&\mathrm{Tr}(p^{\|}\mathbf{S}p^{\perp}p^{\perp}\mathbf{S}p^{\|}) = \mathsf{S}_{12}^2 + \mathsf{S}_{13}^2,\\ &(\mathrm{Tr}(p^{\|}\mathbf{S}p^{\|}))^2 + \tfrac{1}{2}(\mathrm{Tr}(p^{\perp}\mathbf{S}p^{\perp}))^2 = \mathsf{S}_{11}^2 + \tfrac{1}{2}(\mathsf{S}_{22}+\mathsf{S}_{33})^2,\end{aligned}$$

and

$$\mathrm{Tr}(p^{\perp}\mathbf{S}p^{\perp}p^{\perp}\mathbf{S}p^{\perp}) - \tfrac{1}{2}(\mathrm{Tr}(p^{\perp}\mathbf{S}p^{\perp}))^2 = 2\mathsf{S}_{23}^2 + \tfrac{1}{2}(\mathsf{S}_{22}-\mathsf{S}_{33})^2,$$

so that $-\frac{1}{T}\boldsymbol{\Pi}{:}\boldsymbol{v}$ is nonnegative and is zero if and only if S is zero.

We finally show that the entropy production term involving the heat flux and the diffusion velocities

$$\Upsilon_v = -\Big(\boldsymbol{Q} - \sum_{k\in\mathfrak{S}} \rho_k h_k \boldsymbol{V}_k\Big)\cdot\frac{\boldsymbol{\partial_x} T}{T^2} - \sum_{k\in\mathfrak{S}} \frac{p}{T}\boldsymbol{V}_k\cdot\boldsymbol{d}_k,$$

is nonnegative. By using expressions (2.5) and (2.8) of $\boldsymbol{V}_k$, $k\in\mathfrak{S}$, and $\boldsymbol{Q}$, we can write

$$\Upsilon_v = \frac{p}{T}\sum_{i\in\mathfrak{S}}\Big[\langle A^{\parallel}\mathsf{x}_i^{\parallel},\mathsf{x}_i^{\parallel}\rangle + \tfrac{1}{2}\langle A^{\perp}\mathsf{x}_i^{\perp},\mathsf{x}_i^{\perp}\rangle + \tfrac{1}{2}\langle A^{\perp}\mathsf{x}_i^{\odot},\mathsf{x}_i^{\odot}\rangle$$
$$+ \tfrac{1}{2}\langle A^{\odot}\mathsf{x}_i^{\perp},\mathsf{x}_i^{\odot}\rangle - \tfrac{1}{2}\langle \mathsf{x}_i^{\perp}, A^{\odot}\mathsf{x}_i^{\odot}\rangle\Big],$$

with $\mathsf{x}_i^{\diamond} = \big(\frac{1}{T}((\boldsymbol{\partial_x} T)^{\diamond})_i, (\boldsymbol{d}_1^{\diamond})_i, \dots, (\boldsymbol{d}_{n^s}^{\diamond})_i\big)^{\mathsf{T}}$, for $\diamond\in\{\parallel,\perp,\odot\}$ and $i\in\mathfrak{C}$. By using assumption (Tr_3), we immediately obtain that the last term involving the heat flux and the diffusion velocities is nonnegative.

2.8 Alternative formulations

The resulting balance equations describing the fluid (2.1)–(2.3) and Maxwell's equations (2.13) (2.15) are not in a conservative form. The source terms contain in particular the electric current density $\boldsymbol{J}$. We now discuss three different formulations of these equations by recombination with Maxwell's equations.

We define the electromagnetic energy per unit mass e^{e} by $\rho e^{\mathrm{e}} = (\varepsilon_0 E^2 + B^2/\mu_0)/2$, the Poynting vector $\boldsymbol{P}$ by $\boldsymbol{P} = (\boldsymbol{E}\wedge\boldsymbol{B})/\mu_0$ and the electromagnetic force tensor $\boldsymbol{T}$ by $\boldsymbol{T} = \varepsilon_0\boldsymbol{E}\otimes\boldsymbol{E} + \boldsymbol{B}\otimes\boldsymbol{B}/\mu_0 - (\varepsilon_0\boldsymbol{E}\cdot\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{B}/\mu_0)/2\,\mathbb{I}$. By using Maxwell's equations, we obtain the electromagnetic momentum conservation equation

$$\partial_t(\varepsilon_0\mu_0\boldsymbol{P}) - \boldsymbol{\partial_x}\cdot\boldsymbol{T} = -q\boldsymbol{E} - \boldsymbol{J}\wedge\boldsymbol{B}, \tag{2.18}$$

and the electromagnetic energy conservation equation

$$\partial_t(\rho e^{\mathrm{e}}) + \boldsymbol{\partial_x}\cdot\boldsymbol{P} = -\boldsymbol{J}\cdot\boldsymbol{E}. \tag{2.19}$$

Combining equation (2.18) with the momentum conservation equation (2.2) yields the total momentum conservation equation

$$\partial_t(\rho\boldsymbol{v} + \varepsilon_0\mu_0\boldsymbol{P}) + \boldsymbol{\partial_x}\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v} + p\mathbb{I} - \boldsymbol{T}) = \rho\boldsymbol{g}, \tag{2.20}$$

and combining equation (2.19) with the kinetic and internal energy conservation equation (2.3) yields the total energy conservation equation

$$\partial_t(\rho e^{\mathrm{t}}) + \boldsymbol{\partial_x}\cdot[(\rho e^{\mathrm{t}} + p)\boldsymbol{v}] + \boldsymbol{\partial_x}\cdot(\boldsymbol{Q} + \boldsymbol{\Pi}\cdot\boldsymbol{v}) = \rho\boldsymbol{g}\cdot\boldsymbol{v}, \tag{2.21}$$

where the total energy per unit mass e^{t} is defined by

$$\rho e^{\mathrm{t}} = \rho e^{\mathrm{f}} + \rho e^{\mathrm{e}} = \rho e + \tfrac{1}{2}\rho\boldsymbol{v}\cdot\boldsymbol{v} + \tfrac{1}{2}(\varepsilon_0\boldsymbol{E}\cdot\boldsymbol{E} + \tfrac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}).$$

Various systems of balance equations coupled with Maxwell's equations can then be considered. These systems are formally equivalent but their mathematical structures are different. The first one corresponds to the original fluid conservation equations

$$\begin{cases} \partial_t \rho_k + \boldsymbol{\partial_x}\cdot(\rho_k \boldsymbol{v}) + \boldsymbol{\partial_x}\cdot(\boldsymbol{\mathcal{F}}_k) = m_k \omega_k, \quad k \in \mathfrak{S}, \\ \partial_t(\rho \boldsymbol{v}) + \boldsymbol{\partial_x}\cdot(\rho \boldsymbol{v}\otimes\boldsymbol{v} + p\mathbb{I}) + \boldsymbol{\partial_x}\cdot\boldsymbol{\Pi} = \rho \boldsymbol{g} + q\boldsymbol{E} + \boldsymbol{J}\wedge\boldsymbol{B}, \\ \partial_t(\rho e^{\mathrm{f}}) + \boldsymbol{\partial_x}\cdot\left[\left(\rho e^{\mathrm{f}} + p\right)\boldsymbol{v}\right] + \boldsymbol{\partial_x}\cdot(\boldsymbol{Q} + \boldsymbol{\Pi}\cdot\boldsymbol{v}) = \rho \boldsymbol{g}\cdot\boldsymbol{v} + \boldsymbol{J}\cdot\boldsymbol{E}. \end{cases} \tag{2.22}$$

A second possibility is the conservative formulation

$$\begin{cases} \partial_t \rho_k + \boldsymbol{\partial_x}\cdot(\rho_k \boldsymbol{v}) + \boldsymbol{\partial_x}\cdot(\boldsymbol{\mathcal{F}}_k) = m_k \omega_k, \quad k \in \mathfrak{S}, \\ \partial_t(\rho_k \boldsymbol{v} + \varepsilon_0\mu_0 \boldsymbol{P}) + \boldsymbol{\partial_x}\cdot(\rho \boldsymbol{v}\otimes\boldsymbol{v} + p\mathbb{I} - \boldsymbol{T}) = \rho \boldsymbol{g}, \\ \partial_t(\rho e^{\mathrm{t}}) + \boldsymbol{\partial_x}\cdot[(\rho e^{\mathrm{t}} + p)\boldsymbol{v}] + \boldsymbol{\partial_x}\cdot(\boldsymbol{Q} + \boldsymbol{\Pi}\cdot\boldsymbol{v}) = \rho \boldsymbol{g}\cdot\boldsymbol{v}. \end{cases} \tag{2.23}$$

A last possibility is of intermediate form

$$\begin{cases} \partial_t \rho_k + \boldsymbol{\partial_x}\cdot(\rho_k \boldsymbol{v}) + \boldsymbol{\partial_x}\cdot(\boldsymbol{\mathcal{F}}_k) = m_k \omega_k, \quad k \in \mathfrak{S}, \\ \partial_t(\rho \boldsymbol{v}) + \boldsymbol{\partial_x}\cdot(\rho \boldsymbol{v}\otimes\boldsymbol{v} + p\mathbb{I}) + \boldsymbol{\partial_x}\cdot\boldsymbol{\Pi} = \rho \boldsymbol{g} + q\boldsymbol{E} + \boldsymbol{J}\wedge\boldsymbol{B}, \\ \partial_t(\rho e^{\mathrm{t}}) + \boldsymbol{\partial_x}\cdot[(\rho e^{\mathrm{t}} + p)\boldsymbol{v}] + \boldsymbol{\partial_x}\cdot(\boldsymbol{Q} + \boldsymbol{\Pi}\cdot\boldsymbol{v}) = \rho \boldsymbol{g}\cdot\boldsymbol{v}. \end{cases} \tag{2.24}$$

The first system (2.22) is not satisfactory since the Hessian of $-\rho s$ with respect to the corresponding conservative variable $(\rho_1, \dots, \rho_{n^s}, \rho \boldsymbol{v}^{\mathsf{T}}, \boldsymbol{E}^{\mathsf{T}}, \boldsymbol{B}^{\mathsf{T}}, \rho e^{\mathrm{f}})^{\mathsf{T}}$ is not positive definite. Applying the existence results of section 4 to this formulation would nevertheless be possible by considering the modified mathematical entropy $-\rho s + \varepsilon_0 \boldsymbol{E}\cdot\boldsymbol{E} + \frac{1}{\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}$.

The formulation (2.23) is at first appealing, since in the absence of external force $\boldsymbol{g} = 0$, the equations are fully conservative. However, this is of no use since the resulting full coupled system still contain source terms in the Maxwell-Ampere equation (2.15). These source terms even involve the solution gradients through the conduction current density $\boldsymbol{j}$. In addition, the corresponding calculations of symmetrized forms are very intricate and complex, but the mathematical structure of the resulting system is identical to that of the third formulation (2.24). Moreover, the corresponding partial normal forms coincide with those of the third formulation (2.24) and lead to the same existence results.

The presented formulation is therefore the third one corresponding to the intermediate form (2.24). The Hessian of the natural entropy $-\rho s$ with respect to the corresponding conservative variable $(\rho_1, \dots, \rho_{n^s}, \rho \boldsymbol{v}^{\mathsf{T}}, \boldsymbol{E}^{\mathsf{T}}, \boldsymbol{B}^{\mathsf{T}}, \rho e^{\mathrm{t}})^{\mathsf{T}}$ is positive definite and the calculations of symmetrized forms are not so complicated as for the second formulation because the equations are less coupled. And by abuse of language, we describe the variables ρ_k, $\rho \boldsymbol{v}$, ρe^{t}, $\boldsymbol{E}$ and $\boldsymbol{B}$ as the conservative variables even if the system written in term of these variables is not into a full conservative form.

3 Quasilinear Form

In this section, we rewrite the system of equations governing reactive ionized magnetized dissipative plasmas as a quasilinear system of second-order partial differential equations in terms of the conservative variable U.

3.1 Vector Notations

In this section, we introduce a compact notation that will be used throughout this paper. We define the conservative variable U by

$$\mathsf{U} = \Big(\varrho^{\mathsf{T}}, \rho\boldsymbol{v}^{\mathsf{T}}, \boldsymbol{E}^{\mathsf{T}}, \boldsymbol{B}^{\mathsf{T}}, \rho e^{\mathrm{t}}\Big)^{\mathsf{T}}, \tag{3.1}$$

and the natural variable Z by

$$\mathsf{Z} = \Big(\varrho^{\mathsf{T}}, \boldsymbol{v}^{\mathsf{T}}, \boldsymbol{E}^{\mathsf{T}}, \boldsymbol{B}^{\mathsf{T}}, T\Big)^{\mathsf{T}}. \tag{3.2}$$

The components of U naturally appear in the system of partial differential equations governing ionized magnetized gas mixtures. On the other hand, the components of the natural variable Z are more practical to use in actual calculations of differential identities.

The conservation equations can be written in the compact form

$$\partial_t \mathsf{U} + \sum_{i\in\mathfrak{C}} \partial_i \mathsf{F}_i + \sum_{i\in\mathfrak{C}} \partial_i \mathsf{F}_i^{\mathrm{diss}} = \Omega^{\boldsymbol{j}}, \tag{3.3}$$

where $\mathfrak{C}$ denotes the set $\{1,2,3\}$, F_i, $i\in\mathfrak{C}$, the convective flux in the i^{th} direction, $\mathsf{F}_i^{\mathrm{diss}}$, $i\in\mathfrak{C}$, the dissipative flux in the i^{th} direction and $\Omega^{\boldsymbol{j}}$ the source term. The exponent $\boldsymbol{j}$ is used here to indicate that the source term depends on the macroscopic gradients throw the current density $\boldsymbol{j}$, because of the Maxwell-Ampère equation (2.15). From the conservation equations (2.1)–(2.3) and Maxwell's equations (2.13) (2.15) it is easily obtained that the source term $\Omega^{\boldsymbol{j}}$ is given by

$$\Omega^{\boldsymbol{j}} = \Big(m_1\omega_1, \dots, m_{n^s}\omega_{n^s}, (\rho\boldsymbol{g} + q\boldsymbol{E} + \boldsymbol{J}\wedge\boldsymbol{B})^{\mathsf{T}}, -\tfrac{1}{\varepsilon_0}\boldsymbol{J}^{\mathsf{T}}, 0_{1,3}, \rho\boldsymbol{g}\cdot\boldsymbol{v}\Big)^{\mathsf{T}}, \tag{3.4}$$

the convective flux F_i by

$$\mathsf{F}_i = \Big(\varrho^{\mathsf{T}} v_i, \rho\boldsymbol{v}^{\mathsf{T}} v_i + p\boldsymbol{e}_i{}^{\mathsf{T}}, -\tfrac{1}{\varepsilon_0\mu_0}(\boldsymbol{e}_i\wedge\boldsymbol{B})^{\mathsf{T}}, (\boldsymbol{e}_i\wedge\boldsymbol{E})^{\mathsf{T}}, (\rho e^{\mathrm{f}} + p)v_i + P_i\Big)^{\mathsf{T}}, \tag{3.5}$$

and the dissipative flux $\mathsf{F}_i^{\mathrm{diss}}$ by

$$\mathsf{F}_i^{\mathrm{diss}} = \mathsf{F}_i^{\mathrm{diff}} + \mathsf{F}_i^{\mathrm{visc}}, \tag{3.6}$$

where $\mathsf{F}_i^{\mathrm{visc}}$, the viscous flux, and $\mathsf{F}_i^{\mathrm{diff}}$, the diffusion flux, are defined by

$$\mathsf{F}_i^{\mathrm{visc}} = (0_{1,n^s}, \boldsymbol{\Pi}_i, 0_{1,3}, 0_{1,3}, \boldsymbol{\Pi}_i\cdot\boldsymbol{v})^{\mathsf{T}} \tag{3.7}$$

and

$$\mathsf{F}_i^{\mathrm{diff}} = \Big(\mathcal{F}_{1i}, \dots, \mathcal{F}_{n^s i}, 0_{1,3}, 0_{1,3}, 0_{1,3}, Q_i\Big)^{\mathsf{T}}. \tag{3.8}$$

For notational convenience, we have denoted by $\boldsymbol{\Pi}_i$ the i^{th} rows extracted from the stress tensor $\boldsymbol{\Pi}$ and by $\boldsymbol{e}_i$, $i\in\mathfrak{C}$, the canonical basis vectors of $\mathbb{R}^3$.

3.2 The map $\mathsf{Z} \mapsto \mathsf{U}$

In order to express the natural variable Z in terms of the conservative variable U, we investigate the map $\mathsf{Z} \mapsto \mathsf{U}$ and its range. We introduce the two open sets $\mathcal{O}_{\mathsf{Z}}$ and $\mathcal{O}_{\mathsf{U}}$ defined by

$$\mathcal{O}_{\mathsf{Z}} = (0,\infty)^{n^s} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times (0,\infty)$$

and

$$\mathcal{O}_{\mathsf{U}} = \{(u_i) \in \mathbb{R}^{n^s+10} : u_1,\ldots,u_{n^s} > 0, u_{n^s+10} > f(u_i)\},$$

where f is the map from $(0,\infty)^{n^s} \times \mathbb{R}^9$ onto $\mathbb{R}$ which reads

$$f(u_i) = \frac{1}{2}\left(\frac{\sum\limits_{i=n^s+1}^{n^s+3} u_i^2}{\sum\limits_{i=1}^{n^s} u_i} + \varepsilon_0 \sum_{i=n^s+4}^{n^s+6} u_i^2 + \frac{1}{\mu_0} \sum_{i=n^s+7}^{n^s+9} u_i^2 \right) + \sum_{i=1}^{n^s} u_i e_i^0.$$

Proposition 3.1 *The map* $\mathsf{Z} \mapsto \mathsf{U}$ *is a* $\mathcal{C}^\infty$ *diffeomorphism from the open set* $\mathcal{O}_{\mathsf{Z}}$ *onto the open set* $\mathcal{O}_{\mathsf{U}}$*. Furthermore,* $\mathcal{O}_{\mathsf{U}}$ *is a convex open set.*

Proof From assumption (Th_1) on the coefficients $c_{v,k}$, $k \in \mathfrak{S}$, we first deduce that the map $\mathsf{Z} \mapsto \mathsf{U}$ is $\mathcal{C}^\infty$ over the domain $\mathcal{O}_{\mathsf{Z}}$. Furthermore, it is straightforward to show that the map is one to one from $\mathcal{O}_{\mathsf{Z}}$ onto $\mathcal{O}_{\mathsf{U}}$ by using the positivity of the coefficients $c_{v,k}$, $k \in \mathfrak{S}$. The matrix $\partial_{\mathsf{Z}}\mathsf{U}$, given in a bloc structure by

$$\partial_Z \mathsf{U} = \begin{bmatrix} \mathbb{I}_{n^s,n^s} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,1} \\ \boldsymbol{v} \otimes \mathsf{u} & \rho\mathbb{I} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & \mathbb{I} & 0_{3,3} & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & \mathbb{I} & 0_{3,1} \\ \mathsf{e}^{\mathrm{f}\mathsf{T}} & \rho\boldsymbol{v}^{\mathsf{T}} & \varepsilon_0 \boldsymbol{E}^{\mathsf{T}} & \frac{1}{\mu_0}\boldsymbol{B}^{\mathsf{T}} & \rho c_v \end{bmatrix},$$

where we have introduced the vector e^{f} defined by $\mathsf{e}^{\mathrm{f}} = \big(e_1^{\mathrm{f}},\ldots,e_{n^s}^{\mathrm{f}}\big)^{\mathsf{T}}$, is readily nonsingular over $\mathcal{O}_{\mathsf{Z}}$, thanks to its triangular structure. From the inverse function theorem, we deduce that the map $\mathsf{Z} \mapsto \mathsf{U}$ is a $\mathcal{C}^\infty$ diffeomorphism onto $\mathcal{O}_{\mathsf{U}}$. The convexity of $\mathcal{O}_{\mathsf{U}}$ is finally a direct consequence of the convexity of f, which is established by evaluating its second derivative. □

3.3 Quasilinear form

Proposition 3.2 *The convective fluxes* $\mathsf{F}_i(\mathsf{U})$*,* $i \in \mathfrak{C}$*, are* $\mathcal{C}^\infty$ *functions of* $\mathsf{U} \in \mathcal{O}_{\mathsf{U}}$*, the dissipative fluxes* $\mathsf{F}_i^{\mathrm{diss}}(\mathsf{U}, \boldsymbol{\partial_x}\mathsf{U})$*,* $i \in \mathfrak{C}$*, are* $\mathcal{C}^\infty$ *functions of* $\mathsf{U} \in \mathcal{O}_{\mathsf{U}}$ *and* $\boldsymbol{\partial_x}\mathsf{U} \in \mathbb{R}^{3(n^s+10)}$*,* $j \in \mathfrak{C}$*, and the source term* $\Omega^{\boldsymbol{j}}(\mathsf{U}, \boldsymbol{j})$ *is a* $\mathcal{C}^\infty$ *function of* $\mathsf{U} \in \mathcal{O}_{\mathsf{U}}$ *and* $\boldsymbol{j} \in \mathbb{R}^3$*. Moreover, the system of partial differential equations* (3.3) *can be rewritten in the form*

$$\partial_t \mathsf{U} + \sum_{i \in \mathfrak{C}} \left[\mathsf{A}_i(\mathsf{U}) + \mathsf{A}_i^{\mathrm{e}}(\mathsf{U})\right] \partial_i \mathsf{U} = \sum_{i,j \in \mathfrak{C}} \partial_i \left[\mathsf{B}_{ij}(\mathsf{U})\, \partial_j \mathsf{U}\right] + \Omega(\mathsf{U}), \tag{3.9}$$

where matrices $\mathsf{A}_i(\mathsf{U})$, $\mathsf{A}_i^{\mathrm{e}}(\mathsf{U})$, $i \in \mathfrak{C}$, $\mathsf{B}_{ij}(\mathsf{U})$, $i,j \in \mathfrak{C}$, *and vector* $\Omega(\mathsf{U})$ *are* $\mathcal{C}^\infty$ *functions of* $\mathsf{U} \in \mathcal{O}_{\mathsf{U}}$. *Moreover we have*

$$\mathsf{F}_i^{\mathrm{diss}}(\mathsf{U}, \boldsymbol{\partial_x}\mathsf{U}) = -\sum_{j\in\mathfrak{C}} \big(\mathsf{B}_{ij}(\mathsf{U})\partial_j \mathsf{U} + \mathsf{F}_{ij}^{\mathrm{e}}(\mathsf{U})\big), \quad i \in \mathfrak{C},$$

$$\Omega^{\boldsymbol{j}}(\mathsf{U}, \boldsymbol{j}) = \Omega(\mathsf{U}) + \sum_{i\in\mathfrak{C}} \mathsf{A}_i^{\mathrm{e},\Omega}(\mathsf{U})\partial_i \mathsf{U},$$

$$\mathsf{A}_i(\mathsf{U}) = \partial_{\mathsf{U}}\mathsf{F}_i(\mathsf{U}), \qquad \mathsf{A}_i^{\mathrm{e}}(\mathsf{U}) = -\sum_{j\in\mathfrak{C}} \partial_{\mathsf{U}}\mathsf{F}_{ij}^{\mathrm{e}}(\mathsf{U}) - \mathsf{A}_i^{\mathrm{e},\Omega}(\mathsf{U}), \quad i \in \mathfrak{C},$$

where the matrices $\mathsf{A}_i^{\mathrm{e},\Omega}(\mathsf{U})$, $i \in \mathfrak{C}$, *are* $\mathcal{C}^\infty$ *functions of* $\mathsf{U} \in \mathcal{O}_{\mathsf{U}}$.

We observe a fundamental difference between neutral and ionized mixtures. For neutral mixtures, the dissipative terms $\mathsf{F}_i^{\mathrm{diss}}$, $i \in \mathfrak{C}$, are linear combinations of gradients and the terms $\mathsf{F}_{ij}^{\mathrm{e}}$, $i,j \in \mathfrak{C}$ vanish [3]. For ionized mixtures, the dissipative terms $\mathsf{F}_i^{\mathrm{diss}}$, $i \in \mathfrak{C}$, still involve linear combinations of gradients but in addition, also contain the zeroth-order terms $\mathsf{F}_{ij}^{\mathrm{e}}$, $i,j \in \mathfrak{C}$, arising from the action of macroscopic electromagnetic forces over the ionized species. Moreover the source term $\Omega^{\boldsymbol{j}}$ depends not only on U but also on the gradient $\boldsymbol{\partial_x}\mathsf{U}$ through the conduction current $\boldsymbol{j}$ appearing in Maxwell's equations.

Proof We first remark that it is easier to differentiate with respect to Z than to U. By using expression (3.5) of vectors F_i, $i \in \mathfrak{S}$, and assumption $(\mathsf{Th_1})$ on the regularity of specific heats $c_{v,k}$, $k \in \mathfrak{S}$, we immediately obtain that vectors $\mathsf{F}_i(\mathsf{Z})$, $i \in \mathfrak{S}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z} \in \mathcal{O}_{\mathsf{Z}}$.

We write $\mathsf{F}_i^{\mathrm{diss}} = \mathsf{F}_i^{\mathrm{diff}} + \mathsf{F}_i^{\mathrm{visc}}$, $i \in \mathfrak{C}$, and we treat the two terms separately, the first one $\mathsf{F}_i^{\mathrm{diss}}$ corresponding to the dissipative term and the second one $\mathsf{F}_i^{\mathrm{visc}}$ to the viscous term. The transport fluxes $\boldsymbol{\Pi}$, $\boldsymbol{\mathcal{F}}_k$, $k \in \mathfrak{S}$ and $\boldsymbol{Q}$ naturally split into a sum of gradients of the natural variable Z and a sum of zeroth order terms. In particular the term $\mathsf{F}_i^{\mathrm{diff}}$ can be written

$$\mathsf{F}_i^{\mathrm{diff}} = -\sum_{j\in\mathfrak{C}} \left(\widehat{\mathsf{B}}_{ij}^{\mathrm{diff}}\partial_j \mathsf{Z} + \mathsf{F}_{ij}^{\mathrm{e}}\right), \tag{3.10}$$

where the matrices $\widehat{\mathsf{B}}_{ij}^{\mathrm{diff}}$ and the vectors $\mathsf{F}_{ij}^{\mathrm{e}}$, $i,j \in \mathfrak{C}$, are defined by

$$\widehat{\mathsf{B}}_{ij}^{\mathrm{diff}} = \mathcal{B}_i\mathcal{B}_j\widehat{\mathsf{B}}^{\parallel} + (\delta_{ij} - \mathcal{B}_i\mathcal{B}_j)\widehat{\mathsf{B}}^{\perp} + \mathsf{T}_{ij}(\boldsymbol{\mathcal{B}})\widehat{\mathsf{B}}^{\odot}, \tag{3.11}$$

$$\mathsf{F}_{ij}^{\mathrm{e}} = -\left[\mathcal{B}_i\mathcal{B}_j\mathsf{F}^{\mathrm{e}\parallel} + (\delta_{ij} - \mathcal{B}_i\mathcal{B}_j)\mathsf{F}^{\mathrm{e}\perp} + \mathsf{T}_{ij}(\boldsymbol{\mathcal{B}})\mathsf{F}^{\mathrm{e}\odot}\right](\boldsymbol{E} + \boldsymbol{v}\wedge\boldsymbol{B})_j. \tag{3.12}$$

The matrices $\widehat{\mathsf{B}}^{\parallel}$, $\widehat{\mathsf{B}}^{\perp}$ and $\widehat{\mathsf{B}}^{\odot}$ have the structure

$$\widehat{\mathsf{B}}^{\diamond} = \frac{T}{p}\begin{bmatrix} \widehat{\mathsf{B}}_{\varrho,\varrho}^{\diamond} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,3} & \widehat{\mathsf{B}}_{\varrho,\mathsf{e}}^{\diamond} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ \widehat{\mathsf{B}}_{\mathsf{e},\varrho}^{\diamond} & 0_{1,3} & 0_{1,3} & 0_{1,3} & \widehat{\mathsf{B}}_{\mathsf{e},\mathsf{e}}^{\diamond} \end{bmatrix}, \qquad \diamond \in \{\parallel, \perp, \odot\}. \tag{3.13}$$

The coefficients corresponding to $\diamond = \|$ read

$$\begin{aligned}
\widehat{\mathsf{B}}^{\|}_{\varrho,\varrho} &= D^{\|}_{\varrho}\mathsf{d}^{\mathsf{r}}_{\varrho},\\
\widehat{\mathsf{B}}^{\|}_{\varrho,\mathsf{e}} &= D^{\|}_{\varrho}\left(\tfrac{1}{T^2}\varkappa^{\|}+\tfrac{1}{T}\mathsf{r}\right),\\
\widehat{\mathsf{B}}^{\|}_{\mathsf{e},\varrho} &= (\widehat{\varkappa}^{\|}+\mathsf{h})^{\mathsf{T}}D^{\|}_{\varrho}\mathsf{d}^{\mathsf{r}}_{\varrho},\\
\widehat{\mathsf{B}}^{\|}_{\mathsf{e},\mathsf{e}} &= \frac{p}{T}\lambda^{\|} + (\widehat{\varkappa}^{\|}+\mathsf{h})^{\mathsf{T}}D^{\|}_{\varrho}\left(\tfrac{1}{T^2}\varkappa^{\|}+\tfrac{1}{T}\mathsf{r}\right),
\end{aligned}$$

those corresponding to $\diamond = \perp$

$$\begin{aligned}
\widehat{\mathsf{B}}^{\perp}_{\varrho,\varrho} &= D^{\perp}_{\varrho}\mathsf{d}^{\mathsf{r}}_{\varrho},\\
\widehat{\mathsf{B}}^{\perp}_{\varrho,\mathsf{e}} &= D^{\perp}_{\varrho}\left(\tfrac{1}{T^2}\varkappa^{\perp}+\tfrac{1}{T}\mathsf{r}\right) - \tfrac{1}{T^2}D^{\odot}_{\varrho}\varkappa^{\odot},\\
\widehat{\mathsf{B}}^{\perp}_{\mathsf{e},\varrho} &= (\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}}D^{\perp}_{\varrho}\mathsf{d}^{\mathsf{r}}_{\varrho} - \widehat{\varkappa}^{\odot\mathsf{T}}D^{\odot}_{\varrho}\mathsf{d}^{\mathsf{r}}_{\varrho},\\
\widehat{\mathsf{B}}^{\perp}_{\mathsf{e},\mathsf{e}} &= \frac{p}{T}\lambda^{\perp} + (\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}}D^{\perp}_{\varrho}\left(\tfrac{1}{T^2}\varkappa^{\perp}+\tfrac{1}{T}\mathsf{r}\right)\\
&\quad - \tfrac{1}{T^2}\left(\widehat{\varkappa}^{\odot\mathsf{T}}D^{\perp}_{\varrho}\varkappa^{\odot}+\widehat{\varkappa}^{\odot\mathsf{T}}D^{\odot}_{\varrho}(\varkappa^{\perp}+T\mathsf{r})+(\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}}D^{\odot}_{\varrho}\varkappa^{\odot}\right),
\end{aligned}$$

and those corresponding to $\diamond = \odot$

$$\begin{aligned}
\widehat{\mathsf{B}}^{\odot}_{\varrho,\varrho} &= D^{\odot}_{\varrho}\mathsf{d}^{\mathsf{r}}_{\varrho},\\
\widehat{\mathsf{B}}^{\odot}_{\varrho,\mathsf{e}} &= D^{\odot}_{\varrho}\left(\tfrac{1}{T^2}\varkappa^{\perp}+\tfrac{1}{T}\mathsf{r}\right) + \tfrac{1}{T^2}D^{\perp}_{\varrho}\varkappa^{\odot},\\
\widehat{\mathsf{B}}^{\odot}_{\mathsf{e},\varrho} &= (\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}}D^{\odot}_{\varrho}\mathsf{d}^{\mathsf{r}}_{\varrho} + \widehat{\varkappa}^{\odot\mathsf{T}}D^{\perp}_{\varrho}\mathsf{d}^{\mathsf{r}}_{\varrho},\\
\widehat{\mathsf{B}}^{\odot}_{\mathsf{e},\mathsf{e}} &= \frac{p}{T}\lambda^{\odot} + (\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}}D^{\odot}_{\varrho}\left(\tfrac{1}{T^2}\varkappa^{\perp}+\tfrac{1}{T}\mathsf{r}\right)\\
&\quad + \tfrac{1}{T^2}\left(-\widehat{\varkappa}^{\odot\mathsf{T}}D^{\odot}_{\varrho}\varkappa^{\odot}+\widehat{\varkappa}^{\odot\mathsf{T}}D^{\perp}_{\varrho}(\varkappa^{\perp}+T\mathsf{r})+(\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}}D^{\perp}_{\varrho}\varkappa^{\odot}\right),
\end{aligned}$$

where $\mathsf{r} = (r_1,\dots,r_{n^s})^{\mathsf{T}}$, $\mathsf{h} = (h_1,\dots,h_{n^s})^{\mathsf{T}}$, and where we have defined the $n^s\times n^s$ square matrices $D^{\diamond}_{\varrho}$, $\diamond \in \{\|,\perp,\odot\}$, by $(D^{\diamond}_{\varrho})_{k,l} = D^{\diamond}_{kl}\rho_k\rho_l$, $k,l \in \mathfrak{S}$, and the vectors $\varkappa^{\diamond}$ and $\widehat{\varkappa}^{\diamond}$, $\diamond \in \{\|,\perp,\odot\}$, by $\varkappa^{\diamond}_k = p/\rho_k\,\chi^{\diamond}_k$ and $\widehat{\varkappa}^{\diamond}_k = p/\rho_k\,\widehat{\chi}^{\diamond}_k$, $k \in \mathfrak{S}$.

We finally express the vectors $\mathsf{F}^{\mathrm{e}\|}$, $\mathsf{F}^{\mathrm{e}\perp}$ and $\mathsf{F}^{\mathrm{e}\odot}$ by

$$\mathsf{F}^{\mathrm{e}\diamond} = \frac{1}{p}\left[\mathsf{F}^{\mathrm{e}\diamond}_{\varrho}\ \ 0_{1,3}\ \ 0_{1,3}\ \ 0_{1,3}\ \ \mathsf{F}^{\mathrm{e}\diamond}_{\mathsf{e}}\right]^{\mathsf{T}}, \qquad \diamond \in \{\|,\perp,\odot\}, \tag{3.14}$$

where

$$\begin{aligned}
\mathsf{F}^{\mathrm{e}\diamond}_{\varrho} &= D^{\diamond}_{\varrho}\mathsf{z},\\
\mathsf{F}^{\mathrm{e}\|}_{\mathsf{e}} &= (\widehat{\varkappa}^{\|}+\mathsf{h})^{\mathsf{T}}D^{\|}_{\varrho}\mathsf{z},\\
\mathsf{F}^{\mathrm{e}\perp}_{\mathsf{e}} &= (\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}}D^{\perp}_{\varrho}\mathsf{z} - \widehat{\varkappa}^{\odot\mathsf{T}}D^{\odot}_{\varrho}\mathsf{z},\\
\mathsf{F}^{\mathrm{e}\odot}_{\mathsf{e}} &= (\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}}D^{\odot}_{\varrho}\mathsf{z} + \widehat{\varkappa}^{\odot\mathsf{T}}D^{\perp}_{\varrho}\mathsf{z},
\end{aligned}$$

where $\mathsf{z} = (z_1,\dots,z_{n^s})^{\mathsf{T}}$.

In order to investigate the regularity of $\mathsf{F}_i^{\rm diff}(\mathsf{U},\boldsymbol{\partial_x}\mathsf{U})$, $i\in\mathfrak{C}$, we prove that the matrices $\widehat{\mathsf{B}}_{ij}^{\rm diff}(\mathsf{Z})$ and the vectors $\mathsf{F}_{ij}^{\rm e}(\mathsf{Z})$, $i,j\in\mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_\mathsf{Z}$. We can rewrite matrices $\widehat{\mathsf{B}}_{ij}^{\rm diff}$, $i,j\in\mathfrak{C}$, in the form

$$\widehat{\mathsf{B}}_{ij}^{\rm diff}=\delta_{ij}\widehat{\mathsf{B}}^{\perp}+\mathcal{B}_i\mathcal{B}_j(\widehat{\mathsf{B}}^{\parallel}-\widehat{\mathsf{B}}^{\perp})+\mathsf{T}_{ij}(\boldsymbol{\mathcal{B}})\widehat{\mathsf{B}}^{\odot}.$$

By using (Th_1) and (Tr_1), we immediately obtain that the matrix $\widehat{\mathsf{B}}^{\perp}$ is a $\mathcal{C}^\infty$ function of $\mathsf{Z}\in\mathcal{O}_\mathsf{Z}$, that the matrix $\widehat{\mathsf{B}}^{\parallel}-\widehat{\mathsf{B}}^{\perp}$ reads $B^2\Phi_1(\mathsf{Z})$ and that the matrix $\widehat{\mathsf{B}}^{\odot}$ reads $B\Phi_2(\mathsf{Z})$, where $\Phi_1(\mathsf{Z})$ and $\Phi_2(\mathsf{Z})$ are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_\mathsf{Z}$. These properties imply that the matrices $\widehat{\mathsf{B}}_{ij}^{\rm diff}(\mathsf{Z})$, $i,j\in\mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_\mathsf{Z}$, since

$$\widehat{\mathsf{B}}_{ij}^{\rm diff}(\mathsf{Z})=\delta_{ij}\widehat{\mathsf{B}}^{\perp}(\mathsf{Z})+B_iB_j\Phi_1(\mathsf{Z})+\mathsf{T}_{ij}(\boldsymbol{B})\Phi_2(\mathsf{Z}),$$

thanks to $\boldsymbol{B}=B\boldsymbol{\mathcal{B}}$. Similarly the vectors $\mathsf{F}_{ij}^{\rm e}$, $i,j\in\mathfrak{C}$, can be rewritten in the form

$$\mathsf{F}_{ij}^{\rm e}=-\Big[\delta_{ij}\mathsf{F}^{\rm e\perp}+\mathcal{B}_i\mathcal{B}_j(\mathsf{F}^{\rm e\parallel}-\mathsf{F}^{\rm e\perp})+\mathsf{T}_{ij}(\boldsymbol{\mathcal{B}})\mathsf{F}^{\rm e\odot}\Big]\,(\boldsymbol{E}+\boldsymbol{v}\wedge\boldsymbol{B})_j.$$

By using (Th_1) and (Tr_1), we obtain that the vector $\mathsf{F}^{\rm e\perp}$ is a $\mathcal{C}^\infty$ function of $\mathsf{Z}\in\mathcal{O}_\mathsf{Z}$, the vector $\mathsf{F}^{\rm e\parallel}-\mathsf{F}^{\rm e\perp}$ reads $B^2\Phi_3(\mathsf{Z})$ and the vector $\mathsf{F}^{\rm e\odot}$ reads $B\Phi_4(\mathsf{Z})$, where $\Phi_3(\mathsf{Z})$ and $\Phi_4(\mathsf{Z})$ are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_\mathsf{Z}$. As a consequence we deduce that the vectors $\mathsf{F}_{ij}^{\rm e}(\mathsf{Z})$, $i,j\in\mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_\mathsf{Z}$. The two regularity properties of matrices $\widehat{\mathsf{B}}_{ij}^{\rm diff}$ and vectors $\mathsf{F}_{ij}^{\rm e}$, $i,j\in\mathfrak{C}$, yields finally that the diffusive flux $\mathsf{F}_i^{\rm diff}(\mathsf{Z},\boldsymbol{\partial_x}\mathsf{Z})$, $i\in\mathfrak{C}$, is a $\mathcal{C}^\infty$ function of $\mathsf{Z}\in\mathcal{O}_\mathsf{Z}$ and $\boldsymbol{\partial_x}\mathsf{Z}\in\mathbb{R}^{3(n^s+10)}$, and that the matrices $\partial_\mathsf{Z}\mathsf{F}_{ij}^{\rm e}$, $i,j\in\mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_\mathsf{Z}$.

With regard to $\mathsf{F}_i^{\rm visc}$, we write

$$\mathsf{F}_i^{\rm visc}=-\sum_{j\in\mathfrak{C}}\widehat{\mathsf{B}}_{ij}^{\rm visc}\partial_j\mathsf{Z},\quad\text{with}\quad\widehat{\mathsf{B}}_{ij}^{\rm visc}=\kappa\widehat{\mathsf{B}}_{ij}^{\rm div}+\sum_{\alpha=1}^{5}\eta_\alpha\widehat{\mathsf{B}}_{ij}^{\eta_\alpha}.\tag{3.15}$$

After some algebra, we obtain the following expressions for the matrices $\widehat{\mathsf{B}}_{ij}^{\triangle}$, $\triangle\in\{\mathrm{div},\eta_1\ldots\eta_5\}$,

$$\widehat{\mathsf{B}}_{ij}^{\triangle}=\begin{bmatrix}0_{n^s,n^s}&0_{n^s,3}&0_{n^s,3}&0_{n^s,3}&0_{n^s,1}\\0_{3,n^s}&\widehat{\mathsf{B}}_{ij,\boldsymbol{v}}^{\triangle}&0_{3,3}&0_{3,3}&0_{3,1}\\0_{3,n^s}&0_{3,3}&0_{3,3}&0_{3,3}&0_{3,1}\\0_{3,n^s}&0_{3,3}&0_{3,3}&0_{3,3}&0_{3,1}\\0_{1,n^s}&\boldsymbol{v}^{\mathsf{T}}\widehat{\mathsf{B}}_{ij,\boldsymbol{v}}^{\triangle}&0_{1,3}&0_{1,3}&0_{1,1}\end{bmatrix},\qquad\triangle\in\{\mathrm{div},\eta_1\ldots\eta_5\},\tag{3.16}$$

with

$$\begin{aligned}\widehat{\mathsf{B}}_{ij,\boldsymbol{v}}^{\rm div}&=\boldsymbol{e}_i\otimes\boldsymbol{e}_j,\\ \widehat{\mathsf{B}}_{ij,\boldsymbol{v}}^{\eta_1}&=\delta_{ij}\mathbb{I}+\boldsymbol{e}_j\otimes\boldsymbol{e}_i-\tfrac{2}{3}\boldsymbol{e}_i\otimes\boldsymbol{e}_j,\end{aligned}$$

$$
\begin{aligned}
\widehat{\mathsf{B}}^{\eta_2}_{ij,\boldsymbol{v}} &= 2\delta_{ij}\mathbf{T}(\boldsymbol{\mathcal{B}}) + \mathbf{T}(\boldsymbol{e}_i)\mathbf{T}(\boldsymbol{\mathcal{B}})\mathbf{T}(\boldsymbol{e}_j) + 2\boldsymbol{e}_i{}^{\mathsf{T}}\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{e}_j\mathbb{I},\\
\widehat{\mathsf{B}}^{\eta_3}_{ij,\boldsymbol{v}} &= 2\mathcal{B}_i\mathcal{B}_j\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}} - \tfrac{2}{3}\boldsymbol{e}_i\otimes\boldsymbol{e}_j + 2\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{e}_j\otimes\boldsymbol{e}_i\mathbf{T}(\boldsymbol{\mathcal{B}}) - \mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{e}_i\otimes\boldsymbol{e}_j\mathbf{T}(\boldsymbol{\mathcal{B}}),\\
\widehat{\mathsf{B}}^{\eta_4}_{ij,\boldsymbol{v}} &= -4\mathcal{B}_i\mathcal{B}_j\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}} + \mathcal{B}_i\mathcal{B}_j\mathbb{I} + \delta_{ij}\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}} + \mathcal{B}_i\boldsymbol{e}_j\otimes\boldsymbol{\mathcal{B}} + \mathcal{B}_j\boldsymbol{\mathcal{B}}\otimes\boldsymbol{e}_i,\\
\widehat{\mathsf{B}}^{\eta_5}_{ij,\boldsymbol{v}} &= -2\mathcal{B}_i\mathcal{B}_j\mathbf{T}(\boldsymbol{\mathcal{B}}) - \mathbf{T}(\boldsymbol{e}_i)\mathbf{T}(\boldsymbol{\mathcal{B}})\mathbf{T}(\boldsymbol{e}_j) - 2\boldsymbol{e}_i{}^{\mathsf{T}}\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{e}_j\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}}.
\end{aligned}
$$

To obtain that $\mathsf{F}_i^{\rm visc}(\mathsf{Z},\boldsymbol{\partial_x}\mathsf{Z})$, $i\in\mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_{\mathsf{Z}}$, $\boldsymbol{\partial_x}\mathsf{Z}\in\mathbb{R}^{3(n^s+10)}$, we only have to prove that the matrices $\widehat{\mathsf{B}}^{\rm visc}_{ij}(\mathsf{Z})$, $i,j\in\mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_{\mathsf{Z}}$, by using expression (3.15). Assumption $(\mathsf{Tr_1})$ and expressions of matrices $\widehat{\mathsf{B}}^{\triangle}_{ij}$, $i,j\in\mathfrak{C}$, $\triangle\in\{\mathrm{div},\eta_1\ldots\eta_5\}$ immediately yield that matrices $\widehat{\mathsf{B}}^{\rm visc}_{ij}(\mathsf{Z})$, $i,j\in\mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_{\mathsf{Z}}$, $\boldsymbol{B}\neq 0$. To prove that those matrices are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_{\mathsf{Z}}$, even at $\boldsymbol{B}=0$, we remark that

$$
\begin{aligned}
\kappa\widehat{\mathsf{B}}^{\rm div}_{ij,\boldsymbol{v}} &= \kappa\boldsymbol{e}_i\otimes\boldsymbol{e}_j,\\
\eta_1\widehat{\mathsf{B}}^{\eta_1}_{ij,\boldsymbol{v}} &= \eta_1\left[\delta_{ij}\mathbb{I} + \boldsymbol{e}_j\otimes\boldsymbol{e}_i - \tfrac{2}{3}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\right],\\
\eta_2\widehat{\mathsf{B}}^{\eta_2}_{ij,\boldsymbol{v}} &= \eta_2\left[2\delta_{ij}\mathbf{T}(\boldsymbol{\mathcal{B}}) + \mathbf{T}(\boldsymbol{e}_i)\mathbf{T}(\boldsymbol{\mathcal{B}})\mathbf{T}(\boldsymbol{e}_j) + 2\boldsymbol{e}_i{}^{\mathsf{T}}\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{e}_j\mathbb{I}\right],\\
\eta_3\widehat{\mathsf{B}}^{\eta_3}_{ij,\boldsymbol{v}}+\eta_4\widehat{\mathsf{B}}^{\eta_4}_{ij,\boldsymbol{v}} &= \eta_3\left[-\tfrac{2}{3}\boldsymbol{e}_i\otimes\boldsymbol{e}_j + 2\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{e}_j\otimes\boldsymbol{e}_i\mathbf{T}(\boldsymbol{\mathcal{B}}) - \mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{e}_i\otimes\boldsymbol{e}_j\mathbf{T}(\boldsymbol{\mathcal{B}})\right]\\
&\quad + \eta_4\left[\mathcal{B}_i\mathcal{B}_j\mathbb{I} + \delta_{ij}\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}} + \mathcal{B}_i\boldsymbol{e}_j\otimes\boldsymbol{\mathcal{B}} + \mathcal{B}_j\boldsymbol{\mathcal{B}}\otimes\boldsymbol{e}_i\right]\\
&\quad + 2(\eta_3-2\eta_4)\mathcal{B}_i\mathcal{B}_j\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}},\\
\eta_5\widehat{\mathsf{B}}^{\eta_5}_{ij,\boldsymbol{v}} &= -\eta_5\left[2\mathcal{B}_i\mathcal{B}_j\mathbf{T}(\boldsymbol{\mathcal{B}}) + \mathbf{T}(\boldsymbol{e}_i)\mathbf{T}(\boldsymbol{\mathcal{B}})\mathbf{T}(\boldsymbol{e}_j) + 2\boldsymbol{e}_i{}^{\mathsf{T}}\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{e}_j\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}}\right].
\end{aligned}
$$

By using assumption $(\mathsf{Tr_1})$, in particular that $\eta_1=\varphi_1(B^2)$, $\eta_2=B\varphi_2(B^2)$, $\eta_3=B^2\varphi_3(B^2)$, $\eta_4=B^2\varphi_4(B^2)$, $\eta_5=B^3\varphi_5(B^2)$ and $2\eta_4-\eta_3=B^4\varphi_6(B^2)$, where φ_α, $\alpha\in\{1,\ldots,6\}$, are $\mathcal{C}^\infty([0,\infty),\mathbb{R})$ functions, we so deduce that matrices $\widehat{\mathsf{B}}^{\rm visc}_{ij}(\mathsf{Z})$, $i,j\in\mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z}\in\mathcal{O}_{\mathsf{Z}}$.

Moreover, by using expression (3.4) of vector $\Omega^{\boldsymbol{j}}$, assumption $(\mathsf{Th_1})$ on the regularity of specific heats $c_{v,k}$, $k\in\mathfrak{S}$, and assumption $(\mathsf{Th_4})$ on the regularity of the rate constant $\mathcal{K}_r^{\rm s}$, we immediately obtain that vector $\Omega^{\boldsymbol{j}}(\mathsf{Z},\boldsymbol{j})$ is a $\mathcal{C}^\infty$ function of $\mathsf{Z}\in\mathcal{O}_{\mathsf{Z}}$ and $\boldsymbol{j}\in\mathbb{R}^3$. After some algebra, we can rewrite $\Omega^{\boldsymbol{j}}$ in the form

$$
\Omega^{\boldsymbol{j}}(\mathsf{Z},\boldsymbol{j}) = \Omega(\mathsf{Z}) + \sum_{i\in\mathfrak{C}}\widehat{\mathsf{A}}_i^{\mathrm{e},\Omega}(\mathsf{Z})\partial_i\mathsf{Z},
$$

where the vector Ω is given by

$$
\Omega = \big(m_1\omega_1,\ldots,m_{n^s}\omega_{n^s},\Omega_{\boldsymbol{v}}^{\mathsf{T}},\Omega_{\boldsymbol{E}}^{\mathsf{T}},0_{1,3},\rho\boldsymbol{g}\cdot\boldsymbol{v}\big)^{\mathsf{T}}, \tag{3.17}
$$

with

$$
\begin{aligned}
\Omega_{\boldsymbol{v}} &= \rho\boldsymbol{g} + \left[q\,\mathbb{I} + \sum_{k,l\in\mathfrak{S}} B\frac{q_kq_l}{p}\left(D^{\odot}_{kl}(\mathbb{I}-\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}}) - D^{\perp}_{kl}\mathbf{T}(\boldsymbol{\mathcal{B}})\right)\right](\boldsymbol{E}+\boldsymbol{v}\wedge\boldsymbol{B}),\\
\Omega_{\boldsymbol{E}} &= -\frac{1}{\varepsilon_0}q\boldsymbol{v} - \frac{1}{\varepsilon_0}\sum_{k,l\in\mathfrak{S}}\frac{q_kq_l}{p}\left(D^{\parallel}_{kl}\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}} + D^{\perp}_{kl}(\mathbb{I}-\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}}) + D^{\odot}_{kl}\mathbf{T}(\boldsymbol{\mathcal{B}})\right)(\boldsymbol{E}+\boldsymbol{v}\wedge\boldsymbol{B}),
\end{aligned}
$$

and matrices $\widehat{\mathsf{A}}_i^{\mathrm{e},\Omega}$, $i \in \mathfrak{C}$, have the structure

$$\widehat{\mathsf{A}}_i^{\mathrm{e},\Omega} = \begin{bmatrix} 0_{n^s,n^s} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,1} \\ \widehat{\mathsf{A}}^{\mathrm{e},\Omega}_{i,\boldsymbol{v},\varrho} & 0_{3,3} & 0_{3,3} & 0_{3,3} & \widehat{\mathsf{A}}^{\mathrm{e},\Omega}_{i,\boldsymbol{v},\mathsf{e}} \\ \widehat{\mathsf{A}}^{\mathrm{e},\Omega}_{i,\boldsymbol{E},\varrho} & 0_{3,3} & 0_{3,3} & 0_{3,3} & \widehat{\mathsf{A}}^{\mathrm{e},\Omega}_{i,\boldsymbol{E},\mathsf{e}} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{1,n^s} & 0_{1,3} & 0_{1,3} & 0_{1,3} & 0 \end{bmatrix}, \tag{3.18}$$

with

$$\begin{aligned}
\widehat{\mathsf{A}}^{\mathrm{e},\Omega}_{i,\boldsymbol{v},\varrho} &= -\tfrac{BT}{p}\left[\boldsymbol{e}_i^{\perp}\left(\mathsf{z}^{\mathsf{T}} D_\varrho^{\odot}\mathsf{d}_\varrho^{\mathsf{r}}\right) - \boldsymbol{e}_i^{\odot}\left(\mathsf{z}^{\mathsf{T}} D_\varrho^{\perp}\mathsf{d}_\varrho^{\mathsf{r}}\right)\right],\\
\widehat{\mathsf{A}}^{\mathrm{e},\Omega}_{i,\boldsymbol{v},\mathsf{e}} &= -\tfrac{B}{p}\mathsf{z}^{\mathsf{T}}\Big[-D_\varrho^{\perp}(\tfrac{\varkappa^{\perp}}{T}+\mathsf{r})\boldsymbol{e}_i^{\odot} + D_\varrho^{\perp}\tfrac{\varkappa^{\odot}}{T}\boldsymbol{e}_i^{\perp} + D_\varrho^{\odot}(\tfrac{\varkappa^{\perp}}{T}+\mathsf{r})\boldsymbol{e}_i^{\perp} + D_\varrho^{\odot}\tfrac{\varkappa^{\odot}}{T}\boldsymbol{e}_i^{\odot}\Big],\\
\widehat{\mathsf{A}}^{\mathrm{e},\Omega}_{i,\boldsymbol{E},\varrho} &= \tfrac{T}{\varepsilon_0 p}\Big[\boldsymbol{e}_i^{\parallel}\Big(\mathsf{z}^{\mathsf{T}} D_\varrho^{\parallel}\mathsf{d}_\varrho^{\mathsf{r}}\Big) + \boldsymbol{e}_i^{\perp}\left(\mathsf{z}^{\mathsf{T}} D_\varrho^{\perp}\mathsf{d}_\varrho^{\mathsf{r}}\right) + \boldsymbol{e}_i^{\odot}\left(\mathsf{z}^{\mathsf{T}} D_\varrho^{\odot}\mathsf{d}_\varrho^{\mathsf{r}}\right)\Big],\\
\widehat{\mathsf{A}}^{\mathrm{e},\Omega}_{i,\boldsymbol{E},\mathsf{e}} &= \tfrac{B}{\varepsilon_0 p}\mathsf{z}^{\mathsf{T}}\Big[D_\varrho^{\parallel}(\tfrac{\varkappa^{\parallel}}{T}+\mathsf{r})\boldsymbol{e}_i^{\parallel} + D_\varrho^{\perp}(\tfrac{\varkappa^{\perp}}{T}+\mathsf{r})\boldsymbol{e}_i^{\perp} + \\
&\qquad D_\varrho^{\perp}\tfrac{\varkappa^{\odot}}{T}\boldsymbol{e}_i^{\odot} + D_\varrho^{\odot}(\tfrac{\varkappa^{\perp}}{T}+\mathsf{r})\boldsymbol{e}_i^{\odot} - D_\varrho^{\odot}\tfrac{\varkappa^{\odot}}{T}\boldsymbol{e}_i^{\perp}\Big].
\end{aligned}$$

Concerning the regularity of the vector $\Omega(\mathsf{Z})$, its expression (3.17) immediately yields that it is a $\mathcal{C}^\infty$ function of $\mathsf{Z} \in \mathcal{O}_{\mathsf{Z}}$, $\boldsymbol{B} \neq 0$, by using assumption $(\mathsf{Tr_1})$. Moreover, the equalities $D_{kl}^{\perp} - D_{kl}^{\parallel} = B^2\phi_{kl}^{\perp}(B^2)$ and $D_{kl}^{\odot} = B\phi_{kl}^{\odot}(B^2)$, where $\phi_{kl}^{\perp}$ and $\phi_{kl}^{\odot}$, $k,l \in \mathfrak{S}$, are $\mathcal{C}^\infty([0,\infty),\mathbb{R})$ functions, imply that the vectors $\Omega_{\boldsymbol{v}}$, $\Omega_{\boldsymbol{E}}$, and consequently Ω, are $\mathcal{C}^\infty$ functions of $\mathsf{Z} \in \mathcal{O}_{\mathsf{Z}}$. These equalities combining with $\chi_k^{\perp} - \chi_k^{\parallel} = B^2\psi_k^{\perp}(B^2)$, $\chi_k^{\odot} = B\psi_k^{\odot}(B^2)$, where $\psi_k^{\perp}$ and $\psi_k^{\odot}$ are $\mathcal{C}^\infty([0,\infty),\mathbb{R})$ functions, also yield that the matrices $\widehat{\mathsf{A}}_i^{\mathrm{e},\Omega}(\mathsf{Z})$, $i \in \mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{Z} \in \mathcal{O}_{\mathsf{Z}}$.

We then investigate the change of variables $\mathsf{Z} \mapsto \mathsf{U}$. Matrices A_i, $\mathsf{A}_i^{\mathrm{e},\Omega}$, $i \in \mathfrak{C}$, and B_{ij}, $i,j \in \mathfrak{C}$, read

$$\mathsf{A}_i = \partial_{\mathsf{U}}\mathsf{F}_i = \widehat{\mathsf{A}}_i\,\partial_{\mathsf{U}}\mathsf{Z}, \quad \mathsf{A}_i^{\mathrm{e},\Omega} = \widehat{\mathsf{A}}_i^{\mathrm{e},\Omega}\,\partial_{\mathsf{U}}\mathsf{Z}, \quad \mathsf{B}_{ij} = \widehat{\mathsf{B}}_{ij}\,\partial_{\mathsf{U}}\mathsf{Z},$$

where the matrix $\partial_{\mathsf{U}}\mathsf{Z}$ is given by

$$\partial_{\mathsf{U}}\mathsf{Z} = \begin{bmatrix} \mathbb{I}_{n^s,n^s} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,1} \\ -\frac{1}{\rho}\boldsymbol{v}\otimes\mathsf{u} & \frac{1}{\rho}\mathbb{I} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & \mathbb{I} & 0_{3,3} & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & \mathbb{I} & 0_{3,1} \\ -\frac{1}{\rho c_v}\mathsf{e}^{\mathrm{r}\mathsf{T}} & -\frac{1}{\rho c_v}\boldsymbol{v}^{\mathsf{T}} & -\frac{\varepsilon_0}{\rho c_v}\boldsymbol{E}^{\mathsf{T}} & -\frac{1}{\mu_0\rho c_v}\boldsymbol{B}^{\mathsf{T}} & \frac{1}{\rho c_v} \end{bmatrix},$$

with $\mathsf{e}^{\mathrm{r}} = (e_1^{\mathrm{r}},\ldots,e_{n^s}^{\mathrm{r}})^{\mathsf{T}}$ and $e_k^{\mathrm{r}} = e_k - \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}$, $k \in \mathfrak{S}$. Matrices $\widehat{\mathsf{A}}_i = \partial_{\mathsf{Z}}\mathsf{F}_i$, $i \in \mathfrak{S}$, are given by

$$\widehat{\mathsf{A}}_i = \begin{bmatrix} v_i\mathbb{I}_{n^s,n^s} & \varrho\otimes\boldsymbol{e}_i & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,1} \\ v_i\boldsymbol{v}\otimes\mathsf{u} + \boldsymbol{e}_i\otimes\mathsf{r}T & \rho\left(v_i\mathbb{I} + \boldsymbol{v}\otimes\boldsymbol{e}_i\right) & 0_{n^s,3} & 0_{n^s,3} & nR\boldsymbol{e}_i \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & -\frac{1}{\varepsilon_0\mu_0}\mathbf{T}(\boldsymbol{e}_i) & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & \mathbf{T}(\boldsymbol{e}_i) & 0_{3,3} & 0_{3,1} \\ v_i\mathsf{h}^{\mathrm{p}\mathsf{T}} & \rho(v_i\boldsymbol{v} + h^{\mathrm{p}}\boldsymbol{e}_i)^{\mathsf{T}} & -\frac{1}{\mu_0}(\boldsymbol{e}_i\wedge\boldsymbol{B})^{\mathsf{T}} & \frac{1}{\mu_0}(\boldsymbol{e}_i\wedge\boldsymbol{E})^{\mathsf{T}} & v_i\rho c_p \end{bmatrix}, \tag{3.19}$$

where we have used $c_p = c_v + R$, $\mathsf{r} = (r_1, \ldots, r_{n^s})^{\mathsf{T}}$, $\mathsf{h}^{\mathrm{p}} = (h_1^{\mathrm{p}}, \ldots, h_{n^s}^{\mathrm{p}})^{\mathsf{T}}$ and $h_k^{\mathrm{p}} = e_k^{\mathrm{f}} + r_k T$, $k \in \mathfrak{S}$.

By using the regularity of matrices $\widehat{\mathsf{A}}_i$, $\widehat{\mathsf{A}}_i^{\mathrm{e},\Omega}$, $i \in \mathfrak{C}$, $\widehat{\mathsf{B}}_{ij}$, $i,j \in \mathfrak{C}$, and $\partial_{\mathsf{U}}\mathsf{Z}$ given in proposition 3.1, we immediately obtain that matrices $\mathsf{A}_i(U)$, $\mathsf{A}_i^{\mathrm{e},\Omega}(\mathsf{U})$, $i \in \mathfrak{C}$, $\mathsf{B}_{ij}(U)$, $i,j \in \mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{U} \in \mathcal{O}_{\mathsf{U}}$. The regularity of the matrices $\partial_{\mathsf{U}}\mathsf{F}_{ij}^{\mathrm{e}}$, $i,j \in \mathfrak{C}$, and $\mathsf{A}_i^{\mathrm{e},\Omega}$, $i \in \mathfrak{C}$, yields then that the matrices $\mathsf{A}_i^{\mathrm{e}}$, $i \in \mathfrak{C}$, are $\mathcal{C}^\infty$ functions of $\mathsf{U} \in \mathcal{O}_{\mathsf{U}}$. □

4 Local existence for an abstract system

In this section, we investigate partial symmetrization and partial normal forms for an abstract second-order quasilinear system. We then use an existence theorem of Vol'pert and Hudjaev [6] which applies to the corresponding system.

4.1 Partial Symmetrization and Partial Entropy

Symmetric forms are a fundamental step towards existence results for systems of partial differential equations, in particular for hyperbolic-parabolic systems [6, 9, 10]. In the framework of isotropic hyperbolic-parabolic systems, existence of a conservative symmetric formulation is equivalent to the existence of a mathematical entropy [11, 3, 12].

We generalize in this section the notion of symmetrization for hyperbolic-parabolic systems, to include convective fluxes with weaker symmetry properties. We introduce two notions and establish the equivalence between them : partial symmetrization and existence of a partial entropy.

We consider an abstract second-order quasilinear system in the form

$$\partial_t \mathsf{U}^* + \sum_{i\in\mathfrak{C}^*} \left[\partial_{\mathsf{U}^*}\mathsf{F}_i^*(\mathsf{U}^*) + \mathsf{A}_i^{*\mathrm{e}}(\mathsf{U}^*)\right] \partial_i \mathsf{U}^* = \sum_{i,j\in\mathfrak{C}^*} \partial_i \left[\mathsf{B}_{ij}^*(\mathsf{U}^*)\partial_j \mathsf{U}^*\right] + \Omega^*(\mathsf{U}^*), \quad (4.1)$$

where $\mathsf{U}^* \in \mathcal{O}_{\mathsf{U}^*}$, $\mathcal{O}_{\mathsf{U}^*}$ is an open convex set of $\mathbb{R}^{n^*}$, and $\mathfrak{C}^* = \{1, \ldots, d\}$ is the set of direction indexes of $\mathbb{R}^d$. Note that the superscript * is used to distinguish between the abstract second-order system (4.1) of size n^* in $\mathbb{R}^d$ and the particular multicomponent reactive magnetized flows system (3.9) of size n^s+10 in $\mathbb{R}^3$. We assume that the following properties hold for system (4.1)

(Edp$_1$) *The convective fluxes F_i^*, $i \in \mathfrak{C}^*$, the Jacobian matrices $\mathsf{A}_i^{*\mathrm{e}}$, $i \in \mathfrak{C}^*$, the dissipation matrices B_{ij}^*, $i,j \in \mathfrak{C}^*$, and the source term Ω^* are smooth functions of the variable $\mathsf{U}^* \in \mathcal{O}_{\mathsf{U}^*}$.*

We give the definition of a partial symmetric form for the system (4.1).

Definition 4.1 *Assume that* $\mathsf{V}^* \mapsto \mathsf{U}^*$ *is a diffeomorphism from* $\mathcal{O}_{\mathsf{V}^*}$ *onto* $\mathcal{O}_{\mathsf{U}^*}$*, and consider the system in the* V^* *variable*

$$\widetilde{\mathsf{A}}_0^*(\mathsf{V}^*)\partial_t \mathsf{V}^* + \sum_{i\in\mathfrak{C}^*} \Big[\widetilde{\mathsf{A}}_i^*(\mathsf{V}^*)+\widetilde{\mathsf{A}}_i^{*\mathrm{e}}(\mathsf{V}^*)\Big]\,\partial_i \mathsf{V}^* = \sum_{i,j\in\mathfrak{C}^*} \partial_i \Big[\widetilde{\mathsf{B}}_{ij}^*(\mathsf{V}^*)\partial_j \mathsf{V}^*\Big] + \widetilde{\Omega}^*(\mathsf{V}^*), \tag{4.2}$$

where $\widetilde{\mathsf{A}}_0^* = \partial_{\mathsf{V}^*}\mathsf{U}^*$*,* $\widetilde{\mathsf{A}}_i^* = \mathsf{A}_i^*\,\partial_{\mathsf{V}^*}\mathsf{U}^* = \partial_{\mathsf{V}^*}\mathsf{F}_i^*$*,* $\widetilde{\mathsf{A}}_i^{*\mathrm{e}} = \mathsf{A}_i^{*\mathrm{e}}\,\partial_{\mathsf{V}^*}\mathsf{U}^*$*,* $\widetilde{\mathsf{B}}_{ij}^* = \mathsf{B}_{ij}^*\,\partial_{\mathsf{V}^*}\mathsf{U}^*$*,* $\widetilde{\Omega}^* = \Omega^*$*. The system* (4.2) *is said to be of the partial symmetric form if the matrix coefficients satisfy the following properties.*

(S_1) *The matrix* $\widetilde{\mathsf{A}}_0^*(\mathsf{V}^*)$ *is symmetric positive definite for* $\mathsf{V}^* \in \mathcal{O}_{\mathsf{V}^*}$*.*

(S_2) *The matrices* $\widetilde{\mathsf{A}}_i^*(\mathsf{V}^*)$*,* $i \in \mathfrak{C}^*$*, are symmetric for* $\mathsf{V}^* \in \mathcal{O}_{\mathsf{V}^*}$*.*

(S_3) *The matrix* $\widetilde{\mathsf{B}}^*(\mathsf{V}^*;\boldsymbol{\xi}) = \sum_{i,j\in\mathfrak{C}^*} \widetilde{\mathsf{B}}_{ij}^*(\mathsf{V}^*)\xi_i\xi_j$*,* $\boldsymbol{\xi} \in \Sigma^{d-1}$*, where* Σ^{d-1} *is the unit sphere in* d *dimensions, satisfies* $\mathsf{X}^{\mathsf{T}}\ \widetilde{\mathsf{B}}^*(\mathsf{V}^*;\boldsymbol{\xi})\ \mathsf{X} \geqslant 0$*, for* $\mathsf{X} \in \mathbb{R}^{n^*}$*,* $\mathsf{V}^* \in \mathcal{O}_{\mathsf{V}^*}$ *and* $\boldsymbol{\xi} \in \Sigma^{d-1}$*.*

Properties (S_1-S_2) are the same as those of neutral mixtures. Property (S_3) is a generalization of the property on dissipative matrices for neutral gases. The matrix $\widetilde{\mathsf{B}}^*$ is not symmetric in the general but its symmetric part is positive definite. There is no assumption on the matrices $\widetilde{\mathsf{A}}_i^{*\mathrm{e}}$, $i \in \mathfrak{C}$. These assumptions will be introduced in the following.

We then define the partial entropy function.

Definition 4.2 *A real-valued smooth function* $\sigma^*(\mathsf{U}^*)$ *defined on a convex set* $\mathcal{O}_{\mathsf{U}^*}$ *is said to be a partial entropy function for the system* (4.1) *if the following properties hold.*

(E_1) *The function* σ^* *is a strictly convex function on* $\mathcal{O}_{\mathsf{U}^*}$ *in the sense that the Hessian matrix is positive definite on* $\mathcal{O}_{\mathsf{U}^*}$*.*

(E_2) *There exists real-valued smooth functions* $\mathfrak{q}_i^*(\mathsf{U}^*)$ *such that*

$$(\partial_{\mathsf{U}^*}\sigma^*)\ \mathsf{A}_i^* = \partial_{\mathsf{U}^*}\mathfrak{q}_i^*, \quad i \in \mathfrak{C}^*, \quad \mathsf{U}^* \in \mathcal{O}_{\mathsf{U}^*}.$$

(E_3) *The matrix* $\widetilde{\mathsf{B}}^*(\mathsf{U}^*;\boldsymbol{\xi}) = \sum_{i,j\in\mathfrak{C}^*} \mathsf{B}_{ij}^*(\mathsf{U}^*)\big(\partial^2_{\mathsf{U}^*}\sigma^*(\mathsf{U}^*)\big)^{-1}\xi_i\xi_j$*,* $\boldsymbol{\xi} \in \Sigma^{d-1}$*, satisfies* $\mathsf{X}^{\mathsf{T}}\ \widetilde{\mathsf{B}}^*(\mathsf{U}^*;\boldsymbol{\xi})\ \mathsf{X} \geqslant 0$*, for* $\mathsf{X} \in \mathbb{R}^{n^*}$*,* $\mathsf{U}^* \in \mathcal{O}_{\mathsf{U}^*}$ *and* $\boldsymbol{\xi} \in \Sigma^{d-1}$*.*

We then establish the equivalence theorem between conservative partial symmetrizability and the existence of a partial entropy function for the system (4.1).

Theorem 4.3 *The system* (4.1) *can be partially symmetrized on the open convex set* $\mathcal{O}_{\mathsf{U}^*}$ *if and only if the system admits a partial entropy function* σ^* *on* $\mathcal{O}_{\mathsf{U}^*}$*. In this situation, the symmetrizing variable* V^* *can be expressed in terms of the gradient of the entropy function* σ^* *as* $\mathsf{V}^* = (\partial_{\mathsf{U}^*}\sigma^*)^{\mathsf{T}}$*.*

Proof Assume first that there exists a partial entropy σ^*, and let $\mathsf{V}^* = (\partial_{\mathsf{U}^*}\sigma^*)^{\mathsf{T}}$ be the symmetrizing variable. The map $\mathsf{U}^* \mapsto \mathsf{V}^*$ is then a diffeomorphism since $\mathcal{O}_{\mathsf{U}^*}$ is convex and $\partial_{\mathsf{U}^*}\mathsf{V}^* = \partial^2_{\mathsf{U}^*}\sigma^*$ is positive definite. We can thus define the smooth functions

$$\widehat{\sigma}^*(\mathsf{V}^*) = \mathsf{U}^{*\mathsf{T}}\mathsf{V}^* - \sigma^*(\mathsf{U}^*) \quad \text{and} \quad \widehat{\mathfrak{q}}_i^*(\mathsf{V}^*) = \mathsf{F}_i^{*\mathsf{T}}\mathsf{V}^* - \mathfrak{q}_i^*(\mathsf{U}^*), \quad i \in \mathfrak{C}^*.$$

Differentiating these equalities then yields the relations $(\partial_{\mathsf{V}^*}\widehat{\sigma}^*)^{\mathsf{T}} = \mathsf{U}^*$ and $(\partial_{\mathsf{V}^*}\widehat{\mathfrak{q}}_i^*)^{\mathsf{T}} = \mathsf{F}_i^*$, making use of property (E_2). By using properties $(\mathsf{E}_1\text{-}\mathsf{E}_3)$, we obtain that $\widetilde{\mathsf{A}}_0^* = \partial_{\mathsf{V}^*}\mathsf{U}^* = (\partial_{\mathsf{U}^*}\mathsf{V}^*)^{-1} = (\partial^2_{\mathsf{U}^*}\sigma^*)^{-1}$ and $\widetilde{\mathsf{A}}_i^* = \partial_{\mathsf{V}^*}\mathsf{F}_i^* = \partial^2_{\mathsf{V}^*}\widehat{\mathfrak{q}}_i^*$, $i \in \mathfrak{C}^*$, so that the matrix $\widetilde{\mathsf{A}}_0^*$ is symmetric definite positive and the matrices $\widetilde{\mathsf{A}}_i^*$, $i \in \mathfrak{C}^*$, are symmetric. Moreover, we directly get from properties $(\mathsf{E}_1\text{-}\mathsf{E}_3)$ that the matrices $\widetilde{\mathsf{B}}_{ij}^* = \mathsf{B}_{ij}^*(\partial^2_{\mathsf{U}^*}\sigma^*)^{-1}$, $i,j \in \mathfrak{C}^*$, are such that property (S_3) holds.

Conversely, assume that the system can be partially symmetrized in the sense of definition 4.1. Since $\partial_{\mathsf{V}^*}\mathsf{U}^*$ and $\partial_{\mathsf{V}^*}\mathsf{F}_i^*$, $i \in \mathfrak{C}^*$, are symmetric and $\mathcal{O}_{\mathsf{V}^*}$ is simply connected, there exists $\widehat{\sigma}^*$ and $\widehat{\mathfrak{q}}_i^*$, $i \in \mathfrak{C}^*$, define over $\mathcal{O}_{\mathsf{V}^*}$, such that $(\partial_{\mathsf{V}^*}\widehat{\sigma}^*)^{\mathsf{T}} = \mathsf{U}^*$ and $(\partial_{\mathsf{V}^*}\widehat{\mathfrak{q}}_i^*)^{\mathsf{T}} = \mathsf{F}_i^*$, $i \in \mathfrak{C}^*$. We can thus define the functions

$$\sigma^*(\mathsf{U}^*) = \mathsf{U}^{*\mathsf{T}}\mathsf{V}^* - \widehat{\sigma}^*(\mathsf{V}^*) \quad \text{and} \quad \mathfrak{q}_i^*(\mathsf{U}^*) = \mathsf{F}_i^{*\mathsf{T}}\mathsf{V}^* - \widehat{\mathfrak{q}}_i^*(\mathsf{V}^*), \quad i \in \mathfrak{C}^*.$$

Differentiating these identities, and using properties $(\mathsf{S}_1\text{-}\mathsf{S}_3)$, it is then straightforward to establish that σ^* is an entropy with fluxes $\mathfrak{q}_i^*$, $i \in \mathfrak{C}^*$, such that $\mathsf{V}^* = (\partial_{\mathsf{U}^*}\sigma^*)^{\mathsf{T}}$. □

4.2 Partial normal forms

In this section, we investigate partial normal forms for the abstract system (4.1). We assume that this system satisfies

$(\mathsf{Edp_2})$ *The system* (4.1) *admits a partial entropy function σ^* on the open convex set $\mathcal{O}_{\mathsf{U}^*}$.*

Introducing the symmetrizing variable $\mathsf{V}^* = (\partial_{\mathsf{U}^*}\sigma^*)^{\mathsf{T}}$, the corresponding partially symmetric system (4.2) then satisfies properties $(\mathsf{S}_1\text{-}\mathsf{S}_3)$. Introducing a new variable W^*, associated with a diffeomorphism from $\mathcal{O}_{\mathsf{W}^*}$ onto $\mathcal{O}_{\mathsf{V}^*}$, and multiplying the conservative partially symmetric form (4.2) on the left side by the transpose of the matrix $\partial_{\mathsf{W}^*}\mathsf{V}^*$, we get a new system in the variable W^*.

Definition 4.4 *Consider a system in partially symmetric form, as in Definition 4.1, and a diffeomorphism $\mathsf{W}^* \mapsto \mathsf{V}^*$ from $\mathcal{O}_{\mathsf{W}^*}$ onto $\mathcal{O}_{\mathsf{V}^*}$. The system in the new variable* W^*

$$\overline{\mathsf{A}}_0^*\partial_t\mathsf{W}^* + \sum_{i\in\mathfrak{C}^*}\left(\overline{\mathsf{A}}_i^* + \overline{\mathsf{A}}_i^{*\mathrm{e}}\right)\partial_i\mathsf{W}^* = \sum_{i,j\in\mathfrak{C}^*}\partial_i\left(\overline{\mathsf{B}}_{ij}^*\partial_j\mathsf{W}^*\right) + \overline{\mathcal{T}}^* + \overline{\Omega}^*, \qquad (4.3)$$

where

$$\begin{aligned}
\overline{\mathsf{A}}_0^* &= (\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathsf{T}}\,\widetilde{\mathsf{A}}_0^*\,(\partial_{\mathsf{W}^*}\mathsf{V}^*), & \overline{\mathsf{B}}_{ij}^* &= (\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathsf{T}}\,\widetilde{\mathsf{B}}_{ij}^*\,(\partial_{\mathsf{W}^*}\mathsf{V}^*),\\
\overline{\mathsf{A}}_i^* &= (\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathsf{T}}\,\widetilde{\mathsf{A}}_i^*\,(\partial_{\mathsf{W}^*}\mathsf{V}^*), & \overline{\Omega}^* &= (\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathsf{T}}\,\widetilde{\Omega}^*,\\
\overline{\mathsf{A}}_i^{*\mathrm{e}} &= (\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathsf{T}}\,\widetilde{\mathsf{A}}_i^{*\mathrm{e}}\,(\partial_{\mathsf{W}^*}\mathsf{V}^*), & \overline{\mathcal{T}}^* &= -\sum_{i,j\in\mathfrak{C}^*}\partial_i(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathsf{T}}\,\widetilde{\mathsf{B}}_{ij}^*\,(\partial_{\mathsf{W}^*}\mathsf{V}^*)\,\partial_j\mathsf{W}^*,
\end{aligned}$$

satisfy

($\overline{\mathsf{S}}_1$) *The matrix* $\overline{\mathsf{A}}_0^*(\mathsf{W}^*)$ *is symmetric positive definite for* $\mathsf{W}^*\in\mathcal{O}_{\mathsf{W}^*}$.

($\overline{\mathsf{S}}_2$) *The matrices* $\overline{\mathsf{A}}_i^*(\mathsf{W}^*)$, $i\in\mathfrak{C}^*$, *are symmetric for* $\mathsf{W}^*\in\mathcal{O}_{\mathsf{W}^*}$.

($\overline{\mathsf{S}}_3$) *The matrix* $\overline{\mathsf{B}}^*(\mathsf{W}^*,\boldsymbol{\xi})=\sum_{i,j\in\mathfrak{C}^*}\overline{\mathsf{B}}_{ij}^*(\mathsf{W}^*)\xi_i\xi_j$, $\boldsymbol{\xi}\in\Sigma^{d-1}$, *satisfies* $\mathsf{X}^{\mathsf{T}}\,\overline{\mathsf{B}}^*(\mathsf{W}^*,\boldsymbol{\xi})\,\mathsf{X}\geqslant 0$, *for* $\mathsf{X}\in\mathbb{R}^{n^*}$, $\mathsf{W}^*\in\mathcal{O}_{\mathsf{W}^*}$ *and* $\boldsymbol{\xi}\in\Sigma^{d-1}$.

This system is said to be of the partial normal form if there exists a partition of $\{1,\dots,n^*\}$ *into* $\mathrm{I}=\{1,\dots,n_0^*\}$ *and* $\mathrm{II}=\{n_0^*{+}1,\dots,n^*\}$, *such that the following properties hold.*

(Nor_1) *The matrices* $\overline{\mathsf{A}}_0^*$, $\overline{\mathsf{A}}_i^{*\mathrm{e}}$, $i\in\mathfrak{C}^*$, *and* $\overline{\mathsf{B}}_{ij}^*$, $i,j\in\mathfrak{C}^*$, *have the bloc structure*

$$\overline{\mathsf{A}}_0^*=\begin{pmatrix}\overline{\mathsf{A}}_0^{*\mathrm{I},\mathrm{I}} & 0\\ 0 & \overline{\mathsf{A}}_0^{*\mathrm{II},\mathrm{II}}\end{pmatrix},\quad \overline{\mathsf{A}}_i^{*\mathrm{e}}=\begin{pmatrix}0 & \overline{\mathsf{A}}_i^{*\mathrm{eI},\mathrm{II}}\\ \overline{\mathsf{A}}_i^{*\mathrm{eII},\mathrm{I}} & \overline{\mathsf{A}}_i^{*\mathrm{eII},\mathrm{II}}\end{pmatrix},\quad \overline{\mathsf{B}}_{ij}^*=\begin{pmatrix}0 & 0\\ 0 & \overline{\mathsf{B}}_{ij}^{*\mathrm{II},\mathrm{II}}\end{pmatrix}.$$

(Nor_2) *The matrix* $\overline{\mathsf{B}}^{*\mathrm{II},\mathrm{II}}(\mathsf{W}^*,\boldsymbol{\xi})=\sum_{i,j\in\mathfrak{C}}\overline{\mathsf{B}}_{ij}^{*\mathrm{II},\mathrm{II}}(\mathsf{W}^*)\xi_i\xi_j$, $\boldsymbol{\xi}\in\Sigma^{d-1}$ *satisfies* $\mathsf{X}^{\mathsf{T}}\,\overline{\mathsf{B}}^{*\mathrm{II},\mathrm{II}}(\mathsf{W}^*,\boldsymbol{\xi})\,\mathsf{X}>0$, *for* $\mathsf{X}\in\mathbb{R}^{n^*}$, $\mathsf{X}\neq 0$, $\mathsf{W}^*\in\mathcal{O}_{\mathsf{W}^*}$ *and* $\boldsymbol{\xi}\in\Sigma^{d-1}$.

(Nor_3) *We have* $\overline{\mathcal{T}}^*(\mathsf{W}^*,\boldsymbol{\partial_x}\mathsf{W}^*)=\Big(\overline{\mathcal{T}}_{\mathrm{I}}^*(\mathsf{W}^*,\boldsymbol{\partial_x}\mathsf{W}_{\mathrm{II}}^*)^{\mathsf{T}},\overline{\mathcal{T}}_{\mathrm{II}}^*(\mathsf{W}^*,\boldsymbol{\partial_x}\mathsf{W}^*)^{\mathsf{T}}\Big)^{\mathsf{T}}$,

where we have used the vector and the matrix block structure induced bu the partitioning of $\{1,\dots,n^*\}$, *so that we have* $\mathsf{W}^*=(\mathsf{W}_{\mathrm{I}}^{*\mathsf{T}},\mathsf{W}_{\mathrm{II}}^{*\mathsf{T}})^{\mathsf{T}}$, *for instance.*

We then introduce the nullspace invariance and the nullspace consistency properties which are a sufficient condition for system (4.2) to be recast into a partial normal form. These properties generalize the nullspace invariance property in the case for neutral gases [9, 3].

(Edp_3) **(nullspace invariance)** *The nullspace of the symmetric part of the matrix* $\widetilde{\mathsf{B}}^*(\mathsf{V}^*,\boldsymbol{\xi})=\sum_{i,j\in\mathfrak{C}^*}\widetilde{\mathsf{B}}_{ij}^*(\mathsf{V}^*)\xi_i\xi_j$, *denoted by* N, *does not depend on* $\mathsf{V}^*\in\mathcal{O}_{\mathsf{V}^*}$ *and* $\boldsymbol{\xi}\in\Sigma^{d-1}$, *and we denote by* n_0^* *its dimension.*

(Edp_4) **(nullspace consistency)** *Denoting by* N *the nullspace of the symmetric part of the matrix* $\widetilde{\mathsf{B}}^*(\mathsf{V}^*,\boldsymbol{\xi})$, *we have*

$$\widetilde{\mathsf{B}}_{ij}^*(\mathsf{V}^*)N=\widetilde{\mathsf{B}}_{ij}^{*\,\mathsf{T}}(\mathsf{V}^*)N=0,\quad i,j\in\mathfrak{C}^*,\quad N^{\mathsf{T}}\widetilde{\mathsf{A}}_i^{*\mathrm{e}}(\mathsf{V}^*)N=0,\quad i\in\mathfrak{C}^*,$$

that is to say $\widetilde{\mathsf{B}}_{ij}^*(\mathsf{V}^*)\mathsf{X}=0$, $\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{B}}_{ij}^*(\mathsf{V}^*)=0$, *and* $\mathsf{Y}^{\mathsf{T}}\widetilde{\mathsf{A}}_i^{*\mathrm{e}}(\mathsf{V}^*)\mathsf{X}=0$ *for* $\mathsf{X},\mathsf{Y}\in N$.

In the following of this section, we assume that these properties holds. In order to characterize more easily partial normal forms for partial symmetric systems satisfying the nullspace consistency property, we introduce the auxiliary variables $\mathsf{U}^{*\prime}$ and $\mathsf{V}^{*\prime}$, depending linearly on U^* and V^*, respectively. The dissipation matrices corresponding to these auxiliary variables have nonzero coefficients only in the lower right bloc of size $n^*-n_0^*$, and moreover, the Jacobian matrices $\widetilde{\mathsf{A}}_i^{*\mathrm{e}\prime}$, $i \in \mathfrak{C}$, have zero coefficients in the upper left bloc of size n_0^*. Partial normal forms are then equivalently – and more easily – obtained from the $\mathsf{V}^{*\prime}$ symmetric equation.

Lemma 4.5 *Consider a system* (4.2) *that is partial symmetric in the sense of definition 4.1. Denote by σ^* the associated partial entropy function and by $\mathsf{V}^* = (\partial_{\mathsf{U}^*}\sigma^*)^{\mathsf{T}}$ the partial symmetrizing variable, and assume that the nullspace consistency property is satisfied over $\mathcal{O}_{\mathsf{V}^*}$. Further consider any constant nonsingular matrix P of dimension n^*, such that its first n_0^* columns span the nullspace N. Then the auxiliary variable $\mathsf{U}^{*\prime} = \mathsf{P}^{\mathsf{T}}\mathsf{U}^*$ satisfies the equation*

$$\partial_t \mathsf{U}^{*\prime} + \sum_{i\in\mathfrak{C}^*} \left(\mathsf{A}_i^{*\prime} + \mathsf{A}_i^{*\mathrm{e}\prime}\right) \partial_i \mathsf{U}^{*\prime} = \sum_{i,j\in\mathfrak{C}^*} \partial_i \left(\mathsf{B}_{ij}^{*\prime}\partial_j \mathsf{U}^{*\prime}\right) + \Omega^{*\prime}, \tag{4.4}$$

where $\mathsf{A}_i^{*\prime} = \mathsf{P}^{\mathsf{T}}\mathsf{A}_i^*(\mathsf{P}^{\mathsf{T}})^{-1}$, $\mathsf{A}_i^{*\mathrm{e}\prime} = \mathsf{P}^{\mathsf{T}}\mathsf{A}_i^{*\mathrm{e}}(\mathsf{P}^{\mathsf{T}})^{-1}$, $\mathsf{B}_{ij}^{*\prime} = \mathsf{P}^{\mathsf{T}}\mathsf{B}_{ij}^*(\mathsf{P}^{\mathsf{T}})^{-1}$, and $\Omega^{*\prime} = \mathsf{P}^{\mathsf{T}}\Omega^*$. The corresponding partial entropy is then the functional $\sigma^{*\prime}(\mathsf{U}^{*\prime}) = \sigma^*((\mathsf{P}^{\mathsf{T}})^{-1}\mathsf{U}^{*\prime})$, and the associated partial entropic variable $\mathsf{V}^{*\prime} = (\partial_{\mathsf{U}^{*\prime}}\sigma^{*\prime})^{\mathsf{T}}$ is given by $\mathsf{V}^{*\prime} = \mathsf{P}^{-1}\mathsf{V}^*$ and satisfies the equation

$$\partial_t \mathsf{V}^{*\prime} + \sum_{i\in\mathfrak{C}^*} \left(\widetilde{\mathsf{A}}_i^{*\prime} + \widetilde{\mathsf{A}}_i^{*\mathrm{e}\prime}\right) \partial_i \mathsf{V}^{*\prime} = \sum_{i,j\in\mathfrak{C}^*} \partial_i \left(\widetilde{\mathsf{B}}_{ij}^{*\prime}\partial_j \mathsf{V}^{*\prime}\right) + \widetilde{\Omega}^{*\prime}, \tag{4.5}$$

where $\widetilde{\mathsf{A}}_0^{*\prime} = \mathsf{P}^{\mathsf{T}}\widetilde{\mathsf{A}}_0^*\mathsf{P}$, $\widetilde{\mathsf{A}}_i^{*\prime} = \mathsf{P}^{\mathsf{T}}\widetilde{\mathsf{A}}_i^*\mathsf{P}$, $\widetilde{\mathsf{A}}_i^{*\mathrm{e}\prime} = \mathsf{P}^{\mathsf{T}}\widetilde{\mathsf{A}}_i^{*\mathrm{e}}\mathsf{P}$, $\widetilde{\mathsf{B}}_{ij}^{*\prime} = \mathsf{P}^{\mathsf{T}}\widetilde{\mathsf{B}}_{ij}^*\mathsf{P}$, and $\widetilde{\Omega}^{*\prime} = \mathsf{P}^{\mathsf{T}}\widetilde{\Omega}^*$. In particular, the matrices $\widetilde{\mathsf{A}}_i^{*\mathrm{e}\prime}$, $i \in \mathfrak{C}^*$, and $\widetilde{\mathsf{B}}_{ij}^{*\prime}$, $i,j \in \mathfrak{C}^*$, are in the form

$$\widetilde{\mathsf{A}}_i^{*\mathrm{e}\prime} = \begin{pmatrix} 0 & \widetilde{\mathsf{A}}_i^{*\mathrm{e}\prime\mathrm{I},\mathrm{II}} \\ \widetilde{\mathsf{A}}_i^{*\mathrm{e}\prime\mathrm{II},\mathrm{I}} & \widetilde{\mathsf{A}}_i^{*\mathrm{e}\prime\mathrm{II},\mathrm{II}} \end{pmatrix}, \qquad \widetilde{\mathsf{B}}_{ij}^{*\prime} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\mathsf{B}}_{ij}^{*\prime\mathrm{II},\mathrm{II}} \end{pmatrix}, \tag{4.6}$$

and the matrix $\widetilde{\mathsf{B}}^{*\prime\mathrm{II},\mathrm{II}}(\mathsf{V}^{*\prime},\boldsymbol{\xi}) = \sum_{i,j\in\mathfrak{C}^*} \widetilde{\mathsf{B}}_{ij}^{*\prime\mathrm{II},\mathrm{II}}(\mathsf{V}^{*\prime})\xi_i\xi_j$, $\boldsymbol{\xi} \in \Sigma^{d-1}$, satisfies $\mathsf{X}^{\mathsf{T}}\,\widetilde{\mathsf{B}}^{*\prime\mathrm{II},\mathrm{II}}(\mathsf{V}^{*\prime},\boldsymbol{\xi})\,\mathsf{X} > 0$, for $\mathsf{X} \in \mathbb{R}^{n^*-n_0^*}$, $\mathsf{X} \neq 0$, $\mathsf{V}^{*\prime} \in \mathcal{O}_{\mathsf{V}^{*\prime}}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$. Finally, the partial normal form (4.3) *is equivalently obtained by multiplying the V^* equation* (4.2) *by $(\partial_{\mathsf{W}^*}\mathsf{V}^*)^{\mathsf{T}}$ or the $\mathsf{V}^{*\prime}$ equation* (4.5) *by $(\partial_{\mathsf{W}^*}\mathsf{V}^{*\prime})^{\mathsf{T}}$.*

Proof Equation (4.4) is easily established by multiplying (4.1) on the left by P^{T}. This also yields the matrix relations $\mathsf{A}_i^{*\prime} = \mathsf{P}^{\mathsf{T}}\mathsf{A}_i^*(\mathsf{P}^{\mathsf{T}})^{-1}$, $\mathsf{A}_i^{*\mathrm{e}\prime} = \mathsf{P}^{\mathsf{T}}\mathsf{A}_i^{*\mathrm{e}}(\mathsf{P}^{\mathsf{T}})^{-1}$, $\mathsf{B}_{ij}^{*\prime} = \mathsf{P}^{\mathsf{T}}\mathsf{B}_{ij}^*(\mathsf{P}^{\mathsf{T}})^{-1}$, and $\Omega^{*\prime} = \mathsf{P}^{\mathsf{T}}\Omega^*$. It is also easily checked that the functional $\sigma^{*\prime}(\mathsf{U}^{*\prime}) = \sigma^*((\mathsf{P}^{\mathsf{T}})^{-1}\mathsf{U}^{*\prime})$ is the corresponding partial entropy. From the definition $\mathsf{V}^{*\prime} = (\partial_{\mathsf{U}^{*\prime}}\sigma^{*\prime})^{\mathsf{T}}$ and the chain rule, we then get that $\mathsf{V}^{*\prime} = \mathsf{P}^{-1}\mathsf{V}^*$ and

(4.5) is obtained as in (4.1)–(4.2). Since $\widetilde{\mathsf{B}}^{*\prime} = \mathsf{P}^{\mathsf{T}}\widetilde{\mathsf{B}}^{*}\mathsf{P}$ and the first n_0^* columns of P span N, we next deduce that $\widetilde{\mathsf{B}}^{*\prime}$ is in the form

$$\widetilde{\mathsf{B}}^{*\prime} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{\mathsf{B}}^{*\prime \mathrm{II},\mathrm{II}} \end{pmatrix},$$

and similarly, all matrices $\widetilde{\mathsf{B}}^{*\prime}_{ij}$, $i,j \in \mathfrak{C}^*$, and $\widetilde{\mathsf{A}}^{*\mathrm{e}\prime}_{i}$, $i \in \mathfrak{C}^*$, are also in the form (4.6). Moreover, the matrix $\widetilde{\mathsf{B}}^{*\prime \mathrm{II},\mathrm{II}}(\mathsf{V}^{*\prime},\boldsymbol{\xi})$, satisfies $\mathsf{X}^{\mathsf{T}}\, \widetilde{\mathsf{B}}^{*\prime \mathrm{II},\mathrm{II}}(\mathsf{V}^{*\prime},\boldsymbol{\xi})\,\mathsf{X} > 0$, for $\mathsf{X} \in \mathbb{R}^{n^*-n_0^*}$, $\mathsf{X} \neq 0$, $\mathsf{V}^{*\prime} \in \mathcal{O}_{\mathsf{V}^{*\prime}}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$, since the $n^*-n_0^*$ last columns of P span a subspace complementary to N. □

Normal forms for partial symmetrizable systems satisfying the nullspace consistency property are now completely characterized in the following theorem, in terms of the auxiliary variables $\mathsf{U}^{*\prime}$ and $\mathsf{V}^{*\prime}$.

Theorem 4.6 *Keeping the assumptions and notations of Lemma 4.5, any partial normal form of the system* (4.2) *is given by a change of variables in the form*

$$\mathsf{W}^* = (\psi_{\mathrm{I}}(\mathsf{U}^{*\prime}_{\mathrm{I}}), \phi_{\mathrm{II}}(\mathsf{V}^{*\prime}_{\mathrm{II}}))^{\mathsf{T}}, \tag{4.7}$$

where ψ_{I} and ϕ_{II} are two diffeomorphisms of $\mathbb{R}^{n_0^*}$ and $\mathbb{R}^{n^*-n_0^*}$, respectively. Furthermore, we have

$$\overline{\mathcal{T}}^*(\mathsf{W}^*, \boldsymbol{\partial_x}\mathsf{W}^*) = \Big(0, \overline{\mathcal{T}}^*_{\mathrm{II}}(\mathsf{W}^*, \boldsymbol{\partial_x}\mathsf{W}^*_{\mathrm{II}})\Big)^{\mathsf{T}}.$$

Proof The proof is exactly the same as in [3] for neutral gases, with the same notation. We only have to be investigate the matrices $\overline{\mathsf{A}}_i^{*\mathrm{e}}$ which vanish for neutral gases, but the treatment is the same as for the matrices $\overline{\mathsf{B}}^*_{ij}$, using the block structure (4.6). □

4.3 An existence theorem in $V_l(\mathbb{R}^d)$

The abstract system of second order partial differential equation (4.1) rewritten in the partial normal form (4.3) can be split into a hyperbolic subsystem and a parabolic subsystem

$$\begin{cases} \overline{\mathsf{A}}_0^{*\mathrm{I},\mathrm{I}}\,\partial_t \mathsf{W}^*_{\mathrm{I}} = -\displaystyle\sum_{i\in\mathfrak{C}^*} \overline{\mathsf{A}}_i^{*\mathrm{I},\mathrm{I}}\,\partial_i \mathsf{W}^*_{\mathrm{I}} + \overline{\Gamma}^*_{\mathrm{I}}, \\ \overline{\mathsf{A}}_0^{*\mathrm{II},\mathrm{II}}\,\partial_t \mathsf{W}^*_{\mathrm{II}} = -\displaystyle\sum_{i\in\mathfrak{C}^*} \left(\overline{\mathsf{A}}_i^{*\mathrm{II},\mathrm{I}} + \overline{\mathsf{A}}_i^{*\mathrm{eII},\mathrm{I}}\right)\partial_i \mathsf{W}^*_{\mathrm{I}} + \displaystyle\sum_{i,j\in\mathfrak{C}^*} \partial_i\left(\overline{\mathsf{B}}_{ij}^{*\mathrm{II},\mathrm{II}}\partial_j \mathsf{W}^*_{\mathrm{II}}\right) + \overline{\Gamma}^*_{\mathrm{II}}, \end{cases} \tag{4.8}$$

where

$$\overline{\Gamma}^*_{\mathrm{I}} = \overline{\Omega}^*_{\mathrm{I}} - \sum_{i\in\mathfrak{C}^*}(\overline{\mathsf{A}}_i^{*\mathrm{eI},\mathrm{II}} + \overline{\mathsf{A}}_i^{*\mathrm{I},\mathrm{II}})\partial_i \mathsf{W}^*_{\mathrm{II}}, \quad \overline{\Gamma}^*_{\mathrm{II}} = \overline{\Omega}^*_{\mathrm{II}} + \overline{\mathcal{T}}^*_{\mathrm{II}} - \sum_{i\in\mathfrak{C}^*}(\overline{\mathsf{A}}_i^{*\mathrm{eII},\mathrm{II}} + \overline{\mathsf{A}}_i^{*\mathrm{II},\mathrm{II}})\partial_i \mathsf{W}^*_{\mathrm{II}}.$$

We then consider the Cauchy problem for the system (4.8) with initial conditions

$$\mathsf{W}_{\mathrm{I}}^*(0,x) = \mathsf{W}_{\mathrm{I}}^{*0}(x), \quad \mathsf{W}_{\mathrm{II}}^*(0,x) = \mathsf{W}_{\mathrm{II}}^{*0}(x). \tag{4.9}$$

These equations are considered in the strip $\bar{Q}_\Theta$ where Θ is positive and $Q_t = (0,t)\times\mathbb{R}^d$, for $t > 0$. The unknown vectors $\mathsf{W}_{\mathrm{I}}^*$ and $\mathsf{W}_{\mathrm{II}}^*$ are assumed to be in the convex open sets $\mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*} \subset \mathbb{R}^{n_0^*}$ and $\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*} \subset \mathbb{R}^{n^*-n_0^*}$.

We will use the classical functional spaces $L_p(\mathbb{R}^d)$ with norm

$$\|\phi\|_{0,p} = \left(\int_{\mathbb{R}^d} |\phi(x)|^p \,\mathrm{d}x\right)^{1/p}, \quad \text{if } 1 \leqslant p < \infty, \text{ and } \quad \|\phi\|_{0,\infty} = \sup_{\mathbb{R}^d} |\phi(x)|,$$

the Sobolev spaces $W_p^l(\mathbb{R}^d)$, $1 \leqslant p \leqslant \infty$, with norm

$$\|\phi\|_{l,p} = \sum_{k\in[\![0,l]\!]} |\phi|_{k,p}, \quad |\phi|_{k,p} = \sum_{|\beta|=k} \|\partial^\beta \phi\|_{0,p},$$

and the Vol'Pert spaces $V_l(\mathbb{R}^d)$ with norm [6]

$$\|\phi\|_l = |\phi|_{0,\infty} + \sum_{k\in[\![1,l]\!]} |\phi|_{k,2}.$$

We extend these definitions to vector functions by using the Euclidean norm of $\mathbb{R}^d$. According to the Sobolev inequalities, there is an embedding of $W_2^l(\mathbb{R}^d)$ into $W_\infty^k(\mathbb{R}^d)$ for $l > k + d/2$ and an embedding of $W_2^l(\mathbb{R}^d)$ into $V_l(\mathbb{R}^d)$ for $l > d/2$.

In the following, $\mathcal{L}$ denotes an arbitrary fixed positive continuous convex function, on the open convex set $\mathcal{O}_{\mathsf{W}^*} = \mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}\times\mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$, which grows without bound as any finite point of the boundary of $\mathcal{O}_{\mathsf{W}^*}$ is approached.

The following theorem of Vol'pert and Hudjaev [6] shows that, in a certain strip, there exists a solution which preserves the smoothness of the initial condition.

Theorem 4.7 *Suppose that the system* (4.8)-(4.9) *satisfies the following assumptions where l denotes an integer such that $l > d/2 + 3$.*

(Ex$_1$) *The initial conditions $\mathsf{W}_{\mathrm{I}}^{*0}$, $\mathsf{W}_{\mathrm{II}}^{*0}$ satisfy $\sup_{x\in\mathbb{R}^d} \mathcal{L}(\mathsf{W}_{\mathrm{I}}^{*0}(x), \mathsf{W}_{\mathrm{II}}^{*0}(x)) < +\infty$ and $\mathsf{W}_{\mathrm{I}}^{*0}$ and $\mathsf{W}_{\mathrm{II}}^{*0}$ are in the space $V_l(\mathbb{R}^d)$.*

(Ex$_2$) *The matrix coefficients $\overline{\mathsf{A}}_0^{*\mathrm{I,I}}$, $\overline{\mathsf{A}}_0^{*\mathrm{II,II}}$, $i \in \mathfrak{C}^*$, $\overline{\mathsf{B}}_{ij}^{*\mathrm{II,II}}$, $i,j \in \mathfrak{C}^*$, depend only on $\mathsf{W}_{\mathrm{I}}^*$ and on $\mathsf{W}_{\mathrm{II}}^*$. The matrix coefficients $\overline{\mathsf{A}}_i^{*\mathrm{I,I}}$, $\overline{\mathsf{A}}_i^{*\mathrm{II,I}}$, $\overline{\mathsf{A}}_i^{*\mathrm{eII,I}}$, $i \in \mathfrak{C}^*$ and the vector coefficients $\overline{\Gamma}_{\mathrm{I}}^*$, $\overline{\Gamma}_{\mathrm{II}}^*$ depend on $\mathsf{W}_{\mathrm{I}}^*$, $\mathsf{W}_{\mathrm{II}}^*$ and on $\boldsymbol{\partial_x}\mathsf{W}_{\mathrm{II}}^*$.*

(Ex$_3$) *The matrix coefficients $\overline{\mathsf{A}}_0^{*\mathrm{I,I}}(w_{\mathrm{I}}, w_{\mathrm{II}})$, $\overline{\mathsf{A}}_0^{*\mathrm{II,II}}(w_{\mathrm{I}}, w_{\mathrm{II}})$ and $\overline{\mathsf{B}}_{ij}^{*\mathrm{II,II}}(w_{\mathrm{I}}, w_{\mathrm{II}})$, $i,j \in \mathfrak{C}^*$, have continuous derivative of order l with respect to $w_{\mathrm{I}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$ and $w_{\mathrm{II}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$.*

($\mathsf{Ex_4}$) *The matrix coefficients* $\overline{\mathsf{A}}_i^{*\mathrm{I},\mathrm{I}}(w_{\mathrm{I}}, w_{\mathrm{II}}, \xi)$, $\overline{\mathsf{A}}_i^{*\mathrm{II},\mathrm{I}}(w_{\mathrm{I}}, w_{\mathrm{II}}, \xi)$, $\overline{\mathsf{A}}_i^{*\mathrm{eII},\mathrm{I}}(w_{\mathrm{I}}, w_{\mathrm{II}}, \xi)$, $i \in \mathfrak{C}^*$, *and the vector coefficients* $\overline{\Gamma}_{\mathrm{I}}^*(w_{\mathrm{I}}, w_{\mathrm{II}}, \xi)$ *and* $\overline{\Gamma}_{\mathrm{II}}^*(w_{\mathrm{I}}, w_{\mathrm{II}}, \xi)$ *have continuous derivative of order* l *with respect to* $w_{\mathrm{I}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$, $w_{\mathrm{II}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$ *and* $\boldsymbol{\xi} \in \mathbb{R}^{d\times(n^*-n_0^*)}$.

($\mathsf{Ex_5}$) *The matrix coefficients* $\overline{\mathsf{A}}_0^{*\mathrm{I},\mathrm{I}}(w_{\mathrm{I}}, w_{\mathrm{II}})$ *and* $\overline{\mathsf{A}}_0^{*\mathrm{II},\mathrm{II}}(w_{\mathrm{I}}, w_{\mathrm{II}})$ *are symmetric and positive definite for* $w_{\mathrm{I}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$ *and* $w_{\mathrm{II}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$.

($\mathsf{Ex_6}$) *The matrix coefficients* $\overline{\mathsf{A}}_i^{*\mathrm{I},\mathrm{I}}(w_{\mathrm{I}}, w_{\mathrm{II}}, \xi)$, $i \in \mathfrak{C}^*$, *are symmetric for* $w_{\mathrm{I}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$, $w_{\mathrm{II}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$ *and* $\xi \in \mathbb{R}^{d\times(n^*-n_0^*)}$.

($\mathsf{Ex_7}$) *The matrices* $\overline{\mathsf{A}}_0^{*\mathrm{I},\mathrm{I}}(w_{\mathrm{I}}, w_{\mathrm{II}})$, $\overline{\mathsf{A}}_0^{*\mathrm{II},\mathrm{II}}(w_{\mathrm{I}}, w_{\mathrm{II}})$ *and the vectors* $\overline{\Gamma}_{\mathrm{I}}^*(w_{\mathrm{I}}, w_{\mathrm{II}}, 0)$ *and* $\overline{\Gamma}_{\mathrm{II}}^*(w_{\mathrm{I}}, w_{\mathrm{II}}, 0)$ *have continuous derivatives to order* $l+3$ *in* $w_{\mathrm{I}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*}$ *and* $w_{\mathrm{II}} \in \mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$.

($\mathsf{Ex_8}$) *For any compact subset* K *of* $\mathcal{O}_{\mathsf{W}^*} = \mathcal{O}_{\mathsf{W}_{\mathrm{I}}^*} \times \mathcal{O}_{\mathsf{W}_{\mathrm{II}}^*}$, *there exists* $\alpha > 0$ *such that for any smooth function* $w = (w_{\mathrm{I}}, w_{\mathrm{II}})$ *from* $\mathbb{R}^d$ *to* $\mathbb{R}^{n^*}$ *with value in* K *we have*

$$\int_{\mathbb{R}^d} \sum_{i,j\in\mathfrak{C}^*} (\partial_i \phi_{\mathrm{II}})^{\mathsf{T}}\ \overline{\mathsf{B}}_{ij}^{*\mathrm{II},\mathrm{II}}(w(x))\,(\partial_j \phi_{\mathrm{II}})\ \mathrm{d}x \geqslant \alpha \int_{\mathbb{R}^d} \sum_{i\in\mathfrak{C}^*} (\partial_i \phi_{\mathrm{II}})^{\mathsf{T}}\,(\partial_i \phi_{\mathrm{II}})\ \mathrm{d}x, \tag{4.10}$$

where ϕ_{II} *is any function in* $W_2^1(\mathbb{R}^d)$ *with* $n^*-n_0^*$ *components.*

Then there exists t_0, $0 < t_0 \leqslant \Theta$, *such that the Cauchy problem* (4.8), (4.9), *admits a unique solution* $(\mathsf{W}_{\mathrm{I}}^{*\mathsf{T}}, \mathsf{W}_{\mathrm{II}}^{*\mathsf{T}})^{\mathsf{T}}$ *defined on* $\bar{Q}_{t_0} = [0, t_0]\times\mathbb{R}^d$, *which is continuous with its derivatives of first order in* t *and second order in* $\boldsymbol{x}$, *and for which the following quantities are finite*

$$\sup_{0\leqslant t\leqslant t_0} \|(\mathsf{W}_{\mathrm{I}}^*(t), \mathsf{W}_{\mathrm{II}}^*(t))\|_l, \quad \sup_{\bar{Q}_{t_0}} \mathcal{L}(\mathsf{W}_{\mathrm{I}}^*, \mathsf{W}_{\mathrm{II}}^*), \tag{4.11}$$

$$\sup_{0\leqslant t\leqslant t_0} \|\partial_t \mathsf{W}_{\mathrm{I}}^*(t)\|_{l-1}, \quad \int_0^{t_0} \Big(\|\partial_t \mathsf{W}_{\mathrm{II}}^*(\tau)\|_{l-1}^2 + \|\mathsf{W}_{\mathrm{II}}^*(\tau)\|_{l+1}^2 \Big)\, \mathrm{d}\tau. \tag{4.12}$$

Moreover, either $t_0 = \Theta$, *or there exists* t_1 *such that the theorem is true for any* $t_0 < t_1$ *and such that for* $t_0 \to t_1^-$, *at least one of the quantities*

$$\|\mathsf{W}_{\mathrm{I}}^*(t_0)\|_{1,\infty} + \|\mathsf{W}_{\mathrm{II}}^*(t_0)\|_{2,\infty}, \quad \sup_{\bar{Q}_{t_0}} \mathcal{L}(\mathsf{W}_{\mathrm{I}}^*, \mathsf{W}_{\mathrm{II}}^*), \tag{4.13}$$

grows without bound, that is to say, the solution can be extended as long as quantities (4.13) *remain finite.*

5 Existence theorem for multicomponent magnetized mixtures

We now apply the general results of section 4 to the system of equations governing multicomponent ionized magnetized reactive flows (3.9).

5.1 Partial Symmetrization

For ionized magnetized mixtures, the existence of an entropy function only yields partial symmetrization of the system, because of the terms $\mathsf{F}^{\mathrm{e}}_{ij}$, $i,j \in \mathfrak{C}$, which prevent full symmetrization. Nevertheless, the achieved symmetrization will be sufficient to establish existence of a solution.

Let us first introduce the mathematical entropy function σ as the opposite of the physical entropy per unit volume. Its expression is given by

$$\sigma = -\sum_{k\in\mathfrak{S}} \rho_k s_k = -\frac{1}{T}\sum_{k\in\mathfrak{S}} \rho_k (h_k - g_k). \tag{5.1}$$

We prove in this subsection that σ is a $\mathcal{C}^\infty$ strictly convex function on the convex open set $\mathcal{O}_{\mathsf{U}}$. It seems then judicious to consider the map $\mathsf{U} \mapsto \mathsf{V} = (\partial_{\mathsf{U}}\sigma)^{\mathsf{T}}$.

Proposition 5.1 *The function σ defined on $\mathcal{O}_{\mathsf{U}}$ to $\mathbb{R}$ is a $\mathcal{C}^\infty$ strictly convex function, in the sense that its Hessian matrix $\partial^2_{\mathsf{U}}\sigma$ is symmetric positive definite.*

Proof From assumption (Th_1) on the coefficients $c_{v,k}$, $k \in \mathfrak{S}$, σ is a $\mathcal{C}^\infty$ function. The differential of σ is

$$\mathrm{d}\sigma = -\frac{\rho c_v}{T}\mathrm{d}T - \sum_{k\in\mathfrak{S}} (s_k - \mathcal{r}_k)\mathrm{d}\rho_k,$$

so that

$$\partial_{\mathsf{Z}}\sigma = \Big(\mathcal{r}_1 - s_1, \ldots, \mathcal{r}_{n^s} - s_{n^s}, 0_{1,3}, 0_{1,3}, 0_{1,3}, -\frac{\rho c_v}{T}\Big).$$

We then obtain $\partial_{\mathsf{U}}\sigma = \partial_{\mathsf{Z}}\sigma\partial_{\mathsf{U}}\mathsf{Z}$ and $\partial^2_{\mathsf{U}}\sigma = \partial_{\mathsf{Z}}(\partial_{\mathsf{U}}\sigma)^{\mathsf{T}}\partial_{\mathsf{U}}\mathsf{Z}$:

$$\partial_{\mathsf{U}}\sigma = \frac{1}{T}\Big(g_1 - \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}, \ldots, g_{n^s} - \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}, \boldsymbol{v}^{\mathsf{T}}, \varepsilon_0\boldsymbol{E}^{\mathsf{T}}, \tfrac{1}{\mu_0}\boldsymbol{B}^{\mathsf{T}}, -1\Big),$$

$$\partial^2_{\mathsf{U}}\sigma = \begin{bmatrix}
\mathsf{d}^{\mathsf{r}}_{\varrho} + \frac{v^2\mathsf{u}\otimes\mathsf{u}}{\rho T} + \frac{\mathsf{e}^{\mathrm{r}}\otimes\mathsf{e}^{\mathrm{r}}}{\rho c_v T^2} & -\frac{\mathsf{u}\otimes\boldsymbol{v}}{\rho T} + \frac{\mathsf{e}^{\mathrm{r}}\otimes\boldsymbol{v}}{\rho c_v T^2} & \frac{\varepsilon_0\mathsf{e}^{\mathrm{r}}\otimes\boldsymbol{E}}{\rho c_v T^2} & \frac{\mathsf{e}^{\mathrm{r}}\otimes\boldsymbol{B}}{\mu_0\rho c_v T^2} & -\frac{\mathsf{e}^{\mathrm{r}}}{\rho c_v T^2} \\
-\frac{\boldsymbol{v}\otimes\mathsf{u}}{\rho T} + \frac{\boldsymbol{v}\otimes\mathsf{e}^{\mathrm{r}}}{\rho c_v T^2} & \frac{\boldsymbol{v}\otimes\boldsymbol{v}}{\rho c_v T^2} + \frac{\mathbb{I}}{\rho T} & \frac{\varepsilon_0\boldsymbol{v}\otimes\boldsymbol{E}}{\rho c_v T^2} & \frac{\boldsymbol{v}\otimes\boldsymbol{B}}{\mu_0\rho c_v T^2} & -\frac{\boldsymbol{v}}{\rho c_v T^2} \\
\frac{\varepsilon_0\boldsymbol{E}\otimes\mathsf{e}^{\mathrm{r}}}{\rho c_v T^2} & \frac{\varepsilon_0\boldsymbol{E}\otimes\boldsymbol{v}}{\rho c_v T^2} & \frac{\varepsilon_0^2\boldsymbol{E}\otimes\boldsymbol{E}}{\rho c_v T^2} + \frac{\varepsilon_0\mathbb{I}_3}{T} & \frac{\varepsilon_0\boldsymbol{E}\otimes\boldsymbol{B}}{\mu_0\rho c_v T^2} & -\frac{\varepsilon_0\boldsymbol{E}}{\rho c_v T^2} \\
\frac{\boldsymbol{B}\otimes\mathsf{e}^{\mathrm{r}}}{\mu_0\rho c_v T^2} & \frac{\boldsymbol{B}\otimes\boldsymbol{v}}{\mu_0\rho c_v T^2} & \frac{\varepsilon_0\boldsymbol{B}\otimes\boldsymbol{E}}{\mu_0\rho c_v T^2} & \frac{\boldsymbol{B}\otimes\boldsymbol{B}}{\mu_0^2\rho c_v T^2} + \frac{\mathbb{I}_3}{\mu_0 T} & -\frac{\boldsymbol{B}}{\mu_0\rho c_v T^2} \\
-\frac{\mathsf{e}^{\mathrm{r}\mathsf{T}}}{\rho c_v T^2} & -\frac{\boldsymbol{v}^{\mathsf{T}}}{\rho c_v T^2} & -\frac{\varepsilon_0\boldsymbol{E}^{\mathsf{T}}}{\rho c_v T^2} & -\frac{\boldsymbol{B}^{\mathsf{T}}}{\mu_0\rho c_v T^2} & \frac{1}{\rho c_v T^2}
\end{bmatrix},$$

where $\mathsf{d}^{\mathsf{r}}_{\varrho}$ is the n^s by n^s diagonal matrix defined by $\mathsf{d}^{\mathsf{r}}_{\varrho} = \mathrm{diag}((\mathcal{r}_k/\rho_k)_{k\in\mathfrak{S}})$ and $\mathsf{e}^{\mathrm{r}} = (e^{\mathrm{r}}_1, \ldots, e^{\mathrm{r}}_{n^s})^{\mathsf{T}}$, $e^{\mathrm{r}}_k = e_k - \frac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}$, $k \in \mathfrak{S}$.

The matrix $\partial^2_{\mathsf{U}}\sigma$ is clearly symmetric. We now prove that it is positive definite. Let be $\mathsf{X} = (\mathsf{x}_{\varrho}{}^{\mathsf{T}}, \mathbf{x}_{\boldsymbol{v}}{}^{\mathsf{T}}, \mathbf{x}_{\boldsymbol{E}}{}^{\mathsf{T}}, \mathbf{x}_{\boldsymbol{B}}{}^{\mathsf{T}}, \mathsf{x}_T)^{\mathsf{T}}$. After some algebra, $\mathsf{X}^{\mathsf{T}}\ \partial^2_{\mathsf{U}}\sigma\ \mathsf{X}$ can be written in the form

$$\begin{aligned}\mathsf{X}^{\mathsf{T}}\ \partial^2_{\mathsf{U}}\sigma\ \mathsf{X} =& \mathsf{x}_\varrho{}^{\mathsf{T}}\mathsf{d}^{\mathrm{r}}_\varrho \mathsf{x}_\varrho + \frac{\varepsilon_0}{T}\mathbf{x}_{\boldsymbol{E}}\cdot\mathbf{x}_{\boldsymbol{E}} + \frac{1}{\mu_0 T}\mathbf{x}_{\boldsymbol{B}}\cdot\mathbf{x}_{\boldsymbol{B}} + \frac{1}{\rho T}(\mathbf{x}_{\boldsymbol{v}} - \mathsf{x}_\varrho\cdot\mathsf{u}\ \boldsymbol{v})\cdot(\mathbf{x}_{\boldsymbol{v}} - \mathsf{x}_\varrho\cdot\mathsf{u}\ \boldsymbol{v})\\ &+ \frac{1}{\rho c_v T^2}\Big[\mathsf{e}^{\mathrm{r}}\cdot\mathsf{x}_\varrho + \boldsymbol{v}\cdot\mathbf{x}_{\boldsymbol{v}} + \varepsilon_0 \boldsymbol{E}\cdot\mathbf{x}_{\boldsymbol{E}} + \tfrac{1}{\mu_0}\boldsymbol{B}\cdot\mathbf{x}_{\boldsymbol{B}} - \mathsf{x}_T\Big]^2\end{aligned}$$

so that the Hessian matrix $\partial^2_{\mathsf{U}}\sigma$ is positive definite. □

We then introduce the partial entropic variables V by

$$\mathsf{V} = \frac{1}{T}\Big(g_1 - \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}, \dots, g_{n^s} - \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}, \boldsymbol{v}^{\mathsf{T}}, \varepsilon_0\boldsymbol{E}^{\mathsf{T}}, \tfrac{1}{\mu_0}\boldsymbol{B}^{\mathsf{T}}, -1\Big)^{\mathsf{T}}. \qquad (5.2)$$

Proposition 5.2 *The map* $\mathsf{U} \mapsto \mathsf{V}$ *is a* $\mathcal{C}^\infty$ *diffeomorphism from the open set* $\mathcal{O}_{\mathsf{U}}$ *onto the open set* $\mathcal{O}_{\mathsf{V}} = \mathbb{R}^{n^s+9}\times(-\infty, 0)$.

Proof For proposition 3.1, we have to prove that the map $\mathsf{Z} \mapsto \mathsf{V}$ is a $\mathcal{C}^\infty$ diffeomorphism from $\mathcal{O}_{\mathsf{Z}}$ onto $\mathcal{O}_{\mathsf{V}}$. It is $\mathcal{C}^\infty$ for assumption (Th_1) on the coefficients $c_{v,k}$, $k \in \mathfrak{S}$. For assumption (Th_1), positivity of $c_{v,k}$, $k \in \mathfrak{S}$ yields that the map $\mathsf{Z} \mapsto \mathsf{V}$ is one to one. We now prove that the map is onto. Let be $\mathsf{V} \in \mathcal{O}_{\mathsf{V}}$, we define

$$\mathsf{Z}_5 = -\frac{1}{\mathsf{V}_5}, \quad \mathsf{Z}_4 = -\mu_0\frac{\mathsf{V}_4}{\mathsf{V}_5}, \quad \mathsf{Z}_3 = -\frac{1}{\varepsilon_0}\frac{\mathsf{V}_3}{\mathsf{V}_5}, \quad \mathsf{Z}_2 = -\frac{\mathsf{V}_2}{\mathsf{V}_5},$$

$$\mathsf{Z}_{1k} = m_k\gamma^{\mathrm{st}}\exp\left[\frac{\mathsf{V}_{1k} - r_k + s^{\mathrm{st}}_k + e^{\mathrm{st}}_k\mathsf{V}_5}{r_k} - \frac{\mathsf{V}_4\cdot\mathsf{V}_4}{2r_k\mathsf{V}_5} + \frac{1}{r_k}\int_{T^{\mathrm{st}}}^{\frac{-1}{\mathsf{V}_5}} c_{v,k}(\tau)\left(\tfrac{1}{\tau} + \mathsf{V}_5\right)\,\mathrm{d}\tau\right],$$

for $k \in \mathfrak{S}$. The range of the vector Z so defined is then V. We now give the expression of the matrix $\partial_{\mathsf{Z}}\mathsf{V}$.

$$\partial_{\mathsf{Z}}\mathsf{V} = \begin{bmatrix} \mathsf{d}^{\mathrm{r}}_\varrho & -\frac{1}{T}\mathsf{u}\otimes\boldsymbol{v} & 0_{n^s,3} & 0_{n^s,3} & -\frac{1}{T^2}\mathsf{e}^{\mathrm{r}} \\ 0_{3,n^s} & \frac{1}{T}\mathbb{I} & 0_{3,3} & 0_{3,3} & -\frac{1}{T^2}\boldsymbol{v} \\ 0_{3,n^s} & 0_{3,3} & \frac{\varepsilon_0}{T}\mathbb{I} & 0_{3,3} & -\frac{\varepsilon_0}{T^2}\boldsymbol{E} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & \frac{1}{\mu_0 T}\mathbb{I} & -\frac{1}{\mu_0 T^2}\boldsymbol{B} \\ 0_{1,n^s} & 0_{1,3} & 0_{1,3} & 0_{1,3} & \frac{1}{T^2} \end{bmatrix}. \qquad (5.3)$$

$\partial_{\mathsf{Z}}\mathsf{V}$ is then an bloc upper triangular matrix and these diagonal terms are positive for assumption (Th_1). From the inverse function theorem, we deduce that the map $\mathsf{Z} \mapsto \mathsf{V}$ is a $\mathcal{C}^\infty$ diffeomorphism onto $\mathcal{O}_{\mathsf{V}}$. □

We then investigate the symmetry properties of the system of partial differential equations governing multicomponent magnetized reactive flows. We obtain in particular a partial symmetrized form of this system, making use of entropic variables.

Theorem 5.3 *The function σ is a partial entropy function for the system* (3.9)*. Furthermore, the change of variables* $\mathsf{U} \mapsto \mathsf{V}$ *transforms the system* (3.9) *into*

$$\widetilde{\mathsf{A}}_0(\mathsf{V})\,\partial_t\mathsf{V} + \sum_{i\in\mathfrak{C}} \Big(\widetilde{\mathsf{A}}_i(\mathsf{V}) + \widetilde{\mathsf{A}}_i^{\mathrm{e}}(\mathsf{V})\Big)\,\partial_i\mathsf{V} = \sum_{i,j\in\mathfrak{C}} \partial_i\Big(\widetilde{\mathsf{B}}_{ij}(\mathsf{V})\,\partial_j\mathsf{V}\Big) + \widetilde{\Omega}(\mathsf{V}), \quad (5.4)$$

with $\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{V}}\mathsf{U}$, $\widetilde{\mathsf{A}}_i = \mathsf{A}_i\partial_{\mathsf{V}}\mathsf{U}$, $\widetilde{\mathsf{A}}_i^{\mathrm{e}} = \mathsf{A}_i^{\mathrm{e}}\partial_{\mathsf{V}}\mathsf{U}$, $\widetilde{\mathsf{B}}_{ij} = \mathsf{B}_{ij}\partial_{\mathsf{V}}\mathsf{U}$, $\widetilde{\Omega} = \Omega$, *where matrices* $\widetilde{\mathsf{A}}_0(\mathsf{V})$, $\widetilde{\mathsf{A}}_i(\mathsf{V})$, $\widetilde{\mathsf{A}}_i^{\mathrm{e}}(\mathsf{V})$, $i\in\mathfrak{C}$, $\widetilde{\mathsf{B}}_{ij}(\mathsf{V})$, $i,j\in\mathfrak{C}$, *and vector* $\widetilde{\Omega}(\mathsf{V})$ *are* $\mathcal{C}^\infty$ *functions. Moreover, the system* (5.4) *is of the partial symmetric form, that is, the matrices* $\widetilde{\mathsf{A}}_0$, $\widetilde{\mathsf{A}}_i$, $i\in\mathfrak{C}$, *and* $\widetilde{\mathsf{B}}_{ij}$, $i,j\in\mathfrak{C}$, *verify properties* (S_1-S_3).

Proof 3.2 In order to evaluate the matrices $\widetilde{\mathsf{A}}_0$, $\widetilde{\mathsf{A}}_i$, $\widetilde{\mathsf{A}}_i^{\mathrm{e}}$, $i\in\mathfrak{C}$, and $\widetilde{\mathsf{B}}_{ij}$, $i,j\in\mathfrak{C}$, we use notations defining in the proof of the proposition 3.2 and concerning the system written in the natural variable Z,

$$\widehat{\mathsf{A}}_0\partial_t\mathsf{Z} + \sum_{i\in\mathfrak{C}}(\widehat{\mathsf{A}}_i + \widehat{\mathsf{A}}_i^{\mathrm{e}})\partial_i\mathsf{Z} = \sum_{i,j\in\mathfrak{C}} \partial_i(\widehat{\mathsf{B}}_{ij}\partial_j\mathsf{Z}) + \Omega.$$

These matrices are indeed easily calculated by using the natural variable Z and the relations $\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{Z}}\mathsf{U}\partial_{\mathsf{V}}\mathsf{Z}$, $\widetilde{\mathsf{A}}_i = \widehat{\mathsf{A}}_i\partial_{\mathsf{V}}\mathsf{Z}$, $\widetilde{\mathsf{A}}_i^{\mathrm{e}} = \widehat{\mathsf{A}}_i^{\mathrm{e}}\partial_{\mathsf{V}}\mathsf{Z}$, and $\widetilde{\mathsf{B}}_{ij} = \widehat{\mathsf{B}}_{ij}\partial_{\mathsf{V}}\mathsf{Z}$, where the matrix $\partial_{\mathsf{V}}\mathsf{Z}$ reads

$$\partial_{\mathsf{V}}\mathsf{Z} = \begin{bmatrix} \mathsf{d}_{\mathsf{r}}^{\varrho} & \mathsf{v}_{\mathsf{r}}^{\varrho}\otimes\boldsymbol{v} & 0_{n^s,3} & 0_{n^s,3} & \mathsf{d}_{\mathsf{r}}^{\varrho}\mathsf{e}^{\mathrm{r}} \\ 0_{3,n^s} & T\mathbb{I} & 0_{3,3} & 0_{3,3} & T\boldsymbol{v} \\ 0_{3,n^s} & 0_{3,3} & \frac{T}{\varepsilon_0}\mathbb{I} & 0_{3,3} & T\boldsymbol{E} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & \mu_0 T\mathbb{I} & T\boldsymbol{B} \\ 0_{1,n^s} & 0_{1,3} & 0_{1,3} & 0_{1,3} & T^2 \end{bmatrix}, \quad (5.5)$$

with $\mathsf{d}_{\mathsf{r}}^{\varrho}$ the n^s by n^s diagonal matrix defined by $\mathsf{d}_{\mathsf{r}}^{\varrho} = \mathrm{diag}((\rho_k/r_k)_{k\in\mathfrak{S}})$ and $\mathsf{v}_{\mathsf{r}}^{\varrho}$ the vector given by $\mathsf{v}_{\mathsf{r}}^{\varrho} = \mathsf{d}_{\mathsf{r}}^{\varrho}\mathsf{u} = ((\rho_k/r_k)_{k\in\mathfrak{S}})^{\mathsf{T}}$.

The matrix $\widetilde{\mathsf{A}}_0$ can be written in the form

$$\widetilde{\mathsf{A}}_0 = \begin{bmatrix} \mathsf{d}_{\mathsf{r}}^{\varrho} & \mathsf{v}_{\mathsf{r}}^{\varrho}\otimes\boldsymbol{v} & 0_{n^s,3} & 0_{n^s,3} & \mathsf{d}_{\mathsf{r}}^{\varrho}\mathsf{e}^{\mathrm{f}} \\ & \rho T\mathbb{I} + \Sigma_\rho\boldsymbol{v}\otimes\boldsymbol{v} & 0_{n^s,3} & 0_{n^s,3} & (\rho T + \Sigma_e)\boldsymbol{v} \\ & & \frac{T}{\varepsilon_0}\mathbb{I} & 0_{3,3} & T\boldsymbol{E} \\ & & & T\mu_0\mathbb{I} & T\boldsymbol{B} \\ \mathrm{Sym} & & & & \Upsilon_e \end{bmatrix}, \quad (5.6)$$

where we have defined $\Sigma_\rho = \mathsf{u}^{\mathsf{T}}\mathsf{d}_{\mathsf{r}}^{\varrho}\mathsf{u}$, $\Sigma_e = \mathsf{u}^{\mathsf{T}}\mathsf{d}_{\mathsf{r}}^{\varrho}\mathsf{e}^{\mathrm{f}}$ and $\Upsilon_e = \mathsf{e}^{\mathrm{f}\mathsf{T}}\mathsf{d}_{\mathsf{r}}^{\varrho}\mathsf{e}^{\mathrm{f}} + \rho T\boldsymbol{v}\cdot\boldsymbol{v} + \rho c_v T^2 + 2T\rho e^{\mathrm{e}}$. Since this matrix is symmetric, we only give its right upper triangular part and write "Sym" in the lower triangular part. The matrix $\widetilde{\mathsf{A}}_0$ is positive definite since for any vector $\mathsf{X}\in\mathbb{R}^{n^s+10}$, written in the form $\mathsf{X} = (\mathsf{x}_\varrho{}^{\mathsf{T}}, \mathbf{x}_{\boldsymbol{v}}{}^{\mathsf{T}}, \mathbf{x}_{\boldsymbol{E}}{}^{\mathsf{T}}, \mathbf{x}_{\boldsymbol{B}}{}^{\mathsf{T}}, \mathsf{x}_T)^{\mathsf{T}}$,

$$\begin{aligned} \mathsf{X}^{\mathsf{T}}\,\widetilde{\mathsf{A}}_0\,\mathsf{X} \;=\;& \rho c_v T^2\mathsf{x}_T^2 + \tfrac{T}{\varepsilon_0}(\mathbf{x}_{\boldsymbol{E}} + \varepsilon_0\mathsf{x}_T\boldsymbol{E})^{\mathsf{T}}\cdot(\mathbf{x}_{\boldsymbol{E}} + \varepsilon_0\mathsf{x}_T\boldsymbol{E}) \\ & + \rho T(\mathbf{x}_{\boldsymbol{v}} + \mathsf{x}_T\boldsymbol{v})\cdot(\mathbf{x}_{\boldsymbol{v}} + \mathsf{x}_T\boldsymbol{v}) + \mu_0 T(\mathbf{x}_{\boldsymbol{B}} + \tfrac{1}{\mu_0}\mathsf{x}_T\boldsymbol{B})^{\mathsf{T}}\cdot(\mathbf{x}_{\boldsymbol{B}} + \tfrac{1}{\mu_0}\mathsf{x}_T\boldsymbol{B}) \\ & + (\mathsf{x}_\varrho + \mathbf{x}_{\boldsymbol{v}}\cdot\boldsymbol{v}\mathsf{u} + \mathsf{x}_T\mathsf{e}^{\mathrm{f}})^{\mathsf{T}}\mathsf{d}_{\mathsf{r}}^{\varrho}(\mathsf{x}_\varrho + \mathbf{x}_{\boldsymbol{v}}\cdot\boldsymbol{v}\mathsf{u} + \mathsf{x}_T\mathsf{e}^{\mathrm{f}}). \end{aligned}$$

Denoting by $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^{\mathsf{T}}$ an arbitrary vector of $\mathbb{R}^3$, the matrices $\widetilde{\mathsf{A}}_i$, $i \in \mathfrak{C}$, are given by

$$\sum_{i\in\mathfrak{C}} \widetilde{\mathsf{A}}_i \xi_i = \begin{bmatrix} \mathsf{d}_\mathrm{r}^\varrho \boldsymbol{v}\cdot\boldsymbol{\xi} & \boldsymbol{v}\cdot\boldsymbol{\xi}\mathsf{v}_\mathrm{r}^\varrho \otimes \boldsymbol{v} + T\varrho\otimes\boldsymbol{\xi} & 0_{n^s,3} & 0_{n^s,3} & \mathsf{d}_\mathrm{r}^\varrho \mathsf{h}^\mathrm{p} \boldsymbol{v}\cdot\boldsymbol{\xi} \\ & \begin{matrix}\rho T(\boldsymbol{v}\cdot\boldsymbol{\xi}\mathbb{I} + \boldsymbol{v}\otimes\boldsymbol{\xi} + \boldsymbol{\xi}\otimes\boldsymbol{v}) \\ +\Sigma_\rho \boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{v}\otimes\boldsymbol{v}\end{matrix} & 0_{n^s,3} & 0_{n^s,3} & \begin{matrix}(\Sigma_h + \rho T)\boldsymbol{v}\cdot\boldsymbol{\xi}\boldsymbol{v} \\ +\rho T h^\mathrm{p} \boldsymbol{\xi}\end{matrix} \\ & & 0_{3,3} & -\frac{T}{\varepsilon_0}\mathbf{T}(\boldsymbol{\xi}) & \frac{1}{\varepsilon_0\mu_0} T\boldsymbol{B}\wedge\boldsymbol{\xi} \\ & & & 0_{3,3} & -T\boldsymbol{E}\wedge\boldsymbol{\xi} \\ \mathrm{Sym} & & & & \Upsilon_h \boldsymbol{v}\cdot\boldsymbol{\xi} + 2T\boldsymbol{P}\cdot\boldsymbol{\xi} \end{bmatrix}, \tag{5.7}$$

with $\Sigma_h = \mathsf{u}^{\mathsf{T}} \mathsf{d}_\mathrm{r}^\varrho \mathsf{h}^\mathrm{p}$ and $\Upsilon_h = \mathsf{h}^{\mathrm{p}\mathsf{T}} \mathsf{d}_\mathrm{r}^\varrho \mathsf{h}^\mathrm{p} + \rho T \boldsymbol{v}\cdot\boldsymbol{v} + \rho c_p T^2$.

Concerning the matrices $\widetilde{\mathsf{A}}_i^\mathrm{e}$, $i \in \mathfrak{C}$, we have

$$\widetilde{\mathsf{A}}_i^\mathrm{e} = -\sum_{j\in\mathfrak{C}} \partial_\mathsf{Z} \mathsf{F}_{ij}^\mathrm{e} \, \partial_\mathsf{V} \mathsf{Z} - \widetilde{\mathsf{A}}_i^{\mathrm{e},\Omega}, \tag{5.8}$$

where the matrices $\widetilde{\mathsf{A}}_i^{\mathrm{e},\Omega}$, $i \in \mathfrak{C}$, read

$$\widetilde{\mathsf{A}}_i^{\mathrm{e},\Omega} = \begin{bmatrix} 0_{n^s,n^s} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,1} \\ \widehat{\mathsf{A}}_{i,\boldsymbol{v},\varrho}^{\mathrm{e},\Omega} \mathsf{d}_\mathrm{r}^\varrho & \widehat{\mathsf{A}}_{i,\boldsymbol{v},\varrho}^{\mathrm{e},\Omega} \mathsf{v}_\mathrm{r}^\varrho \otimes \boldsymbol{v} & 0_{3,3} & 0_{3,3} & T^2 \widehat{\mathsf{A}}_{i,\boldsymbol{v},\mathsf{e}}^{\mathrm{e},\Omega} + \widehat{\mathsf{A}}_{i,\boldsymbol{v},\varrho}^{\mathrm{e},\Omega} \mathsf{d}_\mathrm{r}^\varrho \mathsf{e}^\mathrm{r} \\ \widehat{\mathsf{A}}_{i,\boldsymbol{E},\varrho}^{\mathrm{e},\Omega} \mathsf{d}_\mathrm{r}^\varrho & \widehat{\mathsf{A}}_{i,\boldsymbol{E},\varrho}^{\mathrm{e},\Omega} \mathsf{v}_\mathrm{r}^\varrho \otimes \boldsymbol{v} & 0_{3,3} & 0_{3,3} & T^2 \widehat{\mathsf{A}}_{i,\boldsymbol{E},\mathsf{e}}^{\mathrm{e},\Omega} + \widehat{\mathsf{A}}_{i,\boldsymbol{E},\varrho}^{\mathrm{e},\Omega} \mathsf{d}_\mathrm{r}^\varrho \mathsf{e}^\mathrm{r} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{1,n^s} & 0_{1,3} & 0_{1,3} & 0_{1,3} & 0_{1,1} \end{bmatrix}, \quad i \in \mathfrak{C}. \tag{5.9}$$

Using the form of matrices $\widehat{\mathsf{B}}_{ij}$, $i,j \in \mathfrak{C}$, we classically decompose dissipation matrices $\widetilde{\mathsf{B}}_{ij}$, $i,j \in \mathfrak{C}$, into diffusion and viscous matrices, $\widetilde{\mathsf{B}}_{ij} = \widetilde{\mathsf{B}}_{ij}^\mathrm{diff} + \widetilde{\mathsf{B}}_{ij}^\mathrm{visc}$, $i,j \in \mathfrak{C}$. Denoting by $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^{\mathsf{T}}$ and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)^{\mathsf{T}}$ arbitrary vectors of $\mathbb{R}^3$, diffusion matrices, $\widetilde{\mathsf{B}}_{ij}^\mathrm{diff}$, $i,j \in \mathfrak{C}$, are given by

$$\sum_{i,j\in\mathfrak{C}} \widetilde{\mathsf{B}}_{ij}^\mathrm{diff} \xi_i \zeta_j = \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}\ \widetilde{\mathsf{B}}^\parallel + (\boldsymbol{\xi}\cdot\boldsymbol{\zeta} - \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta})\ \widetilde{\mathsf{B}}^\perp + \boldsymbol{\xi}^{\mathsf{T}} \mathbf{T}(\boldsymbol{\mathcal{B}}) \boldsymbol{\zeta}\ \widetilde{\mathsf{B}}^\odot, \tag{5.10}$$

with

$$\widetilde{\mathsf{B}}^\diamond = \frac{T}{p} \begin{bmatrix} \widetilde{\mathsf{B}}_{\varrho,\varrho}^\diamond & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,3} & \widetilde{\mathsf{B}}_{\varrho,\mathsf{e}}^\diamond \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ \widetilde{\mathsf{B}}_{\mathsf{e},\varrho}^\diamond & 0_{1,3} & 0_{1,3} & 0_{1,3} & \widetilde{\mathsf{B}}_{\mathsf{e},\mathsf{e}}^\diamond \end{bmatrix}, \qquad \diamond \in \{\parallel, \perp, \odot\}. \tag{5.11}$$

Coefficients with $\diamond =\parallel$ read

$$\widetilde{\mathsf{B}}_{\varrho,\varrho}^\parallel = D_\varrho^\parallel, \quad \widetilde{\mathsf{B}}_{\varrho,\mathsf{e}}^\parallel = D_\varrho^\parallel (\varkappa^\parallel + \mathsf{h}), \quad \widetilde{\mathsf{B}}_{\mathsf{e},\varrho}^\parallel = (\widehat{\varkappa}^\parallel + \mathsf{h})^{\mathsf{T}} D_\varrho^\parallel,$$
$$\widetilde{\mathsf{B}}_{\mathsf{e},\mathsf{e}}^\parallel = pT\lambda^\parallel + (\widehat{\varkappa}^\parallel + \mathsf{h})^{\mathsf{T}} D_\varrho^\parallel (\varkappa^\parallel + \mathsf{h}),$$

those with $\diamond =\perp$

$$\widetilde{\mathsf{B}}_{\varrho,\varrho}^\perp = D_\varrho^\perp, \quad \widetilde{\mathsf{B}}_{\varrho,\mathsf{e}}^\perp = D_\varrho^\perp (\varkappa^\perp + \mathsf{h}) - D_\varrho^\odot \varkappa^\odot, \quad \widetilde{\mathsf{B}}_{\mathsf{e},\varrho}^\perp = (\widehat{\varkappa}^\perp + \mathsf{h})^{\mathsf{T}} D_\varrho^\perp - \widehat{\varkappa}^{\odot\mathsf{T}} D_\varrho^\odot,$$
$$\widetilde{\mathsf{B}}_{\mathsf{e},\mathsf{e}}^\perp = pT\lambda^\perp + (\widehat{\varkappa}^\perp + \mathsf{h})^{\mathsf{T}} D_\varrho^\perp (\varkappa^\perp + \mathsf{h}) - \widehat{\varkappa}^{\odot\mathsf{T}} D_\varrho^\perp \varkappa^\odot - \widehat{\varkappa}^{\odot\mathsf{T}} D_\varrho^\odot (\varkappa^\perp + \mathsf{h}) - (\widehat{\varkappa}^\perp + \mathsf{h})^{\mathsf{T}} D_\varrho^\odot \varkappa^\odot,$$

and those with $\diamond = \odot$

$$\widetilde{\mathsf{B}}^{\odot}_{\varrho,\varrho} = D^{\odot}_{\varrho}, \quad \widetilde{\mathsf{B}}^{\odot}_{\varrho,\mathsf{e}} = D^{\odot}_{\varrho}(\varkappa^{\perp}+\mathsf{h}) + D^{\perp}_{\varrho}\varkappa^{\odot}, \quad \widetilde{\mathsf{B}}^{\odot}_{\mathsf{e},\varrho} = (\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}} D^{\odot}_{\varrho} + \widehat{\varkappa}^{\odot\mathsf{T}} D^{\perp}_{\varrho},$$
$$\widetilde{\mathsf{B}}^{\odot}_{\mathsf{e},\mathsf{e}} = pT\lambda^{\odot} + (\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}} D^{\odot}_{\varrho}(\varkappa^{\perp}+\mathsf{h}) - \widehat{\varkappa}^{\odot\mathsf{T}} D^{\odot}_{\varrho}\varkappa^{\odot} + \widehat{\varkappa}^{\odot\mathsf{T}} D^{\perp}_{\varrho}(\varkappa^{\perp}+\mathsf{h}) + (\widehat{\varkappa}^{\perp}+\mathsf{h})^{\mathsf{T}} D^{\perp}_{\varrho}\varkappa^{\odot}.$$

We then observe that the matrices $\widetilde{\mathsf{B}}^{\mathrm{diff}}_{ij}$, $i,j \in \mathfrak{C}$, do not satisfy the symmetry properties $(\widetilde{\mathsf{B}}^{\mathrm{diff}}_{ij})^{\mathsf{T}} = \widetilde{\mathsf{B}}^{\mathrm{diff}}_{ji}$, $i,j \in \mathfrak{C}$, because the matrices $\widetilde{\mathsf{B}}^{\perp}$ and $\widetilde{\mathsf{B}}^{\odot}$ have no symmetry properties as soon as the magnetic field is not null.

Viscous matrices $\widetilde{\mathsf{B}}^{\mathrm{visc}}_{ij}$, $i,j \in \mathfrak{C}$, are given by

$$\sum_{i,j\in\mathfrak{C}} \widetilde{\mathsf{B}}^{\mathrm{visc}}_{ij}\xi_i\zeta_j = \kappa\widetilde{\mathsf{B}}^{\mathrm{div}}(\boldsymbol{\xi},\boldsymbol{\zeta}) + \sum_{\alpha=1}^{5}\eta_\alpha\widetilde{\mathsf{B}}^{\eta_\alpha}(\boldsymbol{\xi},\boldsymbol{\zeta}), \tag{5.12}$$

with

$$\widetilde{\mathsf{B}}^{\triangle}(\boldsymbol{\xi},\boldsymbol{\zeta}) = \begin{bmatrix} 0_{n^s,n^s} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,3} & 0_{n^s,1} \\ 0_{3,n^s} & \widetilde{\mathsf{B}}^{\triangle}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) & 0_{3,3} & 0_{3,3} & \widetilde{\mathsf{B}}^{\triangle}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta})\boldsymbol{v} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{3,n^s} & 0_{3,3} & 0_{3,3} & 0_{3,3} & 0_{3,1} \\ 0_{1,n^s} & \boldsymbol{v}^{\mathsf{T}}\widetilde{\mathsf{B}}^{\triangle}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) & 0_{1,3} & 0_{1,3} & \boldsymbol{v}^{\mathsf{T}}\widetilde{\mathsf{B}}^{\triangle}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta})\boldsymbol{v} \end{bmatrix}, \tag{5.13}$$

for $\triangle \in \{\mathrm{div}, \eta_\alpha, \alpha = 1,\dots,5\}$. Expressions of these matrices are then

$$\begin{aligned}
&\widetilde{\mathsf{B}}^{\mathrm{div}}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) = T\,\boldsymbol{\xi}\otimes\boldsymbol{\zeta},\\
&\widetilde{\mathsf{B}}^{\eta_1}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) = T\big[\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\mathbb{I} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \tfrac{2}{3}\boldsymbol{\xi}\otimes\boldsymbol{\zeta}\big],\\
&\widetilde{\mathsf{B}}^{\eta_2}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) = T\big[2\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\mathbf{T}(\boldsymbol{\mathcal{B}}) + \mathbf{T}(\boldsymbol{\xi})\mathbf{T}(\boldsymbol{\mathcal{B}})\mathbf{T}(\boldsymbol{\zeta}) + 2\boldsymbol{\xi}^{\mathsf{T}}\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{\zeta}\,\mathbb{I}\big],\\
&\widetilde{\mathsf{B}}^{\eta_3}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) = T\big[2\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}\,\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}} - \tfrac{2}{3}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} + 2\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{\zeta}\otimes\boldsymbol{\xi}\mathbf{T}(\boldsymbol{\mathcal{B}}) - \mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{\xi}\otimes\boldsymbol{\zeta}\mathbf{T}(\boldsymbol{\mathcal{B}})\big],\\
&\widetilde{\mathsf{B}}^{\eta_4}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) = T\big[\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}(\mathbb{I} - 4\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}}) + \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\zeta}\otimes\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}\,\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\xi}\big],\\
&\widetilde{\mathsf{B}}^{\eta_5}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) = T\big[-2\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}\,\mathbf{T}(\boldsymbol{\mathcal{B}}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{T}(\boldsymbol{\mathcal{B}})\mathbf{T}(\boldsymbol{\zeta}) - 2\boldsymbol{\xi}^{\mathsf{T}}\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{\zeta}\,\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}}\big].
\end{aligned}$$

The matrices $\widetilde{\mathsf{B}}^{\mathrm{visc}}_{ij}$, $i,j \in \mathfrak{C}$, do not satisfy the symmetry properties $(\widetilde{\mathsf{B}}^{\mathrm{visc}}_{ij})^{\mathsf{T}} = \widetilde{\mathsf{B}}^{\mathrm{visc}}_{ji}$, because the matrices $\widetilde{\mathsf{B}}^{\triangle}_{\boldsymbol{v}}$, $\triangle \in \{\mathrm{div}, \eta_\alpha, \alpha = 1,\dots,5\}$, are not symmetric.

We consider the vector $\mathsf{X} = (\mathsf{x}_\varrho{}^{\mathsf{T}}, \mathbf{x}_{\boldsymbol{v}}{}^{\mathsf{T}}, \mathbf{x}_{\boldsymbol{E}}{}^{\mathsf{T}}, \mathbf{x}_{\boldsymbol{B}}{}^{\mathsf{T}}, \mathsf{x}_T)^{\mathsf{T}} \in \mathbb{R}^{n^s+10}$. Defining the matrix $\widetilde{\mathsf{B}}^{\mathrm{diff}}(\boldsymbol{\xi}) = \sum_{i.j\in\mathfrak{C}} \widetilde{\mathsf{B}}^{\mathrm{diff}}_{ij}\xi_i\xi_j$, we obtain, after some algebra

$$\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{B}}^{\mathrm{diff}}(\boldsymbol{\xi})\mathsf{X} = \tfrac{T}{p}\Big((\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi})^2\langle A^{\parallel}\tilde{\mathsf{x}},\tilde{\mathsf{x}}\rangle + (1-(\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi})^2)\langle A^{\perp}\tilde{\mathsf{x}},\tilde{\mathsf{x}}\rangle\Big), \tag{5.14}$$

where we have use notations of (Tr_3) for matrices $A^{\parallel}$ and $A^{\perp}$, and we have introduced the vector $\tilde{\mathsf{x}} = (p\mathsf{x}_T, (\mathsf{x}_\varrho + \mathsf{x}_T\mathsf{h})^{\mathsf{T}}\mathsf{d}^{\varrho})^{\mathsf{T}}$, with d^{ϱ} the n^s by n^s diagonal matrix defined by $\mathsf{d}^{\varrho} = \mathrm{diag}\,(\rho_1,\dots,\rho_{n^s})$. Using assumption (Tr_3) give us then the property $\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{B}}^{\mathrm{diff}}(\boldsymbol{\xi})\mathsf{X} \geqslant 0$, for $\boldsymbol{\xi} \in \Sigma^2$.

Defining the matrix $\widetilde{\mathsf{B}}^{\mathrm{visc}}(\boldsymbol{\xi}) = \sum_{i.j\in\mathfrak{C}} \widetilde{\mathsf{B}}^{\mathrm{visc}}_{ij}\xi_i\xi_j$, we obtain, after some algebra

$$\begin{aligned}\tfrac{1}{T}\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{B}}^{\mathrm{visc}}(\boldsymbol{\xi})\mathsf{X} =&\kappa(\boldsymbol{\xi}\cdot\tilde{\mathbf{x}}_{\boldsymbol{v}})^2 + (\eta_1+\eta_4)\left[\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{B}}\,\tilde{\mathbf{x}}_{\boldsymbol{v}}^{\perp}+\tilde{\mathbf{x}}_{\boldsymbol{v}}\cdot\boldsymbol{\mathcal{B}}\,\boldsymbol{\xi}^{\perp}\right]\cdot\left[\boldsymbol{\xi}\cdot\boldsymbol{\mathcal{B}}\,\tilde{\mathbf{x}}_{\boldsymbol{v}}^{\perp}+\tilde{\mathbf{x}}_{\boldsymbol{v}}\cdot\boldsymbol{\mathcal{B}}\,\boldsymbol{\xi}^{\perp}\right]\\ &+\tfrac{\eta_1+\eta_3}{3}(\boldsymbol{\xi}^{\perp}\cdot\tilde{\mathbf{x}}_{\boldsymbol{v}}^{\perp}-2\boldsymbol{\xi}^{\parallel}\cdot\tilde{\mathbf{x}}_{\boldsymbol{v}}^{\parallel})^2+(\eta_1-\eta_3)\left[(\boldsymbol{\xi}^{\perp}\cdot\tilde{\mathbf{x}}_{\boldsymbol{v}}^{\odot})^2+(\boldsymbol{\xi}^{\perp}\cdot\tilde{\mathbf{x}}_{\boldsymbol{v}}^{\perp})^2\right],\end{aligned} \tag{5.15}$$

with $\tilde{\mathbf{x}}_{\boldsymbol{v}} = \mathbf{x}_{\boldsymbol{v}} + \mathsf{x}_T\boldsymbol{v}$. Using assumption (Tr_2) yields the property $\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{B}}^{\mathrm{visc}}(\boldsymbol{\xi})\mathsf{X} \geqslant 0$, for $\boldsymbol{\xi}\in\Sigma^2$.

Finally, we immediately obtain that matrices $\widetilde{\mathsf{A}}_0(\mathsf{V})$, $\widetilde{\mathsf{A}}_i(\mathsf{V})$, $i\in\mathfrak{C}$, $\widetilde{\mathsf{B}}_{ij}(\mathsf{V})$, $i,j\in\mathfrak{C}$, and vectors $\widetilde{\mathsf{F}}^{\mathrm{e}}_{ij}(\mathsf{V})$, $i,j\in\mathfrak{C}$, $\widetilde{\Omega}(\mathsf{V})$ are $\mathcal{C}^\infty$ functions by using the $\mathcal{C}^\infty$ diffeomorphism $\mathsf{U}\mapsto\mathsf{V}$ and the proposition 3.2. □

5.2 Partial normal variable

In this section, we investigate partial normal form for system (3.9). We first establish the nullspace consistency property.

Lemma 5.4 *The nullspace of the symmetric part of the matrix* $\widetilde{\mathsf{B}}(\mathsf{V},\boldsymbol{\xi}) = \sum_{i,j\in\mathfrak{C}}\widetilde{\mathsf{B}}_{ij}(\mathsf{V})\xi_i\xi_j$*, denoted by* N*, does not depend on* $\mathsf{V}\in\mathcal{O}_{\mathsf{V}}$ *and* $\boldsymbol{\xi}\in\Sigma^2$*, and we denote by* n_0 *its dimension. This nullspace is spanned by the column vectors* $(\mathsf{u}^{\mathsf{T}},0_{1,10})^{\mathsf{T}}$ *and* E^{n^s+k}*,* $k=1,\ldots,6$*, where* $(\mathsf{E}^k)_{k=1,\ldots,n^s+10}$ *is the canonical basis of* $\mathbb{R}^{n^s+10}$*. Furthermore, we have*

$$\widetilde{\mathsf{B}}_{ij}(\mathsf{V})N = \widetilde{\mathsf{B}}_{ij}{}^{\mathsf{T}}(\mathsf{V})N = 0, \quad i,j\in\mathfrak{C}, \quad N^{\mathsf{T}}\widetilde{\mathsf{A}}^{\mathrm{e}}_i(\mathsf{V})N = 0, \quad i\in\mathfrak{C}.$$

Proof Using assumptions $(\mathsf{Tr}_2\text{-}\mathsf{Tr}_3)$, expressions (5.14) of $\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{B}}^{\mathrm{diff}}(\boldsymbol{\xi})\mathsf{X}$ and (5.15) of $\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{B}}^{\mathrm{visc}}(\boldsymbol{\xi})\mathsf{X}$ yield that $\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{B}}(\boldsymbol{\xi})\mathsf{X} = 0$ if and only if $\mathsf{x}_T = 0$, if x_ϱ is proportional to the vector u, and if $\boldsymbol{\xi}\cdot\mathbf{x}_{\boldsymbol{v}}^{\parallel} = \boldsymbol{\xi}\cdot\mathbf{x}_{\boldsymbol{v}}^{\perp} = \boldsymbol{\xi}\cdot\mathbf{x}_{\boldsymbol{v}}^{\odot} = \boldsymbol{\xi}\cdot\boldsymbol{\mathcal{B}}\,\mathbf{x}_{\boldsymbol{v}}^{\perp}\cdot\mathbf{x}_{\boldsymbol{v}}^{\perp} = \mathbf{x}_{\boldsymbol{v}}\cdot\boldsymbol{\mathcal{B}}\,\boldsymbol{\xi}^{\perp}\cdot\boldsymbol{\xi}^{\perp} = 0$. As these last relations are equivalent to $\mathbf{x}_{\boldsymbol{v}} = 0$, we deduce that N does not depend on $\mathsf{V}\in\mathcal{O}_{\mathsf{V}}$ and $\boldsymbol{\xi}\in\Sigma^2$ and is spanned by the column vectors $(\mathsf{u}^{\mathsf{T}},0_{1,10})^{\mathsf{T}}$ and E^{n^s+k}, $k=1,\ldots,6$. It is then easily checked that $\widetilde{\mathsf{B}}_{ij}(\mathsf{V})N = \widetilde{\mathsf{B}}_{ij}{}^{\mathsf{T}}(\mathsf{V})N = 0$, $i,j\in\mathfrak{C}$. Moreover, by using assumption (Tr_2) on the nullspace of the matrices $A^{\parallel}$, $A^{\perp}$ and $A^{\odot}$, the expression (3.12) of the vectors $\mathsf{F}^{\mathrm{e}}_{ij}$, $i,j\in\mathfrak{C}$, yields that $N^{\mathsf{T}}\mathsf{F}^{\mathrm{e}}_{ij} = 0$, so that $N^{\mathsf{T}}\partial_Z\mathsf{F}^{\mathrm{e}}_{ij} = 0$, as N is independent on $\mathsf{V}\in\mathcal{O}_{\mathsf{V}}$. By using expression (5.9), we immediately obtain that $\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{A}}^{\mathrm{e},\Omega}_i\mathsf{Y} = 0$, for $\mathsf{X},\mathsf{Y}\in N$. Expression (5.8) then yields that $\mathsf{X}^{\mathsf{T}}\widetilde{\mathsf{A}}^{\mathrm{e}}_i\mathsf{Y} = 0$, for $\mathsf{X},\mathsf{Y}\in N$. □

Making use of the explicit basis of N, we define the matrix P from

$$\mathsf{P} = \begin{bmatrix} 1 & 0_{1,3} & 0_{1,3} & 0_{1,n^s-1} & 0_{1,3} & 0 \\ \check{\mathsf{u}} & 0_{n^s-1,3} & 0_{n^s-1,3} & \mathbb{I}_{n^s-1,n^s-1} & 0_{n^s-1,3} & 0_{n^s-1,1} \\ 0_{3,1} & 0_{3,3} & 0_{3,3} & 0_{3,n^s-1} & \mathbb{I} & 0_{3,1} \\ 0_{3,1} & \mathbb{I} & 0_{3,3} & 0_{3,n^s-1} & 0_{3,3} & 0_{3,1} \\ 0_{3,1} & 0_{3,3} & \mathbb{I} & 0_{3,n^s-1} & 0_{3,3} & 0_{3,1} \\ 0_{1,1} & 0_{1,3} & 0_{1,3} & 0_{1,n^s-1} & 0_{1,3} & 1 \end{bmatrix}, \tag{5.16}$$

with $\check{\mathsf{u}}$ the vector of size n^s-1 defined by $\check{\mathsf{u}} = (1, \dots, 1)^{\mathsf{T}}$. We may then introduce the auxiliary variable $\mathsf{U}' = \mathsf{P}^{\mathsf{T}}\mathsf{U}$ and the corresponding partial entropic variable $\mathsf{V}' = \mathsf{P}^{-1}\mathsf{V}$ given by

$$\mathsf{U}' = \big(\rho, \boldsymbol{E}^{\mathsf{T}}, \boldsymbol{B}^{\mathsf{T}}, \check{\varrho}^{\mathsf{T}}, \rho\boldsymbol{v}^{\mathsf{T}}, \rho e^{\mathrm{t}}\big)^{\mathsf{T}},$$

with $\check{\varrho} = (\rho_2, \dots, \rho_{n^s})^{\mathsf{T}}$, and

$$\mathsf{V}' = \frac{1}{T}\Big(g_1 - \tfrac{1}{2}\boldsymbol{v}\cdot\boldsymbol{v}, \varepsilon_0\boldsymbol{E}^{\mathsf{T}}, \tfrac{1}{\mu_0}\boldsymbol{B}^{\mathsf{T}}, g_2 - g_1, \dots, g_{n^s} - g_1, \boldsymbol{v}^{\mathsf{T}}, -1\Big)^{\mathsf{T}}.$$

From theorem 4.6, normal variables are in the form $\mathsf{W} = (\psi_{\mathrm{I}}(\mathsf{U}'_{\mathrm{I}}), \phi_{\mathrm{II}}(\mathsf{V}'_{\mathrm{II}}))^{\mathsf{T}}$, where U'_{I} is the first seven components of U' and $\mathsf{V}'_{\mathrm{II}}$ the last n^s+3 components of V'. For convenience, we choose the variable W given by

$$\mathsf{W} = \left(\rho, \boldsymbol{E}^{\mathsf{T}}, \boldsymbol{B}^{\mathsf{T}}, \log\frac{\rho_2^{\mathfrak{r}_2}}{\rho_1^{\mathfrak{r}_1}}, \dots, \log\frac{\rho_{n^s}^{\mathfrak{r}_{n^s}}}{\rho_1^{\mathfrak{r}_1}}, \boldsymbol{v}^{\mathsf{T}}, T\right)^{\mathsf{T}}. \tag{5.17}$$

Proposition 5.5 *The map $\mathsf{V} \mapsto \mathsf{W}$ is a $\mathcal{C}^\infty$ diffeomorphism from the open set $\mathcal{O}_{\mathsf{V}}$ onto the open set $\mathcal{O}_{\mathsf{W}} = (0,\infty)\times\mathbb{R}^6\times\mathbb{R}^{n^s-1}\times\mathbb{R}^3\times(0,\infty)$.*

Proof For propositions 3.1 and 5.2, we have to prove that the map $\mathsf{Z} \mapsto \mathsf{W}$ is a $\mathcal{C}^\infty$ diffeomorphism. It is readily $\mathcal{C}^\infty$ and we describe now its range. Let be $\mathsf{W} \in \mathcal{O}_{\mathsf{W}}$. We then define $\mathsf{Z}_{n^s+10} = \mathsf{W}_{n^s+10}$, $\mathsf{Z}_{n^s+6+\mu} = \mathsf{W}_{4+\mu}$, $\mathsf{Z}_{n^s+3+\mu} = \mathsf{W}_{1+\mu}$, $\mathsf{Z}_{n^s+\mu} = \mathsf{W}_{n^s+6+\mu}$, $\mu \in [\![1,3]\!]$, Z_1 by the following equation

$$\mathsf{Z}_1 + \sum_{k=2}^{n^s} \mathsf{Z}_1^{\mathfrak{r}_1/\mathfrak{r}_k} \exp(\mathsf{W}_{6+k}/\mathfrak{r}_k) = \mathsf{W}_1,$$

which admits a unique positive solution and $\mathsf{Z}_k = (\mathsf{Z}_1^{\mathfrak{r}_1} \exp \mathsf{W}_{6+k})^{1/\mathfrak{r}_k}$, $2 \leqslant k \leqslant n^s$. A direct calculation yields the expression of matrix $\partial_{\mathsf{Z}}\mathsf{W}$,

$$\partial_{\mathsf{Z}}\mathsf{W} = \begin{bmatrix} 1 & \check{\mathsf{u}}^{\mathsf{T}} & 0_{1,3} & 0_{1,6} & 0_{1,1} \\ 0_{6,1} & 0_{6,n^s-1} & 0_{6,3} & \mathbb{I}_{6,6} & 0_{6,1} \\ -\frac{\mathfrak{r}_1}{\rho_1}\check{\mathsf{u}} & -\check{\mathsf{d}}_{\varrho}^{\mathsf{r}} & 0_{n^s-1,3} & 0_{n^s-1,6} & 0_{n^s-1,1} \\ 0_{3,1} & 0_{3,n^s-1} & \mathbb{I}_{3,1} & 0_{3,6} & 0_{3,1} \\ 0_{1,1} & 0_{1,n^s-1} & 0_{1,3} & 0_{1,6} & 1 \end{bmatrix}, \tag{5.18}$$

with $\check{\mathsf{d}}^{\mathsf{r}}_{\varrho}$ the $n^s\!-\!1$ by $n^s\!-\!1$ diagonal matrix given by $\check{\mathsf{d}}^{\mathsf{r}}_{\varrho} = \mathrm{diag}(r_2/\rho_2, \dots, r_{n^s}/\rho_{n^s})$. This matrix is non singular and it's invert is given by

$$\partial_{\mathsf{W}}\mathsf{Z} = \begin{bmatrix} \frac{1}{\Sigma_\rho}\frac{\rho_1}{r_1} & 0_{1,6} & -\frac{1}{\Sigma_\rho}\frac{\rho_1}{r_1}\check{\mathsf{v}}_{\mathsf{r}}^{\varrho\,\mathsf{T}} & 0_{1,3} & 0_{1,1} \\ \frac{1}{\Sigma_\rho}\check{\mathsf{v}}_{\mathsf{r}}^{\varrho} & 0_{n^s-1,6} & \check{\mathsf{d}}_{\mathsf{r}}^{\varrho} - \frac{1}{\Sigma_\rho}\check{\mathsf{v}}_{\mathsf{r}}^{\varrho}\otimes\check{\mathsf{v}}_{\mathsf{r}}^{\varrho} & 0_{n^s-1,3} & 0_{n^s-1,1} \\ 0_{3,1} & 0_{3,6} & 0_{3,n^s-1} & \mathbb{I}_{3,3} & 0_{3,1} \\ 0_{6,1} & \mathbb{I}_{6,6} & 0_{6,n^s-1} & 0_{6,3} & 0_{6,1} \\ 0_{1,1} & 0_{1,6} & 0_{1,n^s-1} & 0_{1,3} & 1 \end{bmatrix}, \tag{5.19}$$

with $\check{\mathsf{v}}_{\mathsf{r}}^{\varrho} = (\rho_2/r_2, \dots, \rho_{n^s}/r_{n^s})^{\mathsf{T}}$. From the inverse function theorem, we deduce that the map $\mathsf{Z} \mapsto \mathsf{W}$ is a $\mathcal{C}^\infty$ diffeomorphism onto $\mathcal{O}_{\mathsf{W}}$. □

In order to separate the hyperbolic and the parabolic variables, we introduce the partitioning of $\{1, \dots, n^s\!+\!10\}$ into $\mathrm{I} = \{1, \dots, n_0\}$ and $\mathrm{II} = \{n_0\!+\!1, \dots, n^s\!+\!10\}$, with $n_0 = 7$, and we use the vector and matrix block structure induced by this partitioning. We have in particular $\mathsf{W} = (\mathsf{W}_{\mathrm{I}}{}^{\mathsf{T}}, \mathsf{W}_{\mathrm{II}}{}^{\mathsf{T}})^{\mathsf{T}}$, where W_{I} corresponds to the hyperbolic variables and W_{II} to the parabolic variables,

$$\mathsf{W}_{\mathrm{I}} = \left(\rho, \boldsymbol{E}^{\mathsf{T}}, \boldsymbol{B}^{\mathsf{T}}\right)^{\mathsf{T}}, \quad \mathsf{W}_{\mathrm{II}} = \left(\log\frac{\rho_2^{r_2}}{\rho_1^{r_1}}, \dots, \log\frac{\rho_{n^s}^{r_{n^s}}}{\rho_1^{r_1}}, \boldsymbol{v}^{\mathsf{T}}, T\right)^{\mathsf{T}}. \tag{5.20}$$

Theorem 5.6 *The change of variables* $\mathsf{V} \mapsto \mathsf{W}$ *transforms the system* (5.4) *into*

$$\overline{\mathsf{A}}_0\partial_t\mathsf{W} + \sum_{i\in\mathfrak{C}}\left(\overline{\mathsf{A}}_i + \overline{\mathsf{A}}_i^{\mathrm{e}}\right)\partial_i\mathsf{W} = \sum_{i,j\in\mathfrak{C}}\partial_i\left(\overline{\mathsf{B}}_{ij}\partial_j\mathsf{W}\right) + \overline{\mathcal{T}} + \overline{\Omega}, \tag{5.21}$$

with

$$\begin{aligned} \overline{\mathsf{A}}_0 &= (\partial_{\mathsf{W}}\mathsf{V})^{\mathsf{T}}\,\widetilde{\mathsf{A}}_0\,(\partial_{\mathsf{W}}\mathsf{V}), & \overline{\mathsf{B}}_{ij} &= (\partial_{\mathsf{W}}\mathsf{V})^{\mathsf{T}}\,\widetilde{\mathsf{B}}_{ij}\,(\partial_{\mathsf{W}}\mathsf{V}), \\ \overline{\mathsf{A}}_i &= (\partial_{\mathsf{W}}\mathsf{V})^{\mathsf{T}}\,\widetilde{\mathsf{A}}_i\,(\partial_{\mathsf{W}}\mathsf{V}), & \overline{\Omega} &= (\partial_{\mathsf{W}}\mathsf{V})^{\mathsf{T}}\,\widetilde{\Omega}, \\ \overline{\mathsf{A}}_i^{\mathrm{e}} &= (\partial_{\mathsf{W}}\mathsf{V})^{\mathsf{T}}\,\widetilde{\mathsf{A}}_i^{\mathrm{e}}\,(\partial_{\mathsf{W}}\mathsf{V}), & \overline{\mathcal{T}} &= -\sum_{i,j\in\mathfrak{C}}\partial_i(\partial_{\mathsf{W}}\mathsf{V})^{\mathsf{T}}\,\widetilde{\mathsf{B}}_{ij}\,(\partial_{\mathsf{W}}\mathsf{V})\,\partial_j\mathsf{W}, \end{aligned}$$

where matrices $\overline{\mathsf{A}}_0(\mathsf{W})$, $\overline{\mathsf{A}}_i(\mathsf{W})$, $\overline{\mathsf{A}}_i^{\mathrm{e}}(\mathsf{W})$, $i \in \mathfrak{C}$, $\overline{\mathsf{B}}_{ij}(\mathsf{W})$, $i,j \in \mathfrak{C}$, *and vectors* $\overline{\Omega}(\mathsf{W})$, $\overline{\mathcal{T}}(\mathsf{W}, \boldsymbol{\partial_x}\mathsf{W})$ *are* $\mathcal{C}^\infty$ *functions of* $\mathsf{W} \in \mathcal{O}_{\mathsf{W}}$ *and* $\boldsymbol{\partial_x}\mathsf{W} \in \mathbb{R}^{3(n^s+10)}$. *Furthermore, the system* (5.21) *is of the partial normal form, that is the matrices* $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i^{\mathrm{e}}$, $i \in \mathfrak{C}$, $\overline{\mathsf{B}}_{ij}$, $i,j \in \mathfrak{C}$, *and vector* $\overline{\mathcal{T}}$ *satisfy property* (Nor_1-Nor_3).

Proof The proof of this theorem is only a direct application of theorem 4.6. We then give expressions for the matrices $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i$, $i \in \mathfrak{C}$, $\overline{\mathsf{B}}_{ij}$, $i,j \in \mathfrak{C}$. The blocs of the matrix $\overline{\mathsf{A}}_0$ read

$$\overline{\mathsf{A}}_0^{\mathrm{I,I}} = \begin{bmatrix} \frac{1}{\Sigma_\rho} & 0_{1,3} & 0_{1,3} \\ & \frac{\varepsilon_0}{T}\mathbb{I}_3 & 0_{3,3} \\ \mathrm{Sym} & & \frac{1}{\mu_0 T}\mathbb{I}_3 \end{bmatrix}, \quad \overline{\mathsf{A}}_0^{\mathrm{II,II}} = \begin{bmatrix} \check{\mathsf{d}}_{\mathsf{r}}^{\varrho} - \frac{1}{\Sigma_\rho}\check{\mathsf{v}}_{\mathsf{r}}^{\varrho}\otimes\check{\mathsf{v}}_{\mathsf{r}}^{\varrho} & 0_{n^s-1,3} & 0_{n^s-1,1} \\ & \frac{\rho}{T}\mathbb{I}_3 & 0_{3,1} \\ \mathrm{Sym} & & \frac{\rho c_v}{T^2} \end{bmatrix}. \tag{5.22}$$

Denoting by $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^{\mathsf{T}}$, an arbitrary vector of $\mathbb{R}^3$, matrices $\overline{\mathsf{A}}_i$, $i \in \mathfrak{C}$, are given by

$$\sum_{i\in\mathfrak{C}} \overline{\mathsf{A}}_i \xi_i = \begin{bmatrix} \frac{\boldsymbol{v}\cdot\boldsymbol{\xi}}{\Sigma_\rho} & 0_{1,3} & 0_{1,3} & 0_{1,n^s-1} & \frac{\rho}{\Sigma_\rho}\boldsymbol{\xi}^{\mathsf{T}} & 0_{1,1} \\ & 0_{3,3} & -\frac{1}{\mu_0 T}\mathbf{T}(\boldsymbol{\xi}) & 0_{3,n^s-1} & 0_{3,3} & 0_{3,1} \\ & & 0_{3,3} & 0_{3,n^s-1} & 0_{3,3} & 0_{3,1} \\ & & & (\check{\mathsf{d}}_{\mathsf{r}}^{\varrho} - \frac{1}{\Sigma_\rho}\check{\mathsf{v}}_{\mathsf{r}}^{\varrho}\otimes\check{\mathsf{v}}_{\mathsf{r}}^{\varrho})\boldsymbol{v}\cdot\boldsymbol{\xi} & \check{\varrho}\otimes\boldsymbol{\xi} - \frac{\rho}{\Sigma_\rho}\check{\mathsf{v}}_{\mathsf{r}}^{\varrho}\otimes\boldsymbol{\xi} & 0_{n^s-1,1} \\ & & & & \frac{\rho\boldsymbol{v}\cdot\boldsymbol{\xi}}{T}\mathbb{I} & \frac{nR}{T}\boldsymbol{\xi} \\ \text{Sym} & & & & & \frac{\rho c_v}{T}\boldsymbol{v}\cdot\boldsymbol{\xi} \end{bmatrix}, \tag{5.23}$$

with $\check{\varrho} = (\rho_2, \dots, \rho_{n^s})^{\mathsf{T}}$.

With regard to $\overline{\mathsf{B}}_{ij}$, we use the split of $\widetilde{\mathsf{B}}_{ij}$ to write $\overline{\mathsf{B}}_{ij} = \overline{\mathsf{B}}_{ij}^{\text{diff}} + \overline{\mathsf{B}}_{ij}^{\text{visc}}$, $i, j \in \mathfrak{C}$. Denoting by $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^{\mathsf{T}}$ and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \zeta_3)^{\mathsf{T}}$ arbitrary vectors of $\mathbb{R}^3$, diffusion matrices $\overline{\mathsf{B}}_{ij}^{\text{diff}}$, $i, j \in \mathfrak{C}$, are given by

$$\sum_{i,j\in\mathfrak{C}} \overline{\mathsf{B}}_{ij}^{\text{diff}} \xi_i \zeta_j = \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}\,\overline{\mathsf{B}}^{\parallel} + (\boldsymbol{\xi}\cdot\boldsymbol{\zeta} - \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta})\overline{\mathsf{B}}^{\perp} + \boldsymbol{\xi}^{\mathsf{T}}\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{\zeta}\,\overline{\mathsf{B}}^{\odot}, \tag{5.24}$$

where the matrices $\overline{\mathsf{B}}^{\parallel}$, $\overline{\mathsf{B}}^{\perp}$ and $\overline{\mathsf{B}}^{\odot}$ have the block structure of property (Nor_1). An explicit calculation also yields

$$\overline{\mathsf{B}}^{\diamond,\text{II},\text{II}} = \frac{T}{p}\begin{bmatrix} \overline{\mathsf{B}}_{\varrho,\varrho}^{\diamond} & 0_{n^s-1,3} & \overline{\mathsf{B}}_{\varrho,\mathsf{e}}^{\diamond} \\ 0_{3,n^s-1} & 0_{3,3} & 0_{3,1} \\ \overline{\mathsf{B}}_{\mathsf{e},\varrho}^{\diamond} & 0_{1,3} & \overline{\mathsf{B}}_{\mathsf{e},\mathsf{e}}^{\diamond} \end{bmatrix}, \qquad \diamond \in \{\parallel, \perp, \odot\}. \tag{5.25}$$

where coefficients with $\diamond = \parallel$ read

$$\begin{aligned}
\overline{\mathsf{B}}_{\varrho,\varrho}^{\parallel} &= \check{\mathbb{I}}\, D_{\varrho}^{\parallel}\, \check{\mathbb{I}}^{\mathsf{T}}, \\
\overline{\mathsf{B}}_{\varrho,\mathsf{e}}^{\parallel} &= \check{\mathbb{I}}\, D_{\varrho}^{\parallel}(\tfrac{1}{T^2}\varkappa^{\parallel} + \tfrac{1}{T}\mathsf{r}), \\
\overline{\mathsf{B}}_{\mathsf{e},\varrho}^{\parallel} &= (\tfrac{1}{T^2}\widehat{\varkappa}^{\parallel} + \tfrac{1}{T}\mathsf{r})^{\mathsf{T}} D_{\varrho}^{\parallel}\, \check{\mathbb{I}}^{\mathsf{T}}, \\
\overline{\mathsf{B}}_{\mathsf{e},\mathsf{e}}^{\parallel} &= \tfrac{p}{T^3}\lambda^{\parallel} + (\tfrac{1}{T^2}\widehat{\varkappa}^{\parallel} + \tfrac{1}{T}\mathsf{r})^{\mathsf{T}} D_{\varrho}^{\parallel}(\tfrac{1}{T^2}\varkappa^{\parallel} + \tfrac{1}{T}\mathsf{r}),
\end{aligned}$$

those with $\diamond = \perp$

$$\begin{aligned}
\overline{\mathsf{B}}_{\varrho,\varrho}^{\perp} &= \check{\mathbb{I}}\, D_{\varrho}^{\perp}\, \check{\mathbb{I}}^{\mathsf{T}}, \\
\overline{\mathsf{B}}_{\varrho,\mathsf{e}}^{\perp} &= \check{\mathbb{I}}\left[D_{\varrho}^{\perp}(\tfrac{1}{T^2}\varkappa^{\perp} + \tfrac{1}{T}\mathsf{r}) - \tfrac{1}{T^2}D_{\varrho}^{\odot}\varkappa^{\odot}\right], \\
\overline{\mathsf{B}}_{\mathsf{e},\varrho}^{\perp} &= \left[(\tfrac{1}{T^2}\widehat{\varkappa}^{\perp} + \tfrac{1}{T}\mathsf{r})^{\mathsf{T}} D_{\varrho}^{\perp} - \tfrac{1}{T^2}\widehat{\varkappa}^{\odot\mathsf{T}} D_{\varrho}^{\odot}\right]\check{\mathbb{I}}^{\mathsf{T}}, \\
\overline{\mathsf{B}}_{\mathsf{e},\mathsf{e}}^{\perp} &= \tfrac{p}{T^3}\lambda^{\perp} + (\tfrac{1}{T^2}\widehat{\varkappa}^{\perp} + \tfrac{1}{T}\mathsf{r})^{\mathsf{T}} D_{\varrho}^{\perp}(\tfrac{1}{T^2}\varkappa^{\perp} + \tfrac{1}{T}\mathsf{r}) \\
&\quad - \tfrac{1}{T^4}(\widehat{\varkappa}^{\odot\mathsf{T}} D_{\varrho}^{\perp}\varkappa^{\odot} + \widehat{\varkappa}^{\odot\mathsf{T}} D_{\varrho}^{\odot}(\varkappa^{\perp} + T\mathsf{r}) + (\widehat{\varkappa}^{\perp} + T\mathsf{r})^{\mathsf{T}} D_{\varrho}^{\odot}\varkappa^{\odot}),
\end{aligned}$$

and those with $\diamond = \odot$

$$
\begin{aligned}
\overline{\mathsf{B}}^{\odot}_{\varrho,\varrho} &= \check{\mathbb{I}} D^{\odot}_{\varrho}\, \check{\mathbb{I}}^{\mathsf{T}},\\
\overline{\mathsf{B}}^{\odot}_{\varrho,\mathsf{e}} &= \check{\mathbb{I}}\left[D^{\odot}_{\varrho}(\tfrac{1}{T^2}\varkappa^{\perp}+\tfrac{1}{T}\mathsf{r}) + \tfrac{1}{T^2}D^{\perp}_{\varrho}\varkappa^{\odot}\right],\\
\overline{\mathsf{B}}^{\odot}_{\mathsf{e},\varrho} &= \left[(\tfrac{1}{T^2}\widehat{\varkappa}^{\perp}+\tfrac{1}{T}\mathsf{r})^{\mathsf{T}}D^{\odot}_{\varrho} + \tfrac{1}{T^2}\widehat{\varkappa}^{\odot\mathsf{T}}D^{\perp}_{\varrho}\right]\check{\mathbb{I}}^{\mathsf{T}},\\
\overline{\mathsf{B}}^{\odot}_{\mathsf{e},\mathsf{e}} &= \tfrac{p}{T^3}\lambda^{\odot} + (\tfrac{1}{T^2}\widehat{\varkappa}^{\perp}+\tfrac{1}{T}\mathsf{r})^{\mathsf{T}}D^{\odot}_{\varrho}(\tfrac{1}{T^2}\varkappa^{\perp}+\tfrac{1}{T}\mathsf{r})\\
&\quad + \tfrac{1}{T^4}(-\widehat{\varkappa}^{\odot\mathsf{T}}D^{\odot}_{\varrho}\varkappa^{\odot}+\widehat{\varkappa}^{\odot\mathsf{T}}D^{\perp}_{\varrho}(\varkappa^{\perp}+T\mathsf{r})+(\widehat{\varkappa}^{\perp}+T\mathsf{r})^{\mathsf{T}}D^{\perp}_{\varrho}\varkappa^{\odot}),
\end{aligned}
$$

where we have introduce $\check{\mathbb{I}}$, the $n^s\!-\!1$ by n^s rectangular matrix defined by blocks by $\check{\mathbb{I}} = [0_{n^s-1,1}\ \mathbb{I}_{n^s-1,n^s-1}]$.

Viscous matrices $\overline{\mathsf{B}}^{\mathrm{visc}}_{ij}$, $i,j \in \mathfrak{C}$, are given by

$$
\sum_{i,j\in\mathfrak{C}} \overline{\mathsf{B}}^{\mathrm{visc}}_{ij}\xi_i\zeta_j = \kappa\overline{\mathsf{B}}^{\mathrm{div}}(\boldsymbol{\xi},\boldsymbol{\zeta}) + \sum_{\alpha=1}^{5}\eta_\alpha\overline{\mathsf{B}}^{\eta_\alpha}(\boldsymbol{\xi},\boldsymbol{\zeta}), \tag{5.26}
$$

with

$$
\overline{\mathsf{B}}^{\triangle}(\boldsymbol{\xi},\boldsymbol{\zeta}) = \frac{1}{T}\begin{bmatrix} 0_{n^s-1,n^s-1} & 0_{n^s-1,3} & 0_{n^s-1,1}\\ 0_{3,n^s-1} & \overline{\mathsf{B}}^{\triangle}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) & 0_{3,1}\\ 0_{1,n^s-1} & 0_{1,3} & 0 \end{bmatrix}, \tag{5.27}
$$

for $\triangle \in \{\mathrm{div}, \eta_\alpha, 1 \leqslant \alpha \leqslant 5\}$. Expressions of these matrices are then

$$
\begin{aligned}
\overline{\mathsf{B}}^{\mathrm{div}}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) &= \boldsymbol{\xi}\otimes\boldsymbol{\zeta},\\
\overline{\mathsf{B}}^{\eta_1}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) &= \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\mathbb{I} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \tfrac{2}{3}\boldsymbol{\xi}\otimes\boldsymbol{\zeta},\\
\overline{\mathsf{B}}^{\eta_2}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) &= 2\boldsymbol{\xi}\cdot\boldsymbol{\zeta}\mathbf{T}(\boldsymbol{\mathcal{B}}) + \mathbf{T}(\boldsymbol{\xi})\mathbf{T}(\boldsymbol{\mathcal{B}})\mathbf{T}(\boldsymbol{\zeta}) + 2\boldsymbol{\xi}^{\mathsf{T}}\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{\zeta}\,\mathbb{I},\\
\overline{\mathsf{B}}^{\eta_3}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) &= 2\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}\,\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}} - \tfrac{2}{3}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} + 2\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{\zeta}\otimes\boldsymbol{\xi}\mathbf{T}(\boldsymbol{\mathcal{B}}) - \mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{\xi}\otimes\boldsymbol{\zeta}\mathbf{T}(\boldsymbol{\mathcal{B}}),\\
\overline{\mathsf{B}}^{\eta_4}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) &= \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}(\mathbb{I} - 4\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}}) + \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\zeta}\otimes\boldsymbol{\mathcal{B}} + \boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}\,\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\xi},\\
\overline{\mathsf{B}}^{\eta_5}_{\boldsymbol{v}}(\boldsymbol{\xi},\boldsymbol{\zeta}) &= -2\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\xi}\,\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\zeta}\,\mathbf{T}(\boldsymbol{\mathcal{B}}) - \mathbf{T}(\boldsymbol{\xi})\mathbf{T}(\boldsymbol{\mathcal{B}})\mathbf{T}(\boldsymbol{\zeta}) - 2\boldsymbol{\xi}^{\mathsf{T}}\mathbf{T}(\boldsymbol{\mathcal{B}})\boldsymbol{\zeta}\,\boldsymbol{\mathcal{B}}\otimes\boldsymbol{\mathcal{B}}.
\end{aligned}
$$

Finally, we immediately obtain that the matrices $\overline{\mathsf{A}}_0(\mathsf{W})$,$\overline{\mathsf{A}}_i(\mathsf{W})$, $\overline{\mathsf{A}}^{\mathrm{e}}_i$, $i \in \mathfrak{C}$, $\overline{\mathsf{B}}_{ij}(\mathsf{W})$, $i,j \in \mathfrak{C}$, and the vectors $\overline{\Omega}(\mathsf{W})$, $\overline{\mathcal{T}}(\mathsf{W}, \boldsymbol{\partial_x}\mathsf{W})$ are $\mathcal{C}^\infty$ functions by using the $\mathcal{C}^\infty$ diffeomorphism $\mathsf{V} \mapsto \mathsf{W}$ and the proposition 5.3. □

Proposition 5.7 *According to the hyperbolic variables* W_{I} *and the parabolic variables* W_{II}*, we can then split system* (5.21) *into hyperbolic and parabolic parts,*

$$
\begin{cases}
\overline{\mathsf{A}}^{\mathrm{I,I}}_0\,\partial_t\mathsf{W}_{\mathrm{I}} = -\displaystyle\sum_{i\in\mathfrak{C}}\overline{\mathsf{A}}^{\mathrm{I,I}}_i\,\partial_i\mathsf{W}_{\mathrm{I}} + \overline{\Gamma}_{\mathrm{I}},\\
\overline{\mathsf{A}}^{\mathrm{II,II}}_0\,\partial_t\mathsf{W}_{\mathrm{II}} = -\displaystyle\sum_{i\in\mathfrak{C}}\left(\overline{\mathsf{A}}^{\mathrm{II,I}}_i+\overline{\mathsf{A}}^{\mathrm{eII,I}}_i\right)\partial_i\mathsf{W}_{\mathrm{I}} + \displaystyle\sum_{i,j\in\mathfrak{C}}\partial_i\left(\overline{\mathsf{B}}^{\mathrm{II,II}}_{ij}\partial_j\mathsf{W}_{\mathrm{II}}\right) + \overline{\Gamma}_{\mathrm{II}},
\end{cases} \tag{5.28}
$$

where

$$\overline{\Gamma}_{\rm I} = \overline{\Omega}_{\rm I} - \sum_{i\in\mathfrak{C}}(\overline{\mathsf{A}}_i^{\rm eI,II} + \overline{\mathsf{A}}_i^{\rm I,II})\mathsf{W}_{\rm II}, \quad \overline{\Gamma}_{\rm II} = \overline{\Omega}_{\rm II} + \overline{\mathcal{T}}_{\rm II} - \sum_{i\in\mathfrak{C}}(\overline{\mathsf{A}}_i^{\rm eII,II} + \overline{\mathsf{A}}_i^{\rm II,II})\mathsf{W}_{\rm II}.$$

Moreover, the matrices $\overline{\mathsf{A}}_0^{\rm I,I}(\mathsf{W})$, $\overline{\mathsf{A}}_0^{\rm II,II}(\mathsf{W})$, $\overline{\mathsf{A}}_i^{\rm I,I}(\mathsf{W})$, $\overline{\mathsf{A}}_i^{\rm II,I}(\mathsf{W})$, $\overline{\mathsf{A}}_i^{\rm eII,I}(\mathsf{W})$, $i \in \mathfrak{C}$, $\overline{\mathsf{B}}_{ij}^{\rm II,II}(\mathsf{W})$, $i,j \in \mathfrak{C}$, *are* $\mathcal{C}^\infty$ *functions of* $\mathsf{W} \in \mathcal{O}_{\mathsf{W}}$, *and the vectors* $\overline{\Gamma}_{\rm I}(\mathsf{W}, \boldsymbol{\partial_x}\mathsf{W}_{\rm II})$, $\overline{\Gamma}_{\rm II}(\mathsf{W}, \boldsymbol{\partial_x}\mathsf{W}_{\rm II})$ *are* $\mathcal{C}^\infty$ *functions of* $\mathsf{W} \in \mathcal{O}_{\mathsf{W}}$ *and* $\boldsymbol{\partial_x}\mathsf{W}_{\rm II} \in \mathbb{R}^{3(n^s+3)}$.

Proof It is a direct application of theorem 5.6. □

5.3 Existence theorem

We now apply Theorem 4.7 to the system modeling multicomponent reactive magnetized flows.

Theorem 5.8 (Local Existence) *Consider the Cauchy problem for the system* (5.28) *in* $\mathbb{R}^3$

$$\begin{cases} \overline{\mathsf{A}}_0^{\rm I,I}\ \partial_t \mathsf{W}_{\rm I} = -\sum_{i\in\mathfrak{C}} \overline{\mathsf{A}}_i^{\rm I,I}\ \partial_i \mathsf{W}_{\rm I} + \overline{\Gamma}_{\rm I}, \\ \overline{\mathsf{A}}_0^{\rm II,II}\ \partial_t \mathsf{W}_{\rm II} = -\sum_{i\in\mathfrak{C}} \left(\overline{\mathsf{A}}_i^{\rm II,I} + \overline{\mathsf{A}}_i^{\rm eII,I}\right)\partial_i \mathsf{W}_{\rm I} + \sum_{i,j\in\mathfrak{C}} \partial_i\left(\overline{\mathsf{B}}_{ij}^{\rm II,II}\partial_j \mathsf{W}_{\rm II}\right) + \overline{\Gamma}_{\rm II}, \end{cases} \tag{5.28}$$

with initial conditions

$$\mathsf{W}(0,x) = \mathsf{W}^0(x), \quad x \in \mathbb{R}^3, \tag{5.29}$$

where $\mathsf{W}^0 \in V_l(\mathbb{R}^3)$, $\inf_{\mathbb{R}^3} \rho^0 > 0$ *and* $\inf_{\mathbb{R}^3} T^0 > 0$.

Then there exits $t_0 > 0$, *such that* (5.28), (5.29) *admit an unique solution* $\mathsf{W} = (\mathsf{W}_{\rm I}{}^{\mathsf{T}}, \mathsf{W}_{\rm II}{}^{\mathsf{T}})^{\mathsf{T}}$ *with* $\mathsf{W}(t,x) \in \mathcal{O}_{\mathsf{W}}$ *defined on the strip* $\bar{Q}_{t_0} = [0,t_0]\times\mathbb{R}^3$, *continuous in* $\bar{Q}_{t_0}$ *with its derivatives of first order in* t *and second order in* x, *and for which the following inequalities hold :*

$$\sup_{0\leqslant t\leqslant t_0}\left[\|\rho(t)\|_l + \sum_{k=2}^{n^s}\left\|\log\frac{\rho_k^{r_k}}{\rho_1^{r_1}}(t)\right\|_l + \sum_{i\in\mathfrak{C}}\left(\|v_i(t)\|_l + \|E_i(t)\|_l + \|B_i(t)\|_l\right) + \|T(t)\|_l\right] < +\infty,$$

$$\inf_{\bar{Q}_{t_0}} \rho(t,x) > 0, \qquad \inf_{\bar{Q}_{t_0}} T(t,x) > 0,$$

$$\sup_{0\leqslant t\leqslant t_0}\left[\|\partial_t\rho(t)\|_{l-1} + \sum_{i\in\mathfrak{C}}\left(\|\partial_t E_i(t)\|_{l-1} + \|\partial_t B_i(t)\|_{l-1}\right)\right] < +\infty,$$

$$\int_0^{t_0}\Bigg[\sum_{k=2}^{n^s}\left\|\partial_t\log\frac{\rho_k^{r_k}}{\rho_1^{r_1}}(\tau)\right\|_{l-1}^2 + \sum_{i\in\mathfrak{C}}\|\partial_t v_i(\tau)\|_{l-1}^2 + \|\partial_t T(\tau)\|_{l-1}^2 + \\ \sum_{k=2}^{n^s}\left\|\log\frac{\rho_k^{r_k}}{\rho_1^{r_1}}(\tau)\right\|_{l+1}^2 + \sum_{i\in\mathfrak{C}}\|v_i(\tau)\|_{l+1}^2 + \|T(\tau)\|_{l+1}^2\Bigg)\,\mathrm{d}\tau < +\infty.$$

Moreover, either t_0 is as large as one wants, or there exists t_1 such that the theorem is true for any $t_0 < t_1$ and such that for $t_0 \to t_1^-$, either the following quantity,

$$\|\rho(t_0)\|_{1,\infty} + \sum_{k=2}^{n^s} \left\| \log \frac{\rho_k^{\eta_k}}{\rho_1^{\eta_1}}(t_0) \right\|_{2,\infty} + \|T(t_0)\|_{2,\infty} + \sum_{i\in\mathfrak{C}} \Big(\|v_i(t_0)\|_{2,\infty} + \|E_i(t_0)\|_{1,\infty} + \|B_i(t_0)\|_{1,\infty} \Big), \quad (5.30)$$

or $\sup_{\bar{Q}_{t_0}} 1/T$ *is unbounded.*

The proof of this theorem rests on the theoretical study of Vol'pert and Hudjaev [6], in particular on theorem 4.7. We have to verify its various assumptions, after having defined function $\mathcal{L}$ by $\mathcal{L}(\mathsf{W}_{\rm I}, \mathsf{W}_{\rm II}) = 1/\rho + 1/T$. Moreover, we prove that $\inf_{\mathbb{R}^3} \rho(t, \boldsymbol{x}) > 0$ as long as (5.30) remains finite, so that only T may reach the boundary of $\mathcal{O}_{\mathsf{W}}$.

Proof We define the function $\mathcal{L}$ by $\mathcal{L}(\mathsf{W}_{\rm I}, \mathsf{W}_{\rm II}) = 1/\rho + 1/T$. As we assume that $\mathsf{W}^0 \in V_l(\mathbb{R}^3)$, $\inf_{\mathbb{R}^3} \rho^0 > 0$ and $\inf_{\mathbb{R}^3} T^0 > 0$, we immediately obtain that property (Ex_1) holds. Proposition 5.7 implies readily properties (Ex_2-Ex_4) and (Ex_7) on the regularity of matrices and vectors. Properties (Ex_5-Ex_6) concerning symmetry of matrices $\overline{\mathsf{A}}_0^{\rm I,I}$, $\overline{\mathsf{A}}_0^{\rm II,II}$ and $\overline{\mathsf{A}}_i^{\rm I,I}$, $i \in \mathfrak{C}$, are obtained by using properties (S_1-S_2) obtained in theorem 5.6. Moreover, property (Nor_2) yields readily that property (Ex_8) holds.

Finally, we note that from the conservation of ρ, we have

$$\rho(t, \boldsymbol{x}) \geqslant \inf_{\mathbb{R}^3} \rho^0(\boldsymbol{x}) \exp\left[-\int_0^t \|\boldsymbol{\partial_x}\cdot \boldsymbol{v}(s)\|_{0,\infty} \mathrm{d}s \right],$$

and thus $\inf_{\mathbb{R}^3} \rho(t, \boldsymbol{x}) > 0$ as long as (5.30) remains finite, so that only T may reach the boundary of $\mathcal{O}_{\mathsf{W}}$. □

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