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Abstract

Higher order entropies are kinetic entropy estimators for fluid models. These quantities are quadratics in the velocity and temperature derivatives and have temperature dependent coefficients. We investigate governing equations for higher order entropies and related a priori estimates in the natural situation where viscosity and thermal conductivity depend on temperature. We establish conditionnal entropicity, that is, positivity of higher order derivatives source terms in these governing equations provided that $\|\log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty}$ is small enough. The temperature factors renormalizing temperature and velocity derivatives then yield majorization of lower order convective terms only when the temperature dependence of transport coefficients is taken into account according to the kinetic theory. In this situation, we obtain entropic principles for higher order entropies of arbitrary order. As an application, we investigate a priori estimates and global existence of solutions when the initial values $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small enough in appropriate spaces.

AMS Subject Classification Numbers : 35K, 76N, 82A40

1. Introduction

We investigate higher order entropies of fluid models and related a priori estimates. Higher order entropies are kinetic entropy estimators for fluid models. Higher order entropy estimates may also be interpreted as a generalization of Bernstein method for systems, as a natural rescaling of solution derivatives by temperature weights, or as kinetic Fisher information estimators. For simple fluid models, these quantities are quadratic or polynomial with respect to velocity and temperature derivatives and have temperature dependent coefficients. They are investigated in this paper in the situation of incompressible flows spanning the whole space with temperature dependent thermal conductivity and viscosity. The cases of compressible flows or zero Mach number flows are beyond the scope of the present paper.

As a preliminary study, we consider second order entropies for fluid models with constant transport coefficients. We derive a governing equation for second order kinetic entropy correctors and investigate when higher order derivative terms, which appear as sources, have a sign. Unconditional positivity of these source terms—unconditional entropicity—is established for a restricted family of second order entropy correctors. The temperature weights renormalizing solution derivatives, however, do not yield majorization of the lower order terms arising from convection. As a consequence of the preliminary analysis, we need to investigate conditional entropicity properties as well as to modify the renormalizing temperature weights by taking into account the natural temperature dependence of transport coefficients.

Temperature dependence of viscosity and thermal conductivity is a consequence of the kinetic theory of gases. Away from small temperatures, these coefficients essentially behave like a power of temperature with a common exponent $\varkappa$. In this situation, we derive a balance equation for kinetic entropy correctors of arbitrary order. These higher order kinetic entropy correctors are quadratic or polynomial in the velocity and temperature derivatives with temperature dependent coefficients. We obtain conditional entropicity, that is, positivity of higher order derivative source terms when $\|\log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty}$ is small enough. The lower order convective terms are then majorized thanks to the temperature dependence of transport coefficients as given by the kinetic theory of gases, that is, only when $\varkappa \geq 1/2$. In order to establish these estimates, we use the Coifman-Meyer inequalities for multilinear operators and weighted interpolation inequalities for intermediate derivatives with weights in Muckenhoupt classes. We next investigate higher order kinetic entropy estimators obtained by summing up a zeroth order entropy estimator with kinetic entropy correctors and we obtain entropic principles. As a typical application, we establish a global existence theorem provided that $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small enough in appropriate spaces, which may be interpreted heuristically as an existence theorem for small Mach number flows.

In Section 2 we discuss the concept of higher order entropy. In Section 3 we investigate second order entropies for fluid models with constant transport coefficients. In Section 4, we establish various weighted inequalities and in Section 5 we consider higher order entropies in the natural situation of variable transport coefficients. Finally, in Section 6, we concentrate on global solutions for small Mach number flows.

2. Higher order entropies

The notion of entropy has been shown to be of fundamental importance in fluid modeling from both a physical and mathematical point of view [Bol95] [CC70] [FK72] [Kaw84] [Cer88] [DL89] [GM98] [Gio99] [Gol00] [Vil01] [DV03]. We discuss heuristically, in this section, a concept of higher order mathematical entropies for fluid models.

2.1. Entropic interpretation of Bernstein method

For parabolic—or elliptic—scalar equations, a priori gradient estimates can be obtained by using Bernstein method [Ber38] [LSU68]. More specifically, consider—as a simple exemple—the heat equation

$$\partial_t u - \Delta u = 0. \tag{2.1}$$

Defining $\zeta = |\partial_x u|^2 = \partial_x u{\cdot}\partial_x u$, we then have

$$\partial_t \zeta - \Delta\zeta + 2|\partial_x^2 u|^2 = 0, \tag{2.2}$$

and the higher order source term $|\partial_x^2 u|^2 = \partial_x^2 u{:}\partial_x^2 u = \sum_{ij}(\partial_{ij}u)^2$ is generally discarded so that one obtains inequalities like $\partial_t\zeta - \Delta\zeta \leq 0$ and the maximum principle can then be used [Ber38] [LSU68]. Similarly, classical techniques which consists in multiplying the heat equation by either the laplacian Δu or the time derivative $\partial_t u$ lead to the same type of estimates as (2.2) after integration by parts.

In order to generalize heuristically this technique to systems, we cannot rely on the dissipative term $-\Delta\zeta = \partial_x{\cdot}(\partial_x\zeta)$. Indeed, for systems, dissipative fluxes and gradients are not anymore related by diagonal matrices. However, we may focus on the source term whose principal part $|\partial_x^2 u|^2$ has a sign. The structure of (2.2) then appears to be formally similar to that of an entropy balance, where ζ plays the rôle of a generalized entropy, even though there also exist zeroth order entropies like u^2. In the next section, we introduce a kinetic framework supporting this entropic interpretation.

2.2. Enskog second order kinetic entropy corrector

We consider, for the sake of simplicity, a single monatomic dilute gas. The state of the gas is described by a distribution function $f(t,x,c)$ governed by Boltzmann's equation, where t is time, x the n-dimensional spatial coordinate, and c the molecular velocity [CC70] [FK72] [Cer88] [Gio99] [Gol00] [Vil00] [DV03]. Approximate solutions of Boltzmann's equation are obtained from a first order Enskog—formal—expansion

$$f = f^{(0)}\big(1 + \varepsilon\phi^{(1)} + \mathcal{O}(\varepsilon^2)\big), \tag{2.3}$$

where $f^{(0)}$ is the local Maxwellian distribution, $\phi^{(1)}$ the perturbation associated with the Navier-Stokes regime and ε the usual formal expansion parameter. The perturbation $\phi^{(1)}$ depends linearly on the temperature and velocity gradients and is the solution of a linearized Boltzmann equation [CC70] [FK72] [Gio99]. The compressible Navier-Stokes equations—or the zero Mach number equations—can then be obtained upon taking moments of Boltzmann's equation [CC70] [FK72] [Gol00].

A fundamental property is that the kinetic entropy defined by

$$S^{\mathrm{kin}} = -k_{\mathrm{B}} \int_{\mathbb{R}^n} f\big(\log f - 1\big)dc, \tag{2.4}$$

where k_{B} is the Boltzmann constant, obeys the H theorem, that is, the second principle of thermodynamics [CC70] [FK72] [Gio99] [Vil01] [DV03]. The expansion of $\mathcal{S}^{\mathrm{kin}}$ induced by a second order Enskog expansion, however, can be written

$$\mathcal{S}^{\mathrm{kin}} = S^{(0)} + \varepsilon^2 S^{(2)} + \mathcal{O}(\varepsilon^3), \tag{2.5}$$

where $S^{(0)}$ is the usual zeroth order macroscopic entropy evaluated from Maxwellian distributions and where $S^{(2)}$ reads

$$S^{(2)} = -\frac{k_{\mathrm{B}}}{2} \int_{\mathbb{R}^n} f^{(0)} (\phi^{(1)})^2 dc, \tag{2.6}$$

so that $-S^{(2)}$ is quadratic in the temperature and velocity gradients and is a natural candidate for deriving a balance equation like (2.2). For compressible monatomic gases, after detailed calculations, one can establish that

$$S^{(2)} = -\frac{1}{\rho} \big(\overline{\lambda} |\partial_x T|^2 + \tfrac{1}{2} \overline{\eta}\, |d|^2 \big), \tag{2.7}$$

where T denotes the absolute temperature, ρ the density, v the gas velocity, d the strain rate tensor $d = \partial_x v + \partial_x v^t - \frac{2}{n}(\partial_x \cdot v)\, I$ and $|d|^2 = \sum_{ij} d_{ij}^2$, and where the scalar coefficients $\overline{\lambda}$ and $\overline{\eta}$ only depend on temperature. In a first approximation, using a single term in orthogonal polynomial expansions of perturbed distribution functions, one can establish that $\overline{\lambda} = (1/2rc_p)\lambda^2/T^3$ and $\overline{\eta} = (1/4r)\eta^2/T^2$ where c_p is the constant pressure specific heat per unit mass, r the gas constant per unit mass, λ the thermal conductivity, η the shear viscosity, and the actual values of the numerical factors in front of $\overline{\lambda}$ and $\overline{\eta}$ are evaluated here for $n = 3$. In the special case of Maxwellian gases, such a calculation has already been performed by Boltzmann [Bol95].

2.3. Zeroth order entropy dissipation rate

A second kinetic interpretation can be obtained from the zeroth order entropy balance equation

$$\partial_t S^{(0)} + \partial_x \cdot (v S^{(0)}) + \partial_x \cdot F^{(0)} = \mathfrak{v}^{(0)}, \tag{2.8}$$

where $F^{(0)}$ is the zeroth order dissipative entropy flux and $\mathfrak{v}^{(0)}$ the zeroth order entropy production given by

$$\mathfrak{v}^{(0)} = \frac{\lambda}{T^2} |\partial_x T|^2 + \frac{1}{2} \frac{\eta}{T}\, |d|^2. \tag{2.9}$$

This entropy production is quadratic in the macroscopic variable gradients with temperature dependent coefficients. It also appears as a natural norm of the system and a natural candidate for deriving a balance equation like (2.2). Denoting the linearized Boltmann equation by $\mathcal{J}\phi^{(1)} = \psi^{(1)}$, the second order entropy and the entropy production are essentially in the form $\langle \phi^{(1)}, \phi^{(1)} \rangle$ and $\langle \phi^{(1)}, \psi^{(1)} \rangle$, respectively, where $\langle \xi, \zeta \rangle = \int_{\mathbb{R}^n} f^{(0)} \xi \zeta dc$.

2.4. Enskog second order kinetic information corrector

The Sobolev logarithmic inequality majorizes the relative entropy of f with respect to $f^{(0)}$ by the relative Fisher information of f with respect to $f^{(0)}$ [Vil01]. Here $f^{(0)}$ is the Maxwellian distribution with the same local macroscopic properties as f

$$f^{(0)} = \frac{\rho}{\mathfrak{m}}\Big(\frac{\mathfrak{m}}{2\pi k_{\mathrm{B}} T}\Big)^{\frac{n}{2}} \exp\Big(-\frac{\mathfrak{m}(c-v)^2}{2k_{\mathrm{B}}T}\Big),$$

where $\mathfrak{m}$ denotes the particle mass. After a rescaling 'à la Boltzmann', the Sobolev logarithmic inequality can be written in the form

$$0 \leq k_{\mathrm{B}} \int_{\mathbb{R}^n} \big(f/f^{(0)}\big) \log\big(f/f^{(0)}\big) f^{(0)}\, dc \leq \frac{k_{\mathrm{B}}^2 T}{2\mathfrak{m}} \int_{\mathbb{R}^n} \frac{\big|\partial_c(f/f^{(0)})\big|^2}{f/f^{(0)}} f^{(0)}\, dc, \tag{2.10}$$

and one can establish that

$$\mathcal{S}^{\mathrm{kin}} - S^{(0)} = -k_{\mathrm{B}} \int_{\mathbb{R}^n} \big(f/f^{(0)}\big) \log\big(f/f^{(0)}\big) f^{(0)}\, dc,$$

so that the relative entropy of f with respect to f^0 coincides with $S^{(0)} - \mathcal{S}^{\mathrm{kin}}$. The relative Fisher information thus appears as an estimator of kinetic entropy deviation. One can also establish that the relative Fisher information is given by

$$\mathcal{I}^{\mathrm{kin}} - \mathcal{I}^{(0)} = k_{\mathrm{B}} \int_{\mathbb{R}^n} \frac{\big|\partial_c(f/f^{(0)})\big|^2}{f/f^{(0)}} f^{(0)}\, dc,$$

where $\mathcal{I}^{\mathrm{kin}} = k_{\mathrm{B}} \int_{\mathbb{R}^n}(|\partial_c f|^2/f)\, dc$ and $\mathcal{I}^{(0)} = k_{\mathrm{B}} \int_{\mathbb{R}^n}(|\partial_c f^0|^2/f^0)\, dc$ denote the kinetic and zeroth order Fisher informations. Substituting a second order Enskog expansion in the logarithmic Sobolev inequality (2.10), the leading order term of the left hand side is $-S^{(2)}$ and the leading order term $I^{(2)}$ of the right hand side reads

$$I^{(2)} = \frac{k_{\mathrm{B}} T}{2\mathfrak{m}} \mathcal{I}^{(2)} = \frac{k_{\mathrm{B}}^2 T}{2\mathfrak{m}} \int_{\mathbb{R}^n} \big|\partial_c \phi\big|^2 f^{(0)}\, dc,$$

and is a natural candidate for deriving a balance equation like (2.2). For compressible monatomic gases, after detailed calculations, one can establish that

$$I^{(2)} = \frac{1}{\rho}\big(\,\overline{\lambda}|\partial_x T|^2 + \tfrac{1}{2}\overline{\eta}\,|d|^2\,\big), \tag{2.11}$$

where, again, $\overline{\lambda}$ and $\overline{\eta}$, only depend on temperature. In a first approximation, using a single term in orthogonal polynomial expansions of perturbed distribution functions, one can establish that $\overline{\lambda} = \big((10r/c_p + 3)/2c_p\big)\lambda^2/T^3$ and $\overline{\eta} = (1/2r)\eta^2/T^2$, where the actual values of the numerical factors in front of $\overline{\lambda}$ and $\overline{\eta}$ are evaluated here for $n = 3$. The second order information corrector $I^{(2)}$ is thus similar to the second order entropy corrector $-S^{(2)}$ and to the zeroth order entropy production rate $\mathfrak{v}^{(0)}$.

Remark 2.1. Logarithmic Sobolev inequalities have been investigated in a probabilistic framework by Patrick Cattiaux [Cat04]. In this situation, the relative Fisher information has been shown to represent a relative entropy in a path space, that is, in the space of particle trajectories [Cat04]. This further supports the idea that these quadratic quantities represent an entropy. ∎

2.5. Enskog higher order entropy correctors

Higher order Enskog expansions $f/f^{(0)} = 1 + \varepsilon\phi^{(1)} + \cdots + \epsilon^{2k}\phi^{(2k)} + \mathcal{O}(\varepsilon^{2k+1})$ actually induce higher order expansions for $\mathcal{S}^{\mathrm{kin}}$

$$\mathcal{S}^{\mathrm{kin}} - S^{(0)} = \varepsilon^2 S^{(2)} + \varepsilon^3 S^{(3)} + \cdots + \varepsilon^{2k} S^{(2k)} + \mathcal{O}(\varepsilon^{2k+1}),$$

where $S^{(l)}$ is a sum of terms in the form $k_{\mathrm{B}} \int_{\mathbb{R}^n} \prod_{1\leq i\leq l} \big(\phi^{(i)}\big)^{\nu_i} f^{(0)}\, dc$ with nonnegative integers $\nu_i \geq 0$, $1 \leq i \leq l$, such that $l = \sum_{1\leq i\leq l} i\nu_i$.

On the other hand, in the absence of external forces acting on the particles, $\phi^{(l)}$ is a sum of products of solution derivatives with a total number of l derivations $\phi^{(l)} = \big(\frac{\eta}{\rho\sqrt{rT}}\big)^l \sum_{\alpha\beta\delta} \mathrm{c}_{\alpha\beta\delta} \big(\frac{\partial^\alpha T}{T}\big)^{\nu_\alpha} \big(\frac{\partial^\beta \rho}{\rho}\big)^{\nu_\beta} \big(\frac{\partial^\delta v}{\sqrt{rT}}\big)^{\nu_\delta}$, where the summation is over all multiindices α, β and δ, and all nonnegative integers ν_α, ν_β, and ν_δ, with $|\alpha| \geq 1$, $|\beta| \geq 1$, $|\delta| \geq 1$, and $\nu_\alpha|\alpha| + \nu_\beta|\beta| + \nu_\delta|\delta| = l$ and where the coefficients $\mathrm{c}_{\alpha\beta\delta}$ are tensors in $\mathcal{C} = (c-v)/\sqrt{2rT}$ multiplied by smooth scalar functions of $|\mathcal{C}|^2$ and $\log T$. This result is established by examining the successive construction of $\phi^{(l)}$ from $\phi^{(1)}, \ldots, \phi^{(l-1)}$ applying l^{th} time the—generalized—inverse of the linearized collision operator which scales as $\eta/\rho rT$ and has isotropicity properties [FK72] [Cer88]. After integration with respect to $\mathcal{C}$, $S^{(2k)}$ is found in the form

$$S^{(2k)} = r\rho \Big(\frac{\eta}{\rho\sqrt{rT}}\Big)^{2k} \sum_{\alpha\beta\delta} c_{\alpha\beta\delta} \Big(\frac{\partial^\alpha T}{T}\Big)^{\nu_\alpha} \Big(\frac{\partial^\beta \rho}{\rho}\Big)^{\nu_\beta} \Big(\frac{\partial^\delta v}{\sqrt{rT}}\Big)^{\nu_\delta}, \tag{2.12}$$

where the summation is over all multiindices α, β and δ, and all nonnegative integers ν_α, ν_β, and ν_δ, with $|\alpha| \geq 1$, $|\beta| \geq 1$, $|\delta| \geq 1$, and $\nu_\alpha|\alpha| + \nu_\beta|\beta| + \nu_\delta|\delta| = 2k$ and where the coefficients $c_{\alpha\beta\delta}$ are smooth bounded scalar functions of $\log T$. After integrations by parts with respect to the spatial variables in the integrals associated with $\int_{\mathbb{R}^n} S^{(2k)}\, dx$, in order to eliminate spatial derivatives of order strictly greater than k, and by using interpolation inequalities, one obtains that the quantity $|\int_{\mathbb{R}^n} S^{(2k)}\, dx|$ is essentially controled by the integral of

$$\gamma^{[k]} = r\rho \Big(\frac{\eta}{\rho\sqrt{rT}}\Big)^{2k} \Big(\big|\frac{\partial^k T}{T}\big|^2 + \big|\frac{\partial^k v}{\sqrt{rT}}\big|^2 + \big|\frac{\partial^k \rho}{\rho}\big|^2\Big), \tag{2.13}$$

or equivalently of

$$\widetilde{\gamma}^{[k]} = r\rho \Big(\frac{\eta}{\rho\sqrt{rT}}\Big)^{2k} \big(|\partial^k \log T|^2 + |\partial^k (v/\sqrt{rT})|^2 + |\partial^k \log\rho|^2\big). \tag{2.14}$$

This suggests the quantities $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as $(2k)^{\mathrm{th}}$ order kinetic entropy correctors—or kinetic entropy deviation estimators. Furthermore, $|\int_{\mathbb{R}^n} S^{(2k-1)}\, dx|$ is controled by $\int_{\mathbb{R}^n} \gamma^{[k]}\, dx$ and $\int_{\mathbb{R}^n} \gamma^{[k-1]}\, dx$.

A similar analysis can also be conducted for the Fisher information and suggests the same quantities $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as higher order kinetic information correctors. Moreover, denoting by $\gamma^{[0]}$ or $\widetilde{\gamma}^{[0]}$ zeroth order entropy estimators, and upon summation, we obtain the $(2k)^{\rm th}$ order kinetic entropy estimators $\gamma^{[0]} + \cdots + \gamma^{[k]}$ and $\tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]}$.

A parallel can be made with the heat equation, for which the quantity $\zeta^{[k]} = |\partial^k u|^2$ can be considered as a $(2k)^{\rm th}$ order entropy corrector, and with Bernstein method applied to parabolic scalar equations with variables coefficients, for which one considers sums of squares of derivatives [LSU68].

2.6. Temperature scaling

Scaling properties of scalar partial differential equations are of fundamental importance for investigating the behavior of solutions like asymptotic expansions, singular limits, boundary layers, or even existence of solutions with the concept of renormalized solutions [DL89] [Lio96].

Considering systems of partial differential equations, however, a possible rescaling method could be to use functions of a single scalar quantity to rescale all solution components and solution derivatives. For fluid models, a natural candidate of such a scalar quantity appears to be temperature. In particular, higher order entropies provide a natural scaling of solution derivatives in terms of powers of temperature.

Remark 2.2. There are also ρ factors at the denominator of the corresponding derivatives $\partial^k\rho$ in (2.12)(2.13). Similarly, for multicomponent flows, entropy production associated with diffusion is essentially in the form $\sum_{1\leq i\leq n_{\rm s}} \int_{\mathbb{R}^n} (pD/T)|\partial_x \rho_i|^2/\rho_i \, dx$, where D is a typical diffusion coefficient, ρ_i a typical concentration of the i$^{\rm th}$ species and $n_{\rm s}$ the number of species in the mixture [Gio99]. ∎

2.7. Persistence of kinetic entropy and small Mach numbers

Various thermodynamic theories have already considered entropies differing from that of zeroth order, that is, entropies depending on macroscopic variable gradients. These generalized entropies have been associated notably with Burnett type equations and extended thermodynamics. In both situations, new macroscopic equations are correspondingly obtained, which are of higher order than Navier-Stokes type equations.

On the contrary, in this work, we want to investigate the properties of solutions of a given fluid model, that is, of a given second order system of partial differential equations. In particular, we do not consider composite quantities like $S^{(0)} + S^{(2)}$ as the system entropy, since we typically deals with fluid equations for which the zeroth order entropy $S^{(0)}$ is already of fundamental importance as imposed by the hyperbolic-parabolic structure of these equations [Kaw84] [Gio99]. We only want to use quantities like $S^{(0)}$, $S^{(0)}+S^{(2)}$ and more generally like $\gamma^{[0]}+\cdots+\gamma^{[l]}$, or $\tilde{\gamma}^{[0]}+\cdots+\tilde{\gamma}^{[l]}$, $0 \leq l \leq k$, as a mean to obtain further information on solutions of the fluid model. These quantities should thus be considered as families of mathematical entropy estimators—of kinetic origin—and we will establish that they indeed satisfy conditional entropic principles for solutions of Navier-Stokes type equations.

Enskog expansion is associated with small Knudsen numbers ${\rm Kn} = l/L$, where l is a typical mean free path and L a hydrodynamic length. On the other hand, we are interested in fluid models which take into account dissipative effects like viscosity and heat conduction and the corresponding characteristic length L is such that the Reynolds number $\mathrm{Re} = \rho v L/\eta$ is of order unity, that is, $L = \eta/\rho v$. As a consequence, since $\rho\bar{c}l = \eta$ [FK72], where $\bar{c}$ is a typical sound velocity, we obtain that the Knudsen number $\mathrm{Kn} = \mathrm{Kn}\,\mathrm{Re} = \rho v l/\eta = v/\bar{c}$ is equal to a typical Mach number $\mathrm{Ma} = v/\bar{c}$. Therefore, since $\mathrm{Ma} \simeq \mathrm{Kn}$, assuming that the Mach number is small is equivalent with the underlying kinetic assumption of a small Knudsen number and we expect the Mach number to play a fundamental role in the analysis [Gol00].

3. Preliminary study

We investigate in this section some elementary properties of second order entropy correctors in the simplified situation of incompressible fluids with constant transport coefficients.

3.1. Incompressible model

We consider a fluid governed by the incompressible Navier-Stokes equations

$$\partial_x\cdot v = 0, \tag{3.1}$$

$$\partial_t(\rho v) + \partial_x\cdot(\rho v\otimes v + pI) + \partial_x\cdot \varPi = 0, \tag{3.2}$$

$$\partial_t(\rho e) + \partial_x\cdot(\rho e v) + \partial_x\cdot \mathcal{Q} = -\varPi{:}\partial_x v, \tag{3.3}$$

where ρ is the constant density, v the velocity, p the pressure, I the unit tensor, $\varPi$ the viscous tensor, e the internal energy per unit mass, and $\mathcal{Q}$ the heat flux vector. The viscous tensor is given by $\varPi = -\eta d$ where $d = \partial_x v + \partial_x v^t$ is the strain rate tensor and η the shear viscosity, the heat flux by $\mathcal{Q} = -\lambda\partial_x T$ where λ is the thermal conductivity, and the energy per unit mass e is taken for simplicity in the form $e = c_v T$, where c_v is the specific heat per unit mass. All the coefficients c_v, λ, and η, are taken to be constant in this section.

Our aim is not to study various boundary conditions and we only consider the case of functions defined on $\mathbb{R}^n$, with $n \geq 2$, that are 'constant at infinity'. From Galilean invariance and incompressibility, we can choose that v and p vanish at infinity. We only consider smooth solutions of the Navier-Stokes equations, that is, taken into account the eventual temperature dependence of the system coefficients as in Section 5, we assume that

$$v, T - T_\infty \in C\big([0,\bar{t}\,], H^l\big) \cap C^1\big([0,\bar{t}\,], H^{l-2}\big) \cap L^2\big([0,\bar{t}\,], H^{l+1}\big), \tag{3.4}$$

where l is an integer such that $l \geq [n/2] + 2$, that is, $l > n/2 + 1$, and $\bar{t}$ is some positive time. We will establish in Section 6 that these solutions are as smooth as expected from initial data. In the simpler case of constant coefficients, smoothness properties hold as soon as it is established that $v \in C([0,\bar{t}\,], L^n)$ [Lio96]. Existence of such smooth solutions can be established locally in time, or globally in time for small initial data. We will also assume that T is positive and bounded away from zero $T \geq T_{\min}$ where $T_{\min} > 0$ and this property is easily established as soon as it holds at initial time $T_0 \geq T_{\min}$ thanks to the nonnegativity of viscous heat dissipation [Lio96]. We consider as usual the momentum equation as projected on the space of divergence-free L^2 functions. More specifically, we introduce the Leray projector $\mathbb{P}$ defined on $L^2(\mathbb{R}^n)^n$ by

$$\mathbb{P} = \mathbb{I} + R\otimes R, \tag{3.5}$$

where $R = (R_1, \dots, R_n)^t$ and $R_i = (-\Delta)^{-1/2}\partial_i$, $1 \le i \le n$, are the Riesz transforms, so that $(\mathbb{P}v)_i = v_i + \sum_{1\le j\le n} R_i R_j v_j$, $1 \le i \le n$ [Lio96] [Lem02]. It is well known that $\mathbb{P}$ is a continuous projector in any Sobolev spaces H^s, $s \in \mathbb{R}$, and $\mathbb{P}$ is also continuous in L^s for $1 < s < \infty$ [Lio96] [Lem02]. Since the viscosity η is constant, the momentum conservation equation is easily rewritten as $\partial_t(\rho v) - \eta\Delta v = -\mathbb{P}\big(\partial_x\cdot(\rho v\otimes v)\big)$, which is equivalent to defining the pressure from

$$p = \sum_{1\le i,j\le n} R_i R_j(\rho\, v_i v_j), \tag{3.6}$$

and we have $p \in C\big([0,\bar{t}\,], H^l\big) \cap C^1\big([0,\bar{t}\,], H^{l-2}\big) \cap L^2\big([0,\bar{t}\,], H^{l+1}\big)$.

3.2. Second order entropy corrector γ

As is traditional in mathematics, we change the sign of entropy, and thus of second order entropies, and we define γ as one of the equivalent expressions $-S^{(2)}$, $\mathfrak{v}^{(0)}$ or $I^{(2)}$. Specializing formally expressions (2.7), (2.9), or (2.11) to the situation of incompressible gases we are led to consider $\gamma = \overline{\lambda}|\partial_x T|^2 + \frac{1}{2}\overline{\eta}|d|^2$ where $d = \partial_x v + \partial_x v^t$, and we will use the coefficients $\overline{\lambda} = \mathrm{A}_\lambda/T^{1+a}$ and $\overline{\eta} = \mathrm{A}_\eta/T^a$, so that

$$\gamma = \frac{\mathrm{A}_\lambda}{T^{1+a}}|\partial_x T|^2 + \frac{1}{2}\frac{\mathrm{A}_\eta}{T^a}|d|^2, \tag{3.7}$$

where $\mathrm{A}_\lambda > 0$, $\mathrm{A}_\eta > 0$, and $a > 0$, are positive parameters at our disposal. Kinetic theory suggests values $a \in (0, 2]$, e.g., for small Mach number flows or incompressible flows. We will assume in the following that $a \in (0, 1]$ since we want to control $\log T$ from the second order entropy corrector γ.

Remark 3.1. A natural scaling associated with the temperature weights of $-S^{(2)}$, $\mathfrak{v}^{(0)}$, $I^{(2)}$, and γ is that v scales as $\sqrt{T}$. ∎

Remark 3.2. The second order entropy corrector γ corresponds to $\gamma^{[k]}$ in (2.13) with $k = 1$ if we replace d by $\partial_x v$. These modifications are unessential and a similar analysis can be conducted for $\gamma^{[1]}$ as for γ. ∎

Remark 3.3. In the definition of higher order entropies, we have confined ourselves to weights in the form of power functions of temperature but more general functions of temperature could also be considered, as well as functions of entropy. ∎

3.3. Balance equation for γ

We write the balance equation for γ in the form

$$\partial_t\gamma + \partial_x\cdot(v\gamma) + \partial_x\cdot\varphi + \pi + \Sigma + \omega = 0, \tag{3.8}$$

where φ represents a dissipative flux and $\pi + \Sigma + \omega$ a source term. We expect π to be nonnegative and composed of higher order derivative terms, Σ to be composed of higher order derivative split terms and is to be controled with π, and ω to be composed of lower order derivative terms arising from convection and is to be majorized by π.

Proposition 3.4. *Let (v, T) be a smooth solution of the incompressible Navier-Stokes equations. Then we may take*

$$\varphi = \frac{a\mathrm{A}_\eta\lambda}{2\rho c_v}\frac{|d|^2\,\partial_x T}{T^{1+a}} - \frac{\mathrm{A}_\eta\eta}{\rho}\frac{d{:}\partial_x d}{T^a} + \frac{(1+a)\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{|\partial_x T|^2\,\partial_x T}{T^{2+a}} - \frac{2\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{\partial_x^2 T\cdot\partial_x T}{T^{1+a}}$$

$$\begin{aligned}\pi =& \frac{2\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{|\partial_x^2 T|^2}{T^{1+a}} + \frac{(1+a)(2+a)\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{|\partial_x T|^4}{T^{3+a}}\\ &+ \Big(\frac{(1+a)\mathrm{A}_\lambda\eta}{2\rho c_v} + \frac{a(1+a)\mathrm{A}_\eta\lambda}{2\rho c_v}\Big)\frac{|d|^2\,|\partial_x T|^2}{T^{2+a}} + \frac{a\mathrm{A}_\eta\eta}{4\rho c_v}\frac{|d|^4}{T^{1+a}} + \frac{\mathrm{A}_\eta\eta}{\rho}\frac{|\partial_x d|^2}{T^a},\end{aligned}$$

$$\Sigma = -\frac{4(1+a)\mathrm{A}_\lambda\lambda}{\rho c_v}\frac{\partial_x^2 T{:}\partial_x T\otimes\partial_x T}{T^{2+a}} - \Big(\frac{2\mathrm{A}_\lambda\eta}{\rho c_v} + \frac{a\mathrm{A}_\eta\lambda}{\rho c_v} + \frac{a\mathrm{A}_\eta\eta}{\rho}\Big)\frac{\partial_x d\,\vdots\, d\otimes\partial_x T}{T^{1+a}}$$

$$\omega = \mathrm{A}_\lambda\frac{d{:}\partial_x T\otimes\partial_x T}{T^{1+a}} + 2\frac{\mathrm{A}_\eta}{\rho}\frac{d{:}\partial_x^2 p}{T^a} + 2\mathrm{A}_\eta\frac{d{:}(\partial_x v\cdot\partial_x v)}{T^a}.$$

In this proposition, for the sake of conciseness, we have introduced convenient compact notation. For a and b vectors, we denote by $a\otimes b$ the matrix with elements a_ib_j, $1\le i,j\le n$. For a matrix and b vector we denote by $a\otimes b$ the third order tensor with elements $a_{ij}b_k$, $1\le i,j,k\le n$. For a and b matrices we denote by $a{:}b$ the quantity $\sum_{ij} a_{ij}b_{ij}$ and $|a|^2 = a{:}a$. For a and b third order tensors, like $\partial_x d$ or $d\otimes\partial_x T$, we denote by $a\,\vdots\, b$ the quantity $\sum_{ijk} a_{ijk}b_{ijk}$ and we define $|a|^2 = a\,\vdots\, a$. Some expressions would be ambiguous for general tensors, but these ambiguities are easily resolved thanks to symmetry properties of multiple derivatives.

Of course, the decomposition (3.8) is not unique since various integration by parts may be performed and terms may be exchanged between φ, π, and Σ. In particular, all expressions involving tensor full contractions—like for instance $\partial_x^2 T{:}\partial_x^2 T$—can be replaced by similar expressions involving partial contractions—as $(\Delta T)^2$ for instance. Some of these expressions are derived in Section 3.4 where we investigate the sign of the higher order derivative terms $\int_{\mathbb{R}^n}(\pi+\Sigma)\,dx$.

Proof. In order to derive a balance equation for γ, we evaluate its time differential in terms of temperature and velocity gradients. To this aim, letting $\overline{\lambda} = \mathrm{A}_\lambda/T^{1+a}$, and $\overline{\eta} = \mathrm{A}_\eta/T^a$, we write that

$$\begin{aligned}\partial_t\gamma + \sum_{1\le l\le n} v_l\partial_l\gamma =& \Big(\partial_T\overline{\lambda}|\partial_x T|^2 + \tfrac{1}{2}\partial_T\overline{\eta}|d|^2\Big)(\partial_t T + \sum_{1\le l\le n} v_l\partial_l T)\\ &+ 2\overline{\lambda}\sum_{1\le i\le n}\partial_i T(\partial_t\partial_i T + \sum_{1\le l\le n} v_l\partial_l\partial_i T) + \overline{\eta}\sum_{1\le i,j\le n} d_{ij}(\partial_t d_{ij} + \sum_{1\le l\le n} v_l\partial_l d_{ij}).\end{aligned}$$

Upon using the governing equations we obtain

$$\begin{aligned}\partial_t\gamma + \sum_{1\le l\le n} v_l\partial_l\gamma &= \Big(\partial_T\overline{\lambda}|\partial_x T|^2 + \tfrac12\partial_T\overline{\eta}|d|^2\Big)\frac{1}{\rho c_v}\Big(\lambda\partial_x\cdot\partial_x T + \tfrac12\eta|d|^2\Big)\\
&+ 2\overline{\lambda}\Big(\sum_{1\le i\le n}\partial_i T\partial_i\big(\frac{1}{\rho c_v}(\lambda\partial_x\cdot\partial_x T + \tfrac12\eta|d|^2)\big) - \sum_{1\le i,l\le n}\partial_i T\partial_i v_l\partial_l T\Big)\\
&+ 2\overline{\eta}\Big(\sum_{1\le i,j\le n} d_{ij}\partial_j\big(\frac{1}{\rho}(\eta\partial_x\cdot\partial_x v_i - \partial_i p)\big) - \sum_{1\le i,j,l\le n} d_{ij}\partial_j v_l\partial_l v_i\Big).\end{aligned}\tag{3.9}$$

The governing equation for γ is then obtained after various integrations by parts. More specifically, let us denote by $\mathcal{T}^\partial$, $\mathcal{T}^\lambda$, and $\mathcal{T}^\eta$, the terms appearing in the rigth hand side of Equation (3.9). The contributions in $\mathcal{T}^\partial$ in the form $|\partial_x T|^2|d|^2$ and $|d|^4$ are left unchanged whereas the contributions in the form $|\partial_x T|^2\,\partial_x\cdot\partial_x T$ and $|d|^2\,\partial_x\cdot\partial_x T$ are integrated by parts. This yields in particular a term in the form $|\partial_x T|^4$. The two first terms of $\mathcal{T}^\lambda$ and $\mathcal{T}^\eta$ are integrated by parts, thereby eliminating third order derivatives, whereas the second term of $\mathcal{T}^\lambda$ is left unchanged. Finally, the third term of $\mathcal{T}^\lambda$ and the second and third of $\mathcal{T}^\eta$ yield the lower order convective contributions of ω. ∎

3.4. Unconditional entropicity

Integrating the γ balance equation (3.8), all the flux terms are eliminated, and we obtain the identity

$$\partial_t\int_{\mathbb{R}^n}\gamma\,dx + \int_{\mathbb{R}^n}(\pi+\Sigma)\,dx = -\int_{\mathbb{R}^n}\omega\,dx. \tag{3.10}$$

Our aim in this section is to study the sign of $\int_{\mathbb{R}^n}(\pi+\Sigma)\,dx$. More specifically, we investigate the entropicity property

$$\frac1c\int_{\mathbb{R}^n}\pi\,dx \le \int_{\mathbb{R}^n}(\pi+\Sigma)\,dx \le c\int_{\mathbb{R}^n}\pi\,dx, \tag{3.11}$$

where c denotes a positive constant. This inequality implies in particular that

$$\partial_t\int_{\mathbb{R}^n}\gamma\,dx + \frac1c\int_{\mathbb{R}^n}\pi\,dx \le \int_{\mathbb{R}^n}|\omega|\,dx, \tag{3.12}$$

and majorization of the remaining term $\int_{\mathbb{R}^n}|\omega|\,dx$ is discussed in Section 3.5.

Proposition 3.5. *Assume that the parameter a associated with γ is such that*

$$0 < a < \inf\Big(\frac{4n-1}{2n^2+1}, 2\big(\frac{\lambda}{\eta c_v} + \frac{\eta c_v}{\lambda}\big)^{-1}\Big). \tag{3.13}$$

Then there exists positive constants A_λ and A_η such that unconditional entropicity holds, that is, (3.11) *holds for any functions (v,T) such that $v, T-T_\infty \in H^2(\mathbb{R}^n)\cap A(\mathbb{R}^n)$, and $T \ge T_{\min} > 0$, where A denotes the Wiener algebra. On the other hand, unconditional entropicity does not hold when a is close to unity.*

Proof. By a density argument, and thanks to classical interpolation inequalities, it is sufficient to consider the situation where $v, T - T_\infty \in \mathcal{D}(\mathbb{R}^n)$, $T \geq T_{\min} > 0$. We first consider the terms of $\int_{\mathbb{R}^n} (\pi + \Sigma)\, dx$ which only involve temperature gradients. Regrouping these terms, we have to investigate the sign of

$$z^{[T]} = \int_{\mathbb{R}^n} \Big(2 \frac{|\partial_x^2 T|^2}{T^{1+a}} - 4(1+a) \frac{\partial_x^2 T : \partial_x T \otimes \partial_x T}{T^{2+a}} + (1+a)(2+a) \frac{|\partial_x T|^4}{T^{3+a}} \Big) dx. \quad (3.14)$$

A direct application of the binomial formula is not very useful since it would only yield entropicity with respect to temperature gradients for $-1 \leq a \leq 0$. In order to obtain entropicity for posivite values of a we use the polar decomposition of the Hessian matrix $\partial_x^2 T$. Defining for short

$$z^{[2]} = \frac{\partial_x^2 T}{T^{(1+a)/2}}, \qquad z^{[1]} = \frac{\partial_x T \otimes \partial_x T}{T^{(3+a)/2}}, \quad (3.15)$$

we have

$$z^{[T]} = \int_{\mathbb{R}^n} \big(2|z^{[2]}|^2 - 4(1+a) z^{[2]} : z^{[1]} + (1+a)(2+a) |z^{[1]}|^2 \big)\, dx. \quad (3.16)$$

On the other hand, using integrations by parts, one easily establishes that

$$\int_{\mathbb{R}^n} \big(\mathrm{tr} z^{[2]} \big)^2\, dx = \int_{\mathbb{R}^n} \big(|z^{[2]}|^2 - 3(1+a) z^{[2]} : z^{[1]} + (1+a)(2+a) |z^{[1]}|^2 \big)\, dx, \quad (3.17)$$

$$\int_{\mathbb{R}^n} \mathrm{tr} z^{[2]} \mathrm{tr} z^{[1]}\, dx = \int_{\mathbb{R}^n} \big(-2 z^{[2]} : z^{[1]} + (2+a) |z^{[1]}|^2 \big)\, dx, \quad (3.18)$$

and we also have $z^{[1]} : z^{[1]} = \big(\mathrm{tr} z^{[1]} \big)^2$. We have denoted by $\mathrm{tr} A$ the trace of a matrix A and we define $A = \widehat{A} + (\mathrm{tr} A / n) \mathbb{I}$ where $\mathbb{I}$ is the unit matrix and $\mathrm{tr}(\widehat{A}) = 0$. After some manipulations, using (3.17) (3.18) and polar decompositions, we obtain

$$(1 - \frac{1}{n}) \int_{\mathbb{R}^n} |z^{[2]}|^2\, dx = \int_{\mathbb{R}^n} \Big(|\widehat{z}^{[2]}|^2 - \frac{3(1+a)}{n+2} \widehat{z}^{[2]} : \widehat{z}^{[1]} + \frac{(1+a)(2+a)}{n+2} |\widehat{z}^{[1]}|^2 \Big)\, dx,$$

$$(1 + \frac{2}{n}) \int_{\mathbb{R}^n} z^{[2]} : z^{[1]}\, dx = \int_{\mathbb{R}^n} \Big(\widehat{z}^{[2]} : \widehat{z}^{[1]} + \frac{(2+a)}{n-1} |\widehat{z}^{[1]}|^2 \Big)\, dx,$$

$$(1 - \frac{1}{n}) \int_{\mathbb{R}^n} |z^{[1]}|^2\, dx = \int_{\mathbb{R}^n} |\widehat{z}^{[1]}|^2\, dx.$$

These relations imply—after some algebra—that

$$(1 - \frac{1}{n})(1 + \frac{2}{n}) z^{[T]} = \int_{\mathbb{R}^n} \Big(2(1 + \frac{2}{n}) |\widehat{z}^{[2]}|^2 - (1+a)(4 + \frac{2}{n}) \widehat{z}^{[2]} : \widehat{z}^{[1]} \\ + (1+a)(2+a) |\widehat{z}^{[1]}|^2 \Big)\, dx.$$

From the binomial formula, there exists $\delta > 0$ such that

$$\delta\Big(\int_{\mathbb{R}^n} |\widehat{z}^{[2]}|^2\,dx + \int_{\mathbb{R}^n} |\widehat{z}^{[1]}|^2\,dx\Big) \le z^{[T]}, \tag{3.19}$$

provided that

$$(1+a)^2(4+\frac{2}{n})^2 - 8(1+\frac{2}{n})(1+a)(2+a) < 0,$$

that is, provided $a < (4n-1)/(2n^2+1)$. The inequality (3.19) then implies that for some positive constant δ we have

$$\delta\Big(\int_{\mathbb{R}^n} |z^{[2]}|^2\,dx + \int_{\mathbb{R}^n} |z^{[1]}|^2\,dx\Big) \le z^{[T]}. \tag{3.20}$$

We now have to consider the remaining terms of $\int_{\mathbb{R}^n}(\pi + \Sigma)\,dx$ involving velocity gradients

$$\begin{aligned} z^{[v]} = & \frac{\mathrm{A}_\eta \eta}{\rho}\int_{\mathbb{R}^n} \frac{|\partial_x d|^2}{T^a} - \Big(2\frac{\mathrm{A}_\lambda \eta}{\rho c_v} + \frac{a\mathrm{A}_\eta \lambda}{\rho c_v} + \frac{a\mathrm{A}_\eta \eta}{\rho}\Big)\int_{\mathbb{R}^n} \frac{\partial_x d{:}d\otimes\partial_x T}{T^{1+a}} \\ & + \Big(\frac{(1+a)\mathrm{A}_\lambda \eta}{2\rho c_v} + \frac{a(1+a)\mathrm{A}_\eta \lambda}{2\rho c_v}\Big)\int_{\mathbb{R}^n} \frac{|d|^2\,|\partial_x T|^2}{T^{2+a}} + \frac{a\mathrm{A}_\eta \eta}{4\rho c_v}\int_{\mathbb{R}^n} \frac{|d|^4}{T^{1+a}}. \end{aligned}$$

Using the binomial formula, the existence of $\delta > 0$ such that

$$\delta\Big(\int_{\mathbb{R}^n} \frac{|\partial_x d|^2}{T^a} + \int_{\mathbb{R}^n} \frac{|d|^2\,|\partial_x T|^2}{T^{2+a}} + \int_{\mathbb{R}^n} \frac{|d|^4}{T^{1+a}}\Big) \le z^{[v]},$$

is a consequence of

$$\Big(2\frac{\mathrm{A}_\lambda \eta}{\rho c_v} + \frac{a\mathrm{A}_\eta \lambda}{\rho c_v} + \frac{a\mathrm{A}_\eta \eta}{\rho}\Big)^2 - 4\frac{\mathrm{A}_\eta \eta}{\rho}\Big(\frac{(1+a)\mathrm{A}_\lambda \eta}{2\rho c_v} + \frac{a(1+a)\mathrm{A}_\eta \lambda}{2\rho c_v}\Big) < 0.$$

Defining $\zeta = \mathrm{A}_\eta c_v / 2\mathrm{A}_\lambda$ and $\xi = \lambda/\eta c_v$, this is equivalent to

$$a^2\zeta^2(1+\xi^2) + a\xi\zeta\big(\xi^{-1} + 2(1-\zeta)\big) + 1 - \zeta < 0, \tag{3.21}$$

and it implies that $\zeta = \mathrm{A}_\eta c_v / 2\mathrm{A}_\lambda > 1$. In this situation $\zeta > 1$ there are two roots of the left hand side of (3.21), one negative $\underline{a}(\xi,\zeta)$ and one positive $\overline{a}(\xi,\zeta)$. Keeping in mind that a has to be positive, we must have

$$a < \overline{a}(\xi,\zeta) = \frac{-\big(\xi^{-1} + 2(1-\zeta)\big) + \big(4\zeta^2 + \zeta(4\xi^{-2} - 4\xi^{-1} - 4) + 4\xi^{-1} - 3\xi^{-2}\big)^{1/2}}{2\zeta(\xi + \xi^{-1})}.$$

Noting that $\overline{a}(\xi,1) = 0$ and $\overline{a}(\xi,\infty) = 2/(\xi + \xi^{-1})$, we obtain for large ζ the sufficient condition $0 < a < 2/(\xi + \xi^{-1})$ and the first part of Proposition 3.5 is proved.

We assume now that $a = 1$ and establish that unconditional entropicity does not hold, keeping in mind that $n \ge 2$. To this purpose, it is sufficient to let $v = 0$ and to only consider the terms involving temperature derivatives. Denoting $\tau = \log T$, it is easily checked that

$$z^{[T]} = \int_{\mathbb{R}^n} \Big(2|\partial_x^2 \tau|^2 - 4\partial_x^2\tau : \partial_x\tau \otimes \partial_x\tau \Big) dx. \tag{3.22}$$

Let $\psi \in \mathcal{D}(\mathbb{R})$ be a C^∞ function with $\psi(s) > 0$ for $|s| < 1$ and $\psi(s) = 0$ for $|s| \geq 1$, and consider $\zeta(x) = \prod_i \psi(x_i)$. Since the contributions $\int_{\mathbb{R}^n} \partial_k^2 \zeta \, (\partial_k\zeta)^2 \, dx$, $1 \leq k \leq n$, in the sum $\int_{\mathbb{R}^n} \partial_x^2\zeta : \partial_x\zeta \otimes \partial_x\zeta \, dx$ vanish and since $\partial_{jk}\zeta \, \partial_j\zeta \, \partial_k\zeta$ is nonnegative and nonzero for $j \neq k$, it is easily checked that $\int_{\mathbb{R}^n} \partial_x^2\zeta : \partial_x\zeta \otimes \partial_x\zeta \, dx$ is strictly positive. Since the two terms scale differently in (3.22), letting $\tau - \tau_\infty = \Lambda\zeta$, that is, $T = T_\infty \exp(\Lambda\zeta)$, there exists $\Lambda > 0$ large enough such that the quantity $z^{[T]}$ is negative, and we also have $v, T - T_\infty \in \mathcal{D}(\mathbb{R}^n)$. By a continuity argument with respect to a, using the same fixed τ, $z^{[T]}$ remains negative for a close to unity and the proof is complete. ∎

Remark 3.6. Many refinements of Proposition 3.5 are feasible but are beyond the scope of this work. Note that we have investigated entropicity independently of the fact that (v, T) is a solution of the governing equations and independently of any constraint on (v, T). This is in contrast with Section 5 where we will impose constraints on the norms of $\log T$ and $v/\sqrt{T}$ for entropicity. The ratio $(4n-1)/(2n^2+1)$ can be written $1 - 2(n-1)^2/(2n^2+1)$, is always smaller than unity—keeping in mind that $n \geq 2$—and is $11/19$ for $n = 3$. ∎

Remark 3.7. Inequalities like

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^4}{T^{3+a}} \, dx \leq c \int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a}} \, dx, \tag{3.23}$$

hold whenever $a \neq -2$, $T - T^\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ and $T \geq T_{\min} > 0$. It can be established by a density argument and upon considering $\partial_x \cdot \big(|\partial_x T|^2 \partial_x T / T^{(2+a)} \big)$. The particular case $a = -1$ has been investigated by Lions and Villani [LV95]. Different inequalities will be established in Section 4 with powers of $\| \log T \|_{BMO}$ as multiplicative factors in the right hand side of (3.23). ∎

Proposition 3.5 shows that unconditional entropicity—unconditional positiveness of higher order derivative source terms $\int_{\mathbb{R}^n} (\pi + \Sigma) \, dx$—only holds for a restricted family of second order entropy correctors. In particular, unconditional entropicity does not hold for the natural logarithmic scaling $a = 1$. An inescapable consequence is that only conditional entropicity will allow stronger and more satisfactory results. Conditional entropicity will be investigated in Section 5 for generalized entropies of arbitrary order with temperature dependent transport coefficients.

3.5. Estimates of convective terms

In order to estimate convective terms, we need to express velocity gradients in terms of the strain rate tensor.

Proposition 3.8. *For any $v \in H^1$ and any index pair (i, j) we have* [Tin72]

$$2\partial_j v_i = d_{ij} - \sum_{1 \le l \le n} R_l R_j d_{li} + \sum_{1 \le l \le n} R_l R_i d_{lj}, \tag{3.24}$$

where $R_i = (-\Delta)^{-1/2}\partial_i$ are the Riesz transforms, $1 \le i \le n$, and we also have

$$2\partial_j \partial_k v_i = \partial_k d_{ij} + \partial_j d_{ik} - \partial_i d_{jk}. \tag{3.25}$$

In the following proposition, we obtain a typical estimate of $\int_{\mathbb{R}^n} |\omega| \, dx$ in terms of $\int_{\mathbb{R}^n} \pi \, dx$ and $\int_{\mathbb{R}^n} |d|^2 \, dx$.

Proposition 3.9. *Assume that $v, T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $T \ge T_{\min} > 0$, and $a \le 1/3$. Then the following estimate holds*

$$\int_{\mathbb{R}^n} |\omega| \, dx \le c\Big(\int_{\mathbb{R}^n} \pi \, dx\Big)^{1/2} \Big(\int_{\mathbb{R}^n} |d|^2 \, dx\Big)^{1/2} \sup_{\mathbb{R}^n} T^{(1-a)/2}. \tag{3.26}$$

Proof. We have $\omega = \overline{\lambda} d{:}\partial_x T \otimes \partial_x T + 2(\overline{\eta}/\rho) d{:}\partial_x^2 p + 2\overline{\eta} d{:}\partial_x v \cdot \partial_x v$ and we examine each term at a time. The first term $\overline{\lambda} d{:}\partial_x T \otimes \partial_x T$ can directly be estimated by using Holder inequality

$$\int_{\mathbb{R}^n} \frac{|d{:}\partial_x T \otimes \partial_x T|}{T^{1+a}} \, dx \le c\Big(\int_{\mathbb{R}^n} \pi \, dx\Big)^{1/2} \Big(\int_{\mathbb{R}^n} |d|^2 \, dx\Big)^{1/2} \sup T^{(1-a)/2}.$$

In order to estimate $|2\overline{\eta} d{:}\partial_x v \cdot \partial_x v|$, we use the expression of $\partial_x v$ in terms of d, and we obtain a sum of terms in the form

$$\int_{\mathbb{R}^n} \frac{|d| \, \big|\mathcal{R}(d)\big| \, \big|\mathcal{R}'(d)\big|}{T^a} \, dx$$

where $\mathcal{R}$ and $\mathcal{R}'$ are products of Riesz transforms. Upon introducing temperature factors as

$$\int_{\mathbb{R}^n} \frac{|d|}{T^{\frac{1+a}{4}}} \big|\mathcal{R}\big(\frac{d}{T^{\frac{1+a}{4}}} T^{\frac{1+a}{4}}\big)\big| \, \big|\mathcal{R}'(d)\big| T^{(1-3a)/4} \, dx \tag{3.27}$$

and applying Holder inequality with exponents

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{\infty} + \frac{1}{2} + \frac{1}{\infty} = 1$$

we obtain the desired estimate provided that $a \le 1/3$. Upon using the expression of $\partial_x^2 p$ in terms of velocity gradients and the expression of velocity gradients in terms of the strain rate tensor, we can finally express $2(\overline{\eta}/\rho) d{:}\partial_x^2 p$ as a sum of terms in the form

$$\int_{\mathbb{R}^n} \frac{d \, \mathcal{R}\big(\mathcal{R}'(d) \, \mathcal{R}''(d)\big)}{T^a} \, dx$$

where $\mathcal{R}$, $\mathcal{R}'$, and $\mathcal{R}''$ are products of Riesz transforms so that the pressure term can be treated as the term $|2\overline{\eta}d{:}\partial_x v\cdot\partial_x v|$. ∎

3.6. Temperature weights

The main difficulties of the Navier-Stokes equations arise from the nonlinear convective terms $v\cdot\partial_x\phi$, where ϕ stands for v or T. These terms introduce nonlinearities through a multiplication by the velocity v of the gradient $\partial_x\phi$ appearing in the ϕ equation. On the other hand, the natural scaling associated with the temperature weights of γ is that v scales as $\sqrt{T}$. Therefore, we expect extra temperature factors in the form $\sqrt{T}$ to appear when estimating $\int|\omega|\,dx$ in terms of $\int\pi\,dx$ and $\int\gamma\,dx$, as inherited from original nonlinearities. Indeed, a direct consequence of Proposition 3.9 is that

$$\int_{\mathbb{R}^n}|\omega|\,dx \le c\Big(\int_{\mathbb{R}^n}\pi\,dx\Big)^{1/2}\Big(\int_{\mathbb{R}^n}\gamma\,dx\Big)^{1/2}\sup_{\mathbb{R}^n}T^{1/2}, \tag{3.28}$$

and the problem is now to control these $\sup_{\mathbb{R}^n}T$ factors. A first possibility could be to use (3.26) with $a=1$ and the natural kinetic energy estimates of $\int_0^t\int_{\mathbb{R}^n}|d|^2\,dxdt$, assuming that we can eliminate the limitation $a\le 1/3$ in Proposition 3.9. However, this seems hopeless since unconditional entropicity does to hold for $a=1$. More generally, larger values of a promote majorization of convective terms, but inhibit unconditional entropicity, and, conversely, smaller values of a promote unconditional entropicity, but prevent majorization of convective terms. Another possibility could be to estimate $\sup_{\mathbb{R}^n}T$ in terms of $\int_{\mathbb{R}^n}\pi\,dx$ and $\int_{\mathbb{R}^n}\gamma\,dx$, but this is not possible since $T_\infty>0$ and only $T-T_\infty$ can be estimated in this manner, as for instance for $n=3$

$$\sup_{\mathbb{R}^n}|T-T_\infty| \le c\Big(\int_{\mathbb{R}^n}\pi\,dx\int_{\mathbb{R}^n}\gamma\,dx\Big)^{\frac{1}{2(1-a)}}. \tag{3.29}$$

Therefore, it appears that, in the estimates of Proposition 3.9, the powers of the solution derivatives are not a difficulty thanks to the terms $|\partial_x T|^4$, $|d|^4$, $|\partial_x^2 T|^2$, and $|\partial_x d|^2$, but difficulties arise, however, in the temperature exponents appearing at the denominator of the convective term ω which are too small in comparison to those appearing in the higher derivative terms π associated with transport fluxes.

A natural phenomenon which reduces the temperature exponents appearing at the denominator of the higher order derivative terms of π is the temperature dependence of transport coefficients. When λ and η scale as $T^{\varkappa}$, all temperature exponents at the denominators of π are decreased by $\varkappa$ whereas those of ω are unchanged. The corresponding system of partial differential equations is investigated in the following sections using somewhat different methods. The direct techniques used in this section do not apply anymore because of a new pressure term $p_\eta=-\sum_{1\le i,j\le n}R_iR_j(\eta d_{ij})$ due to the derivatives of viscosity with respect to temperature, which vanishes for constant η. This pressure term introduces extra contributions in Σ which are not simply controled by those of π. Furthermore, the simple direct method of this section cannot be used for the higher order entropy correctors $\gamma^{[k]}$ or $\tilde{\gamma}^{[k]}$ when $k\ge 2$.

Remark 3.10. The same discussion can be conducted in a periodic framework and yields the same conclusions. In this situation, we also have estimates in the form

$$\int_{\Omega} |\omega|\, dx \le c\Big(\int_{\Omega} \pi\, dx\Big)^{3/4}\Big(\int_{\Omega} T^{3-a}\, dx\Big)^{1/4}, \tag{3.30}$$

where the periodic domain Ω is a product of intervals, but the quantity $\int_{\Omega} T^{3-a}\, dx$ cannot be estimated in terms of $\int_{\Omega} \pi\, dx$ and $\int_{\Omega} \gamma\, dx$. Only the difference $T-\overline{T}$, where $\overline{T}$ denotes the average of T over the periodic domain Ω, can be estimated in terms of $\int_{\Omega} \pi\, dx$ and $\int_{\Omega} \gamma\, dx$. ∎

Remark 3.11. Assuming that $\partial_x T/T \in L^2 \cap L^4$ when $n=3$ implies that $\log T$ has a finite limit at infinity [Gal94] so that T_∞ must be positive. In other words, it does not make sense to try to rescale with $T_\infty = 0$. ∎

Remark 3.12. The convective term ω, after a few integrations by parts, can also be written as a sum of terms proportional to the velocity v. This does not improve the estimates of $\int_{\mathbb{R}^n} |\omega|\, dx$ since the gain obtained with the v factor are compensated by the loss of one derivative factor. ∎

4. Weighted inequalities

We collect in this section various weighted inequalities that will be used in the following in order to investigate the situation of temperature dependent transport coefficients.

4.1. Differential identities

As usual, if α_i, $1 \le i \le n$, are nonnegative integers and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is the corresponding multiindex, we denote by ∂^α the differential operator $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and by $|\alpha|$ its order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The derivative of superpositions has been investigated in particular by Vol'pert and Hudjaev [VH72] and the following proposition is established by induction of $|\alpha|$.

Lemma 4.1. *Let f and g be smooth functions and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex. Then we have*

$$\partial^\alpha(fg) = \sum_{0 \le \beta \le \alpha} c_{\alpha\beta}\, \partial^\beta f \partial^{(\alpha-\beta)} g, \tag{4.1}$$

where $c_{\alpha\beta} = \alpha!/\beta!(\alpha-\beta)!$ are nonnegative integer coefficients with $\beta! = \beta_1! \cdots \beta_n!$ and where we write $0 \le \beta \le \alpha$ when $0 \le \beta_i \le \alpha_i$, $1 \le i \le n$.

Furthermore, let f and g be smooth scalar functions, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex with $|\alpha| \ge 1$. The partial derivatives of the superposition $g \circ f$ can be written in the form

$$\partial^\alpha(g \circ f) = \sum_{\sigma\mu} c_{\sigma\mu}\, \partial^\sigma g \prod_{1 \le |\beta| \le |\alpha|} \big(\partial^\beta f\big)^{\mu_\beta}, \tag{4.2}$$

where $c_{\sigma\mu}$ are nonnegative integer coefficients, and the sum is over $1 \leq \sigma \leq |\alpha|$, $\mu = (\mu_\beta)_{1\leq|\beta|\leq|\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, such that

$$\sum_{1\leq|\beta|\leq|\alpha|} \mu_\beta = \sigma, \qquad \sum_{1\leq|\beta|\leq|\alpha|} \beta\, \mu_\beta = \alpha, \tag{4.3}$$

so that we have in particular $\sum_\beta |\beta|\, \mu_\beta = |\alpha|$.

A natural scaling induced by higher order entropies is that v scales as $\sqrt{T}$. As a consequence, we introduce the rescaled unknowns τ and w defined by

$$\tau = \log T, \qquad w = \frac{v}{\sqrt{T}}, \tag{4.4}$$

which will naturally appear in higher order entropy estimates. In particular, we will need the following differential indentities, easily established by induction on $|\alpha|$.

Lemma 4.2. *Let T be smooth and positive and α be a multiindex. Then we have*

$$\frac{\partial^\alpha T}{T} = \sum_\mu c_\mu \prod_{1\leq|\beta|\leq|\alpha|} (\partial^\beta \tau)^{\mu_\beta} = \partial^\alpha \tau + \sum_\mu c_\mu \prod_{1\leq|\beta|\leq|\alpha|-1} (\partial^\beta \tau)^{\mu_\beta}, \tag{4.5}$$

where $\mu = (\mu_\beta)_{1\leq|\beta|\leq|\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and c_μ are nonnegative integer coefficients. The sum is extended over the μ such that

$$\sum_{1\leq|\beta|\leq|\alpha|} \beta\, \mu_\beta = \alpha,$$

so that we have in particular $\sum_\beta |\beta|\ \mu_\beta = |\alpha|$, and the only term with $|\beta| = |\alpha|$ corresponds to $\partial^\alpha \tau$. Conversely, we have

$$\partial^\alpha \tau = \sum_\mu c'_\mu \prod_{1\leq|\beta|\leq|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta} = \frac{\partial^\alpha T}{T} + \sum_\mu c'_\mu \prod_{1\leq|\beta|\leq|\alpha|-1} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta}, \tag{4.6}$$

where c'_μ are integer coefficients and the sum is extended over the same set of μ.

Lemma 4.3. *Let T and v be smooth, T be positive, i with $1 \leq i \leq n$, and α be a multiindex. Then we have*

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \sum_{\mu\tilde{\alpha}} c_{\mu\tilde{\alpha}} \prod_{1\leq|\beta|\leq|\alpha|} (\partial^\beta \tau)^{\mu_\beta}\ \partial^{\tilde{\alpha}} w_i, \tag{4.7}$$

where $\mu = (\mu_\beta)_{1\leq|\beta|\leq|\alpha|}$, $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, $\tilde{\alpha} \in \mathbb{N}^n$, $c_{\mu\tilde{\alpha}}$ are nonnegative integer coefficients, and the sum is extended over the μ and $\tilde{\alpha}$, such that

$$0 \leq \tilde{\alpha} \leq \alpha, \qquad \sum_{1\leq|\beta|\leq|\alpha|} \beta\, \mu_\beta + \tilde{\alpha} = \alpha.$$

More precisely, isolating the only term $\partial^\alpha w_i$ corresponding to $\tilde\alpha = \alpha$ and all the terms corresponding to $\tilde\alpha = (0, \ldots, 0)$, we have

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \partial^\alpha w_i + \sum_{\mu\tilde\alpha} c_{\mu\tilde\alpha} \prod_{1\le|\beta|\le|\alpha|} (\partial^\beta \tau)^{\mu_\beta}\ \partial^{\tilde\alpha} w_i\ + \sum_{\mu} c_{\mu 0} \prod_{1\le|\beta|\le|\alpha|} (\partial^\beta \tau)^{\mu_\beta}\ w_i, \tag{4.8}$$

where the $\tilde\alpha$ in the middle sum are such that $1 \le |\tilde\alpha| < |\alpha|$. Conversely, we have

$$\partial^\alpha w_i = \sum_{\mu\tilde\alpha} c'_{\mu\tilde\alpha} \prod_{1\le|\beta|\le|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta}\ \frac{\partial^{\tilde\alpha} v_i}{\sqrt{T}}, \tag{4.9}$$

and more precisely

$$\partial^\alpha w_i = \frac{\partial^\alpha v_i}{\sqrt{T}} + \sum_{\mu\tilde\alpha} c'_{\mu\tilde\alpha} \prod_{1\le|\beta|\le|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta}\ \frac{\partial^{\tilde\alpha} v_i}{\sqrt{T}}\ + \sum_{\mu} c'_{\mu 0} \prod_{1\le|\beta|\le|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta}\ \frac{v_i}{\sqrt{T}}, \tag{4.10}$$

where $c'_{\mu\tilde\alpha}$ are integer coefficients and the sums are extended over the same sets.

4.2. Weighted operators

We investigate the norm of weighted Calderón-Zygmund operators in Lebesgue spaces [CF74] [GCR85] [MC90] [MC91]. A natural condition associated with weights has been shown to be the Muckenhoupt property A_p, where $1 \le p \le \infty$ [CF74] [CRW76] [GCR85] [GT02a] [GT02b] [GT02c] [Mey90b].

Definition 4.4. *Let $\mathcal{G} : \mathcal{D}(\mathbb{R}^n) \longrightarrow \mathcal{D}'(\mathbb{R}^n)$ be a continuous linear operator and denote by K the restriction of the distribution kernel associated with $\mathcal{G}$ to the open set $x \neq y$ of $\mathbb{R}^n \times \mathbb{R}^n$. We say that $\mathcal{G}$ is a Calderón-Zygmund operator when the following properties are satisfied.*

- *(i) K is a locally integrable function and there exists c_0 such that for $x \neq y$ $|K(x, y)| \le c_0 |x - y|^{-n}$.*
- *(ii) There exists $\delta \in (0, 1]$ and c_1 such that for $x \neq y$ and $|x' - x| \le \frac{1}{2}|x - y|$ we have $|K(x', y) - K(x, y)| \le c_1 |x' - x|^\delta |x - y|^{-n-\delta}$.*
- *(iii) Similarly if $x \neq y$ and $|y' - y| \le \frac{1}{2}|x - y|$ we have $|K(x, y') - K(x, y)| \le c_1 |y' - y|^\delta |x - y|^{-n-\delta}$.*
- *(iv) $\mathcal{G}$ can be extended into a continuous linear operator over $L^2(\mathbb{R}^n)$ with norm lower or equal to c_2.*

Definition 4.5. *Let $g \in L^1_{loc}(\mathbb{R}^n)$ be positive and locally integrable and let $1 < p < \infty$. We say that g satisfies the Muckenhoupt condition A_p if*

$$[g]_{A_p} = \sup_Q \Big(\frac{1}{|Q|} \int_Q g\, dx \Big) \Big(\frac{1}{|Q|} \int_Q g^{-\frac{1}{p-1}}\, dx \Big)^{p-1} < \infty, \tag{4.11}$$

where the supremum is taken over all cubes.

For detailed studies about the Muckenhoupt property we refer to the book of Garcia-Cuerva and Rubio de Francia [GCR85]. We have in particular $A_p \cap A_q = A_{\min(p,q)}$ and the weights of A_p have their logarithms in BMO [GCR85]. A locally summable function f belongs to the space $BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q \left|f(x) - \bar{f}_Q\right| dx < \infty,$$

where the supremum is taken over all cubes Q and where $\bar{f}_Q = 1/|Q| \int_Q f(x)\, dx$ denotes the average of f over Q [Mey90a]. The function space BMO has been introduced by John and Niremberg [JN61] and naturally arises when estimating the norms of the weighted operators $T^\theta R_i T^{-\theta}$ where $R_i = (-\Delta)^{-1/2}\partial_i$, $1 \le i \le n$, are Riesz transforms, or when using the Coifman and Meyer inequalities [MC90] [MC91]. The space BMO and its dual $\mathcal{H}^1$ have already been used in the context of the Navier-Stokes equations [Lio96] [KT00] [Lem02].

Theorem 4.6. *Let $\mathcal{G}$ be a Calderón-Zygmund operator, let $1 < p < \infty$, and let g be a weight in A_p. Then the operator $\mathcal{G}$ is bounded in $L^p\big(g dx\big)$, or equivalently, the operator $g^{1/p}\mathcal{G}g^{-1/p}$ is bounded in L^p, with norm lower than $\mathcal{C}(c_0, c_1, c_2, n, p, [g]_{A_p})$, where $\mathcal{C}$ only depends on c_0, c_1, c_2, n, p, and $[g]_{A_p}$.*

Proof. We refer to Meyer [Mey90a] [Mey90b], Garcia-Cuerva and Rubio de Francia [GCR85], and Coifman and Fefferman [CF74]. A careful examination of the above mentioned references reveals that the constant $\mathcal{C}$ only depends on c_0, c_1, c_2, n, p, and $[g]_{A_p}$. ∎

Theorem 4.7. *There exists constants $b(n)$ and $B(n)$ such that for any $\theta \in \mathbb{R}$, any $u \in BMO$, and any $1 < p < \infty$, the conditions*

$$|\theta| \|u\|_{BMO} < b(n)/2, \qquad |\theta| \|u\|_{BMO} < (p-1)b(n)/2,$$

implies that $\exp(\theta u) \in A_p$ *and*

$$\big[\exp(\theta u)\big]_{A_p} \le \big(1 + B(n)\big)^p.$$

Proof. From a result of John and Niremberg [JN61], there exists positive constants $b(n)$ and $B(n)$, depending only on n, such that for any $u \in BMO$, any cube Q and any positive s the following inequality holds

$$\mu_Q(s) = \frac{1}{|Q|} \text{mes}\{\, x \in Q,\ |u - \bar{u}_Q| > s \,\} \le B(n) \exp\Big(-\frac{s\, b(n)}{\|u\|_{BMO}}\Big)$$

where $\bar{u}_Q$ denotes the average of u over Q. Using the identity

$$\frac{1}{|Q|} \int_Q f(|u - \bar{u}_Q|)\, dx = \int_0^\infty \mu_Q(s)\, df(s),$$

valid for increasing continuously differentiable functions f such that $f(0) = 0$, with $f(s) = \exp(b's) - 1$ and $0 < b'\|u\|_{BMO} < b(n)/2$, we obtain that

$$\frac{1}{|Q|}\int_Q \exp\bigl(b'|u - \bar{u}_Q|\bigr)\,dx \le 1 + \frac{B(n)b'\|u\|_{BMO}}{b(n) - b'\|u\|_{BMO}} \le 1 + B(n).$$

Therefore, we deduce that

$$\sup_Q\Bigl(\frac{1}{|Q|}\int_Q \exp(\theta u)\,dx\Bigr)\Bigl(\frac{1}{|Q|}\int_Q \exp(-\frac{\theta u}{p-1})\,dx\Bigr)^{p-1} \le$$

$$\sup_Q\Bigl(\frac{1}{|Q|}\int_Q \exp(|\theta|\,|u - \bar{u}_Q|)\,dx\Bigr)\Bigl(\frac{1}{|Q|}\int_Q \exp(\frac{|\theta|}{p-1}\,|u - \bar{u}_Q|)\,dx\Bigr)^{p-1} \le \bigl(1 + B(n)\bigr)^p,$$

provided that $|\theta|\|u\|_{BMO} < b(n)/2$ and $|\theta|\|u\|_{BMO} < (p-1)b(n)/2$. ∎

As a direct application of Theorems 4.6 and 4.7 we investigate operator with weights in the form $\exp(\theta u)$ where $\theta \in \mathbb{R}$ and $u \in BMO$.

Corollary 4.8. *Let $\mathcal{G}$ be a Calderón-Zygmund operator and $1 < p < \infty$. There exists constants $\delta(n,p)$ and $\mathcal{C}(c_0, c_1, c_2, n, p)$, depending only on (n,p), and (c_0, c_1, c_2, n, p), respectively, such that for any $\theta \in \mathbb{R}$ and $u \in BMO$, the condition $|\theta|\|u\|_{BMO} < \delta(n,p)$ implies that the operator $\mathcal{G}$ is bounded in $L^p\bigl(\exp(\theta u)dx\bigr)$, or equivalently, that the operator $\exp(\theta u/p)\mathcal{G}\exp(-\theta u/p)$ is bounded in L^p, with norm lower than $\mathcal{C}(c_0, c_1, c_2, n, p)$.*

4.3. Multilinear estimates

We first investigate weighted multilinear estimates, with weights in A_p, and we denote by $A(\mathbb{R}^n)$ the Wiener algebra in $\mathbb{R}^n$.

Theorem 4.9. *Let k, l be positive integers, and α^j, $1 \le j \le l$, be multiindices such that $|\alpha^j| \ge 1$, $1 \le j \le l$, and $k = \sum_{1\le j\le l} |\alpha^j|$. Let $1 < p < \infty$, $g \in A_p$, and $u_1, \dots, u_l$, be such that there exist constants $u_{j,\infty}$ with $u_j - u_{j,\infty} \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, and such that $g^{\frac{1}{p}}\partial^k u_j \in L^p$, $1 \le j \le l$. There exists a constant $c = c(k, n, p, [g]_{A_p})$ only depending on k, n, p, and $[g]_{A_p}$, such that*

$$\bigl\|g^{1/p}\prod_{1\le j\le l}\partial^{\alpha^j}u_j\bigr\|_{L^p} \le c\Bigl(\sum_{1\le j\le l}\|u_j\|_{BMO}\Bigr)^{l-1}\Bigl(\sum_{1\le j\le l}\bigl\|g^{1/p}\partial^k u_j\bigr\|_{L^p}\Bigr), \tag{4.12}$$

where we define

$$\bigl\|g^{1/p}\partial^k v\bigr\|_{L^p}^p = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\int_{\mathbb{R}^n} g\bigl|\partial^\alpha v\bigr|^p\,dx,$$

using the multinomial coefficients [Tau63] [Com70]

$$\binom{k}{\alpha} = \frac{k!}{\alpha!} = \frac{k!}{\alpha_1 \cdots \alpha_n!}.$$

Proof. We use the Coifman-Meyer theory of multilinear operators [MC90] [MC91]. We can first assume that $u_{j,\infty} = 0$, $1 \le j \le l$, since these constants do not modify the norms in (4.12). Since the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, we can assume that $u_j \in \mathcal{S}(\mathbb{R}^n)$, $1 \le j \le l$. In this situation, we can write that

$$\prod_{1\le j\le l} \partial^{\alpha^j} u_j(x) = \mathrm{Cte} \int \prod_{1\le j\le l} \exp\bigl(\mathrm{i}x\cdot\xi^j\bigr)(\xi^j)^{\alpha^j}\, \hat{u}_j(\xi^j)\, d\xi^j,$$

where $\xi^j \in \mathbb{R}^n$, $1 \le j \le l$, and $\hat{u}_j$ denotes the Fourier transform of u_j. We introduce $\varPhi \in C^\infty[0,\infty)$ such that $\sup(\varPhi) \subset [0,1]$, $0 \le \varPhi \le 1$, $\varPhi = 1$ over $[0,1/2]$, and we set $\varPsi = 1 - \varPhi$. We further define $\varPhi_i = \varPhi(\sum_{i+1\le j\le l} |\xi^j|^2/|\xi^i|^2)$ and $\varPsi_i = 1 - \varPhi_i$, for $1 \le i \le l-1$, and we have the partition of unity

$$1 = \varPhi_1 + \varPsi_1\varPhi_2 + \varPsi_1\varPsi_2\varPhi_3 + \cdots + \prod_{1\le j\le l-2} \varPsi_j\, \varPhi_{l-1} + \prod_{1\le j\le l-1} \varPsi_j.$$

We multiply this partition of unity by the product $\prod_{1\le i\le l}(\xi^j)^{\alpha^j}$ and each factor is rewritten in the form

$$\prod_{1\le j\le l} (\xi^j)^{\alpha^j} \prod_{1\le j\le i-1} \varPsi_j\, \varPhi_i = |\xi^i|^k \prod_{1\le j\le l} \Bigl(\frac{\xi^j}{|\xi^i|}\Bigr)^{\alpha^j} \prod_{1\le j\le i-1} \varPsi_j\, \varPhi_i,$$

with the convention that $\varPhi_l = 1$. Denoting by $\mathcal{H}_i$ the multilinear operator associated with the kernel

$$\zeta_i = \prod_{1\le j\le l} \Bigl(\frac{\xi^j}{|\xi^i|}\Bigr)^{\alpha^j} \prod_{1\le j\le i-1} \varPsi_j\, \varPhi_i,$$

we have obtained that

$$\prod_{1\le j\le l} \partial^{\alpha^j} u_j = \sum_{1\le i\le l} \mathcal{H}_i\bigl(u_1, \ldots, u_{i-1}, (-\Delta)^{\frac{k}{2}} u_i, u_{i+1}, \ldots, u_l\bigr).$$

We claim that the operator $\mathcal{H}_i$, where $1 \le i \le l$ is fixed, satisfy the assumptions of the Coifman and Meyer Theorem (Theorem 2, page 434, Section III.XIII.4 of [MC90] [MC91]), so that it can be extended into a continuous operator over $BMO^{i-1} \times L^2 \times BMO^{l-i}$. Indeed, for $1 \le i \le l-1$, the kernel ζ_i of $\mathcal{H}_i$ is nonzero only when $\prod_{1\le j\le i-1} \varPsi_j \varPhi_i$ is nonzero, that is, only when

$$\begin{cases} \displaystyle\sum_{i+1\le j\le l} |\xi^j|^2 \le |\xi^i|^2, & \\ \displaystyle |\xi^{i-k}|^2 \le 2 \sum_{i-k+1\le j\le l} |\xi^j|^2, & 1 \le k \le i-1. \end{cases}$$

These conditions imply that $|\xi^{i-k}|^2 \le 4^k|\xi^i|^2$, $1 \le k \le i-1$, and $\sum_{j\ne i} |\xi^j|^2 \le 4^i|\xi^i|^2$. On the other hand, for $i = l$, we have $\varPhi_l = 1$, and the kernel ζ_l of $\mathcal{H}_l$ is nonzero only when $\prod_{1\le j\le l-1} \varPsi_j$ is nonzero, that is, only when

$$|\xi^{l-k}|^2 \le 2 \sum_{l-k+1\le j\le l} |\xi^j|^2, \qquad 1 \le k \le l-1,$$

and these conditions imply that $|\xi^{l-k}|^2 \leq 4^k|\xi^l|^2$, $1 \leq k \leq l-1$, and $\sum_{j\neq l}|\xi^j|^2 \leq 4^l|\xi^l|^2$. As a consequence, for any $1 \leq i \leq l$, the kernel ζ_i of $\mathcal{H}_i$ is bounded and smooth for $(\xi_1,\ldots,\xi_l) \neq (0,\ldots,0)$. Furthermore, for $\beta = (\beta^1,\ldots,\beta^l)$, $\beta^j \in \mathbb{N}^n$, we have

$$|\partial^\beta \zeta_i| \leq C\big(|\xi^1| + \cdots + |\xi^l|\big)^{-|\beta|},$$

where $|\beta| = |\beta^1| + \cdots + |\beta^l|$. Finally, we have $\zeta_i(\xi_1,\ldots,\xi_l) = 0$ whenever $\xi_j = 0$ for any $j \neq i$. Therefore, from the Coifman and Meyer theorem we obtain that for $\nu \in L^2(\mathbb{R}^n)$

$$\big\|\mathcal{H}_i(u_1,\ldots,u_{i-1},\nu,u_{i+1},\ldots,u_l)\big\|_{L^2} \leq c\prod_{j\neq i}\|u_j\|_{BMO}\,\|\nu\|_{L^2},$$

and that for u_j, $j \neq i$ fixed, the operator

$$\nu \longrightarrow \mathcal{H}_i(u_1,\ldots,u_{i-1},\nu,u_{i+1},\ldots,u_l)$$

is a Calderón-Zygmund operator. From the results of Coifman-Meyer (Theorem 2, page 434, Section III.XIII.4 of [MC90] [MC91]), we also obtain that the distribution kernel K_i associated with $\mathcal{H}_i$ is such that

$$|K_i(x,y)| \leq c\prod_{j\neq i}\|u_j\|_{BMO}|x-y|^{-n}$$

with similar inequalities for the derivatives. Therefore, the operator $\mathcal{H}_i$ rescaled by $\prod_{j\neq i}\|u_j\|_{BMO}$ satisfies the properties (i)–(iv) of Definition 4.4 with $\delta = 1$ and with constants c_0, c_1 and c_2 depending only on k and n. As a consequence, as soon as the weight g satisfies the Muckenhoupt condition A_p, we have

$$\big\|g^{1/p}\mathcal{H}_i(u_1,\ldots,u_{i-1},\nu,u_{i+1},\ldots,u_l)\big\|_{L^p} \leq c\prod_{j\neq i}\|u_j\|_{BMO}\,\|g^{1/p}\nu\|_{L^p},$$

where c only depends on n, k, p and $[g]_{A_p}$. Summing over i, we have obtained that

$$\Big\|g^{1/p}\prod_{1\leq j\leq l}\partial^{\alpha^j}u_j\Big\|_{L^p} \leq c\sum_{1\leq i\leq l}\prod_{\substack{1\leq j\leq l\\ j\neq i}}\|u_j\|_{BMO}\,\big\|g^{1/p}(-\Delta)^{\frac{k}{2}}u_i\big\|_{L^p}.$$

The proof is then complete upon noting that there exists a constant c only depending on n, k, p and $[g]_{A_p}$ such that

$$\big\|g^{1/p}(-\Delta)^{\frac{k}{2}}u_i\big\|_{L^p} \leq c\big\|g^{1/p}\partial^k u_i\big\|_{L^p}.$$

This is obvious when k is even since then $k = 2l$ and $(-\Delta)^{k/2} = (-\Delta)^l$ whereas for k odd $k = 2l+1$ we have $(-\Delta)^{k/2} = (-\Delta)^l(-\Delta)^{1/2}$ and from $\sum_{1\leq j\leq n}R_j^2 = -I$ we obtain that $-(-\Delta)^{k/2} = \sum_{1\leq j\leq n}(-\Delta)^l(-\Delta)^{1/2}R_j^2 = \sum_{1\leq j\leq n}R_j(-\Delta)^l\partial_j$ and thus $\big\|g^{1/p}(-\Delta)^{k/2}\phi\big\|_{L^p} \leq \sum_{1\leq j\leq n}\big\|g^{1/p}(-\Delta)^l\partial_j\phi\big\|_{L^p}$ from the continuity of $g^{1/p}R_jg^{-1/p}$ in L^p, $1 \leq j \leq n$, which is valid if $g \in A_p$ as established in Theorem 4.6. ∎

Remark 4.10. The definition of $\|\partial^k \mathsf{v}\|_{L^p}^p$ using the multinomial coefficients yields in particular that for $p = 2$

$$\|\partial^k \mathsf{v}\|_{L^2}^2 = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \int_{\mathbb{R}^n} \big(\partial^\alpha \mathsf{v}\big)^2 \, dx = \sum_{1 \le i_1, \dots, i_k \le n} \|\partial_{i_1} \cdots \partial_{i_k} \mathsf{v}\|_{L^2}^2, \tag{4.13}$$

so that it is compatible with the classical definition $|\partial^2 \mathsf{v}|^2 = \sum_{ij} (\partial_i \partial_j \mathsf{v})^2$ already used in Section 3.3. This natural definition also simplifies the analytic form of higher order entropies governing equations. ∎

Remark 4.11. The space of smooth functions with compact support $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$—for the norm $\|\cdot\|_{H^k} + \|\cdot\|_{BMO}$ of course—if and only if $k \ge n/2$. Indeed, for $k < n/2$, $\mathcal{D}(\mathbb{R}^n)$ is not even dense in $H^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and counterexemples are classically found in the form of series of needles. On the other hand, for $k = n/2$, we have $H^k(\mathbb{R}^n) \cap BMO(\mathbb{R}^n) = H^k(\mathbb{R}^n)$, whereas for $k > n/2$, $H^k(\mathbb{R}^n)$ is included in the Wiener algebra $A(\mathbb{R}^n)$. We have introduced the natural simplifying assumption $u_j - u_{j,\infty} \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ since it will be sufficient for our applications and since for $k < n/2$, $\mathcal{D}(\mathbb{R}^n)$ is dense in $H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ and $A(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n) \subset BMO(\mathbb{R}^n)$. Extending these inequalities to $H^k(\mathbb{R}^n) \cap BMO(\mathbb{R}^n)$ when $k < n/2$ by using the notion of strict convergence [MC90] is not relevant to our study. ∎

4.4. Weighted interpolation inequalities

We generalize here some Nirenberg interpolation inequalities for intermediate derivatives [Nir59] with weights satisfying the Muckenhoupt properties.

Theorem 4.12. *Let k, j be nonnegative integers, let $1 < q < \infty$, $1 < r < \infty$, and assume that $k \ge 1$ and $0 \le j \le k$. Further assume that g is a weight of the muckenhoupt class $A_r \cap A_q = A_{\min(q,r)}$ and let p be such that*

$$\frac{1}{p} = \frac{k-j}{k} \frac{1}{q} + \frac{j}{k} \frac{1}{r}. \tag{4.14}$$

Then for any v such that $\mathsf{v} \in L^q(gdx)$ and $\partial^k \mathsf{v} \in L^r(gdx)$, the intermediate derivative $\partial^j \mathsf{v}$ is in $L^p(gdx)$ and there exists a constant $\mathcal{C}$ only depending on n, k, q, r, $[g]_{A_q}$ and $[g]_{A_r}$ such that

$$\Big(\int_{\mathbb{R}^n} g |\partial^j \mathsf{v}|^p \, dx \Big)^{\frac{1}{p}} \le \mathcal{C} \Big(\int_{\mathbb{R}^n} g |\mathsf{v}|^q \, dx \Big)^{(1-\frac{j}{k})\frac{1}{q}} \Big(\int_{\mathbb{R}^n} g |\partial^k \mathsf{v}|^r \, dx \Big)^{\frac{j}{k}\frac{1}{r}}. \tag{4.15}$$

Proof. By induction on k, the proof of (4.15) is easily reduced to the special case $j = 1$ and $k = 2$. In this situation, we have $2/p = 1/q + 1/r$ and defining $p' = p/2$ we have $1/2 < p' < \infty$ and

$$\frac{1}{p'} = \frac{1}{q} + \frac{1}{r}. \tag{4.16}$$

Inequality (4.15) can then be rewritten in terms of the square of the gradient

$$\Big(\int_{\mathbb{R}^n} g\big(|\partial \mathsf{v}|^2\big)^{p'}\,dx\Big)^{\frac{1}{p'}} \leq c\Big(\int_{\mathbb{R}^n} g|\mathsf{v}|^q\,dx\Big)^{\frac{1}{q}}\Big(\int_{\mathbb{R}^n} g|\partial^2\mathsf{v}|^r\,dx\Big)^{\frac{1}{r}}. \tag{4.17}$$

In order to estimate the square of the gradient $|\partial\mathsf{v}|^2$ we consider any pair of indices i_1 and i_2, any functions u_1, u_2, in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, and we write—as in the proof of Theorem 4.9—that

$$\partial_{i_1}u_1(x)\;\partial_{i_2}u_2(x) = \mathrm{Cte}\int \exp\big(\mathrm{i}x\cdot(\xi^1+\xi^2)\big)\xi^1_{i_1}\xi^2_{i_2}\hat{u}_1(\xi^1)\hat{u}_2(\xi^2)\,d\xi^1 d\xi^2, \tag{4.18}$$

where $\xi^1,\xi^2\in\mathbb{R}^n$, and $\hat{u}_j$ is the Fourier transform of u_j, $j=1,2$. Introducing again a partition of unity $\Phi_1+\Psi_1=1$ as in Theorem 4.9, we can write that

$$\partial_{i_1}u_1\;\partial_{i_2}u_2 = \mathcal{H}_1\big(u_1,(-\Delta)u_2\big)+\mathcal{H}_2\big((-\Delta)u_1,u_2\big), \tag{4.19}$$

where $\mathcal{H}_1$ is the multilinear operator associated with the symbol $\Phi_1\;\;\xi^1_{i_1}\xi^2_{i_2}/|\xi^1|^2$ and $\mathcal{H}_2$ the multilinear operator associated with $\Psi_1\;\;\xi^1_{i_1}\xi^2_{i_2}/|\xi^2|^2$. Using the results of Grafakos and Torres [GT02a] [GT02b] [GT02c] we deduce that the operators $\mathcal{H}_1$ et $\mathcal{H}_2$ are multilinear Calderón-Zygmund operators (Proposition 6 of [GT02a] or Section 2 of [GT02c]). On the other hand, the weight g also belongs to the class A_∞, that is, there exists constants $C>0$ and $\epsilon\in(0,1]$ such that for any cube Q and any measurable set $E\subset Q$ we have

$$\frac{g(E)}{g(Q)}\leq C\Big(\frac{|E|}{|Q|}\Big)^{\epsilon}$$

where $g(E)=\int_E g(x)\,dx$ and $|E|$ denotes the Lebesgue measure of E. More specifically, for any $1<s<\infty$ and $g\in A_s$, we have $g\in A_\infty$ where the constants C and ϵ only depend on s and $[g]_{A_s}$ [CF74] [GCR85]. As a consequence, we can use the weighted inequalities established by Grafakos and Torres (Corollary 3 of [GT02b] or Corollary 5 of [GT02c]) taking into account that $1<r<\infty$, $1<q<\infty$, $1/2<p'<\infty$, (4.16) and letting $u_1=u_2=v$, and the interpolation constant $\mathcal{C}$ finally only depends on n, k, q, r, $[g]_{A_q}$ and $[g]_{A_r}$. ■

We now consider the case $q=\infty$ by combining the interpolation inequality of Theorem 4.12 with the multilinear estimates of Theorem 4.9.

Theorem 4.13. *Let k, j be nonnegative integers, $1<r<\infty$, assume that $k\geq 1$, and $1\leq j\leq k$. Further assume that g is a weight in the muckenhoupt class A_r and let p be such that*

$$\frac{1}{p}=\frac{j}{kr}. \tag{4.20}$$

Then for any v such that $\mathsf{v}-\mathsf{v}_\infty\in H^k(\mathbb{R}^n)\cap A(\mathbb{R}^n)$, where v_∞ is a constant, and such that $\partial^k\mathsf{v}\in L^r(gdx)$, the intermediate derivative $\partial^j\mathsf{v}$ is in $L^p(gdx)$ and there exists a constant $\mathcal{C}$ only depending on n, k, r, and $[g]_{A_r}$ such that

$$\Big(\int_{\mathbb{R}^n} g|\partial^j\mathsf{v}|^{\frac{rk}{j}}\,dx\Big)^{\frac{j}{rk}} \leq \mathcal{C}\,\|\mathsf{v}\|_{BMO}^{1-\frac{j}{k}}\,\Big(\int_{\mathbb{R}^n} g|\partial^k\mathsf{v}|^r\,dx\Big)^{\frac{j}{rk}}. \tag{4.21}$$

Proof. Letting $p=r$, $|\alpha^j|=1$, and $k=l$ in Theorem 4.9, we deduce that

$$\Big(\int_{\mathbb{R}^n} g|\partial \nu|^{rk}\,dx\Big)^{\frac{1}{rk}} \le c\|\nu\|_{BMO}^{1-\frac{1}{k}} \Big(\int_{\mathbb{R}^n} g|\partial^k \nu|^{r}\,dx\Big)^{\frac{1}{rk}}, \tag{4.22}$$

and this yields (4.21) for $j=1$. For $1<j<k$ we can then interpolate $\partial^j\nu$ between $\partial^1\nu$ and $\partial^k\nu$ and combine inequality (4.15) of Theorem 4.12 with (4.22). ∎

4.5. Weighted products of derivatives

We first investigate products of derivatives of the rescaled unknowns τ and w, with powers of temperature as natural weights.

Theorem 4.14. *Let $k\ge 1$ be an integer, $\bar\theta>0$ be positive, $1<p<\infty$, τ be such that $\tau-\tau_\infty\in H^k(\mathbb{R}^n)\cap A(\mathbb{R}^n)$ for some constant τ_∞. There exist positive constants $\delta(n,k,\bar\theta,p)$ and $c(n,k,p)$, only depending on $(n,k,\bar\theta,p)$ and (n,k,p), respectively, such that if $\|\tau\|_{BMO}<\delta$, then for any real θ with $|\theta|\le\bar\theta$, any integer $l\ge 1$, and any multiindices α^j, $1\le j\le l$, with $|\alpha^j|\ge 1$, $1\le j\le l$, and $\sum_{1\le j\le l}|\alpha^j|=k$, whenenver $\exp(\theta\tau/p)\partial^k\tau\in L^p(\mathbb{R}^n)$, we have the estimates*

$$\Big\|e^{\frac{\theta\tau}{p}}\prod_{1\le j\le l}\big(\partial^{\alpha^j}\tau\big)\Big\|_{L^p} \le c\|\tau\|_{BMO}^{l-1}\,\big\|e^{\frac{\theta\tau}{p}}\partial^k\tau\big\|_{L^p}. \tag{4.23}$$

Further assuming that $w\in H^k(\mathbb{R}^n)\cap A(\mathbb{R}^n)$, $e^{\theta\tau/p}\partial^k w\in L^p(\mathbb{R}^n)$, and $0\le\bar l\le l$, we have

$$\Big\|e^{\frac{\theta\tau}{p}}\prod_{1\le j\le \bar l}\big(\partial^{\alpha^j}\tau\big)\prod_{\bar l+1\le j\le l}\big(\partial^{\alpha^j}w\big)\Big\|_{L^p} \le c\Big(\|\tau\|_{BMO}+\|w\|_{BMO}\Big)^{l-1}$$
$$\times\Big(\big\|e^{\frac{\theta\tau}{p}}\partial^k\tau\big\|_{L^p}+\big\|e^{\frac{\theta\tau}{p}}\partial^k w\big\|_{L^p}\Big), \tag{4.24}$$

where we have naturally defined

$$\big\|e^{\frac{\theta\tau}{p}}\partial^k w\big\|_{L^p}^p=\sum_{1\le i\le n}\big\|e^{\frac{\theta\tau}{p}}\partial^k w_i\big\|_{L^p}^p=\sum_{\substack{|\alpha|=k\\1\le i\le n}}\frac{k!}{\alpha!}\int_{\mathbb{R}^n}e^{\theta\tau}|\partial^\alpha w_i|^p\,dx,$$

and where, in the left hand member of (4.24)*, with a slight abuse of notation, we have denoted by w any of its components $w_1,\dots,w_n$.*

Proof. This is a direct application of Theorems 4.7 and 4.9 since for $\bar\theta\|\tau\|_{BMO}<b(n)/2$ and $\bar\theta\|\tau\|_{BMO}<(p-1)b(n)/2$ we have $[e^{\theta\tau}]_{A_p}\le\big(1+B(n)\big)^p$. ∎

We now estimates products of derivatives of temperature and velocity components rescaled by the proper temperature factors.

Theorem 4.15. *Let $k\ge 1$ be an integer, $\bar\theta>0$ be positive , $1<p<\infty$, T be such that $T\ge T_{\min}>0$ and $T-T_\infty\in H^k(\mathbb{R}^n)\cap A(\mathbb{R}^n)$ for some positive T_∞. There exist positive constants $\delta(n,k,\bar\theta,p)$ and $c(n,k,p)$, only depending on $(n,k,\bar\theta,p)$ and (n,k,p),* *respectively, such that if* $\|\log T\|_{BMO} < \delta$*, then for any real* θ *such that* $|\theta| \leq \bar{\theta}$*, any integer* $l \geq 1$*, and any multiindices* α^j*,* $1 \leq j \leq l$*, with* $|\alpha^j| \geq 1$*,* $1 \leq j \leq l$*, and* $\sum_{1\leq j\leq l} |\alpha^j| = k$*, whenenver* $T^{\theta/p}(\partial^k T)/T \in L^p(\mathbb{R}^n)$*, we have the estimates*

$$\Big\|T^{\frac{\theta}{p}} \prod_{1\leq j\leq l} \Big(\frac{\partial^{\alpha^j} T}{T}\Big)\Big\|_{L^p} \leq c\big\|\log T\big\|_{BMO}^{l-1} \,\big\|T^{\frac{\theta}{p}} \frac{\partial^k T}{T}\big\|_{L^p}. \tag{4.25}$$

Further assuming $v \in H^k(\mathbb{R}^n) \cap A(\mathbb{R}^n)$ *and* $\|\log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty} < \delta(n, k, \bar{\theta}, p)$*, whenever* $T^{\theta/p}(\partial^k v)/\sqrt{T} \in L^p(\mathbb{R}^n)$*, we have for* $0 \leq \bar{l} \leq l$

$$\Big\|T^{\frac{\theta}{p}} \prod_{1\leq j\leq \bar{l}} \Big(\frac{\partial^{\alpha^j} T}{T}\Big) \prod_{\bar{l}+1\leq j\leq l} \Big(\frac{\partial^{\alpha^j} v}{\sqrt{T}}\Big)\Big\|_{L^p} \leq c\Big(\big\|\log T\big\|_{BMO} + \big\|\frac{v}{\sqrt{T}}\big\|_{L^\infty}\Big)^{l-1}$$

$$\times\Big(\big\|T^{\frac{\theta}{p}} \frac{\partial^k T}{T}\big\|_{L^p} + \big\|T^{\frac{\theta}{p}} \frac{\partial^k v}{\sqrt{T}}\big\|_{L^p}\Big), \tag{4.26}$$

where, in the left hand member, with a slight abuse of notation, we have denoted by v *any of its components* $v_1, \ldots, v_n$*.*

Proof. Assume that $\delta < b(n)/2\bar{\theta}$, and $\delta < (p-1)b(n)/2\bar{\theta}$ so that all the weights $T^\theta = \exp(\theta \log T)$ satisfy the Muckenhoupt condition A_p as soon as $\|\log T\|_{BMO} < \delta$, and are such that $[T^\theta]_{A_p} \leq \big(1 + B(n)\big)^p$. Let $l \geq 1$ be an integer, and α^j, $1 \leq j \leq l$, be multiindices with $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $\sum_{1\leq j\leq l} |\alpha^j| = k$. From Theorem 4.14 applied with $\tau = \log T$ we have

$$\big\|T^{\frac{\theta}{p}} \prod_{1\leq j\leq l} \partial^{\alpha^j} \tau\big\|_{L^p} \leq c\| \log T\|_{BMO}^{l-1} \,\big\|T^{\frac{\theta}{p}} \partial^k \tau\big\|_{L^p},$$

where $c = c(n, k, p)$, and, thus, we only have to estimate integrals like

$$\Big\|T^{\frac{\theta}{p}} \Big(\prod_{1\leq j\leq l} \big(\frac{\partial^{\alpha^j} T}{T}\big) - \prod_{1\leq j\leq l} \partial^{\alpha^j} \tau\Big)\Big\|_{L^p}.$$

Thanks to the differential indentities established in Lemma 4.3, we can write that

$$\prod_{1\leq j\leq l} \big(\frac{\partial^{\alpha^j} T}{T}\big) = \prod_{1\leq j\leq l} \partial^{\alpha^j} \tau \; + \sum_{\mu^1\cdots\mu^l} \prod_{1\leq j\leq l} c_{\mu^j} \prod_{1\leq|\beta|\leq|\alpha^j|} (\partial^\beta \tau)^{\mu^j_\beta}, \tag{4.27}$$

where $\mu^j = (\mu^j_\beta)_{1\leq|\beta|\leq|\alpha^j|}$, with $\mu^j_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and where c_{μ^j} are nonnegative integer coefficients. In addition, the μ^j are such that $\sum_{1\leq|\beta|\leq|\alpha^j|} |\beta| \; \mu^j_\beta = |\alpha^j|$, so that we have in particular

$$\sum_{\substack{1\leq|\beta|\leq|\alpha^j|\\ 1\leq j\leq l}} |\beta| \; \mu^j_\beta = \sum_{1\leq j\leq l} |\alpha^j| = k.$$

When $k = l$, all derivatives must be of first order so that the sum in the right hand side of (4.27) is absent. On the other hand, when $l < k$, in each term of this sum, there are always at least $l + 1$ derivative factors in the product, since the only term with exactly l factors has been isolated, and at most k derivative factors. From the multilinear estimates of Theorem 4.9 applied to each of these terms we obtain

$$\Big\| T^{\frac{\theta}{p}} \Big(\prod_{1\le j\le l} \big(\frac{\partial^{\alpha^j} T}{T} \big) - \prod_{1\le j\le l} \partial^{\alpha^j}\tau \Big) \Big\|_{L^p} \le c \left(\big\|\log T\big\|_{BMO}^{l} + \cdots + \big\|\log T\big\|_{BMO}^{k-1} \right) \big\| T^{\frac{\theta}{p}} \partial^k \tau \big\|_{L^p}.$$

Therefore, assuming $\delta < 1$, we have established for any $1 \le l \le k$ that

$$\Big\| T^{\frac{\theta}{p}} \Big(\prod_{1\le j\le l} \big(\frac{\partial^{\alpha^j} T}{T} \big) - \prod_{1\le j\le l} \partial^{\alpha^j}\tau \Big) \Big\|_{L^p} \le c \,\big\|\log T\big\|_{BMO}^{l} \,\big\| T^{\frac{\theta}{p}} \partial^k \tau \big\|_{L^p}. \tag{4.28}$$

We now consider the special case $l = 1$ and we sum the above estimates (4.28) over all α with $|\alpha| = k$. This yields

$$\Big\| T^{\frac{\theta}{p}} \big(\frac{\partial^k T}{T} - \partial^k \tau \big) \Big\|_{L^p} \le c \big\|\log T\big\|_{BMO} \,\big\| T^{\frac{\theta}{p}} \partial^k \tau \big\|_{L^p},$$

where $c = c(n, k, p)$ so that for $c(n, k, p)\|\log T\|_{BMO} < 1/2$ we have

$$\tfrac{1}{2} \,\big\| T^{\frac{\theta}{p}} \partial^k \tau \big\|_{L^p} \;\le\; \big\| T^{\frac{\theta}{p}} \frac{\partial^k T}{T} \big\|_{L^p} \;\le\; \tfrac{3}{2} \,\big\| T^{\frac{\theta}{p}} \partial^k \tau \big\|_{L^p}. \tag{4.29}$$

Reinserting then this inequality (4.29) in inequality (4.28) completes the proof of (4.25).

The proof of inequality (4.26) is similar, and it is found in particular that when $\|\log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty} < \delta(n, k, \bar{\theta}, p)$ we have

$$\Big\| T^{\frac{\theta}{p}} \big(\frac{\partial^k v}{\sqrt{T}} - \partial^k w \big) \Big\|_{L^p} \;\le\; c \left(\big\|\log T\big\|_{BMO} + \big\| \frac{v}{\sqrt{T}} \big\|_{L^\infty} \right) \left(\big\| T^{\frac{\theta}{p}} \partial^k \tau \big\|_{L^p} + \big\| T^{\frac{\theta}{p}} \partial^k w \big\|_{L^p} \right), \tag{4.30}$$

where the terms proportional to w in relations (4.8) have been taken into account with the factors $\|v/\sqrt{T}\|_{L^\infty}$. ■

Remark 4.16. As a special case of Theorem 4.15 we obtain that for $T - T_\infty \in H^2(\mathbb{R}^n) \cap A(\mathbb{R}^n)$, $T \ge T_{\min} > 0$ and $\|\log T\|_{BMO}$ small enough, we have

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^4}{T^{3+a}}\, dx \le c \|\log T\|_{BMO}^2 \int_{\mathbb{R}^n} \frac{|\partial_x^2 T|^2}{T^{1+a}}\, dx. \tag{4.31}$$

This inequality differs from that of Remark 3.7 by the factors $\|\log T\|_{BMO}$. ■

5. Higher order entropy estimates

We investigate in this section higher order entropy estimates for incompressible flows spanning the whole space in the natural situation of temperature dependent viscosity and thermal conductivity. We first discuss the temperature dependence of transport coefficients as obtained from the kinetic theory of gases. We then derive a governing equation for kinetic entropy correctors of arbitrary order and establish conditional entropicity properties, that is entropicity holds whenever $\| \log T \|_{BMO} + \| v/\sqrt{T} \|_{L^\infty}$ is small enough. We then discuss majorization of lower order convective terms and the scales of temperature weights used to renormalize the solution derivatives. Combining these results, we finally investigate entropic principles for higher order kinetic entropy estimators.

5.1. Temperature dependent coefficients

Thermal conductivity and viscosity of a gas depend on temperature

$$\lambda = \lambda(T), \qquad \eta = \eta(T), \tag{5.1}$$

as shown by the kinetic theory [CC70] [FK72] [Gio99]. When one term Sonine polynomial expansions are used to evaluate perturbed distribution functions, the coefficients λ/c_v and η are found in particular in the form

$$\lambda/c_v = \frac{\mathfrak{a}_\lambda T^{1/2}}{\Omega^{(2,2)*}}, \qquad \eta = \frac{\mathfrak{a}_\eta T^{1/2}}{\Omega^{(2,2)*}},$$

where $\mathfrak{a}_\lambda$ and $\mathfrak{a}_\eta$ are constants and $\Omega^{(2,2)*}$ is a reduced collision integral, and the ratio $\lambda/c_v\eta$ is then a constant. For the rigid sphere model for instance, we have exactly $\lambda/c_v = \mathfrak{a}_\lambda T^{1/2}$ and $\eta = \mathfrak{a}_\eta T^{1/2}$. Similarly, for particles interacting as point centers of repulsion with an interaction potential $V = c/\mathrm{r}^\nu$, where r is the distance between two particles, one establishes that $\Omega^{(2,2)*}$ is proportional to $T^{-2/\nu}$ so that we have $\lambda/c_v = \mathfrak{a}_\lambda T^\varkappa$, and $\eta = \mathfrak{a}_\eta T^\varkappa$ with $\varkappa = 1/2 + 2/\nu$ [CC70] [FK72]. The temperature exponent $\varkappa$ then varies from $\varkappa = 1/2$ for rigid spheres with $\nu = \infty$ up to $\varkappa = 1$ for Maxwell molecules with $\nu = 4$. More generally, consider particles interacting with a Lennard-Jones ν-ν' potential

$$V = 4\varepsilon\Big(\big(\frac{\sigma}{\mathrm{r}}\big)^\nu - \big(\frac{\sigma}{\mathrm{r}}\big)^{\nu'} \Big),$$

where V denotes the interaction potential, σ the collision diameter, ε the potential well depth, and ν, ν' are intergers with $\nu > \nu'$ and typical values $\nu = 12$, $\nu' = 6$ [CC70] [FK72]. Collision integrals like $\Omega^{(2,2)*}$ then only depend on the reduced temperature $k_{\mathrm{B}}T/\varepsilon$, and, when $k_{\mathrm{B}}T/\varepsilon$ is large, the repulsive part $\mathrm{r}^{-\nu}$ is dominant, whereas when $k_{\mathrm{B}}T/\varepsilon$ is small the attractive part $\mathrm{r}^{-\nu'}$ is dominant [CC70]. As a consequence, like for point centers of repulsion, collision integrals behave like $T^{s'}$ with $s' = 1/2 + 2/\nu'$ for small T and like T^s with $s = 1/2 + 2/\nu$ for large T [CC70]. In particular, the logarithm $\log \Omega^{(2,2)*}$ has linear asymptotes as function of $\log T$, and $d^k \log \Omega^{(2,2)*}/d(\log T)^k$ is bounded for any $k \geq 1$. As a consequence, $\log \eta$ and $\log \lambda$ have parallel linear asymptotes as function of $\log T$, and $d^k \log \eta / d(\log T)^k$ and $d^k \log \lambda / d(\log T)^k$ are bounded for any $k \geq 1$, or equivalently, $(1/\eta) T^k d^k \eta / dT^k$ and $(1/\lambda) T^k d^k \lambda / dT^k$ are bounded for any $k \geq 1$.

Similar results are also obtained when more than one term are taken into account in orthogonal polynomial expansions of perturbed distribution functions. Indeed, all collision integrals $\Omega^{(i,j)*}$, $i, j \geq 1$, have a common temperature behavior, that is, all ratios of collision integrals are bounded, as for instance for Lennard-Jones or Stockmayer potentials [FK72] [Gio99]. These collision integrals are then used to define the coefficients of the transport linear systems which thus share a common temperature scaling. As a consequence, the transport coefficients, which are obtained as solution of transport linear systems, inherit a common temperature scaling [Gio99]. The same conclusion is also reached with polyatomic molecules when Wang-Chang-Uhlenbeck-Sonine polynomial expansions are used [Gio99]. As a consequence, the relevant mathematical assumptions are that all transport coefficients have a common temperature scaling in such a way $\lambda / c_v \eta$ remains positive and bounded, and $d^k \log \eta / d(\log T)^k$ and $d^k \log \lambda / d(\log T)^k$ are bounded for any $k \geq 1$.

On the other hand, in our particular application, using the maximum principle for temperature yields a uniform lower bound for T, only depending on initial data. Therefore, we may assume that $T \geq T_{\min}$, where $T_{\min}$ is fixed and positive. In this situation, the behavior of transport coefficients for small temperatures is not relevant. In other words, only the repulsive part of the interaction potential between particles plays a role and we may assume that such a behavior is asymptotically that of point centers of repulsion as we have discussed for Lennard-Jones potentials. Therefore, from a mathematical point of view, since we are not interested in small temperatures, we may simplify the assumptions about the temperature dependence of transport coefficients and assume that λ and η are $C^\infty(0, \infty)$, that there exist $\varkappa$, $\underline{\mathfrak{a}} > 0$, and $\overline{\mathfrak{a}} > 0$ with

$$\underline{\mathfrak{a}}\, T^\varkappa \leq \lambda / c_v \leq \overline{\mathfrak{a}}\, T^\varkappa, \qquad \underline{\mathfrak{a}}\, T^\varkappa \leq \eta \leq \overline{\mathfrak{a}}\, T^\varkappa, \tag{5.2}$$

and that, for any integer $\sigma \geq 1$, there exists $\overline{\mathfrak{a}}_\sigma > 0$ with

$$T^\sigma \, |\partial_T^\sigma \lambda| \leq \overline{\mathfrak{a}}_\sigma \, T^\varkappa, \qquad T^\sigma \, |\partial_T^\sigma \eta| \leq \overline{\mathfrak{a}}_\sigma \, T^\varkappa. \tag{5.3}$$

Kinetic theory suggests that $1/2 \leq \varkappa \leq 1$ but the situations where $0 \leq \varkappa < 1/2$ or $\varkappa > 1$ are still interesting to investigate from a mathematical point of view.

Remark 5.1. Assumptions on transport coefficients valid for all temperature may be written

$$\underline{c}\,\zeta \leq \lambda / c_v \leq \overline{c}\,\zeta, \qquad \underline{c}\,\zeta \leq \eta \leq \overline{c}\,\zeta,$$
$$T^\sigma \, |\partial_T^\sigma \lambda| \leq \overline{c}_\sigma \, \zeta, \qquad T^\sigma \, |\partial_T^\sigma \eta| \leq \overline{c}_\sigma \, \zeta, \qquad \sigma \geq 1,$$

where $\underline{c}$, $\overline{c}$, and $\overline{c}_\sigma$, $\sigma \geq 1$, are positive constants. The function ζ is a smooth function of T such that $T^\sigma \, |\partial_T^\sigma \zeta| \leq \overline{c}_\sigma \, \zeta$, $\sigma \geq 1$. For Lennard-Jones ν-ν' potentials, we can take for instance $\zeta = T^s$ for large T and $\zeta = T^{s'}$ for small T, with $s = 1/2 + 2/\nu$ and $s' = 1/2 + 2/\nu'$ [CC70]. We have made in this paper the simpler choice $\zeta = T^s = T^\varkappa$ since we can exclude small temperatures. Kinetic theory also indicates that $\mathfrak{a} T^{1/2} \leq \zeta$ for some positive constant $\mathfrak{a}$. It is interesting to note that with an interaction potential which is infinite at small interparticle distances, we always have $\nu = \infty$ so that $\zeta = T^{1/2}$ for large temperatures. ∎

5.2. Governing equations

With variable transport coefficients, the fluid governing equations can be written

$$\partial_x \cdot v = 0, \tag{5.4}$$

$$\partial_t(\rho v) + \partial_x \cdot (\rho v \otimes v + pI) - \partial_x \cdot \big(\eta(T)\, d\big) = 0, \tag{5.5}$$

$$\partial_t(\rho e) + \partial_x \cdot (\rho e v) - \partial_x \cdot \big(\lambda(T)\, \partial_x T\big) = \tfrac{1}{2}\eta(T) d{:}d, \tag{5.6}$$

where ρ is the constant density, v the velocity, p the pressure, $d = \partial_x v + \partial_x v^t$ the strain rate tensor, $\eta(T)$ the viscosity, e the internal energy per unit mass, and $\lambda(T)$ the thermal conductivity. The energy per unit mass e is still taken for simplicity in the form $e = c_v T$ where c_v is a constant.

We again consider the case of functions defined on $\mathbb{R}^n$ with $n \geq 2$, that are 'constant at infinity', and we only consider solutions such that

$$v, T - T_\infty \in C\big([0,\bar{t}\,], H^l\big) \cap C^1\big([0,\bar{t}\,], H^{l-2}\big) \cap L^2\big([0,\bar{t}\,], H^{l+1}\big), \tag{5.7}$$

where l is an integer such that $l \geq [n/2]+2$, that is, $l > n/2+1$, and $\bar{t}$ is some positive time. We also assume that T is positive and bounded away from zero $T \geq T_{\min}$ where $T_{\min}$ is positive. It will be shown in the next section that these solutions are smooth when the initial data is smooth, whenever they exist. Since the viscosity η is not anymore a constant, the momentum conservation equation is rewritten in the form $\partial_t(\rho v) = \mathbb{P}\big(\partial_x \cdot (-\rho v \otimes v + \eta(T) d)\big)$, which is equivalent to defining the pressure by

$$p = \sum_{1 \leq i,j \leq n} R_i R_j \big(\rho v_i v_j - \eta d_{ij}\big). \tag{5.8}$$

We also have $p \in C\big([0,\bar{t}\,], H^l\big) \cap C^1\big([0,\bar{t}\,], H^{l-2}\big) \cap L^2\big([0,\bar{t}\,], H^{l+1}\big)$ from (5.8) and from the identity $\partial_k p = \sum_{1 \leq i,j \leq n} R_k R_j \big(\rho v_i \partial_i v_j - 2\partial_T \eta \partial_i T \partial_j v_i\big)$.

Remark 5.2. In the special case where $\lambda = \mathfrak{a}_\lambda T^\varkappa$, $\eta = \mathfrak{a}_\eta T^\varkappa$, and c_v is constant, if $v(t,x)$ and $T(t,x)$ are a solution of the Navier-Stokes equations (5.4)–(5.6), then

$$\xi\, v(\xi^{2(1-\varkappa)}t, \xi^{(1-2\varkappa)}x), \qquad \xi^2\, T(\xi^{2(1-\varkappa)}t, \xi^{(1-2\varkappa)}x),$$

are also a solution for any positive ξ. The special situation $\varkappa = 0$ corresponds to the usual rescaled solutions [Che92] [Lem02]. Note that space and time are not stretched in the same direction when $1/2 < \varkappa < 1$. ▪

Remark 5.3. All the results obtained in this section and the following are also valid if the internal energy e per unit mass is taken to be $e = e_0 + \int_0^T c_v(s)\, ds$ with a heat capacity coefficient c_v depending on temperature in such a way that

$$\underline{c} \leq c_v \leq \overline{c}, \qquad T^\sigma\, |\partial_T^\sigma c_v| \leq \overline{c}_\sigma, \qquad \sigma \geq 1,$$

where $\underline{c} > 0$, $\overline{c} > 0$, and $\overline{c}_\sigma > 0$, $\sigma \geq 1$, are positive constants. We will not explicit the corresponding results for the sake of simplicity. ▪

5.3. Higher order kinetic entropy estimators

Specializing formally expression (2.13) to the situation of incompressible gases, we define the $(2k)^{\mathrm{th}}$ order kinetic entropy corrector $\gamma^{[k]}$ by

$$\gamma^{[k]} = \mathrm{A}_\lambda^{[k]} \frac{|\partial^k T|^2}{T^{1+a_k}} + \mathrm{A}_\eta^{[k]} \frac{|\partial^k v|^2}{T^{a_k}}, \tag{5.9}$$

with

$$|\partial^k T|^2 = \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\partial^\alpha T)^2, \qquad |\partial^k v|^2 = \sum_{1 \le i \le n} |\partial^k v_i|^2,$$

where $k!/\alpha!$ are the multinomial coefficients [Tau63] [Com70], and $\mathrm{A}_\lambda^{[k]} > 0$, $\mathrm{A}_\eta^{[k]} > 0$, $a_k \in \mathbb{R}$, are parameters at our disposal. We do not assume anymore that a_k is positive since some negative values will naturally appear in the discussion. Similarly, following (2.14), we also define the $(2k)^{\mathrm{th}}$ order kinetic entropy corrector $\tilde{\gamma}^{[k]}$ by

$$\tilde{\gamma}^{[k]} = \exp\big((1-a_k)\tau\big) \Big(\mathrm{A}_\lambda^{[k]} |\partial^k \tau|^2 + \mathrm{A}_\eta^{[k]} |\partial^k w|^2 \Big), \tag{5.10}$$

where $\tau = \log T$ and $w = v/\sqrt{T}$. The entropy correctors $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ will be shown to have similar properties and both may be used to derive a priori estimates.

In order to recast the zeroth order entropy balance into a more convenient form, we also define $\gamma^{[0]} = \tilde{\gamma}^{[0]}$, for $0 < a_0 \le 1$, by

$$\gamma^{[0]} = \tilde{\gamma}^{[0]} = (\mathrm{A}_\lambda^{[0]} + \mathrm{A}_\eta^{[0]}) \zeta^{[0]}, \tag{5.11}$$

where $\mathrm{A}_\lambda^{[0]} > 0$, $\mathrm{A}_\eta^{[0]} > 0$, and

$$\zeta^{[0]} = \begin{cases} \dfrac{T - T_\infty}{T_\infty} - \log\big(\dfrac{T}{T_\infty}\big) + \frac{1}{2} \dfrac{v^2}{c_v T_\infty}, & \text{if } a_0 = 1, \\[2ex] \dfrac{T - T_\infty}{T_\infty^{a_0}} - \dfrac{T^{1-a_0} - T_\infty^{1-a_0}}{1 - a_0} + \frac{1}{2} \dfrac{v^2}{c_v T_\infty^{a_0}}, & \text{if } 0 < a_0 < 1. \end{cases}$$

Finally, we introduce the $(2k)^{\mathrm{th}}$ order kinetic entropy estimators defined by

$$\Gamma^{[k]} = \gamma^{[0]} + \cdots + \gamma^{[k]}, \qquad k \ge 0, \tag{5.12}$$

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]}, \qquad k \ge 0, \tag{5.13}$$

which will play an important rôle. Strictly speaking, we should term $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ "mathematical $(2k)^{\mathrm{th}}$ order partial entropies" or "$(2k)^{\mathrm{th}}$ order kinetic entropy correctors" or $(2k)^{\mathrm{th}}$ order kinetic entropy deviation estimators" and $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "mathematical $(2k)^{\mathrm{th}}$ order entropies", or "$(2k)^{\mathrm{th}}$ order kinetic entropy estimators". We have also seen in Section 2 that all these quantities can also be associated with Fisher information. However, we will often informally term $\gamma^{[k]}$, $\tilde{\gamma}^{[k]}$, $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "mathematical $(2k)^{\mathrm{th}}$ order entropies" or simply "higher order entropies". Our aim is now to establish balance equations for $\gamma^{[k]}$ and $\widetilde{\gamma}^{[k]}$, to use these equations to derive a priori estimates, and to establish that $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ satisfy conditional entropic principles.

Remark 5.4. Replacing $\partial_x v$ by d in the definition of $\gamma^{[k]}$ would yield

$$\widehat{\gamma}^{[k]} = \mathrm{A}_\lambda^{[k]} \frac{|\partial^k T|^2}{T^{1+a_k}} + \tfrac{1}{2}\mathrm{A}_\eta^{[k]} \frac{|\partial^{k-1} d|^2}{T^{a_k}}, \tag{5.14}$$

which coincides for $k=1$ with the quantity γ introduced in Section 3. However, the definitions (5.9) and (5.14) are equivalent for $k \geq 2$ from (3.25) and yield similar results for $k=1$ from the expression (3.24) of $\partial_x v$ in terms of d and the continuity of $T^\theta R_i T^{-\theta}$ for $\|\log T\|_{BMO}$ small enough. ∎

5.4. Balance equation for $\gamma^{[k]}$

We investigate the $\gamma^{[k]}$ balance equation for incompressible fluids with temperature dependent transport coefficients.

Proposition 5.5. *Let $k \geq 1$ be an integer and (v, T) be a smooth solution of the incompressible Navier-Stokes equations* (5.4)–(5.6)*. Then the following balance equation holds*

$$\partial_t \gamma^{[k]} + \partial_x\!\cdot(v\gamma^{[k]}) + \partial_x\!\cdot\varphi_\gamma^{[k]} + \pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]} = 0, \tag{5.15}$$

where $\varphi_\gamma^{[k]}$ is a dissipative flux and $\pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]}$ a source term. The quantity $\pi_\gamma^{[k]}$ contains higher order derivative nonnegative terms, $\Sigma_\gamma^{[k]}$ higher order derivative split terms to be controled by $\pi_\gamma^{[k]}$, and $\omega_\gamma^{[k]}$ lower order derivative terms of convective origin to be majorized by $\pi_\gamma^{[k]}$. The term $\pi_\gamma^{[k]}$ can be taken as

$$\pi_\gamma^{[k]} = \frac{2\lambda \mathrm{A}_\lambda^{[k]}}{\rho c_v} \frac{|\partial^{k+1} T|^2}{T^{1+a_k}} + \frac{2\eta \mathrm{A}_\eta^{[k]}}{\rho} \frac{|\partial^{k+1} v|^2}{T^{a_k}}, \tag{5.16}$$

in such a way that

$$2\underline{b}_k\, \gamma^{[k+1]} \leq \pi_\gamma^{[k]}\, T^{-(a_{k+1}-a_k+\varkappa)} \leq 2\overline{b}_k\, \gamma^{[k+1]}, \tag{5.17}$$

where $\rho \underline{b}_k = \underline{\mathfrak{a}} \min(\mathrm{A}_\lambda^{[k]}/\mathrm{A}_\lambda^{[k+1]}, \mathrm{A}_\eta^{[k]}/\mathrm{A}_\eta^{[k+1]})$ and $\rho \overline{b}_k = \overline{\mathfrak{a}} \max(\mathrm{A}_\lambda^{[k]}/\mathrm{A}_\lambda^{[k+1]}, \mathrm{A}_\eta^{[k]}/\mathrm{A}_\eta^{[k+1]})$. The term $\Sigma_\gamma^{[k]}$ is in the form

$$\begin{aligned}\Sigma_\gamma^{[k]} = &\sum_{\sigma\nu\mu} c_{\sigma\nu\mu} T^{\sigma-\varkappa} \partial_T^\sigma \lambda\, \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu} c_{\sigma\nu\mu} T^{\sigma-\varkappa} \partial_T^\sigma \eta\, \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} \\ &+ \sum_{\sigma\nu\mu\mathcal{R}} c_{\sigma\nu\mu\mathcal{R}} \Pi_\nu^{(k+1)} \mathcal{R}\big(T^{\sigma-\varkappa} \partial_T^\sigma \eta\, \Pi_\mu^{(k+1)}\big),\end{aligned} \tag{5.18}$$

where the sums are over $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1\leq|\alpha|\leq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\leq|\alpha|\leq k+1}$, $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and for $\mathcal{R}$ singular operator in the form $T^{-\theta} R_i R_j T^\theta$ with $\theta = (a_k + \varkappa)/2$ and $1 \leq i, j \leq n$. The products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = T^{(1-a_k+\varkappa)/2} \prod_{1\leq|\alpha|\leq k+1} \Big(\frac{\partial^\alpha T}{T}\Big)^{\nu_\alpha} \prod_{1\leq|\alpha|\leq k+1} \Big(\frac{\partial^\alpha v}{\sqrt{T}}\Big)^{\nu'_\alpha}, \tag{5.19}$$

where v denotes—with a slight abuse of notation—any of its components $v_1, \dots, v_n$, and μ and ν must be such that

$$\sum_{1\le|\alpha|\le k+1} |\alpha|(\nu_\alpha + \nu'_\alpha) = k+1, \qquad \sum_{1\le|\alpha|\le k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1,$$

$$\sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) \le 1,$$

so that there is at most one derivative of $(k+1)^{\text{th}}$ order in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$. In particular, one of the terms $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is always split between two or more derivatives factors. Furthermore the term $\omega_\gamma^{[k]}$ is given by

$$\omega_\gamma^{[k]}\, T^{-(1-2\varkappa+a_{k-1}-a_k)/2} = \sum_{\nu\mu} c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_{\nu\mu\mathcal{R}} c_{\nu\mu\mathcal{R}}\Pi_\nu^{(k)}\mathcal{R}\big(\Pi_\mu^{(k+1)}\big), \qquad (5.20)$$

where we use similar notation for $\Pi_\nu^{(k)}$ as for $\Pi_\mu^{(k+1)}$ and the summation extends over

$$\sum_{1\le|\alpha|\le k} |\alpha|(\nu_\alpha + \nu'_\alpha) = k, \qquad \sum_{1\le|\alpha|\le k} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1,$$

so that in particular $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha) = 0$ and there are always at least two factors in the product $\Pi_\mu^{(k+1)}$, and where the singular operator $\mathcal{R}$ is in the form $T^{-\theta}R_iR_jT^\theta$ with $\theta = (1+a_k-\varkappa)/2$ and $1 \le i,j \le n$. Finally the flux $\varphi_\gamma^{[k]} = (\varphi_{\gamma 1}^{[k]}, \dots, \varphi_{\gamma n}^{[k]})$ is given by the following formula with $\mathcal{R}$ taken as in (5.18)

$$\begin{aligned}\varphi_{\gamma l}^{[k]}\, T^{-(a_{k-1}-a_k)/2} &= \sum_{\sigma\nu\mu l} c_{\sigma\nu\mu l} T^{\sigma-\varkappa}\partial_T^\sigma\lambda\, \Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu l} c_{\sigma\nu\mu l} T^{\sigma-\varkappa}\partial_T^\sigma\eta\, \Pi_\nu^{(k)}\Pi_\mu^{(k+1)}\\ &\quad + \sum_{\sigma\nu\mu\mathcal{R}l} c_{\sigma\nu\mu\mathcal{R}l}\Pi_\nu^{(k)}\mathcal{R}\big(T^{\sigma-\varkappa}\partial_T^\sigma\eta\, \Pi_\mu^{(k+1)}\big). \qquad (5.21)\end{aligned}$$

Proof. The proof is lengthy but present no serious difficulties other than notational. We have

$$\begin{aligned}\partial_t\gamma^{[k]} &+ \Big((1+a_k)\mathrm{A}_\lambda^{[k]}\frac{|\partial^k T|^2}{T^{2+a_k}} + a_k\mathrm{A}_\eta^{[k]}\frac{|\partial^k v|^2}{T^{1+a_k}}\Big)\partial_t T\\ &- 2\mathrm{A}_\lambda^{[k]}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha T\partial^\alpha\partial_t T}{T^{1+a_k}} - 2\mathrm{A}_\eta^{[k]}\sum_{\substack{1\le i\le n\\|\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i\partial^\alpha\partial_t v_i}{T^{a_k}} = 0,\end{aligned}$$

so that from the governing equations

$$
\begin{aligned}
&\partial_t \gamma^{[k]} + (1+a_k)\mathrm{A}_\lambda^{[k]} \frac{|\partial^k T|^2}{T^{2+a_k}} \Big(\frac{1}{\rho c_v} (\partial_x\cdot(\lambda\partial_x T) + \tfrac{1}{2}\eta|d|^2) - v\cdot\partial_x T \Big) \\
&\qquad + a_k \mathrm{A}_\eta^{[k]} \frac{|\partial^k v|^2}{T^{1+a_k}} \Big(\frac{1}{\rho c_v} (\partial_x\cdot(\lambda\partial_x T) + \tfrac{1}{2}\eta|d|^2) - v\cdot\partial_x T \Big) \\
&\qquad - 2\mathrm{A}_\lambda^{[k]} \sum_{|\alpha|=k} \frac{k!}{\alpha!} \frac{\partial^\alpha T}{T^{1+a_k}} \partial^\alpha \Big(\frac{1}{\rho c_v} (\partial_x\cdot(\lambda\partial_x T) + \tfrac{1}{2}\eta|d|^2) - v\cdot\partial_x T \Big) \\
&\qquad - 2\mathrm{A}_\eta^{[k]} \sum_{\substack{1\le i\le n\\ |\alpha|=k}} \frac{k!}{\alpha!} \frac{\partial^\alpha v_i}{T^{a_k}} \partial^\alpha \Big(\frac{1}{\rho} (\partial_x\cdot(\eta d_{i\cdot}) - \partial_i p) - v\cdot\partial_x v_i \Big) = 0,
\end{aligned}
$$

and we denote by $\mathcal{T}^\partial$ the two first sums of the left hand side, by $\mathcal{T}^\lambda$ the third sum and by $\mathcal{T}^\eta$ the last sum. We first examine separately higher order derivative contributions associated with each sum $\mathcal{T}^\partial$, $\mathcal{T}^\lambda$, and $\mathcal{T}^\eta$. The lower order convective terms in $\mathcal{T}^\partial$, $\mathcal{T}^\lambda$, and $\mathcal{T}^\eta$, are examined all together at the end.

The terms in $\mathcal{T}^\partial$ associated with $\partial_x\cdot(\lambda\partial_x T)$ are integrated by parts. They yield flux contributions and source terms in the form

$$
-\sum_{1\le l\le n} \partial_l \Big(\frac{(1+a_k)\mathrm{A}_\lambda^{[k]}}{\rho c_v} \frac{|\partial^k T|^2}{T^{2+a_k}} + \frac{a_k \mathrm{A}_\eta^{[k]}}{\rho c_v} \frac{|\partial^k v|^2}{T^{1+a_k}} \Big) \lambda \partial_l T,
$$

which are easily rewritten as sums of terms like $c_{\sigma\nu\mu} T^{\sigma-\varkappa} \partial_T^\sigma \lambda \; \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\text{th}}$ order. One the other hand, the terms of $\mathcal{T}^\partial$ associated with $|d|^2$ are left unchanged and have the same structure.

The most important contribution in $\mathcal{T}^\lambda$ is that associated with $\partial^\alpha\partial_l(\lambda\partial_l T)$, $1\le l\le n$. The corresponding terms are integrated by parts and yield sources in the form

$$
+\frac{2\mathrm{A}_\lambda^{[k]}}{\rho c_v} \sum_{\substack{1\le l\le n\\ |\alpha|=k}} \frac{k!}{\alpha!} \partial_l \Big(\frac{\partial^\alpha T}{T^{1+a_k}} \Big) \partial^\alpha (\lambda \partial_l T).
$$

After expanding the derivatives, the above sum can be written

$$
\sum_{\substack{1\le l\le n\\ |\alpha|=k}} \frac{k!}{\alpha!} \Big(\frac{\partial^\alpha\partial_l T}{T^{1+a_k}} - (1+a_k) \frac{\partial^\alpha T \partial_l T}{T^{2+a_k}} \Big) \Big(\lambda \partial^\alpha\partial_l T + \sum_{\tilde\alpha\sigma\nu} c_{\alpha\tilde\alpha\nu} \partial^{\alpha-\tilde\alpha}\partial_l T \partial_T^\sigma \lambda \prod_\beta (\partial^\beta T)^{\nu_\beta} \Big),
$$

where the summations and products extend over $1\le l\le n$, $|\alpha|=k$, $0\le\tilde\alpha\le\alpha$, $\tilde\alpha\neq 0$, $1\le\sigma\le|\tilde\alpha|$, $\sum_\beta |\beta|\nu_\beta = |\tilde\alpha|$, $1\le|\beta|\le|\tilde\alpha|$, and $\sum_\beta \nu_\beta = \sigma$. We can now extract for $\pi_\gamma^{[k]}$ the term in the form $\lambda(\partial^\alpha\partial_l T)^2$ which can be written

$$
\frac{2\lambda\mathrm{A}_\lambda^{[k]}}{\rho c_v} \sum_{\substack{1\le l\le n\\ |\alpha|=k}} \frac{k!}{\alpha!} \frac{(\partial^\alpha\partial_l T)^2}{T^{1+a_k}} = \frac{2\lambda\mathrm{A}_\lambda^{[k]}}{\rho c_v} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} \frac{(\partial^\alpha T)^2}{T^{1+a_k}},
$$

thanks to the properties of multinomial coefficients [Tau63] [Com70]. All other terms are of admissible form for $\Sigma_\gamma^{[k]}$, that is, in the form $c_{\sigma\nu\mu}T^{\sigma-\varkappa}\partial_T^\sigma\lambda\,\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\rm th}$ order since $\sum_\beta|\beta|\nu_\beta+1+|\alpha-\tilde\alpha|=k+1$. More specifically, we can factorize T^{-a_k} in the first parenthesis, $T^{1+\varkappa}$ in the second, and all the terms involving derivatives of $\partial_T^\sigma\lambda$ are multiplied and divided by T^σ thanks to $\sum_\beta\nu_\beta=\sigma$.

The contributions in $\mathcal{T}^\lambda$ associated with $|d|^2$ are treated in a similar way. Indeed, we decompose each multiindex α with $|\alpha|=k$ into $\alpha=\tilde\alpha+e_{i_\alpha}$ where $|\tilde\alpha|=k-1$, i_α is chosen arbitrarily with $\alpha_{i_\alpha}\neq 0$, and $e_1,\ldots,e_n$ denotes the canonical basis of $\mathbb{N}^n$, so that we have $\partial^\alpha=\partial^{\tilde\alpha}\partial_{i_\alpha}$. We can then integrate these terms by parts and obtain sources in the form

$$+\frac{\mathrm{A}_\lambda^{[k]}}{\rho c_v}\sum_{\substack{1\le i,j\le n\\ |\alpha|=k}}\partial_{i_\alpha}\Big(\frac{\partial^\alpha T}{T^{1+a_k}}\Big)\partial^{\tilde\alpha}\big(\eta d_{ij}^2\big).$$

Upon expending the derivatives with the help of the differential indentities established in the previous section, all these terms are of admissible form for $\Sigma_\gamma^{[k]}$.

We now consider the sum $\mathcal{T}^\eta$ and its most important contribution is that corresponding to $\partial^\alpha\partial_x\cdot(\eta d)$ which reads

$$\frac{2\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\le i,l\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i\partial^\alpha\partial_l(\eta d_{il})}{T^{a_k}}.$$

We then use the identity $\sum_l\partial_l(\eta d_{il})=\sum_l\partial_l(\eta\partial_l v_i)+\sum_l\partial_l\eta\partial_i v_l$ and focus on the contributions of the terms $\partial_l(\eta\partial_l v_i)$. The contributions associated with $\partial_l\eta\partial_i v_l$ are of admissible form for $\Sigma_\gamma^{[k]}$ after one integration by parts using $\alpha=\tilde\alpha+e_{i_\alpha}$ and the corresponding details are omitted. After integration by parts we obtain sources in the form

$$+\frac{2\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\le i,l\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\partial_l\Big(\frac{\partial^\alpha v_i}{T^{a_k}}\Big)\partial^\alpha(\eta\partial_l v_i),$$

and after expanding the derivatives, the sum can be written

$$\sum_{\substack{1\le i,l\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\Big(\frac{\partial^\alpha\partial_l v_i}{T^{a_k}}-a_k\frac{\partial^\alpha v_i\partial_l T}{T^{1+a_k}}\Big)\Big(\eta\partial^\alpha\partial_l v_i+\sum_{\tilde\alpha\sigma\nu}c_{\alpha\tilde\alpha\nu}\partial^{\alpha-\tilde\alpha}\partial_l v_i\partial_T^\sigma\eta\prod_\beta(\partial^\beta T)^{\nu_\beta}\Big),$$

where the summations and products extend over $1\le i,l\le n$, $|\alpha|=k$, $0\le\tilde\alpha\le\alpha$, $\tilde\alpha\neq 0$, $1\le\sigma\le|\tilde\alpha|$, $\sum_\beta|\beta|\nu_\beta=|\tilde\alpha|$, $1\le|\beta|\le|\tilde\alpha|$, and $\sum_\beta\nu_\beta=\sigma$. We can now extract the term in the form $\eta(\partial^\alpha\partial_l v_i)^2$ for $\pi_\gamma^{[k]}$ which is rewritten as

$$\frac{2\eta\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\le i,l\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{(\partial^\alpha\partial_l v_i)^2}{T^{a_k}}=\frac{2\eta\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\le i\le n\\ |\alpha|=k+1}}\frac{(k+1)!}{\alpha!}\frac{(\partial^\alpha v_i)^2}{T^{a_k}},$$

thanks to the properties of multinomial coefficients. All the other terms are of admissible form for $\Sigma_\gamma^{[k]}$, that is, in the form $c_{\sigma\nu\mu}T^{\sigma-\varkappa}\partial_T^\sigma\eta\ \Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\text{th}}$ order. The pressure can be splitted as (5.8) and we consider here the contribution associated with $-\sum_{i,j}R_iR_j\big(\eta d_{ij}\big)$. After one integration by parts, we obtain the sources

$$+\frac{2\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\le i,l,m\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\partial_i\Big(\frac{\partial^\alpha v_i}{T^{a_k}}\Big)\partial^\alpha R_lR_m(\eta d_{lm}).$$

Since Riesz transforms and derivatives commute, the sum can be written, after expanding derivatives

$$-\sum_{\substack{1\le i,l,m\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}a_k\frac{\partial^\alpha v_i\partial_iT}{T^{1+a_k}}R_lR_m\Big(\sum_{\tilde\alpha\sigma\nu}c_{\alpha\tilde\alpha\nu}\partial^{\alpha-\tilde\alpha}d_{lm}\partial_T^\sigma\eta\prod_\beta(\partial^\beta T)^{\nu_\beta}\Big),$$

where the summations and products extend over $1\le i,l,m\le n$, $|\alpha|=k$, $0\le\tilde\alpha\le\alpha$, $1\le\sigma\le|\tilde\alpha|$, $\sum_\beta|\beta|\nu_\beta=|\tilde\alpha|$, $1\le|\beta|\le|\tilde\alpha|$, and $\sum_\beta\nu_\beta=\sigma$. After some algebra, all these terms are written as $c_{\sigma\nu\mu\mathcal{R}}\Pi_\nu^{(k+1)}\mathcal{R}\big(T^{\sigma-\varkappa}\partial_T^\sigma\eta\ \Pi_\mu^{(k+1)}\big)$ where $\mathcal{R}=T^{-\theta}R_lR_mT^\theta$ with $\theta=(a_k+\varkappa)/2$.

Lower order convective terms first yield the contributions

$$\begin{aligned}&-(1+a_k)\mathrm{A}_\lambda^{[k]}\frac{|\partial^kT|^2}{T^{2+a_k}}\big(v\cdot\partial_xT\big)-a_k\mathrm{A}_\eta^{[k]}\frac{|\partial^kv|^2}{T^{1+a_k}}\big(v\cdot\partial_xT\big)\\&+2\mathrm{A}_\lambda^{[k]}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{1+a_k}}\partial^\alpha\big(v\cdot\partial_xT\big)+2\mathrm{A}_\eta^{[k]}\sum_{\substack{1\le i\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}}\partial^\alpha\big(v\cdot\partial_xv_i\big),\end{aligned}$$

and the terms proportional to v are easily recast in the form $v\cdot\partial_x\gamma^{[k]}$, so that the only remaining contributions are the sources

$$+2\mathrm{A}_\lambda^{[k]}\sum_{\substack{|\alpha|=k\\ 1\le l\le n}}\sum_{\substack{0\le\beta\le\alpha\\ 1\le|\beta|}}c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{1+a_k}}\partial^\beta v_l\partial^{(\alpha-\beta)}\partial_lT$$

$$+2\mathrm{A}_\eta^{[k]}\sum_{\substack{1\le i,l\le n\\ |\alpha|=k}}\sum_{\substack{0\le\beta\le\alpha\\ 1\le|\beta|}}c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}}\partial^\beta v_l\partial^{(\alpha-\beta)}\partial_lv_i,$$

which are easily rewritten in the form $c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}T^{(1-2\varkappa+a_{k-1}-a_k)/2}$. We finally have to consider the contributions to $\omega_\gamma^{[k]}$ due to the pressure term $\sum_{i,j}R_iR_j\big(\rho v_iv_j\big)$, which read

$$+\frac{2\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\le i,l,m\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}}\partial^\alpha\partial_i\big(R_lR_m(v_lv_m)\big).$$

We now use $\partial_i\partial_{i_\alpha} R_l R_m = R_i R_{i_\alpha}\partial_m\partial_n$, where $\alpha = \tilde{\alpha} + e_{i_\alpha}$, and $\sum_{mn}\partial_m\partial_n(v_m v_n) = \sum_{mn}\partial_m v_n \partial_n v_m$, and we obtain

$$+\frac{2\mathrm{A}_\eta^{[k]}}{\rho}\sum_{\substack{1\le i,l,m\le n\\ |\alpha|=k}} \frac{\partial^\alpha v_i}{T^{a_k}} R_i R_{i_\alpha}\big(\partial^{\tilde{\alpha}}(\partial_m v_n \partial_m v_n),\big)$$

and these terms can be written $c_{\nu\mu\mathcal{R}} T^{(1-2\varkappa+a_{k-1}-a_k)/2}\Pi_\nu^{(k)}\mathcal{R}\big(\Pi_\mu^{(k+1)}\big)$ where $\mathcal{R} = T^{-\theta}R_i R_j T^\theta$ with $\theta = (1+a_k-\varkappa)/2$. ■

5.5. Balance equation for $\tilde{\gamma}^{[k]}$

We will also need in the following the $\tilde{\gamma}^{[k]}$ balance equation that we correspondingly write in terms of the auxiliary variables w and τ.

Proposition 5.6. *Let $k \ge 1$ be an integer and (v, T) be a smooth solution of the incompressible Navier-Stokes equations* (5.4)*–*(5.6)*. Then the following balance equation holds*

$$\partial_t\tilde{\gamma}^{[k]} + \partial_x\cdot(v\tilde{\gamma}^{[k]}) + \partial_x\cdot\varphi_{\tilde{\gamma}}^{[k]} + \pi_{\tilde{\gamma}}^{[k]} + \Sigma_{\tilde{\gamma}}^{[k]} + \omega_{\tilde{\gamma}}^{[k]} = 0, \tag{5.22}$$

where $\varphi_{\tilde{\gamma}}^{[k]}$ is a dissipative flux and $\pi_{\tilde{\gamma}}^{[k]} + \Sigma_{\tilde{\gamma}}^{[k]} + \omega_{\tilde{\gamma}}^{[k]}$ a source term. The term $\pi_{\tilde{\gamma}}^{[k]}$ can be taken as

$$\pi_{\tilde{\gamma}}^{[k]} = e^{(1-a_k)\tau}\Big(\frac{2\lambda\mathrm{A}_\lambda^{[k]}}{\rho c_v}\,|\partial^{k+1}\tau|^2 + \frac{2\eta\mathrm{A}_\eta^{[k]}}{\rho}|\partial^{k+1}w|^2\Big), \tag{5.23}$$

in such a way that

$$2\underline{b}_k\,\tilde{\gamma}^{[k+1]} \le \pi_{\tilde{\gamma}}^{[k]}\,e^{-(a_{k+1}-a_k+\varkappa)\tau} \le 2\overline{b}_k\,\tilde{\gamma}^{[k+1]}, \tag{5.24}$$

where $\underline{b}_k$ and $\overline{b}_k$ are as in Proposition 5.5. The term $\Sigma_{\tilde{\gamma}}^{[k]}$ is in the form

$$\Sigma_{\tilde{\gamma}}^{[k]} = \sum_{\sigma\nu\mu} c_{\sigma\nu\mu}e^{-\varkappa\tau}\partial_\tau^\sigma\lambda\;\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu} c_{\sigma\nu\mu}e^{-\varkappa\tau}\partial_\tau^\sigma\eta\;\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$$
$$+\sum_{\sigma\nu\mu\iota\mathcal{R}} c_{\sigma\nu\mu\iota\mathcal{R}}\Pi_\nu^{(k+1)}\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)} + \frac{\mathrm{A}_\eta^{[k]}}{\rho}\Big(\frac{\lambda}{c_v}-\eta\Big)e^{(1-a_k)\tau}w\cdot\partial^{k+1}w\,\partial^{k+1}\tau, \tag{5.25}$$

where we have isolated the new terms which contain two derivatives of $(k+1)^{\text{th}}$ order

$$w\cdot\partial^{k+1}w\,\partial^{k+1}\tau = \sum_{\substack{|\alpha|=k\\ 1\le i\le n}}\frac{k!}{\alpha!}w_i\partial^\alpha w_i\partial^\alpha\tau.$$

For the two first contributions in (5.25) *composed of strictly differential terms, the sums are over $0\le\sigma\le k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1\le|\alpha|\le k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\le|\alpha|\le k+1}$, $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha\in\mathbb{N}^n$, with the products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ defined by*

$$\Pi_\nu^{(k+1)} = e^{(1-a_k+\varkappa)\tau/2}\prod_{1\le|\alpha|\le k+1}\big(\partial^\alpha\tau\big)^{\nu_\alpha}\prod_{0\le|\alpha|\le k+1}\big(\partial^\alpha w\big)^{\nu'_\alpha}, \tag{5.26}$$

where w denotes—with a slight abuse of notation—any of its components $w_1, \dots, w_n$. As in the definition of $\Sigma_{\gamma}^{[k]}$, μ and ν must be such that $\sum_{1\le|\alpha|\le k+1} |\alpha|(\nu_\alpha + \nu'_\alpha) = k+1$, $\sum_{1\le|\alpha|\le k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1$, $\sum_{|\alpha|=k+1}(\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) \le 1$, so that there is at most one derivative of $(k+1)^{\mathrm{th}}$ order in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$. In particular, one of the terms $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is always split between two or more derivatives factors. Note also that the products over the velocity factors extends up to $|\alpha| = 0$ in contrast with the $\gamma^{[k]}$ balance equation. The non strictly differential terms $\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}$ are defined by

$$\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)} = \widetilde{\Pi}_\mu^{(k+1,l)}\mathcal{R}\Big(e^{-\varkappa\tau}\partial_\tau^\sigma\eta\, \widetilde{\Pi}_\iota^{(k+1,k+1-l)}\Big) \tag{5.27}$$

with

$$\begin{cases} \widetilde{\Pi}_\mu^{(k+1,l)} = e^{(1-a_k+\varkappa)\frac{l\tau}{2(k+1)}} \displaystyle\prod_{1\le|\alpha|\le k+1} (\partial^\alpha\tau)^{\mu_\alpha} \prod_{0\le|\alpha|\le k+1} (\partial^\alpha w)^{\mu'_\alpha}, \\ \widetilde{\Pi}_\iota^{(k+1,k+1-l)} = e^{(1-a_k+\varkappa)\frac{(k+1-l)\tau}{2(k+1)}} \displaystyle\prod_{1\le|\alpha|\le k+1} (\partial^\alpha\tau)^{\iota_\alpha} \prod_{0\le|\alpha|\le k+1} (\partial^\alpha w)^{\iota'_\alpha}. \end{cases}$$

The sums are over $\nu = (\nu_\alpha, \nu'_\alpha)_{1\le|\alpha|\le k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\le|\alpha|\le k}$, $\iota = (\iota_\alpha, \iota_\alpha)_{1\le|\alpha|\le k}$, and $0 \le \sigma \le k$, where ν, μ, and ι must be such that

$$\sum_{1\le|\alpha|\le k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = l, \qquad \sum_{1\le|\alpha|\le k+1} |\alpha|(\iota_\alpha + \iota'_\alpha) = k+1-l,$$

$$\sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha + \iota_\alpha + \iota'_\alpha) \le 1,$$

for some $0 \le l \le k$, so that there is at most one derivative of $(k+1)^{\mathrm{th}}$ order in the product $\Pi_\nu^{(k+1)}\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}$, and $\mathcal{R}$ singular operator in the form $e^{-\theta\tau}R_iR_je^{\theta\tau}$, $1 \le i, j \le n$, with

$$\theta = \frac{a_k + \varkappa}{2} + \frac{l}{2(k+1)}(1 - a_k + \varkappa).$$

Note that, in contrast with the $\gamma^{[k]}$ balance equation, the integral operator $\mathcal{R}$ breaks into two pieces the term $\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}$. Furthermore the term $\omega_{\tilde{\gamma}}^{[k]}$ is given by

$$\omega_{\tilde{\gamma}}^{[k]}\, e^{-(1-2\varkappa+a_{k-1}-a_k)\tau/2} = \sum_{\nu\mu} c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_{\nu\mu\iota\mathcal{R}} c_{\nu\mu\iota\mathcal{R}}\Pi_\nu^{(k)}\Pi_{\mu\iota\mathcal{R}}^{(k+1)}, \tag{5.28}$$

where we use similar notation for $\Pi_\nu^{(k)}$ as for $\Pi_\mu^{(k+1)}$ and where $\Pi_{\mu l\mathcal{R}}^{(k+1)}$ is given by

$$\Pi_{\mu\iota\mathcal{R}}^{(k+1)} = \widetilde{\Pi}_\mu^{(k+1,l)}\mathcal{R}\Big(\widetilde{\Pi}_\iota^{(k+1,k+1-l)}\Big). \tag{5.29}$$

The summations are over $\sum_{1\le|\alpha|\le k} |\alpha|(\nu_\alpha + \nu'_\alpha) = k$, $\sum_{1\le|\alpha|\le k} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1$, and $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha) = 0$ for the strictly differential terms $\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}$, so that there are always at least two factors in the product $\Pi_\mu^{(k+1)}$. For the non strictly differential *terms* $\Pi_\nu^{(k)}\Pi_{\mu\iota\mathcal{R}}^{(k+1)}$ *we have* $\sum_{1\leq|\alpha|\leq k}\ |\alpha|(\nu_\alpha+\nu'_\alpha)=k$, $\sum_{1\leq|\alpha|\leq k}\ |\alpha|(\mu_\alpha+\mu'_\alpha)=l$, $\sum_{1\leq|\alpha|\leq k}\ |\alpha|(\iota_\alpha+\iota'_\alpha)=k+1-l$, *and* $\sum_{|\alpha|=k+1}(\mu_\alpha+\mu'_\alpha+\iota_\alpha+\iota'_\alpha)=0$ *for some* $0\leq l\leq k$, *so that there are always at least two factors in the product* $\Pi_{\mu l\mathcal{R}}^{(k+1)}$, *and the singular operator* $\mathcal{R}$ *is in the form* $T^{-\theta}R_iR_jT^{\theta}$ *with* $1\leq i,j\leq n$ *and*

$$\theta = 1-\frac{k+1-l}{2(k+1)}(1-a_k+\varkappa).$$

Finally the flux $\varphi_{\tilde\gamma}^{[k]}=(\varphi_{\tilde\gamma,1}^{[k]},\ldots,\varphi_{\tilde\gamma,n}^{[k]})$ *is given by the following formula with* $\mathcal{R}$ *taken as in* (5.27)

$$\begin{aligned}\varphi_{\tilde\gamma,l}^{[k]}\,e^{-(a_{k-1}-a_k)\tau/2} = &\sum_{\sigma\nu\mu l} c_{\sigma\nu\mu l}e^{-\varkappa\tau}\partial_\tau^\sigma\lambda\,\Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu l} c_{\sigma\nu\mu l}T^{-\varkappa\tau}\partial_\tau^\sigma\eta\,\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}\\ &+\sum_{\sigma\nu\mu l\mathcal{R}l} c_{\sigma\nu\mu\iota\mathcal{R}l}\Pi_\nu^{(k+l)}\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}.\end{aligned} \tag{5.30}$$

Proof. The proof is lengthy and tedious but similar to that of Proposition 5.5. The new complications arise from commutators between temperature weights and differential operators. ∎

5.6. Conditional entropicity

We now investigate the control of $\int_{\mathbb{R}^n}|\Sigma_\gamma^{[k]}|\,dx$ by $\int_{\mathbb{R}^n}\pi_\gamma^{[k]}\,dx$, using the weighted inequalities established in Section 4. We will denote by χ the quantity

$$\chi = \|\log T\|_{BMO} + \|\frac{v}{\sqrt{T}}\|_{L^\infty} = \|\tau\|_{BMO}+\|w\|_{L^\infty}, \tag{5.31}$$

which will play a fundamental rôle in the analysis. We will establish in particular that entropicity properties hold for $\gamma^{[k]}$ and $\tilde\gamma^{[k]}$ when χ is small enough.

Proposition 5.7. *Let* $k\geq 1$ *be an integer and* (v,T) *be a smooth solution of the incompressible Navier-Stokes equations* (5.4)–(5.6). *There exists positive constants* $\delta(k,n)$ *and* $c(k,n)$ *such that for* $\chi=\|\log T\|_{BMO}+\|v/\sqrt{T}\|_{L^\infty}<\delta$ *we have the estimates*

$$\int_{\mathbb{R}^n}|\Sigma_\gamma^{[k]}|\,dx \leq c\,\chi\int_{\mathbb{R}^n}\pi_\gamma^{[k]}\,dx. \tag{5.32}$$

Proof. From the expressions (5.18)(5.19) for $\Sigma_\gamma^{[k]}$, since the quantities $T^{\sigma-\varkappa}\partial_T^\sigma\lambda$ and $T^{\sigma-\varkappa}\partial_T^\sigma\eta$ are uniformly bounded from assumptions (5.2) and (5.3), and since the operators $T^\theta R_iR_jT^{-\theta}$ are continuous over L^2 for $\|\log T\|_{BMO}$ small enough, we only have to estimate the L^2 norm of the products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$. However, using Theorem 4.15 with $p=2$, we obtain when $\|\log T\|_{BMO}+\|v/\sqrt{T}\|_{L^\infty}<\delta(k,n)$ the weighted inequalities

$$\|\Pi_\nu^{(k+1)}\|_{L^2}\leq c\chi^{N_\nu-1}\Big(\|T^{\frac{\theta}{2}}\frac{\partial^{k+1}T}{T}\|_{L^2}+\|T^{\frac{\theta}{2}}\frac{\partial^{k+1}v}{\sqrt{T}}\|_{L^2}\Big),$$

with $\theta = 1 - a_k + \varkappa$, $c = c(k,n)$, and

$$N_\nu = \sum_{1\le|\alpha|\le k+1} (\nu_\alpha + \nu'_\alpha).$$

As a consequence, we have

$$\|\Pi_\nu^{(k+1)}\|_{L^2}\|\Pi_\mu^{(k+1)}\|_{L^2} \le c\chi^{N_\nu+N_\mu-2} \int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx,$$

and the proof is complete from

$$N_\nu + N_\mu - 2 = \sum_{1\le|\alpha|\le k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) - 2 \ge 1,$$

since at least one of the products $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is split into two or more derivative factors. ∎

We have a similar result for $\tilde{\gamma}^{[k]}$ which is more technical to establish because of the special structure of the products $\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}$ in (5.27).

Proposition 5.8. *Let $k \ge 1$ be an integer and (v,T) be a smooth solution of the incompressible Navier-Stokes equations (5.4)–(5.6). There exists positive constants $\delta(k,n)$ and $c(k,n)$ such that for $\chi = \|\tau\|_{BMO} + \|w\|_{L^\infty} < \delta$ we have the estimates*

$$\int_{\mathbb{R}^n} |\Sigma_{\tilde{\gamma}}^{[k]}|\, dx \le c\,\chi \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]}\, dx. \tag{5.33}$$

Proof. We use the expression (5.25)–(5.27) for $\Sigma_{\tilde{\gamma}}^{[k]}$. On one hand, for strictly differential terms, the proof is similar to that for $\gamma^{[k]}$. Indeed, the terms $e^{-\varkappa\tau}\partial_\tau^\sigma\lambda\, \Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ or $e^{-\varkappa\tau}\partial_\tau^\sigma\eta\, \Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ are easily majorized since the quantities $e^{-\varkappa\tau}\partial_\tau^\sigma\lambda$ and $e^{-\varkappa\tau}\partial_\tau^\sigma\eta$ are uniformly bounded and since the L^2 norm of the products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ is directly obtained from the multilinear estimates of Theorem 4.14. The fact that there is always a factor χ in the upper bound results from the fact that one of the two products $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is always split. On the other hand, the special contributions involving two derivatives of $(k+1)^{\text{th}}$ order are rewritten in the form

$$e^{-\varkappa\tau}\Big(\frac{\lambda}{c_v} - \eta\Big) e^{(1-a_k+\varkappa)\tau} w \cdot \partial^{k+1} w\, \partial^{k+1}\tau,$$

and are easily taken into account with a $\|w\|_{L^\infty}$ factor.

The new difficulty lies in the non strictly differential terms $\Pi_\nu^{(k+1)}\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}$ which requires evaluating the L^2 norm of $\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}$. These terms $\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}$ are in the form

$$\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)} = \widetilde{\Pi}_\mu^{(k+1,l)} \mathcal{R}\Big(e^{-\varkappa\tau}\partial_\tau^\sigma\eta\, \widetilde{\Pi}_\iota^{(k+1,k+1-l)}\Big) \tag{5.34}$$

with in particular

$$\widetilde{\Pi}_{\mu}^{(k+1,l)} = e^{(1-a_k+\varkappa)\frac{l\tau}{2(k+1)}} \prod_{1\le|\alpha|\le k+1} \big(\partial^\alpha \tau\big)^{\mu_\alpha} \prod_{0\le|\alpha|\le k+1} \big(\partial^\alpha w\big)^{\mu'_\alpha}. \tag{5.35}$$

The sums are over $0 \le \sigma \le k$, $0 \le l \le k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1\le|\alpha|\le k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\le|\alpha|\le k}$, $\iota = (\iota_\alpha, \iota'_\alpha)_{1\le|\alpha|\le k}$, $\mathcal{R}$ singular operator as described in Proposition 5.6 and ν, μ and ι must be such that

$$\sum_{1\le|\alpha|\le k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = l, \qquad \sum_{1\le|\alpha|\le k+1} |\alpha|(\iota_\alpha + \iota'_\alpha) = k+1-l,$$

$$\sum_{|\alpha|=k+1} (\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha + \iota_\alpha + \iota'_\alpha) \le 1,$$

so that there is at most one derivative of $(k+1)^{\text{th}}$ order in the product $\Pi_{\nu}^{(k+1)}\Pi_{\mu l\mathcal{R}\sigma}^{(k+1)}$.

We first split the exponential term in $\widetilde{\Pi}_{\mu}^{(k+1,l)}$ over each derivative factor thanks to the relation $\sum_{1\le|\alpha|\le k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = l$ and we obtain

$$\widetilde{\Pi}_{\mu}^{(k+1,l)} = \prod_{1\le|\alpha|\le k+1} e^{\frac{(1-a_k+\varkappa)\tau|\alpha|\mu_\alpha}{2(k+1)}} \big(\partial^\alpha \tau\big)^{\mu_\alpha} \prod_{0\le|\alpha|\le k+1} e^{\frac{(1-a_k+\varkappa)\tau|\alpha|\mu'_\alpha\tau}{2(k+1)}} \big(\partial^\alpha w\big)^{\mu'_\alpha}. \tag{5.36}$$

Letting $p_\alpha = 2(k+1)/\mu_\alpha|\alpha|$, and $p'_\alpha = 2(k+1)/\mu'_\alpha|\alpha|$, we have

$$\sum_{1\le|\alpha|\le k+1} \Big(\frac{1}{p_\alpha} + \frac{1}{p'_\alpha}\Big) = \frac{l}{2(k+1)},$$

and we can use Hölder inequality to estimate $\big\|\widetilde{\Pi}_{\mu}^{(k+1,l)}\big\|_{L^{\frac{2(k+1)}{l}}}$. To this purpose, from the weighted interpolation inequalities of intermediate derivatives established in Theorem 4.13 applied with $r=2$, $j=|\alpha|$, and k replaced by $k+1$, we obtain that

$$\Big\|e^{\frac{(1-a_k+\varkappa)\tau|\alpha|\mu_\alpha}{2(k+1)}} \big(\partial^\alpha \phi\big)^{\mu_\alpha}\Big\|_{L^{p_\alpha}} \le c\chi^{\mu_\alpha(1-\frac{|\alpha|}{k+1})} \Big(\|e^{\frac{\theta\tau}{2}}\partial^{k+1}\tau\|_{L^2} + \|e^{\frac{\theta\tau}{2}}\partial^{k+1}w\|_{L^2}\Big)^{\frac{\mu_\alpha|\alpha|}{k+1}},$$

where ϕ denotes τ or w. Upon multiplying these inequalities for $1 \le |\alpha| \le k+1$, and from Hölder inequality, we deduce that

$$\big\|\widetilde{\Pi}_{\mu}^{(k+1,l)}\big\|_{L^{\frac{2(k+1)}{l}}} \le c\chi^{N_\mu - \frac{l}{k+1}} \Big(\|e^{\frac{\theta\tau}{2}}\partial^{k+1}\tau\|_{L^2} + \|e^{\frac{\theta\tau}{2}}\partial^{k+1}w\|_{L^2}\Big)^{\frac{l}{k+1}},$$

where $\theta = 1 - a_k + \varkappa$, $c = c(k,n)$, and

$$N_\mu = \sum_{1\le|\alpha|\le k+1} (\mu_\alpha + \mu'_\alpha).$$

We can treat similarly the factor $\widetilde{\Pi}_\iota^{(k+1,k+1-l)}$ to obtain that

$$\big\|\widetilde{\Pi}_\iota^{(k+1,k+1-l)}\big\|_{L^{\frac{2(k+1)}{k+1-l}}} \leq c\chi^{N_\iota - \frac{k+1-l}{k+1}} \Big(\|e^{\frac{\theta\tau}{2}}\partial^{k+1}\tau\|_{L^2} + \|e^{\frac{\theta\tau}{2}}\partial^{k+1}w\|_{L^2}\Big)^{\frac{k+1-l}{k+1}}.$$

Since the operator $\mathcal{R}$ is continuous in $L^{\frac{2(k+1)}{k+1-l}}$ for $\|\log T\|_{BMO}$ small enough, we deduce that

$$\big\|\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}\big\|_{L^2} \leq c\big\|\widetilde{\Pi}_\mu^{(k+1,l)}\big\|_{L^{\frac{2(k+1)}{l}}}\big\|\widetilde{\Pi}_\iota^{(k+1,k+1-l)}\big\|_{L^{\frac{2(k+1)}{k+1-l}}}$$

so that

$$\big\|\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}\big\|_{L^2} \leq c\chi^{N_\mu+N_\iota-1}\Big(\|e^{\frac{\theta\tau}{2}}\partial^{k+1}\tau\|_{L^2} + \|e^{\frac{\theta\tau}{2}}\partial^{k+1}w\|_{L^2}\Big),$$

and finally

$$\big\|\Pi_\nu^{(k+1)}\big\|_{L^2}\big\|\Pi_{\mu\iota\mathcal{R}\sigma}^{(k+1)}\big\|_{L^2} \leq c\chi^{N_\nu+N_\mu+N_\iota-2}\int_{\mathbb{R}^n}\pi_{\tilde{\gamma}}^{[k]}\,dx,$$

and the end of the proof is similar to that of Proposition 5.7 ∎

Corollary 5.9. *Let $k \geq 1$ be an integer and (v,T) be a smooth solution of the incompressible Navier-Stokes equations* (5.4)–(5.6). *There exists positive constants $\delta(k,n)$ and $c(k,n)$, only depending on (k,n), such that for $\chi < \delta$ the following inequalities hold*

$$\partial_t\int_{\mathbb{R}^n}\gamma^{[k]}\,dx + (1-c\,\chi)\int_{\mathbb{R}^n}\pi_\gamma^{[k]}\,dx \leq \int_{\mathbb{R}^n}|\omega_\gamma^{[k]}|\,dx, \tag{5.37}$$

$$\partial_t\int_{\mathbb{R}^n}\tilde{\gamma}^{[k]}\,dx + (1-c\,\chi)\int_{\mathbb{R}^n}\pi_{\tilde{\gamma}}^{[k]}\,dx \leq \int_{\mathbb{R}^n}|\omega_{\tilde{\gamma}}^{[k]}|\,dx. \tag{5.38}$$

In particular, when $\chi \leq 1/2c(k,n)$, entropicity properties hold for $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$.

We have established in Corollary 5.9 that entropicity holds for $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ as long as the quantity χ is small enough. In order to obtain global estimates, we will have to ensure that this quantity χ remains small if it is initially small. It is important to note that this quantity in invariant under the change of scales described in Remark 5.2. It can also be interpreted as involving the natural variables $\log T$ and $v/\sqrt{rT}$ appearing in Maxwellian distributions [Cer88]. Finally, the constraint that χ remains small may also be interpreted as a small Mach number constraint, which is consistent with Enskog expansion [Gol00].

Remark 5.10. For the heat equation (2.1), the quantity $\zeta^{[k]} = |\partial^k u|^2$ can also be considered as a $(2k)^{\text{th}}$ order entropy corrector. The corresponding balance equations can then be written

$$\partial_t\zeta^{[k]} - \Delta\zeta^{[k]} + 2|\partial_x^2 u|^{k+1} = 0.$$

In contrast with the Navier-Stokes system, we observe that unconditional entropicity holds for $\zeta^{[k]}$. ∎

5.7. Estimates of convective terms and of $\gamma^{[0]} = \tilde{\gamma}^{[0]}$

We first investigate majorization of the lower order convective terms $\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx$ by the higher order dissipative terms $\int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx$ and $\int_{\mathbb{R}^n} \pi_\gamma^{[k-1]}\, dx$.

Proposition 5.11. *Let $k \geq 1$ be an integer and (v,T) be a smooth solution of the incompressible Navier-Stokes equations (5.4)–(5.6). There exists positive constants $\delta(k,n)$ and $c(k,n)$ such that for $\chi < \delta$ we have the estimates*

$$\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx \leq c\chi \sup_{\mathbb{R}^n}\{T^{(1-2\varkappa+a_{k-1}-a_k)/2}\} \Big(\int_{\mathbb{R}^n} \pi_\gamma^{[k-1]}\, dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx\Big)^{\frac{1}{2}}, \qquad (5.39)$$

$$\int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k]}|\, dx \leq c\chi \sup_{\mathbb{R}^n}\{e^{(1-2\varkappa+a_{k-1}-a_k)\tau/2}\} \Big(\int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k-1]}\, dx\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]}\, dx\Big)^{\frac{1}{2}}. \qquad (5.40)$$

Proof. From the expression (5.20) and the continuity of the operators $T^\theta R_i R_j T^{-\theta}$ for $\|\log T\|_{BMO}$ small enough, we deduce that

$$\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx \leq c \sup_{\mathbb{R}^n}\{T^{(1-2\varkappa+a_{k-1}-a_k)/2}\}\ \|\Pi_\nu^{(k)}\|_{L^2}\ \|\Pi_\mu^{(k+1)}\|_{L^2},$$

and the estimate (5.39) is a direct consequence of the inequalities established in Section 4 and in the proof of Proposition 5.7, since there are at least two factors in the product $\Pi_\mu^{(k+1)}$. The proof of (5.40) is similar *mutatis mutandis* since the terms $\Pi_{\mu l \mathcal{R}}^{(k+1)}$ can be estimated by using the inequalities of Proposition 5.8. ■

We now recast the classical zeroth order entropic estimate in a convenient form that will be needed to investigate entropic principles associated with $\Gamma^{[k]}$.

Proposition 5.12. *Let $0 < a_0 \leq 1$ and let $\gamma^{[0]}$ be given by (5.11). Then $\gamma^{[0]} \geq 0$ and there exists positive constants $\delta_0 > 0$ and $\underline{b}_0'$ such that for $\chi < \delta_0$ small enough*

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \underline{b}_0' \int_{\mathbb{R}^n} \pi_\gamma^{[0]}\, dx \leq 0, \qquad (5.41)$$

where we define from (5.16)

$$\pi_\gamma^{[0]} = \frac{2\lambda \mathrm{A}_\lambda^{[0]}}{\rho c_v}\ \frac{|\partial^1 T|^2}{T^{1+a_0}} + \frac{2\eta \mathrm{A}_\eta^{[0]}}{\rho}\ \frac{|\partial^1 v|^2}{T^{a_0}}.$$

Equivalently, there exists a positive constant $\underline{b}_0$ such that for $\chi < \delta_0$ we have

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + 2\underline{b}_0 \int_{\mathbb{R}^n} T^{\varkappa+a_1-a_0} \gamma^{[1]}\, dx \leq 0. \qquad (5.42)$$

Proof. We only consider the case $0 < a_0 < 1$ since the case $a_0 = 1$ is similar. It is first easily established that the temperature part of $\gamma^{[0]}$ is nonnegative so that $\gamma^{[0]} \geq 0$.

Dividing the temperature equation by T^{a_0} and integrating over $\mathbb{R}^n$ we obtain after some algebra

$$-\partial_t \int_{\mathbb{R}^n} \frac{T^{1-a_0} - T_\infty^{1-a_0}}{1-a_0}\, dx + \frac{a_0}{\rho c_v} \int_{\mathbb{R}^n} \frac{\lambda |\partial_x T|^2}{T^{1+a_0}}\, dx + \frac{1}{2\rho c_v} \int_{\mathbb{R}^n} \frac{\eta |d|^2}{T^{a_0}}\, dx = 0.$$

On the other hand, dividing the total energy conservation equation by $T_\infty^{a_0}$ and integrating over $\mathbb{R}^n$ we obtain

$$\partial_t \int_{\mathbb{R}^n} \Big(\frac{T - T_\infty}{T_\infty^{a_0}} + \tfrac{1}{2} \frac{v^2}{c_v T_\infty^{a_0}} \Big)\, dx = 0.$$

Finally, from the relations (3.24), we obtain the inequality

$$\int_{\mathbb{R}^n} \frac{|\partial_x v|^2}{T^{a_0 - \varkappa}}\, dx \leq c \int_{\mathbb{R}^n} \frac{|d|^2}{T^{a_0 - \varkappa}}\, dx,$$

for $\|\log T\|_{BMO}$ small enough and combining these estimates completes the proof. ∎

We also recast the classical zeroth order entropic estimate in a convenient form that will be needed to investigate entropic principles associated with $\widetilde{\Gamma}^{[k]}$.

Proposition 5.13. *Let $0 < a_0 \leq 1$ and let $\tilde{\gamma}^{[0]}$ by given be (5.11). Then $\tilde{\gamma}^{[0]} \geq 0$ and there exists positive constants $\delta_0 > 0$, $\underline{b}_0'$, and c such that for $\chi < \delta_0$*

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[0]}\, dx + \underline{b}_0' \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx \leq c\chi \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx, \tag{5.43}$$

where we define from (5.26)

$$\pi_{\tilde{\gamma}}^{[0]} = e^{(1-a_0)\tau} \Big(\frac{2\lambda \mathrm{A}_\lambda^{[0]}}{\rho c_v} |\partial^1 \tau|^2 + \frac{2\eta \mathrm{A}_\eta^{[0]}}{\rho} |\partial^1 w|^2 \Big).$$

Equivalently, there exists positive constants $\delta_0 > 0$, $\underline{b}_0$, and c such that for $\chi < \delta_0$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[0]}\, dx + (2\underline{b}_0 - c\chi) \int_{\mathbb{R}^n} e^{(\varkappa + a_1 - a_0)\tau} \tilde{\gamma}^{[1]}\, dx \leq 0. \tag{5.44}$$

Proof. This is a direct consequence of Proposition 5.12 and of the differential relations

$$\frac{\partial_i v}{\sqrt{T}} = \partial_i w + \tfrac{1}{2} w \partial_i \tau,$$

which yield that $\int_{\mathbb{R}^n} \pi_{\gamma}^{[0]}\, dx$ is minorized by $(1 - c\chi) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx$. ∎

5.8. Natural scale of temperature weights

The estimates established in the previous sections are valid for any positive $\mathrm{A}_\lambda^{[k]}$ and $\mathrm{A}_\eta^{[k]}$, $k \geq 0$, and we now set for simplicity

$$\mathrm{A}_\lambda^{[k]} = 1, \qquad \mathrm{A}_\eta^{[k]} = \frac{1}{r}, \qquad k \geq 0. \tag{5.45}$$

With this simple choice, we note that the constants $\underline{b}_k = \underline{\mathfrak{a}}/\rho$ in Proposition 5.5 and Proposition 5.6 are independent of $k \geq 1$, and we correspondingly denote $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$.

In order to combine the estimates of Corollary 5.9 and Propositions 5.11, 5.12 and 5.13, obtained for various values of $k \geq 0$, we now need to specify the scale of temperature weights a_k, $k \geq 0$, used to renormalize the successive derivatives of T and v. In this section, we impose that the $\sup_{\mathbb{R}^n} T$ factors appearing in the convective terms estimates of Proposition 5.11 disappear, by letting $1 - 2\varkappa + a_{k-1} - a_k = 0$, $k \geq 1$, in such a way that

$$a_k = a_0 + k(1 - 2\varkappa), \qquad k \geq 0. \tag{5.46}$$

This scale fulfills the natural requirement that estimates for $\pi_\gamma^{[k-1]}$ and conditional entropicity for $\gamma^{[k]}$ yield estimates for $\pi_\gamma^{[k]}$. This scale of temperature weights also corresponds to the scale given by the kinetic theory of gases with (2.13)(2.14) since the factor $(\eta/\rho\sqrt{rT})^2 = \eta^2/\rho^2 rT$ yields the temperature exponent $1-2\varkappa$ from assumptions (5.2)(5.3). Therefore, this scale $a_k = a_0 + k(1 - 2\varkappa)$, $k \geq 0$, will be termed the natural scale of temperature weights. It is interesting to note that with this scale, a_k is decreasing with k for physical values of $\varkappa$, that is, for values such that $\varkappa \geq 1/2$. On the other hand, a_k is increasing with k for unphysical values of $\varkappa$, that is, for values such that $0 \leq \varkappa < 1/2$. This means in particular that, in the unphysical situation $0 \leq \varkappa < 1/2$, larger powers of T are needed in order to renormalize higher derivatives.

As a direct consequence of the preceding sections, we obtain the following estimates concerning higher order entropies. A similar proposition can also be established for $\tilde{\gamma}^{[k]}$ but the details are omitted.

Proposition 5.14. *Let (v, T) be a smooth solution of the incompressible Navier-Stokes equations (5.4)–(5.6). Assume that $a_l = a_0 + l(1 - 2\varkappa)$, $l \geq 0$, and let $k \geq 1$ be fixed. There exists positive constants $\delta(k, n)$ and $c(k, n)$ such that for $\chi < \delta$ we have the estimates*

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \big(2\underline{b} - c\chi\big) \int_{\mathbb{R}^n} T^{1-\varkappa}\gamma^{[k+1]}\, dx \leq c\chi \int_{\mathbb{R}^n} T^{1-\varkappa}\gamma^{[k]}\, dx. \tag{5.47}$$

Proof. These estimates are direct consequences of Corollary 5.9 and Proposition 5.11 since $\pi_\gamma^{[k]} \geq 2\underline{b}T^{1-\varkappa}\gamma^{[k+1]}$ for $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$. ∎

We can now combine the inequalities obtained for $k = 1, \dots, l$, in Proposition 5.14 together with the inequality obtained for $k = 0$ in Proposition 5.12, in order to estimate the $(2k)^{\text{th}}$ order kinetic entropy estimator $\Gamma^{[k]} = \gamma^{[0]} + \cdots + \gamma^{[k]}$.

Theorem 5.15. *Let* (v, T) *be a smooth solution of the incompressible Navier-Stokes equations* (5.4)–(5.6)*. Assume that* $a_l = a_0 + l(1-2\varkappa)$*,* $l \geq 0$*, and let* $k \in \mathbb{N}$ *be fixed. There exists positive constants* $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$ *and* $\delta_{\mathrm{N}}(k,n)$ *such that for* $\chi < \delta_{\mathrm{N}}$ *we have*

$$\partial_t \int_{\mathbb{R}^n} \Big(\gamma^{[0]} + \gamma^{[1]} + \cdots + \gamma^{[k]}\Big)\, dx + \underline{b} \int_{\mathbb{R}^n} T^{1-\varkappa} \Big(\gamma^{[1]} + \gamma^{[2]} + \cdots + \gamma^{[k+1]}\Big)\, dx \leq 0. \quad (5.48)$$

Proof. This results upon summing the estimates of Propositions 5.14 and 5.12. ∎

This theorem shows that the $(2k)^{\mathrm{th}}$ order kinetic entropy estimator $\Gamma^{[k]}$ obeys an entropic principle.

5.9. Uniform scale of temperature weights

In this section we still use the simple values $\mathrm{A}_\lambda^{[k]} = 1$ and $\mathrm{A}_\eta^{[k]} = 1/r$, for $k \geq 0$. On the other hand, in contrast with Section 5.6, we impose that the temperature weights are all equal

$$a_k = a_0, \qquad k \geq 0. \quad (5.49)$$

This scale of temperature weights will be termed the uniform scale. It is important to note that the $\sup_{\mathbb{R}^n} T$ factors of the right members of (5.39) and (5.40) in Proposition 5.11 cannot be majorized in terms of the solution derivatives since $T_\infty > 0$. As a consequence, taking into account the natural lower bound for temperature in terms of initial data $T \geq T_{\min} > 0$, controling these $\sup_{\mathbb{R}^n} T$ factors require to have negative exponents in (5.39) and (5.40). Therefore, we must have $1 - 2\varkappa + a_{k-1} - a_k \leq 0$, $k \geq 1$, and thus $a_0 + k(1-2\varkappa) \leq a_k$, for $k \geq 0$, and the natural scale of temperature weights appears to be a lower bound among all the useful scales. In particular, selecting a uniform scale requires that $k(1-2\varkappa) \leq \mathrm{Cte}$, so that we must have $\varkappa \geq 1/2$. In other words, the transport coefficients have to follow the temperature dependence indicated by the kinetic theory in order to use a uniform scale. With this scale, higher order entropy estimates directly yields estimates for higher order derivatives of $\log T$ and $v/\sqrt{T}$. This scale will be used in Section 6 in order to investigate asymptotic stability of equilibrium states for incompressible flows.

Proposition 5.16. *Let* (v, T) *be a smooth solution of the incompressible Navier-Stokes equations* (5.4)–(5.6)*. Assume that* $\varkappa \geq 1/2$*, that* $a_l = a_0$*,* $l \geq 0$*, let* $k \geq 1$ *be fixed, and assume that* $T_{\min} \leq T$*. There exists positive constants* $\delta(k, n, T_{\min})$ *and* $c(k, n, T_{\min})$ *such that for* $\chi < \delta$ *we have the estimates*

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \big(2\underline{b} - c\chi\big) \int_{\mathbb{R}^n} T^{\varkappa} \gamma^{[k+1]}\, dx \leq c\chi \int_{\mathbb{R}^n} T^{\varkappa} \gamma^{[k]}\, dx, \quad (5.50)$$

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[k]}\, dx + \big(2\underline{b} - c\chi\big) \int_{\mathbb{R}^n} T^{\varkappa} \tilde{\gamma}^{[k+1]}\, dx \leq c\chi \int_{\mathbb{R}^n} T^{\varkappa} \tilde{\gamma}^{[k]}\, dx. \quad (5.51)$$

Proof. The proof is similar to that of Proposition 5.14 and the $T_{\min}$ dependence arises from the negative powers of the $\sup_{\mathbb{R}^n} T$ factors. ∎

Theorem 5.17. *Let (v,T) be a smooth solution of the incompressible Navier-Stokes equations* (5.4)–(5.6)*. Assume that $\varkappa \geq 1/2$, that $a_l = a_0$, $l \geq 0$, let $k \in \mathbb{N}$ be fixed, and assume that $T_{\min} \leq T$. There exists positive constants $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$ and $\delta_{\mathrm{U}}(k,n,T_{\min})$ such that for $\chi < \delta_{\mathrm{U}}$ we have the estimates*

$$\partial_t \int_{\mathbb{R}^n} \Big(\gamma^{[0]} + \gamma^{[1]} + \cdots + \gamma^{[k]} \Big)\, dx + \underline{b} \int_{\mathbb{R}^n} T^{\varkappa} \Big(\gamma^{[1]} + \gamma^{[2]} + \cdots + \gamma^{[k+1]} \Big)\, dx \leq 0. \tag{5.52}$$

$$\partial_t \int_{\mathbb{R}^n} \Big(\tilde{\gamma}^{[0]} + \tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[k]} \Big)\, dx + \underline{b} \int_{\mathbb{R}^n} T^{\varkappa} \Big(\tilde{\gamma}^{[1]} + \tilde{\gamma}^{[2]} + \cdots + \tilde{\gamma}^{[k+1]} \Big)\, dx \leq 0. \tag{5.53}$$

Proof. This is a direct consequence of Propositions 5.12, 5.13 and 5.16. ▪

Theorem 5.17 shows that the $(2k)^{\text{th}}$ order kinetic entropy estimators $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ obey entropic principles. These estimates will be used in the next section in the situation of logarithmic scaling $a_k = 1$, $k \geq 0$.

6. Global solutions

We first establish a local existence theorem for incompressible flows spanning the whole space with temperature dependent transport coefficients and also establish that these solutions are smooth as depending on initial data. We next combine the local existence theorem with higher order entropy estimates in order to obtain global existence and asymptotic stability when $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small in appropriate spaces. This existence result may heuristically be interpreted as an existence result for small Mach number flows since we formally have $v/\sqrt{rT} = \mathcal{O}(\mathrm{Ma})$ and $\log(T/T_\infty) = \mathcal{O}(\mathrm{Ma}^2)$. We assume throughout this section that the scale of temperature weights is uniform with $a_k = 1$, $k \geq 0$, and that the transport coefficients λ and η satisfy assumptions (5.2)(5.3) with $\varkappa \geq 1/2$.

6.1. Local existence

We denote by v the combined unknown $\mathrm{v} = (v,T)$, keeping in mind that the momentum conservation equation is considered as projected on the space of divergence free functions. We denote accordingly by v_∞ the equilibrium point $\mathrm{v}_\infty = (0, T_\infty)$ with $v_\infty = 0$ and $T_\infty > 0$. We denote by $\mathcal{O}_{\mathrm{v}} = \mathbb{R}^n \times (0,\infty)$ the natural domain for the variable v, where $n \geq 2$.

Theorem 6.1. *Let $n \geq 2$ and $l \geq [n/2]+2$ be integers and let $b > 0$ be given. Let $\mathcal{O}_0$ be an open bounded convex set such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}_{\mathrm{v}}$, d_1 with $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathrm{v}})$, and define $\mathcal{O}_1 = \{\, \mathrm{v} \in \mathcal{O}_{\mathrm{v}};\ d(\mathrm{v}, \overline{\mathcal{O}}_0) < d_1 \,\}$. There exists $\bar{t} > 0$ small enough, which only depend on $\mathcal{O}_0$, d_1, and b, such that for any v_0 with $\|\mathrm{v}_0 - \mathrm{v}_\infty\|_{H^l} < b$ and $\mathrm{v}_0 \in \overline{\mathcal{O}}_0$, there exists a unique local solution $\mathrm{v} = (v,T)$ to the system* (5.4)–(5.6) *with initial condition*

$$\mathrm{v}(0,x) = \mathrm{v}_0(x), \tag{6.1}$$

such that

$$\mathrm{v}(t,x) \in \mathcal{O}_1, \tag{6.2}$$

and

$$\mathrm{v}-\mathrm{v}_\infty \in C^0\big([0,\bar t], H^l(\mathbb{R}^n)\big)\cap C^1\big([0,\bar t], H^{l-2}(\mathbb{R}^n)\big)\cap L^2\big((0,\bar t\,), H^{l+1}(\mathbb{R}^n)\big). \tag{6.3}$$

In addition, denoting for short $\mathrm{v}(t)=\mathrm{v}(t,\cdot)$*, there exists* $C>0$ *which only depend on* $\mathcal{O}_0$*,* d_1*, and* b*, such that*

$$\sup_{0\le s\le \bar t}\|\mathrm{v}(s)-\mathrm{v}_\infty\|_{H^l}^2+\int_0^{\bar t}\|\mathrm{v}(s)-\mathrm{v}_\infty\|_{H^{l+1}}^2\,ds\le C\|\mathrm{v}_0-\mathrm{v}_\infty\|_{H^l}^2. \tag{6.4}$$

Proof. The following proof is adapted from Kawashima to the situation of incompressible flows [Kaw84] [Gio99] [GG04]. Solutions to the nonlinear system (5.4)–(5.6) are fixed points $\widetilde{\mathrm{v}}=\mathrm{v}$ of the linear system of equations in $\widetilde{\mathrm{v}}=(\widetilde v,\widetilde T)$

$$\begin{cases}\partial_t(\rho\widetilde v)-\mathbb{P}\big(\partial_x\cdot\big(\eta(T)\,\partial_x\widetilde v\big)\big)=\mathbb{P}\big(f_v(\mathrm{v},\partial_x\mathrm{v})\big),\\ \partial_t(\rho c_v\widetilde T)-\partial_x\cdot\big(\lambda(T)\,\partial_x\widetilde T\big)=f_T(\mathrm{v},\partial_x\mathrm{v}),\end{cases} \tag{6.5}$$

with $f_v=-\partial_x\cdot(\rho v\otimes v)+\partial_T\eta(T)\partial_xT\cdot\partial_x v^t$ and $f_T=-\partial_x\cdot(\rho c_v T v)+\frac{1}{2}\eta(T)d{:}d$. Fixed points $\widetilde{\mathrm{v}}=\mathrm{v}$ are investigated in the function space $X_{\bar t}\big(\mathcal{O}_1,M,M_1\big)$, that is defined by $\mathrm{v}\in X_{\bar t}\big(\mathcal{O}_1,M,M_1\big)$ if $\mathrm{v}(t,x)\in\mathcal{O}_1$,

$$\mathrm{v}-\mathrm{v}_\infty\in C^0\big([0,\bar t], H^l(\mathbb{R}^n)\big)\cap L^2\big((0,\bar t\,), H^{l+1}(\mathbb{R}^n)\big),$$

$$\partial_t\mathrm{v}\in C^0\big([0,\bar t], H^{l-2}(\mathbb{R}^n)\big)\cap L^2\big((0,\bar t\,), H^{l-1}(\mathbb{R}^n)\big),$$

$$\sup_{0\le s\le\bar t}\|\mathrm{v}(s)-\mathrm{v}_\infty\|_{H^l}^2+\int_0^{\bar t}\|\mathrm{v}(s)-\mathrm{v}_\infty\|_{H^{l+1}}^2\,ds\le M^2,$$

and

$$\int_0^{\bar t}\|\partial_t\mathrm{v}(s)\|_{H^{l-1}}^2\,ds\le M_1^2.$$

For v in $X_{\bar t}\big(\mathcal{O}_1,M,M_1\big)$, $1\le k\le l$, and $f=(f_v,f_T)$, we have the estimates

$$\begin{aligned}\|\widetilde{\mathrm{v}}(t)-\mathrm{v}_\infty\|_{H^k}^2+\int_0^t\|\widetilde{\mathrm{v}}(s)-\mathrm{v}_\infty\|_{H^{k+1}}^2\,ds&\le C_1^2\exp\big(C_2(t+M_1\sqrt{t}\,)\big)\\ &\quad\times\Big(\|\mathrm{v}_0-\mathrm{v}_\infty\|_{H^k}^2+C_2\int_0^t\|f(s)\|_{H^{k-1}}^2\,ds\Big),\end{aligned} \tag{6.6}$$

where $C_1=C_1(\mathcal{O}_1)$ depends on $\mathcal{O}_1$ and $C_2=C_2(\mathcal{O}_1,M)$ depends on $\mathcal{O}_1$ and M, and is an increasing function of M. These a priori estimates for solutions of the linear equations (6.5) are obtained by deriving the governing equations, multiplying by the derivative of the solution, and using the properties of the Leray projector $\mathbb{P}$ [Kaw84] [Lio96]. On the other hand, existence of such solutions $\widetilde{\mathrm{v}}$ to the linear equations are obtained from a priori estimates by standard arguments like Galerkin approximations.

Furthermore, using the classical estimates

$$\|\psi(\phi) - \psi(0)\|_{H^k} \leq C_0 \|\psi\|_{C^k(\overline{\mathcal{O}}_\phi)} (1 + \|\phi\|_{L^\infty})^{k-1} \|\phi\|_{H^k}, \tag{6.7}$$

where $\mathcal{O}_\phi$ is a convex open set with $\phi(x) \in \mathcal{O}_\phi$, $x \in \mathbb{R}^n$, and increasing eventually the constant $C_2(\mathcal{O}_1, M)$ of (6.6), we obtain for $f = (f_v, f_T)$ that

$$\|f(t)\|^2_{H^{l-1}} \leq C_2 M^2, \qquad 0 \leq t \leq \bar{t}. \tag{6.8}$$

From the governing equations, we also deduce that

$$\int_0^t \|\partial_t \widetilde{\mathrm{v}}(s)\|^2_{H^{l-1}}\, ds \leq C_3^2 \big(\widetilde{M}^2 + t(M^2 + \widetilde{M}^2)\big). \tag{6.9}$$

where $\widetilde{M}$ is defined for $\widetilde{\mathrm{v}}$ as M for v and C_3 depends on $\mathcal{O}_1$ and M, and is an increasing function of M. We now define for any $\alpha \in (0, b]$

$$M_\alpha = 2C_1(\mathcal{O}_1)\alpha, \qquad M_{1\alpha} = 2C_3(\mathcal{O}_1, M_b) 2C_1(\mathcal{O}_1)\alpha.$$

Let then $\bar{t} \leq 3/2$ be small enough such that

$$\exp\big(C_2(\mathcal{O}_1, M_b)(\bar{t} + M_{1b}\sqrt{\bar{t}}\,)\big) \leq 2,$$

$$C_2^2(\mathcal{O}_1, M_b)\bar{t}\big(2C_1(\mathcal{O}_1)\big)^2 \leq 1,$$

and $C_0 M_{1b}\sqrt{\bar{t}} < d_1$, where C_0 is such that $\|\phi\|_{L^\infty} \leq C_0\|\phi\|_{H^{l-1}}$. Then, for any $\alpha \in (0, b]$, any $\mathrm{v} \in X_{\bar{t}}\big(\mathcal{O}_1, M_\alpha, M_{1\alpha}\big)$, any $\mathrm{v}_0(x)$, such that $\mathrm{v}_0 - \mathrm{v}_\infty \in H^l(\mathbb{R}^d)$, $\mathrm{v}_0 \in \overline{\mathcal{O}}_0$, and $\|\mathrm{v}_0 - \mathrm{v}_\infty\|_{H^l} \leq \alpha$, the solution $\widetilde{\mathrm{v}}$ to the linearized equations with the initial data v_0 stays in the same space $X_{\bar{t}}\big(\mathcal{O}_1, M_\alpha, M_{1\alpha}\big)$. More specifically, we obtain from (6.7) and (6.8) that

$$\widetilde{M}^2 \leq 2C_1^2\alpha^2\big(1 + 4C_1^2C_2^2 t\big) \leq 4C_1^2\alpha^2 = M_\alpha^2,$$

and from (6.9) we deduce that $\widetilde{M}_1^2 \leq C_3^2 M_\alpha^2(1 + 2t) \leq M_{1\alpha}^2$. Finally, we also have $\|\widetilde{\mathrm{v}} - \mathrm{v}_0\|_{L^\infty} \leq C_0 M_{1\alpha}\sqrt{\bar{t}} < d_1$, since $C_0 M_{1b}\sqrt{\bar{t}} < d_1$, so that $\widetilde{\mathrm{v}} \in \mathcal{O}_1$.

In order to obtain fixed points, we establish that for $\bar{t}$ small enough, the map $\mathrm{v} \to \widetilde{\mathrm{v}}$ is a contraction in all the spaces $X_{\bar{t}}\big(\mathcal{O}_1, M_\alpha, M_{1\alpha}\big)$, $\alpha \in (0, b]$. Let v and $\widehat{\mathrm{v}}$ be in $X_{\bar{t}}\big(\mathcal{O}_1, M_b, M_{1b}\big)$, let $\mathrm{v}^0(x)$ and $\widehat{\mathrm{v}}^0(x)$ such that $\mathrm{v}_0 - \mathrm{v}_\infty \in H^l(\mathbb{R}^d)$, $\widehat{\mathrm{v}}_0 - \mathrm{v}_\infty \in H^l(\mathbb{R}^d)$, $\mathrm{v}_0, \widehat{\mathrm{v}}_0 \in \overline{\mathcal{O}}_0$, $\|\mathrm{v}_0 - \mathrm{v}_\infty\|_l < \alpha$, $\|\widehat{\mathrm{v}}_0 - \mathrm{v}_\infty\|_l < \alpha$, and define $\delta\mathrm{v} = \mathrm{v} - \widehat{\mathrm{v}}$ and $\widetilde{\delta\mathrm{v}} = \widetilde{\mathrm{v}} - \widetilde{\widehat{\mathrm{v}}}$. Forming the difference between the linearized equations, we obtain that for $\delta\mathrm{v} = (\delta v, \delta T)$ and $\widetilde{\delta\mathrm{v}} = (\widetilde{\delta v}, \widetilde{\delta T})$

$$\begin{cases} \partial_t(\rho\widetilde{\delta v}) - \mathbb{P}\big(\partial_x\cdot\big(\eta(T)\,\partial_x\widetilde{\delta v}\big)\big) = \mathbb{P}\big(\delta f_v\big), \\ \partial_t(\rho c_v\widetilde{\delta T}) - \partial_x\cdot\big(\lambda(T)\,\partial_x\widetilde{\delta T}\big) = \delta f_T, \end{cases} \tag{6.10}$$

where

$$\begin{aligned} \delta f_v =& f_v(\mathrm{v}, \partial_x\mathrm{v}) - f_v(\widehat{\mathrm{v}}, \partial_x\widehat{\mathrm{v}}) + \partial_x\cdot\big((\eta(T) - \eta(\widehat{T}))\partial_x\widetilde{\widehat{v}}\big), \\ \delta f_T =& f_T(\mathrm{v}, \partial_x\mathrm{v}) - f_T(\widehat{\mathrm{v}}, \partial_x\widehat{\mathrm{v}}) + \partial_x\cdot\big((\lambda(T) - \lambda(\widehat{T}))\partial_x\widetilde{\widehat{T}}\big). \end{aligned}$$

These expression now imply that $\|\delta f\|_{H^{l-2}} \leq C_4 \|\delta \mathrm{v}\|_{H^{l-1}}$, where the constant C_4 depends on $\mathcal{O}_1$ and b, since v, $\widetilde{\mathrm{v}}$, $\widehat{\mathrm{v}}$ and $\widetilde{\widehat{\mathrm{v}}}$ are in the space $X_{\bar{t}}\big(\mathcal{O}_1, M_b, M_{1b}\big)$, and thanks to estimates in the form

$$\|\psi(\phi) - \psi(\hat{\phi})\|_{H^k} \leq C_0 \|\psi\|_{C^{k+1}(\overline{\mathcal{O}}_\phi)} (1 + \|\phi\|_{H^k} + \|\hat{\phi}\|_{H^k})^k \, \|\phi - \hat{\phi}\|_{H^k}, \tag{6.11}$$

where $\mathcal{O}_\phi$ is a convex open set with $\phi(x) \in \mathcal{O}_\phi$, $\hat{\phi}(x) \in \mathcal{O}_\phi$, $x \in \mathbb{R}^n$, and k is such that $k \geq [n/2] + 1$. As a consequence, defining

$$|\!|\!|\widetilde{\delta \mathrm{v}}|\!|\!|_{l-1}^2 = \sup_{0 \leq s \leq \bar{t}} \|\widetilde{\delta \mathrm{v}}(s)\|_{H^{l-1}}^2 + \int_0^{\bar{t}} \|\widetilde{\delta \mathrm{v}}(s)\|_{H^l}^2 \, ds,$$

we obtain that

$$|\!|\!|\widetilde{\delta \mathrm{v}}|\!|\!|_{l-1} \;\leq\; C_5 \|\mathrm{v}_0 - \widehat{\mathrm{v}}_0\|_{H^{l-1}}^2 + \; C_5 \bar{t} \sup_{0 \leq s \leq \bar{t}} \|\delta \mathrm{v}(s)\|_{H^{l-1}}^2,$$

where the constant C_5 depends on $\mathcal{O}_1$ and b. Now if $\bar{t}$ is small enough so that $C_5 \bar{t} < 1/4$, by letting $\mathrm{v}_0 = \widehat{\mathrm{v}}_0$, we obtain that $|\!|\!|\widetilde{\delta \mathrm{v}}|\!|\!|_{l-1} \leq \frac{1}{2} |\!|\!|\delta \mathrm{v}|\!|\!|_{l-1}$ so that the map $\mathrm{v} \to \widetilde{\mathrm{v}}$ is a contraction in all the spaces $X_{\bar{t}}\big(\mathcal{O}_1, M_\alpha, M_{1\alpha}\big)$, $\alpha \in (0, b]$. Introducing the iterates v^n starting at the initial condition $\mathrm{v}^0 = \mathrm{v}_0$ and such that $\mathrm{v}^{n+1} = \widetilde{\mathrm{v}}^n$, that is, v^{n+1} is obtained as the solution of linearized equations, then the sequence $\{\mathrm{v}^n\}_{n \geq 0}$ is easily shown to be convergent to a local solution of the nonlinear equations satisfying the estimates (6.4) at order $l-1$. Finally, the estimates (6.4) at order l are recovered since for any $\alpha \in (0, b]$, the space $X_{\bar{t}}\big(\mathcal{O}_1, M_\alpha, M_{1\alpha}\big)$ is invariant, and the proof is complete. ∎

6.2. Properties of the solutions

We establish in this section that the solutions constructed in Theorem 6.1 are as smooth as expected from initial data.

Theorem 6.2. *The solutions obtained in Theorem 6.1 inherit the regularity of* v_0, *that is, for any* $k \geq l$ *such that* $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k$, *we have*

$$\mathrm{v} - \mathrm{v}_\infty \in C^0\big([0, \bar{t}], H^k(\mathbb{R}^n)\big) \cap C^1\big([0, \bar{t}], H^{k-2}(\mathbb{R}^n)\big) \cap L^2\big((0, \bar{t}), H^{k+1}(\mathbb{R}^n)\big). \tag{6.12}$$

In particular, v *is smooth when* $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k(\mathbb{R}^n)$ *for any* $k \in \mathbb{N}$.

Proof. Let $k \geq l$ such that $\mathrm{v}_0 - \mathrm{v}_\infty \in H^k$ and denote by $e^{[k]}$ the quantity $e^{[k]} = |\partial^k T|^2 + |\partial^k v|^2$. We have to estimate $e^{[k]}$ in order to establish (6.12).

A balance equation for $e^{[k]}$ can easily be derived—and is much simpler than that of $\gamma^{[k]}$ of $\tilde{\gamma}^{[k]}$—and written in the form

$$\partial_t e^{[k]} + \partial_x \cdot (v e^{[k]}) + \partial_x \cdot \varphi_e^{[k]} + \pi_e^{[k]} + \Sigma_e^{[k]} + \omega_e^{[k]} = 0. \tag{6.13}$$

The term $\pi_e^{[k]}$ is given by

$$\pi_e^{[k]} = \frac{2\lambda}{\rho c_v} \, |\partial^{k+1} T|^2 + \frac{2\eta}{\rho} |\partial^{k+1} v|^2, \tag{6.14}$$

in such a way that $2\underline{b}\, e^{[k+1]} \leq \pi_e^{[k]}\, T^{-\varkappa} \leq 2\overline{b}\, e^{[k+1]}$ where $\underline{b}$ and $\overline{b}$ are positive constants. The term $\Sigma_e^{[k]}$ is in the form

$$\Sigma_e^{[k]} = \sum_{\sigma\nu\mu} c_{\sigma\nu\mu} \partial_T^\sigma \lambda\, \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu} c_{\sigma\nu\mu} \partial_T^\sigma \eta\, \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)}, \tag{6.15}$$

where the sums are over $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha)_{1\leq|\alpha|\leq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha)_{1\leq|\alpha|\leq k+1}$, $\nu_\alpha, \nu'_\alpha, \mu_\alpha, \mu'_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = \prod_{1\leq|\alpha|\leq k+1} \big(\partial^\alpha T\big)^{\nu_\alpha} \prod_{1\leq|\alpha|\leq k+1} \big(\partial^\alpha v\big)^{\nu'_\alpha}, \tag{6.16}$$

where v denotes any of its components $v_1, \dots, v_n$, and μ and ν must be such that $\sum_{1\leq|\alpha|\leq k+1} |\alpha|(\nu_\alpha + \nu'_\alpha) = k+1$, $\sum_{1\leq|\alpha|\leq k+1} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1$, $\sum_{|\alpha|=k+1}(\nu_\alpha + \nu'_\alpha + \mu_\alpha + \mu'_\alpha) \leq 1$, so that there is at most one derivative of $(k+1)^{\text{th}}$ order in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$. Furthermore the term $\omega_e^{[k]}$ is given by

$$\omega_e^{[k]} = \sum_{\nu\mu} c_{\nu\mu} \Pi_\nu^{(k)} \Pi_\mu^{(k+1)}, \tag{6.17}$$

where we use similar notation for $\Pi_\nu^{(k)}$ as for $\Pi_\mu^{(k+1)}$ and the summation extends over $\sum_{1\leq|\alpha|\leq k} |\alpha|(\nu_\alpha + \nu'_\alpha) = k$, $\sum_{1\leq|\alpha|\leq k} |\alpha|(\mu_\alpha + \mu'_\alpha) = k+1$ so that in particular $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha) = 0$ and there are always at least two factors in the product $\Pi_\mu^{(k+1)}$. Finally the flux $\varphi_e^{[k]} = (\varphi_{e1}^{[k]}, \dots, \varphi_{en}^{[k]})$ is given by the following formula

$$\varphi_{el}^{[k]} = \sum_{\sigma\nu\mu l} c_{\sigma\nu\mu l} \partial_T^\sigma \lambda\, \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} + \sum_{\sigma\nu\mu l} c_{\sigma\nu\mu l} \partial_T^\sigma \eta\, \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} + \sum_\alpha c_\alpha \partial^\alpha v_l \partial^\alpha p. \tag{6.18}$$

Now instead of regrouping the term $\Sigma_e^{[k]}$ with $\pi_e^{[k]}$, as in Corollary 5.9, Proposition 5.14, and 5.16, we regroup it with $\omega_e^{[k]}$ [VH72], thanks to the L^∞ a priori estimates for gradients, available from $l > n/2 + 1$. Whenever the product $\Pi_\nu^{(k+1)}$ is split, we indeed have estimates in the form [VH72]

$$\|\Pi_\nu^{(k+1)}\|_{L^2}^2 \leq c\big(1 + \|\partial T\|_{L^\infty} + \|\partial v\|_{L^\infty}\big)^{2(k-1)} \int_{\mathbb{R}^n} \big(e^{[1]} + \cdots + e^{[k]}\big)\, dx,$$

so that from

$$\partial_t \int_{\mathbb{R}^n} e^{[k]}\, dx + \int_{\mathbb{R}^n} \pi_e^{[k]}\, dx \leq \int_{\mathbb{R}^n} \big(|\Sigma_e^{[k]}| + |\omega_e^{[k]}|\big)\, dx,$$

we obtain that

$$\partial_t \int_{\mathbb{R}^n} e^{[j]}\, dx + \delta \int_{\mathbb{R}^n} e^{[j+1]}\, dx \leq c \int_{\mathbb{R}^n} \big(e^{[1]} + \cdots + e^{[k]}\big)\, dx, \qquad 1 \leq j \leq k,$$

where δ and c depend on the L^∞ estimates of v and $\partial\mathrm{v}$. We can then sum up these inequalities and use Gronwall lemma to conclude that $\int_{\mathbb{R}^n} e^{[k]}\, dx$ and $\int_0^t \int_{\mathbb{R}^n} e^{[k+1]}\, dx\, dt$

remain uniformly bounded over the whole time interval under consideration $[0,\bar{t}\,]$. Finally, when $\mathrm{v}_0-\mathrm{v}_\infty$ is in H^k for any $k\geq 0$, $\mathrm{v}-\mathrm{v}_\infty$ is in $C^0([0,\bar{t}\,],H^k)$ for any k, and we recover the regularity with respect to time from the governing equations so that v is smooth. ■

Remark 6.3. Boundedness of first order spatial derivatives is sufficient to establish the estimates of Theorem 6.2 because the coefficients of the time derivative terms in (5.5)(5.6) are constants. For general quasilinear parabolic systems, one further needs to control second order spatial derivatives, whereas for general nonlinear second order systems, it is also necessary to control third order spatial derivatives [VH72]. ■

In the next propositions, we reformulate for convenience the local existence theorem in terms of the the combined unknown $\mathrm{w}=(w,\tau)$ associated with the renormalized variables w and τ.

Lemma 6.4. *Denote by $\mathcal{F}:\mathbb{R}^n\times(0,\infty)\to\mathbb{R}^{n+1}$ the application defined by $\mathcal{F}(\mathrm{v})=\mathrm{w}$, that is, $\mathcal{F}(v,T)=(w,\tau)=(v/\sqrt{T},\log T)$. Then $\mathcal{F}$ is a C^∞ diffeomorphism and its jacobian matrix reads*

$$\partial_{\mathrm{v}}\mathcal{F}=\begin{pmatrix}\frac{\mathbb{I}}{\sqrt{T}} & -\frac{1}{2}\frac{v}{T^{3/2}}\\ 0 & \frac{1}{T}\end{pmatrix}.$$

In addition, for any $\mathrm{M}_w>0$, $\mathrm{M}_\tau>0$, defining $\widetilde{\mathcal{O}}=(-\mathrm{M}_w,\mathrm{M}_w)^n\times(-\mathrm{M}_\tau,\mathrm{M}_\tau)$, the corresponding open set $\mathcal{O}=\mathcal{F}^{-1}(\widetilde{\mathcal{O}})$ is convex.

Proof. The fact that $\mathcal{F}$ is a C^∞ diffeomorphism is straightforward to establish. Let then $\mathrm{M}_w>0$, $\mathrm{M}_\tau>0$, and assume that $(v,T)\in\mathcal{O}$, $(v',T')\in\mathcal{O}$. By definition, we have $|v_i|\leq \mathrm{M}_w\sqrt{T}$, $|v_i'|\leq \mathrm{M}_w\sqrt{T'}$, for $1\leq i\leq n$, and $e^{-\mathrm{M}_\tau}<T<e^{\mathrm{M}_\tau}$, $e^{-\mathrm{M}_\tau}<T'<e^{\mathrm{M}_\tau}$. For $0<\alpha<1$, we easily obtain then that $e^{-\mathrm{M}_\tau}<\alpha T+(1-\alpha)T'<e^{\mathrm{M}_\tau}$ and

$$|\alpha v_i+(1-\alpha)v_i'|<\mathrm{M}_w(\alpha\sqrt{T}+(1-\alpha)\sqrt{T'}),$$

but we have $\alpha\sqrt{T}+(1-\alpha)\sqrt{T'}\leq\sqrt{\alpha T+(1-\alpha)T'}$ from concavity properties so that $\mathcal{O}$ is convex. ■

Proposition 6.5. *Let $\mathrm{M}_w>0$, $\mathrm{M}_\tau>0$, define $\widetilde{\mathcal{O}}_0=(-\mathrm{M}_w,\mathrm{M}_w)^n\times(-\mathrm{M}_\tau,\mathrm{M}_\tau)$ and $\mathcal{O}_0=\mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$. Let $0<d_1<d(\overline{\mathcal{O}}_0,\partial\mathcal{O}_{\mathrm{v}})$, denote $\mathcal{O}_1=\{\,\mathrm{v}\in\mathcal{O}_{\mathrm{v}};\ d(\mathrm{v},\overline{\mathcal{O}}_0)<d_1\,\}$, and select an arbitrary $b>0$. From Theorem 6.1 we have a local solution built with the paramaters $\mathcal{O}_0$, d_1, and b. This solution is then such that*

$$\mathrm{w}-\mathrm{w}_\infty\in C^0\big([0,\bar{t}\,],H^l(\mathbb{R}^n)\big)\cap C^1\big([0,\bar{t}\,],H^{l-2}(\mathbb{R}^n)\big)\cap L^2\big((0,\bar{t}\,),H^{l+1}(\mathbb{R}^n)\big),\quad(6.19)$$

and there exists $C>0$ which only depend on $\mathcal{O}_0$, d_1, and b, such that

$$\sup_{0\leq s\leq\bar{t}}\|\mathrm{w}(s)-\mathrm{w}_\infty\|_{H^l}^2+\int_0^{\bar{t}}\|\mathrm{w}(s)-\mathrm{w}_\infty\|_{H^{l+1}}^2\,ds\leq C\|\mathrm{w}_0-\mathrm{w}_\infty\|_{H^l}^2.\qquad(6.20)$$

Moreover, the kinetic entropy estimators are such that $\Gamma^{[l]},\widetilde{\Gamma}^{[l]}\in C\big([0,\bar{t}\,],L^1(\mathbb{R}^n)\big)$.

Proof. The set $\mathcal{O}_0 = \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$ is convex and from Theorem 6.1, there exists a local solution built with $\mathcal{O}_0$, d_1 and b. We then have estimates in the form

$$\underline{c}_{\mathrm{V}} \|\mathrm{W} - \mathrm{W}_\infty\|_{H^l} \leq \|\mathrm{V} - \mathrm{V}_\infty\|_{H^l} \leq \overline{c}_{\mathrm{V}} \|\mathrm{W} - \mathrm{W}_\infty\|_{H^l}, \tag{6.21}$$

where $\underline{c}_{\mathrm{V}}$ and $\overline{c}_{\mathrm{V}}$ only depend on $\mathcal{O}_1$ and l thanks to the estimates (6.7). Similarly, the regularity properties are direct consequences of the estimates (6.11). The properties $\Gamma^{[l]}, \widetilde{\Gamma}^{[l]} \in C\big([0,\bar{t}\,], L^1(\mathbb{R}^n)\big)$ are then straighforward to establish. ∎

Lemma 6.6. *There exists a constant $\underline{c}_\Gamma$ only depending on $T_{\min}$ such that for any $k \geq 0$ and any W with $\mathrm{W} - \mathrm{W}_\infty \in H^k$ we have*

$$\underline{c}_\Gamma \|\mathrm{W} - \mathrm{W}_\infty\|_{H^k}^2 \leq \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, dx. \tag{6.22}$$

Proof. This is a direct consequence of $w^2 \leq v^2/T_{\min}$ and of the inequality

$$\frac{T_{\min}}{2T_\infty} |\zeta - \tau_\infty|^2 \leq \exp(\zeta - \tau_\infty) - 1 - (\zeta - \tau_\infty),$$

valid for $\tau_{\min} = \log T_{\min} \leq \zeta$, where $\tau_\infty = \log T_\infty$ and $T_{\min} \leq T_\infty$. ∎

6.3. Global existence

In this section, we investigate global existence of solutions $\mathrm{V} = (v, T)$ for which the quantity $\chi = \|\log T\|_{BMO} + \|v/\sqrt{T}\|_{L^\infty}$ remains small.

Theorem 6.7. *Let $n \geq 2$ and $l \geq [n/2] + 2$ be integers. Assume that the coefficients λ and η satisfy (5.2)(5.3) with $\varkappa \geq 1/2$. There exists $\delta_\Gamma(l, n, T_{\min}) > 0$ such that for T_0 and v_0 satisfying $T_{\min} \leq \inf_{\mathbb{R}^n} T_0$, $\partial_x \cdot v_0 = 0$, $\mathrm{V}_0 - \mathrm{V}_\infty \in H^k$, $k \in \mathbb{N}$, and*

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \leq \delta_\Gamma, \tag{6.23}$$

where $\widetilde{\Gamma}_0^{[l]}$ denotes the functional $\widetilde{\Gamma}^{[l]}$ evaluated at initial conditions, there exists a unique global solution $\mathrm{V} = (v, T)$ such that

$$\begin{cases} \mathrm{V} - \mathrm{V}_\infty, \mathrm{W} - \mathrm{W}_\infty \in C\big([0,\infty), H^l(\mathbb{R}^n)\big) \cap C^1\big([0,\infty), H^{l-2}(\mathbb{R}^n)\big), \\ \partial_x \mathrm{V}, \partial_x \mathrm{W} \in L^2\big((0,\infty), H^l(\mathbb{R}^n)\big), \end{cases} \tag{6.24}$$

and we have the estimates

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx + \underline{b} \int_0^t \int_{\mathbb{R}^n} T^\varkappa \big(\widetilde{\Gamma}^{[l+1]} - \gamma^{[0]}\big)\, dx dt \leq \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx, \tag{6.25}$$

where $\underline{b} = \min(\underline{b}_0, \underline{\mathfrak{a}}/\rho)$. *Furthermore, this solution is smooth and we have*

$$\lim_{t\to\infty} \|\mathrm{V}(t,\cdot) - \mathrm{V}_\infty\|_{L^\infty} = 0. \tag{6.26}$$

Proof. Letting $l_0 = [n/2] + 1$, we have the inequalities

$$\chi = \|\tau\|_{BMO} + \|w\|_{L^\infty} \le \|\tau - \tau_\infty\|_{L^\infty} + \|w\|_{L^\infty} \le c_0 \|\mathrm{W} - \mathrm{W}_\infty\|_{H^{l_0}}.$$

In order to obtain a value of δ_Γ small enough, so that the higher order entropic estimates of Theorem 5.17 hold, we set

$$\delta_\Gamma = \frac{\delta_{\mathrm{U}}^2}{4c_0^2} \underline{c}_\Gamma,$$

where δ_{U} is defined in Theorem 5.17 and $\underline{c}_\Gamma$ in Lemma 6.6, and this value will indeed insure that $\chi \le \delta_{\mathrm{U}}/2$. Corresponding to this value of δ_Γ, we have estimates in the forms $\|\mathrm{W} - \mathrm{W}_\infty\|_{L^\infty} \le c_0'(\delta_\Gamma/\underline{c}_\Gamma)^{1/2}$ and $\|\mathrm{W} - \mathrm{W}_\infty\|_{H^l} \le (\delta_\Gamma/\underline{c}_\Gamma)^{1/2}$. We now select $\mathrm{M}_w > 0$ and $\mathrm{M}_\tau > 0$ large enough such that

$$\{\, z \in \mathbb{R}^{n+1};\ \|z - \mathrm{W}_\infty\| \le c_0'(\delta_\Gamma/\underline{c}_\Gamma)^{1/2} + 1 \,\} \ \subset\ (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau).$$

We next define $\widetilde{\mathcal{O}}_0 = (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau)$, $\mathcal{O}_0 = \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$, and we know that $\mathcal{O}_0$ is convex. Let then $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathrm{V}})$, and define $\mathcal{O}_1 = \{\, z \in \mathcal{O}_{\mathrm{V}};\ d(z, \mathcal{O}_0) < d_1 \,\}$ and $\widetilde{\mathcal{O}}_1 = \mathcal{F}(\mathcal{O}_1)$. Now for functions taking their values in $\mathcal{O}_1$ we have inequalities in the form $\|\mathrm{V} - \mathrm{V}_\infty\|_{H^k} \le \overline{c}_{\mathrm{V}} \|\mathrm{W} - \mathrm{W}_\infty\|_{H^k}$ where $\overline{c}_{\mathrm{V}}$ only depends on k and $\mathcal{O}_1$. We thus obtain the a priori estimate $\|\mathrm{V} - \mathrm{V}_\infty\|_{H^l} \le \overline{c}_{\mathrm{V}}(\delta_\Gamma/\underline{c}_\Gamma)^{1/2}$. We now set $b = \overline{c}_{\mathrm{V}}(\delta_\Gamma/\underline{c}_\Gamma)^{1/2} + 1$ and from Proposition 6.4 we have local solutions over a time interval $[0, \bar{t}\,]$ built with the paramaters $\mathcal{O}_0$, d_1, and b.

Let now T_0 and v_0 satisfy $T_{\min} \le \inf_{\mathbb{R}^n} T_0$, $\partial_x \cdot v_0 = 0$, $\mathrm{V}_0 - \mathrm{V}_\infty \in H^k$, $k \in \mathbb{N}$, and $\int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \le \delta_\Gamma$. Then by construction $\mathrm{V}_0 \in \mathcal{O}_0$ and $\|\mathrm{V} - \mathrm{V}_\infty\|_{H^l} < b$, and we have a local solution over the time interval $[0, \bar{t}\,]$. Letting $\chi(t) = \|\tau(t,\cdot)\|_{BMO} + \|w(t,\cdot)\|_{L^\infty}$ we also have by construction $\chi(0) \le \delta_{\mathrm{U}}/2$ and we claim that for any $t \in [0, \bar{t}\,]$ we also have $\chi(t) \le \delta_{\mathrm{U}}/2$. We introduce the set

$$\mathcal{E} = \{\ s \in\ (0, \bar{t}\,]; \forall t \in [0, s],\ \chi(t) \le (2/3)\delta_{\mathrm{U}}\ \},$$

which is not empty since $t \to \chi(t)$ is continuous and $\chi(0) \le \delta_{\mathrm{U}}/2$. Denoting $\overline{a} = \sup \mathcal{E}$ we have $\chi(t) \le (2/3)\delta_{\mathrm{U}}$ over $[0, \overline{a}]$ so that the entropic estimates of Theorem 5.17 hold and we have

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx \le \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \le \delta_\Gamma, \qquad 0 \le t \le \overline{a}.$$

This now implies that $\chi(t) \le \delta_{\mathrm{U}}/2$ over $[0, \overline{a}]$ so that $\overline{a} = \bar{t}$. From the above a priori estimates, we also obtain that for $t \in [0, \bar{t}\,]$ we have $\|\mathrm{W}(t) - \mathrm{W}_\infty\|_{L^\infty} \le c_0'(\delta_\Gamma/\underline{c}_\Gamma)^{1/2}$, so that $\mathrm{V}(t) \in \mathcal{O}_0$, and $\|\mathrm{V}(t) - \mathrm{V}_\infty\|_{H^l} \le b - 1 < b$, in particular at $t = \bar{t}$. We may now use again the local existence theorem over $[\bar{t}, 2\bar{t}\,]$ and an easy induction shows that the solution is a global solution.

The asymptotic stability is then obtained upon introducing $\varPhi(t)=\|\partial_x \mathrm{w}(t,\cdot)\|^2_{H^{l-2}}$ and establishing that

$$\int_0^\infty |\varPhi(t)|\,dt+\int_0^\infty |\partial_t\varPhi(t)|\,dt\le C\int_{\mathbb{R}^n}\widetilde{\Gamma}_0^{[l]}\,dx.$$

This shows that $\lim_{t\to\infty}\|\partial_x \mathrm{w}(t,\cdot)\|_{H^{l-2}}=0$, and using the interpolation inequality

$$\|\phi\|_{C^{l-([n/2]+2)}}\le C_0\ \|\partial_x^{l-1}\phi\|^a_{L^2}\ \|\phi\|^{1-a}_{L^2},$$

where $1/a=2(l-1)$ we conclude that that $\lim_{t\to\infty}\|\mathrm{w}(t,\cdot)-\mathrm{w}_\infty\|_{C^{l-([n/2]+2)}}=0$, and next that $\lim_{t\to\infty}\|\mathrm{v}(t,\cdot)-\mathrm{v}_\infty\|_{C^{l-([n/2]+2)}}=0$. ■

Remark 6.8. It is also possible to obtain global existence by assuming that both $\chi(0)$ and $\int_{\mathbb{R}^n}\Gamma_0^{[k]}\,dx$ are small enough. ■

Remark 6.9. The asymptotic stability of constant equilibrium states in the w variable seems more natural than in the v variable since the Knudsen and Mach numbers are of the same order of magnitude. A complete analysis of the asymptotic expansions for small Mach and Knudsen numbers, however, is out of the scope of the present paper. ■

7. Conclusion

We have investigated higher order kinetic entropy estimators for incompressible fluid models in the natural situation where viscosity and thermal conductivity depend on temperature. We have establish that entropicity holds for such estimators provided that $\|\log T\|_{BMO}+\|v/\sqrt{T}\|_{L^\infty}$ is small enough. Domination of lower order convective terms has been obtained for the uniform scale of temperature weights when the temperature dependence of transport coefficients is that suggested by the kinetic theory. In this situation, a global existence theorem is established provided that the initial values $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small enough in appropriate spaces. Similar ideas can also be introduced for compressible fluid models as well as zero Mach number models *mutatis mutandis.*

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