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CENTRE DE MATHÉMATIQUES APPLIQUÉES

UMR CNRS 7641

91128 PALAISEAU CEDEX (FRANCE). Tél: 01 69 33 41 50. Fax: 01 69 33 30 11
http://www.cmap.polytechnique.fr/

HIGHER ORDER ENTROPIES FOR COMPRESSIBLE FLUID MODELS

Vincent Giovangigli

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VINCENT GIOVANGIGLI
CMAP, CNRS, Ecole Polytechnique, 91128 Palaiseau cedex, France
vincent.giovangigli@polytechnique.fr

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Abstract

We investigate higher order entropies for compressible fluid models and related a priori estimates. Higher order entropies are kinetic entropy estimators suggested by Enskog expansion of Boltzmann entropy. These quantities are quadratic in the density ρ, velocity v, and temperature T renormalized derivatives. We investigate governing equations of higher order entropy correctors and related differential inequalities in the natural situation where the volume viscosity, the shear viscosity, and the thermal conductivity depend on temperature, essentially in the form $T^{\varkappa}$. Entropic inequalities are established when $\|\log\rho\|_{BMO}$, $\|v/\sqrt{T}\|_{L^\infty}$, $\|\log T\|_{BMO}$, $\|h\partial_x\rho/\rho\|_{L^\infty}$, $\|h\partial_x v/\sqrt{T}\|_{L^\infty}$, $\|h\partial_x T/T\|_{L^\infty}$, and $\|h^2\partial_x^2 T/T\|_{L^\infty}$ are small enough, where $h = 1/\rho T^{\frac{1}{2}-\varkappa}$ is a weight associated with the dependence of the local mean free path on density and temperature. As an example of application, we investigate global existence of solutions when the initial values $\log(\rho_0/\rho_\infty)$, $v_0/\sqrt{T_0}$, and $\log(T_0/T_\infty)$ are small enough in appropriate spaces. *AMS Subject Classification* : 35Q30, 76N10, 82B40.

1 Introduction

The notion of entropy has been shown to be of fundamental importance in fluid modeling from both a physical and mathematical point of view[4, 5, 6, 9, 10, 11, 13, 20, 25, 27, 32, 40, 44]. We have introduced in previous work[14, 15, 16] a notion of kinetic entropy estimators for fluid models, suggested by Enskog expansion of Boltzmann kinetic entropy. Conditional higher order entropic inequalities have been established in the situation of incompressible flows spanning the whole space[14, 15, 16].

In this paper, we investigate higher order entropies for compressible fluid models and related a priori estimates. Higher order entropies are quadratic with respect to the density, velocity, and temperature renormalized derivatives. They are investigated in the situation of compressible flows spanning the whole space with temperature dependent thermal conductivity, shear viscosity and volume viscosity.

We first summarize the mathematical and physical motivations for higher order entropies. Higher order entropy correctors are first suggested by Enskog expansion of Boltzmann kinetic entropy. The corresponding balance equations may be seen as a generalization of Bernstein equations to systems of partial differential equations but expressed with renormalized variables. Higher order kinetic entropy estimators are then obtained upon summing a zeroth order fluid entropy with higher order entropy correctors. These kinetic entropy estimators may also be interpreted as kinetic Fisher information estimators[15].

We derive balance equations of higher order entropy correctors for compressible fluid models with temperature dependent viscosities and thermal conductivity. These transport coefficients essentially behave—away from small temperatures—like a power of temperature $T^{\varkappa}$ with a common exponent $\varkappa$, as given by the kinetic theory of gases. The hyperbolic-parabolic nature of the system of partial differential equations governing compressible fluids further imposes to consider extra correctors associated with density which is a hyperbolic variable.

We establish weighted inequalities in Sobolev and Lebesgue spaces. These inequalities are required in order to establish a priori estimates since we are using renormalized variables with powers of temperature and density as weights and since we also consider flows with temperature dependent thermal conductivity and viscosities. These inequalities assume that a weighted L^∞ norm of the gradients is finite in addition to the L^∞ or BMO norm of the functions. They differ from previous inequalities established for incompressible flows[15] where only the L^∞ or BMO norm of the functions were assumed to be finite. A weighted L^∞ norm of the gradients is required in order to decrease the number of derivation of hyperbolic variables in a priori estimates.

We next investigate entropic estimates by combining higher order entropy correctors balance equations with weighted inequalities. We obtain differential inequalities for higher order entropy correctors when the quantities $\|\log\rho\|_{BMO}$, $\|v/\sqrt{T}\|_{L^\infty}$, $\|\log T\|_{BMO}$, $\|h\partial_x\rho/\rho\|_{L^\infty}$, $\|h\partial_x v/\sqrt{T}\|_{L^\infty}$, $\|h\partial_x T/T\|_{L^\infty}$, and $\|h^2\partial_x^2 T/T\|_{L^\infty}$, are small enough, where ρ denotes the density, v the velocity vector, T the absolute temperature, and $h = 1/T^{\frac{1}{2}-\varkappa}\rho$ is a weight associated with the dependence of the local mean free path on density and temperature. As a consequence, we establish that higher order kinetic entropy estimators—obtained by summing up a zeroth order entropy with kinetic entropy correctors—obey conditional entropic principles. These inequalities for kinetic entropic estimators are the main result of the paper.

As an example of application of higher order entropic estimates we establish a global existence theorem around constant equilibrium states provided that $\log(\rho_0/\rho_\infty)$, $\log(T_0/T_\infty)$, and $v_0/\sqrt{T_0}$ are small enough in appropriate spaces, which may be interpreted heuristically as an existence theorem for small Mach number flows.

In Section 2 we discuss the concept of higher order entropies. In Section 3 we derive higher order entropies governing equations and in Section 4 we establish various weighted inequalities. In Section 5 we establish that higher order entropies satisfy conditional entropic inequalities. Finally, in Section 6, as an example of application, we concentrate on global solutions.

2 Higher order entropies

In this section we briefly motivate the introduction of higher order entropies by discussing Bernstein equations and Enskog expansion of kinetic entropy[14, 15]

2.1 A thermodynamic interpretation of Bernstein equations

For parabolic—or elliptic—scalar equations, a priori estimates for derivatives can be obtained by using Bernstein method[2, 30]. More specifically, consider—as a simple exemple—the heat equation

$$\partial_t u - \Delta u = 0.$$

Defining $\zeta^{[k]} = |\partial^k u|^2 = \sum_{1\le i_1,\cdots,i_k\le n}(\partial_{i_1}\cdots\partial_{i_k}u)^2$, Bernstein equation for the $k^{\rm th}$ derivatives can be written in the form

$$\partial_t\zeta^{[k]} - \Delta\zeta^{[k]} + 2|\partial^{k+1}u|^2 = 0, \tag{2.1}$$

and more generally, for equations with variables coefficients, Bernstein equations are associated with sums of squares of derivatives[30]. With Bernstein method, the higher order derivatives source term $|\partial^{k+1}u|^2$ is discarded, Equation (2.1) then yields $\partial_t\zeta^{[k]} - \Delta\zeta^{[k]} \le 0$, and the maximum principle can be used[2, 30]. However, one may also directly integrate Bernstein equations to get estimates of the integrals $\int_{\mathbb{R}^n}\zeta^{[k]}\,dx$, and this method is still valid if the flux term $\partial_x\cdot(\partial_x\zeta^{[k]})$ is simply a term in divergence form $\partial_x\cdot\varphi^{[k]}$ as may be expected for balance equations associated with squares of derivatives of solutions of a system of partial differential equations. We may therefore try to derive equations similar to that of Bernstein for systems of partial differential equations, with nonnegative source terms. In this perspective, the structure of (2.1) appears to be formally similar to that of an entropy balance, where $\zeta^{[k]}$, $k\ge 1$, play the rôle of generalized entropies, even though there also exist zeroth order entropies like u^2. In the next section, we introduce a kinetic framework supporting this entropic interpretation.

2.2 Enskog expansion of Boltzmann kinetic entropy

In a semi-quantum framework, the state of a polyatomic gas is described by a particle distribution function $f(t,x,c,\mathrm{I})$—governed by Boltzmann equation—where t denotes time, x the n-dimensional cartesian coordinate, c the particle velocity, I the index of the particle quantum state, and $\mathcal{I}$ is the corresponding indexing set[5, 9, 11, 13]. Approximate solutions of Boltzmann's equation can be obtained from a first order Enskog expansion $f = f^{(0)}\big(1+\varepsilon\phi^{(1)}+\mathcal{O}(\varepsilon^2)\big)$ where $f^{(0)}$ is the local Maxwellian distribution, $\phi^{(1)}$ the perturbation associated with the Navier-Stokes regime and ε the usual Enskog formal expansion parameter. The compressible Navier-Stokes equations for polyatomic gases can then be obtained upon taking moments of Boltzmann's equation[6, 11, 13].

The kinetic entropy $S^{\rm kin} = -k_{\rm B}\sum_{{\rm I}\in\mathcal{I}}\int_{\mathbb{R}^n} f\big(\log f - 1\big)dc$, where $k_{\rm B}$ denotes Boltzmann constant, satisfies the H theorem, that is, the second principle of thermodynamics. Enskog expansion $f/f^{(0)} = 1 + \varepsilon\phi^{(1)} + \cdots + \epsilon^{2k}\phi^{(2k)} + \mathcal{O}(\varepsilon^{2k+1})$ then induces expansions for $S^{\rm kin}$ in the form

$$S^{\rm kin} - S^{(0)} = \varepsilon^2 S^{(2)} + \varepsilon^3 S^{(3)} + \cdots + \varepsilon^{2k} S^{(2k)} + \mathcal{O}(\varepsilon^{2k+1}), \tag{2.2}$$

where $S^{(0)}$ is the usual zeroth order fluid entropy evaluated from the Maxwellian distribution $f^{(0)}$ and where $S^{(l)}$ is a sum of terms in the form $k_{\rm B}\sum_{{\rm I}\in\mathcal{I}}\int_{\mathbb{R}^n}\prod_{1\le i\le l}\big(\phi^{(i)}\big)^{\nu_i} f^{(0)}\,dc$ with nonnegative integers $\nu_i \geq 0$, $1 \leq i \leq l$, such that $l = \sum_{1\le i\le l} i\nu_i$. For compressible polyatomic gases after detailed calculations, one can establish that

$$-\rho S^{(2)} = \overline{\lambda}|\partial_x T|^2 + \overline{\kappa}(\partial_x\!\cdot v)^2 + \tfrac{1}{2}\overline{\eta}\,|d|^2, \tag{2.3}$$

where T denotes the absolute temperature, ρ the density, v the gas velocity, $d = \partial_x v + \partial_x v^t - \frac{2}{n}(\partial_x\!\cdot v)\,I$ the nonisotropic part of the strain rate tensor, $|d|^2$ the sum $|d|^2 = \sum_{ij} d_{ij}^2$, and where the scalar coefficients $\overline{\lambda}$, $\overline{\kappa}$, and $\overline{\eta}$ only depend on temperature. In a first approximation, using a single term in orthogonal polynomial expansions of perturbed distribution functions, one can establish that $\overline{\lambda} = (1/2r_{\rm g}c_p)\lambda^2/T^3$, $\overline{\kappa} = (3c_v/4r_{\rm g}c_{int})\kappa^2/T^2$, and $\overline{\eta} = (1/2r_{\rm g})\eta^2/T^2$ where c_p is the constant pressure specific heat per unit mass, c_v the constant volume specific heat per unit mass, $r_{\rm g}$ the gas constant per unit mass, c_{int} the internal specific heat per unit mass, λ the thermal conductivity, η the shear viscosity, κ the volume viscosity, and the actual values of the numerical factors in front of $\overline{\lambda}$, $\overline{\kappa}$, and $\overline{\eta}$ are evaluated here for $n = 3$.

More generally, from the general expression of $\phi^{(l)}$ in the absence of external forces acting on the particles[15], one can establish that

$$S^{(2k)} = \rho r_{\rm g}\,\Big(\frac{\eta}{\rho\sqrt{r_{\rm g}T}}\Big)^{2k}\sum_{\nu} c_\nu \prod_{1\le|\alpha|\le 2k}\Big(\frac{\partial_x^\alpha\rho}{\rho}\Big)^{\nu_\alpha}\,\Big(\frac{\partial_x^\alpha v}{\sqrt{r_{\rm g}T}}\Big)^{\nu'_\alpha}\,\Big(\frac{\partial_x^\alpha T}{T}\Big)^{\nu''_\alpha}, \tag{2.4}$$

where $\nu_\alpha, \nu'_\alpha, \nu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, and $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\le|\alpha|\le 2k}$ must be such that $\sum_{1\le|\alpha|\le 2k}|\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = 2k$ and where the coefficients c_ν are smooth scalar functions of $\log T$ of order unity. After integrations by parts in the integral $\int_{\mathbb{R}^n} S^{(2k)}\,dx$, in order to eliminate spatial derivatives of order strictly greater than k, and by using interpolation inequalities, one obtains that $|\int_{\mathbb{R}^n} S^{(2k)}\,dx|$ is essentially controled by the integral of

$$\gamma^{[k]} = \rho r_{\rm g}\,\Big(\frac{\eta}{\rho\sqrt{r_{\rm g}T}}\Big)^{2k}\,\Big(\Big|\frac{\partial_x^k\rho}{\rho}\Big|^2 + \Big|\frac{\partial_x^k v}{\sqrt{r_{\rm g}T}}\Big|^2 + \frac{c_v}{r_{\rm g}}\Big|\frac{\partial_x^k T}{T}\Big|^2\Big), \tag{2.5}$$

or equivalently of

$$\widetilde{\gamma}^{[k]} = \rho r_{\rm g}\,\Big(\frac{\eta}{\rho\sqrt{r_{\rm g}T}}\Big)^{2k}\,\big(|\partial_x^k\log\rho|^2 + |\partial_x^k(v/\sqrt{r_{\rm g}T})|^2 + \frac{c_v}{r_{\rm g}}|\partial_x^k\log T|^2\big), \tag{2.6}$$

and $|\int_{\mathbb{R}^n} S^{(2k-1)}\,dx|$ is also controled by $\int_{\mathbb{R}^n}\gamma^{[k]}\,dx$ and $\int_{\mathbb{R}^n}\gamma^{[k-1]}\,dx$. This suggests quantities in the form $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as $(2k)^{\rm th}$ order kinetic entropy correctors—or kinetic entropy deviation estimators[15]. Note that, at variance with $S^{(2)}$, it is not clear that $S^{(2k)}$ has a sign, and this is a motivation for using quantities like $\gamma^{[k]}$ and $\widetilde{\gamma}^{[k]}$ rather than $S^{(2k)}$, beyond simplicity. We are therefore looking for *majorizing entropic correctors* that we are free to modify for convenience, e.g., by multiplying the temperature derivatives by the factor $c_v/r_{\rm g}$. These correctors may also be rescaled by mutiplicative constants depending on k and their temperature dependence may be simplified in accordance with that of transport coefficients. Finally, a similar analysis can also be conducted for the Fisher information and suggests the same quantities $\gamma^{[k]}$ or $\widetilde{\gamma}^{[k]}$ as higher order kinetic information correctors.

2.3 Persistence of kinetic entropy

Denoting by $\gamma^{[0]}$ a nonnegative quantity associated with the zeroth order entropy $S^{(0)}$, we investigate kinetic entropy estimators in the form $\gamma^{[0]} + \cdots + \gamma^{[k]}$, with $0 \le k \le l$, for the solutions of a second order system of partial differential equations modeling compressible fluids. For this system, the zeroth order entropy $S^{(0)}$ is already of fundamental importance as imposed by its hyperbolic-parabolic structure and the corresponding symmetrizing properties[13, 19, 25, 27]. Therefore, we only consider the quantities $\gamma^{[0]} + \cdots + \gamma^{[k]}$, $0 \leq k \leq l$, as a family of mathematical entropy estimators—of kinetic origin—and we will establish that they indeed satisfy conditional entropic inequalities for solutions of compressible fluid equations. This will yield incidentally a thermodynamic interpretation of the corresponding weighted Sobolev norms.

This point of view differs from that of thermodynamic theories that have already considered entropies differing from that of zeroth order, that is, entropies depending on transport fluxes or on macroscopic variable gradients. These generalized entropies have been associated notably with Burnett type equations[6, 11] or extended thermodynamics[38]. In both situations, new macroscopic equations are correspondingly obtained, that is, 'extended fluid models', which are systems of partial differential equations of higher orders than Navier-Stokes type equations.

3 Higher order entropies governing equations

We first present the equations governing compressible fluids and then discuss the temperature dependence of transport coefficients as obtained from the kinetic theory of gases. We then derive governing equations for kinetic entropy correctors of arbitrary order.

3.1 Fluid governing equations

The conservation equations governing compressible fluids can be written[13, 32]

$$\partial_t \rho + \partial_x\cdot(\rho v) = 0, \tag{3.1}$$

$$\partial_t(\rho v) + \partial_x\cdot(\rho v\otimes v + pI) + \partial_x\cdot \mathit{\Pi} = 0, \tag{3.2}$$

$$\partial_t(\rho e) + \partial_x\cdot(\rho e v) + \partial_x\cdot Q = -\mathit{\Pi} : \partial_x v - p\,\partial_x\cdot v, \tag{3.3}$$

where t denotes time, x the n dimensional cartesian coordinate, ρ the density, v the velocity, p the pressure, I the unit tensor, $\mathit{\Pi}$ the viscous tensor, e the internal energy per unit mass, and Q the heat flux. In these equation, ∂_t denotes partial derivation with respect to time, $\partial_x = (\partial_1, \ldots, \partial_n)^t$ the usual spatial differential operator, and ^{t} the transposition operator. We assume for the sake of notational simplicity that these governing equations are in reduced form in such a way that the specific gas constant $r_{\rm g}$ is taken to be unity. The pressure is given by the state law $p = \rho T$ where T is the temperature and the energy per unit mass e is taken for simplicity in the form $e = c_v T$ where c_v is a constant.

The viscous tensor and the heat flux can be obtained from the kinetic theory of gases and written in the form

$$\mathit{\Pi} = -\kappa(T)\,\partial_x\cdot v\, I - \eta(T)\big(\partial_x v + \partial_x v^t - \tfrac{2}{n}\partial_x\cdot v\, I\big), \tag{3.4}$$

$$Q = -\lambda(T)\,\partial_x T, \tag{3.5}$$

where $\kappa(T)$ denotes the volume viscosity, $\eta(T)$ the shear viscosity, and $\lambda(T)$ the thermal conductivity. We will denote by $d = \partial_x v + \partial_x v^t - \frac{2}{n}\partial_x\cdot v\, I$ the non isotropic part of the strain rate tensor so that $\mathit{\Pi} = -\kappa\,\partial_x\cdot v\, I - \eta d$. The assumptions on the transport coefficients κ, η, and λ—which are smooth functions of temperature—are specified in Section 3.2.

Our aim is not to study various boundary conditions and we only consider the case of functions defined on $\mathbb{R}^n$ that are 'constant at infinity'. From Galilean invariance, we can also choose that v vanishes at infinity. Therefore we only consider smooth solutions such that

$$\rho - \rho_\infty \in C\big([0,\bar t], W^{l,2}\big) \cap C^1\big([0,\bar t], W^{l-1,2}\big), \tag{3.6}$$

$$v, T - T_\infty \in C\big([0,\bar t], W^{l,2}\big) \cap C^1\big([0,\bar t], W^{l-2,2}\big) \cap L^2\big((0,\bar t), W^{l+1,2}\big), \tag{3.7}$$

where l is an integer such that $l \geq [n/2] + 3$, that is, $l > n/2 + 2$, $\bar t$ is some positive time, $\rho_\infty > 0$ a fixed positive density and $T_\infty > 0$ a fixed positive temperature. We also assume that ρ and T are such that $\rho \geq \rho_{\min}$ and $T \geq T_{\min}$ where $\rho_{\min} > 0$ and $T_{\min} > 0$ are fixed positive constants. Such smooth solutions are known to exist[13, 24, 25, 26, 27, 28, 33, 39, 43] either locally in time or globally when the initial state is close to the constant state $(\rho_\infty, 0, T_\infty)$. We use clasical notation for functional spaces[1, 45] as for instance $W^{k,p} = W^{k,p}(\mathbb{R}^n) = W_0^{k,p}(\mathbb{R}^n)$ for the usual Sobolev space with $k \geq 0$ and $1 \leq p < \infty$, and $W^{-k,p'}$ for its dual where $p' = p/(p-1)$.

Remark 3.1 *In the special case where* $\lambda = \mathfrak{a}_\lambda T^\varkappa$, $\eta = \mathfrak{a}_\eta T^\varkappa$, $\kappa = \mathfrak{a}_\kappa T^\varkappa$, *and* c_v *is constant, if* $\big(\rho(t,x), v(t,x), T(t,x)\big)$ *is a solution of the Navier-Stokes equations* (3.1)–(3.3)*, then*

$$\Big(\xi^{2\varkappa-1}\,\zeta\;\rho(\xi\zeta t,\zeta x),\quad \xi\;v(\xi\zeta\,t,\zeta\,x),\quad \xi^2\;T(\xi\zeta\,t,\zeta\,x)\Big), \tag{3.8}$$

is also a solution for any positive ξ *and* ζ*. For arbitrary transport coefficients, the one parameter family obtained by letting* $\xi = 1$ *is still a family of solutions. The scaling properties of the incompressible case[15] can also be recovered from* (3.8) *by letting* $\zeta = \xi^{1-2\varkappa}$*.*

Remark 3.2 *All the results obtained in this paper are also valid if the internal energy e per unit mass is taken to be* $e = e_0 + \int_0^T c_v(s)\,ds$ *with a heat capacity coefficient* c_v *depending on temperature in such a way that*

$$\underline{c}\ \le c_v \le \overline{c}, \qquad T^\sigma\,|\partial_T^\sigma c_v| \le \overline{c}_\sigma, \qquad \sigma \ge 1,$$

where $\underline{c} > 0$, $\overline{c} > 0$, *and* $\overline{c}_\sigma > 0$, $\sigma \ge 1$, *are positive constants. We will not explicit the corresponding results for the sake of simplicity.*

Remark 3.3 *The dimension n appearing in the coefficient* $2/n$ *of the viscous tensor* (3.4) *is normally the full spatial dimension, that is, the dimension* n' *of the velocity phase space of the associated kinetic model. We may still assume that the spatial dimension of the model has been reduced, that is, the equations are considered in* $\mathbb{R}^n$ *with* $n < n'$*. The full size viscous tensor* Π' *is then a matrix of order* n'*, and the corresponding coefficient is* $2/n'$*. However, if we denote by* Π *the upper left block of size n of* Π'*, that is, the useful part of* Π'*, we may rewrite* Π *in the form*

$$\Pi = -\big(\kappa + (\tfrac{2}{n} - \tfrac{2}{n'})\eta\big)\,\partial_x\!\cdot v\,I - \eta\big(\partial_x v + \partial_x v^t - \tfrac{2}{n}\partial_x\!\cdot v\,I\big), \tag{3.9}$$

where I is the unit tensor in n dimensions. Therefore, using a smaller dimension n instead of the full dimension n' *in the coefficient of the viscous tensor is equivalent to increasing the volume viscosity by the amount* $2\eta\,(n'-n)/nn'$*.*

3.2 Temperature dependent transport coefficients

Thermal conductivity, shear viscosity, and volume viscosity of a polyatomic gas depend on temperature

$$\lambda = \lambda(T), \qquad \eta = \eta(T), \qquad \kappa = \kappa(T), \tag{3.10}$$

as shown by the kinetic theory of gases[6, 11, 13]. When one term Sonine-Wang-Chang-Uhlenbeck polynomial expansions are used to evaluate perturbed distribution functions, the coefficients λ/c_v, η, and κ are found in the form $\lambda/c_v = \mathfrak{a}_\lambda T^{1/2}/\Omega^{(2,2)*}$, $\eta = \mathfrak{a}_\eta T^{1/2}/\Omega^{(2,2)*}$, and $\kappa/\eta = \mathfrak{a}_\kappa c^{\rm int}\xi^{\rm int}/c_v^2$ where $\mathfrak{a}_\lambda$, $\mathfrak{a}_\eta$, and $\mathfrak{a}_\kappa$ are constants, $\Omega^{(2,2)*}$ is a reduced collision integral, $c^{\rm int}$ the internal heat capacity per unit mass, and $\xi^{\rm int}$ a collision number associated with internal energy relaxation. Note in particular that the ratios $\lambda/c_v\eta$ and κ/η are bounded. For the *rough rigid sphere* model for instance, we have exactly[6, 11] $\lambda/c_v = \mathfrak{a}_\lambda T^{1/2}$, $\eta = \mathfrak{a}_\eta T^{1/2}$, and $\kappa = \mathfrak{a}_\kappa T^{1/2}$. Similarly, for particles interacting as point centers of repulsion with an interaction potential $V = c/\mathrm{r}^\nu$, where r is the distance between two particles, one establishes[6, 11] that $\Omega^{(2,2)*}$ is proportional to $T^{-2/\nu}$ so that we have $\lambda/c_v = \mathfrak{a}_\lambda T^\varkappa$, and $\eta = \mathfrak{a}_\eta T^\varkappa$ with $\varkappa = 1/2 + 2/\nu$, and κ inherits the same scaling $\kappa = \mathfrak{a}_\kappa T^\varkappa$ if we assume that $c^{\rm int}$, $\xi^{\rm int}$, and c_v are constants. The temperature exponent $\varkappa$ then varies from $\varkappa = 1/2$ for rigid spheres with $\nu = \infty$ up to $\varkappa = 1$ for Maxwell molecules with $\nu = 4$.

More generally, consider particles interacting with a Lennard-Jones ν-ν' potential $V = 4\varepsilon\big((\sigma/\mathrm{r})^\nu - (\sigma/\mathrm{r})^{\nu'}\big)$ where σ denotes the collision diameter, ε the potential well depth, and ν, ν' are intergers with $\nu > \nu'$ and typical values $\nu = 12$, $\nu' = 6$[6, 11]. Collision integrals like $\Omega^{(2,2)*}$ then only depend on the reduced temperature $k_{\rm B}T/\varepsilon$, and, when $k_{\rm B}T/\varepsilon$ is large, the repulsive part $\mathrm{r}^{-\nu}$ is dominant[6] so that collision integrals behave like T^s with $s = 1/2 + 2/\nu$ for large T. In particular, the logarithm $\log\Omega^{(2,2)*}$ has linear asymptotes as function of $\log T$, and $d^k\log\Omega^{(2,2)*}/d(\log T)^k$ is bounded for any $k \ge 1$. In addition, classical models indicate that $c^{\rm int}$, $\xi^{\rm int}$, and c_v converge towards constants for large temperatures[13]. As a consequence, $\log\lambda$, $\log\eta$, and $\log\kappa$ have parallel linear asymptotes as function of $\log T$, and $d^k\log\lambda/d(\log T)^k$, $d^k\log\eta/d(\log T)^k$, and $d^k\log\kappa/d(\log T)^k$ are bounded for any $k \ge 1$, or equivalently, $(1/\lambda)T^k d^k\lambda/dT^k$, $(1/\eta)T^k d^k\eta/dT^k$, and $(1/\kappa)T^k d^k\kappa/dT^k$ are bounded for any $k \ge 1$.

Similar results are also obtained when more than one term are taken into account in orthogonal polynomial expansions of perturbed distribution functions. Indeed, all collision integrals $\Omega^{(i,j)*}$, $i, j \ge 1$, have a common temperature behavior, that is, all ratios of collision integrals are bounded, as for instance for Lennard-Jones or Stockmayer potentials[11, 13]. These collision integrals are then used to define the coefficients of the transport linear systems which thus share a common temperature scaling. As a consequence, the transport coefficients, which are obtained through solutions of transport linear systems, inherit a common temperature scaling[13].

On the other hand, in our particular application, we are only interested in solutions such that $T \geq T_{\min}$, where $T_{\min}$ is fixed and positive. In this situation, the behavior of transport coefficients for small temperatures is not relevant and only the repulsive part of the interaction potential between particles plays a role. Therefore, from a mathematical point of view, since we are not interested in small temperatures, we assume that λ, η, and κ are $C^\infty(0,\infty)$, that there exist $\varkappa$, $\underline{\mathfrak{a}} > 0$, and $\overline{\mathfrak{a}} > 0$ with

$$\underline{\mathfrak{a}} T^\varkappa \leq \lambda/c_v \leq \overline{\mathfrak{a}} T^\varkappa, \qquad \underline{\mathfrak{a}} T^\varkappa \leq \eta \leq \overline{\mathfrak{a}} T^\varkappa, \qquad \underline{\mathfrak{a}} T^\varkappa \leq \kappa \leq \overline{\mathfrak{a}} T^\varkappa, \tag{3.11}$$

and that, for any integer $\sigma \geq 1$, there exists $\overline{\mathfrak{a}}_\sigma > 0$ with

$$T^\sigma \big(|\partial_T^\sigma \lambda| + |\partial_T^\sigma \eta| + |\partial_T^\sigma \kappa| \big) \leq \overline{\mathfrak{a}}_\sigma \, T^\varkappa. \tag{3.12}$$

Kinetic theory suggests that $1/2 \leq \varkappa \leq 1$ but the situations where $0 \leq \varkappa < 1/2$ or $\varkappa > 1$ are still interesting to investigate from a mathematical point of view.

Remark 3.4 *Theoretical calculations and experimental measurements have shown that the viscosity ratio κ/η is of order unity for polyatomic gases[3, 6, 11]. Using a one or two terms expansion in Sonine-Wang-Chang-Uhlenbeck polynomials for the perturbed distribution associated with volume viscosity, it is established for instance that $\kappa/\eta = \frac{\pi}{4} r_{\mathrm{g}} c^{\mathrm{int}} \xi^{\mathrm{int}} / c_v^2$ for a polyatomic gas. The collision number ξ^{int} associated with internal energy relaxation is usually taken to be a simple decreasing function of temperature and the internal heat capacity per unit mass c^{int} is associated with the various internal energy modes like rotation, vibration or electronic. In particular, the internal heat capacity is such that $c^{\mathrm{int}} \geq r_{\mathrm{g}}$ for linear molecules and $c^{\mathrm{int}} \geq \frac{3}{2} r_{\mathrm{g}}$ for nonlinear molecules solely from rotational degrees of freedom. Volume viscosity also arise in dense gases and in liquids so that its absence in monatomic dilute gases is an exception rather than a rule[3, 11].*

3.3 Higher order kinetic entropy estimators

Following the physical *ansatz* (2.5) and taking into account the simplifications associated with the temperature dependence of transport coeficients (3.11) and with a specific gas constant taken to be unity, we define the $(2k)^{\mathrm{th}}$ order kinetic entropy corrector $\gamma^{[k]}$ by

$$\gamma^{[k]} = \rho h^{2k} \Big(\frac{|\partial^k \rho|^2}{\rho^2} + \frac{|\partial^k v|^2}{T} + c_v \frac{|\partial^k T|^2}{T^2} \Big), \tag{3.13}$$

where $h = 1/T^{\frac{1}{2}-\varkappa}\rho$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is multiindex, we denote as usual by ∂^α the differential operator $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and by $|\alpha|$ its order $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and the square of k^{th} derivatives of a scalar function ϕ, like T, ρ, or v_i, $1 \leq i \leq n$, is defined by

$$|\partial^k \phi|^2 = \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\partial^\alpha \phi)^2 = \sum_{1 \leq i_1, \ldots, i_k \leq n} (\partial_{i_1} \cdots \partial_{i_k} \phi)^2, \tag{3.14}$$

where $k!/\alpha!$ are the multinomial coefficients[8, 41]. Similarly, for a vector function like v we define $|\partial^k v|^2 = \sum_{1 \leq i \leq n} |\partial^k v_i|^2$.

This choice of $\gamma^{[k]}$ yields more convenient higher order entropic estimates. Calculations show that it eliminates various quadratic terms associated with hyperbolic variables thanks to symmetry properties. This choice can also be associated with symmetrized forms of the system of partial differential equations. Denoting $\mathrm{U} = \big(\rho, \rho v, \rho(e + \frac{1}{2}|v|^2)\big)^t$ the conservative variable, $\mathrm{V} = -(\partial_{\mathrm{U}} S^{(0)})^t$ the entropic variable, $\mathrm{Z} = \big(\rho, v, T\big)^t$ the natural variable, which is also a normal variable[19, 27], and defining the matrix $\overline{A}_0 = (\partial_{\mathrm{Z}} \mathrm{V})^t \partial_{\mathrm{U}} \mathrm{V} (\partial_{\mathrm{Z}} \mathrm{V})$ associated with normal forms of the system of partial differenial equations[19, 27], one can rewrite the higher order entropy correctors in the form $\gamma^{[k]} = h^{2k} < \partial^k \mathrm{Z}, \overline{A}_0 \partial^k \mathrm{Z} >$, where h is the weight associated with the dependence of the local mean free path $l = \eta/\rho\sqrt{r_{\mathrm{g}} T}$ on density and temperature. This choice of $\gamma^{[k]}$ can also be associated with a 'spatial gradient' Fisher information with for instance $\gamma^{[1]} = h^2 \sum_{\mathrm{I} \in \mathcal{I}} k_{\mathrm{B}} \int_{\mathbb{R}^n} |\partial_x \log f^{(0)}|^2 f^{(0)} \, dc$, where $f^{(0)}$ is the local Mawellian distribution discussed in Section 2.2.

Remark 3.5 *We define similarly the* p^{th} *power of derivatives* $|\partial^k\phi|^p$ *by*

$$|\partial^k\phi|^p = \sum_{|\alpha|=k} \frac{k!}{\alpha!}\big(\partial^\alpha\phi\big)^p = \sum_{1\leq i_1,\ldots,i_k\leq n} (\partial_{i_1}\cdots\partial_{i_k}\phi)^p, \tag{3.15}$$

and these definitions (3.14)(3.15) *are compatible with the classical definition already used in Section 2.1 when* $p = 2$. *These natural definitions also simplify the analytic form of higher order entropies governing equations. In agreement with* (3.14) *we also set for future use*

$$\partial^k\phi\,\partial^k\psi = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\partial^\alpha\phi\,\partial^\alpha\psi, \qquad \partial^k v\cdot\partial^k\partial_x\rho = \sum_{\substack{|\alpha|=k\\ 1\leq i\leq n}}\frac{k!}{\alpha!}\partial^\alpha v_i\,\partial^\alpha\partial_i\rho. \tag{3.16}$$

In order to recast the zeroth order entropy balance equation into a more convenient form we introduce a modified zeroth order entropy $\gamma^{[0]}$. The mathematical fluid entropy $-S^{(0)}$ can be shown to be a strictly convex function of the conservative variables[13, 27] $\mathrm{U} = \big(\rho, \rho v, \rho(e+\frac{1}{2}v\cdot v)\big)^t$. Denoting by $\mathrm{E}^{\mathrm{tot}} = \rho(e+\frac{1}{2}v\cdot v)$ the total energy per unit volume, we define $\gamma^{[0]} = \mathsf{C}_0\psi^{[0]}$ where $\psi^{[0]}$ is the modified zeroth order entropy

$$\psi^{[0]} = -S^{(0)} + S^{(0)}_\infty + (\partial_\rho S^{(0)})_\infty(\rho-\rho_\infty) + (\partial_{\mathrm{E}^{\mathrm{tot}}} S^{(0)})_\infty(\mathrm{E}^{\mathrm{tot}} - \mathrm{E}^{\mathrm{tot}}_\infty),$$

and C_0 is a positive constant that will be taken large enough. The zeroth order term $\gamma^{[0]}$ is easily rewritten in the form

$$\gamma^{[0]}/\mathsf{C}_0 = \Big(\rho\log\big(\frac{\rho}{\rho_\infty}\big) - \big(\rho-\rho_\infty\big)\Big) + \tfrac{1}{2}\rho\frac{v^2}{T_\infty} + \rho c_v\Big(\frac{T-T_\infty}{T_\infty} - \log\big(\frac{T}{T_\infty}\big)\Big). \tag{3.17}$$

Thanks to the fact that v and T are parabolic variables, we can expect source terms in the form $|\partial^{k+1}T/T|^2$ and $|\partial^{k+1}v/\sqrt{T}|^2$ to appear in the governing equation for $\gamma^{[k]}$—up to weight factors. However, since ρ is a hyperbolic variable, there will be no such corresponding source term $|\partial^{k+1}\rho/\rho|^2$ for density. A priori estimates for density derivatives and more generally of hyperbolic variables derivatives indeed require to introduce extra entropic corrector terms. These extra corrector terms will yield source terms in the form $|\partial^k\rho/\rho|^2$. These terms are similar to the perturbed quadratic terms introduced by Kawashima[25] in order to obtain hyperbolic variable derivatives estimates for linearized equations around equilibrium states and decay estimates[25]. They are used here with renormalized variables, as well as with powers of h as extra weights factors, in order to obtain higher order entropic principles. More specifically, we define the quantity $\gamma^{[k-\frac{1}{2}]}$ by

$$\gamma^{[k-\frac{1}{2}]} = \rho h^{2k-1}\frac{\partial^{k-1}v}{\sqrt{T}}\cdot\frac{\partial^{k-1}\partial_x\rho}{\rho}, \tag{3.18}$$

and we will see that in the $\gamma^{[k-\frac{1}{2}]}$ governing equation there is a source term in the form $|\partial^k\rho/\rho|^2$—up to weight factors. From a physical point of view, we also note that $\gamma^{[k-\frac{1}{2}]}$ is of the general form (2.4) for $S^{(2k-1)}$. Finally, we define the $(2k)^{\mathrm{th}}$ order kinetic entropy estimator by

$$\Gamma^{[k]} = \gamma^{[0]} + \sum_{1\leq i\leq k}(\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]}), \qquad k\geq 0, \tag{3.19}$$

where a is a parameter that will be chosen small enough. The quantities $\gamma^{[i-\frac{1}{2}]}$, $1\leq i\leq k$, are multiplied by the small rescaling factor a in (3.19) so as to not modify the majorizing properties of the correctors $\gamma^{[k]}$, $k\geq 0$.

Similarly, following the physical *antsatz* (2.6), we define the modified $(2k)^{\mathrm{th}}$ order kinetic entropy corrector $\tilde{\gamma}^{[k]}$ by

$$\tilde{\gamma}^{[k]} = \rho h^{2k}\Big(|\partial^k r|^2 + |\partial^k w|^2 + c_v|\partial^k\tau|^2\Big), \tag{3.20}$$

where $r = \log\rho$, $w = v/\sqrt{T}$, and $\tau = \log T$. We correspondingly define

$$\tilde{\gamma}^{[k-\frac{1}{2}]} = \rho h^{2k-1}\partial^{k-1}w\cdot\partial^{k-1}\partial_x r, \tag{3.21}$$

$\tilde{\gamma}^{[0]} = \gamma^{[0]}$, and introduce the modified $(2k)^{\mathrm{th}}$ order kinetic entropy estimators

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \sum_{1\leq i\leq k}(\tilde{\gamma}^{[i]} + a\tilde{\gamma}^{[i-\frac{1}{2}]}), \qquad k\geq 0. \tag{3.22}$$

The entropy correctors $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$, as well as the estimators $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$, will be shown to have similar properties and both may be used to derive a priori estimates. Strictly speaking, we should term $\gamma^{[k]}$ and $\tilde{\gamma}^{[k]}$ "$(2k)^{\rm th}$ order kinetic entropy correctors" or "$(2k)^{\rm th}$ order kinetic entropy deviation estimators", and $\gamma^{[k-\frac{1}{2}]}$ and $\tilde{\gamma}^{[k-\frac{1}{2}]}$ "$(2k-1)^{\rm th}$ order kinetic entropy correctors", and $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "mathematical $(2k)^{\rm th}$ order entropies", or "$(2k)^{\rm th}$ order kinetic entropy estimators". However, we will often informally term $\gamma^{[k]}$, $\tilde{\gamma}^{[k]}$, $\gamma^{[k-\frac{1}{2}]}$, $\tilde{\gamma}^{[k-\frac{1}{2}]}$, $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ "higher order entropies".

Remark 3.6 *Entropic correctors can also be defined by using the derivatives of the strain rate tensor $\partial^{k-1}d$ instead of that of velocity $\partial^k v$. We have chosen to work with the derivatives of velocity $\partial^k v$ for the sake of simplicity. It is also possible to define extra entropic correctors in the form $\rho h^{2k-1}\partial^{k-1}(\partial_x\cdot v)\partial^{k-1}\rho/\sqrt{T}\rho$ and $\rho h^{2k-1}\partial^{k-1}(\partial_x\cdot w)\partial^{k-1}r$ but their properties are similar to that of $\gamma^{[k-\frac{1}{2}]}$ and $\tilde{\gamma}^{[k-\frac{1}{2}]}$. Entropic estimators can also be defined in the form*

$$\Gamma^{[k]} = \gamma^{[0]} + \sum_{1\leq i\leq k} \theta^i\,(\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]}), \qquad k\geq 0, \tag{3.23}$$

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \sum_{1\leq i\leq k} \theta^i\,(\tilde{\gamma}^{[i]} + a\tilde{\gamma}^{[i-\frac{1}{2}]}), \qquad k\geq 0, \tag{3.24}$$

where θ is a fixed parameter smaller that unity, but the corresponding results are similar to the simpler situation $\theta = 1$.

3.4 Balance equation for $\gamma^{[k]}$ and $\gamma^{[k-\frac{1}{2}]}$

Our aim is to establish balance equations for $\gamma^{[k]}$ and $\gamma^{[k-\frac{1}{2}]}$. In Section 5, we will use these equations to derive a priori estimates and to establish that $\Gamma^{[k]}$ satisfies conditional entropic principles.

Proposition 3.7 *Let (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations* (3.1)–(3.5) *with regularity* (3.6)(3.7) *and let $1\leq k\leq l$. Then the following balance equation holds in $\mathcal{D}'\big((0,\bar{t})\times\mathbb{R}^n\big)$ and $L^1\big((0,\bar{t}),W^{l-k-1,1}\big)$*

$$\partial_t\gamma^{[k]} + \partial_x\cdot(v\gamma^{[k]}) + \partial_x\cdot\varphi_\gamma^{[k]} + \pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]} = 0, \tag{3.25}$$

where $\varphi_\gamma^{[k]}\in L^1\big((0,\bar{t}),W^{l-k,1}\big)$ is a flux and $\pi_\gamma^{[k]}$, $\Sigma_\gamma^{[k]}$, $\omega_\gamma^{[k]}\in L^1\big((0,\bar{t}),W^{l-k,1}\big)$ are source terms. The term $\pi_\gamma^{[k]}$ is given by

$$\pi_\gamma^{[k]} = 2g^2h^{2(k+1)}\Big(\frac{\lambda}{T^{\varkappa}}\frac{|\partial^{k+1}T|^2}{T^2} + \frac{\eta}{T^{\varkappa}}\frac{|\partial^{k+1}v|^2}{T} + \frac{\frac{1}{3}\eta+\kappa}{T^{\varkappa}}\frac{|\partial^k(\partial_x\cdot v)|^2}{T}\Big), \tag{3.26}$$

where $g=\rho T^{\frac{1}{2}(1-\varkappa)}$ and $h=1/\rho T^{\frac{1}{2}-\varkappa}$ so that $\pi_\gamma^{[k]}$ only contains the temperature and velocity $(k+1)^{\rm th}$ derivatives squared as expected from the hyperbolic–parabolic nature of system of partial differential equations. The term $\Sigma_\gamma^{[k]}$ is in the form

$$\Sigma_\gamma^{[k]} = \sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi}T^{\sigma-\varkappa}\partial_T^\sigma\phi\;\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)} + \frac{k(1-2\varkappa)\lambda}{c_vT^{\varkappa}}g^2h^{2(k+1)}\frac{|\partial^k\rho|^2}{\rho^2}\frac{\Delta T}{T}, \tag{3.27}$$

where $c_{\sigma\nu\mu\phi}$ are constants and the sum extends over $\phi\in\{\lambda,\eta,\kappa\}$, $0\leq\sigma\leq k$, $\nu=(\nu_\alpha,\nu'_\alpha,\nu''_\alpha)_{1\leq|\alpha|\leq k+1}$, $\mu=(\mu_\alpha,\mu'_\alpha,\mu''_\alpha)_{1\leq|\alpha|\leq k+1}$, $\nu_\alpha,\nu'_\alpha,\nu''_\alpha,\mu_\alpha,\mu'_\alpha,\mu''_\alpha\in\mathbb{N}$, $\alpha\in\mathbb{N}^n$. The products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = g\,h^{k+1}\prod_{1\leq|\alpha|\leq k+1}\Big(\frac{\partial^\alpha\rho}{\rho}\Big)^{\nu_\alpha}\,\Big(\frac{\partial^\alpha v}{\sqrt{T}}\Big)^{\nu'_\alpha}\,\Big(\frac{\partial^\alpha T}{T}\Big)^{\nu''_\alpha}, \tag{3.28}$$

where v denotes—with a slight abuse of notation—any of its components $v_1,\dots,v_n$, and ν must be such that

$$\sum_{1\leq|\alpha|\leq k+1}|\alpha|(\nu_\alpha+\nu'_\alpha+\nu''_\alpha) = k+1, \qquad \sum_{|\alpha|=k+1}\nu_\alpha = 0,$$

so that there is a total number of $k+1$ derivations and there are no derivative of order $k+1$ of density. Moreover, there is at most one derivative of order $k+1$ of temperature or velocity components in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$

$$\sum_{|\alpha|=k+1}(\nu'_\alpha+\nu''_\alpha+\mu'_\alpha+\mu''_\alpha)\leq 1,$$

so that one of the terms $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is split between two or more derivative factors. Furthermore the term $\omega_\gamma^{[k]}$ is given by

$$\omega_\gamma^{[k]} = \sum_{\nu\mu} c_{\nu\mu} \Pi_\nu^{(k)} \Pi_\mu^{(k+1)}, \tag{3.29}$$

where $c_{\nu\mu}$ are constants and we use similar notation for $\Pi_\nu^{(k)}$ as for $\Pi_\mu^{(k+1)}$ and the summation extends over

$$\sum_{1\le|\alpha|\le k} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k, \qquad \sum_{1\le|\alpha|\le k} |\alpha|(\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = k+1.$$

In particular $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = 0$ and there are at least two factors in the product $\Pi_\mu^{(k+1)}$. Finally the flux $\varphi_\gamma^{[k]} = (\varphi_{\gamma 1}^{[k]}, \ldots, \varphi_{\gamma n}^{[k]})$ is in the form

$$\varphi_{\gamma l}^{[k]} = \sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi l} T^{\sigma-\varkappa} \partial_T^\sigma \phi \, h \, \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} + \sum_{\nu\mu} c_{\nu\mu l} \, h \, \Pi_\nu^{(k)} \Pi_\mu^{(k)}.$$

Proof. The proof—given in A—is lengthy and tedious but presents no serious difficulties. □

We investigate the $\gamma^{[k-\frac{1}{2}]}$ balance equation for compressible fluids with temperature dependent transport coefficients.

Proposition 3.8 *Let (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations (3.1)–(3.5) with regularity (3.6)(3.7) and let $1 \le k \le l$. Then the following balance equation holds in $\mathcal{D}'\big((0,\bar{t})\times\mathbb{R}^n\big)$ and $L^1\big((0,\bar{t}), W^{l-k-1,1}\big)$*

$$\partial_t \gamma^{[k-\frac{1}{2}]} + \partial_x\cdot(v\gamma^{[k-\frac{1}{2}]}) + \partial_x\cdot\varphi_\gamma^{[k-\frac{1}{2}]} + \pi_\gamma^{[k-\frac{1}{2}]} + \Sigma_\gamma^{[k-\frac{1}{2}]} + \omega_\gamma^{[k-\frac{1}{2}]} = 0, \tag{3.30}$$

where $\varphi_\gamma^{[k-\frac{1}{2}]} \in L^1\big((0,\bar{t}), W^{l-k,1}\big)$ is a flux and $\pi_\gamma^{[k-\frac{1}{2}]}$, $\Sigma_\gamma^{[k-\frac{1}{2}]}$, $\omega_\gamma^{[k-\frac{1}{2}]} \in L^1\big((0,\bar{t}), W^{l-k,1}\big)$ are source terms. The term $\pi_\gamma^{[k-\frac{1}{2}]}$ is given by

$$\pi_\gamma^{[k-\frac{1}{2}]} = g^2 h^{2k} \frac{|\partial^k \rho|^2}{\rho^2}, \tag{3.31}$$

where $g = \rho T^{\frac{1}{2}(1-\varkappa)}$ and $h = 1/\rho T^{\frac{1}{2}-\varkappa}$ so that $\pi_\gamma^{[k-\frac{1}{2}]}$ will help to complete the missing gradient terms in $\pi_\gamma^{[k-1]}$. The term $\Sigma_\gamma^{[k-\frac{1}{2}]}$ is in the form

$$\Sigma_\gamma^{[k-\frac{1}{2}]} = \sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi} T^{\sigma-\varkappa} \partial_T^\sigma \phi \, \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} - \frac{\kappa + \frac{4}{3}\eta}{T^\varkappa} g^2 h^{2k+1} \frac{\partial^k(\partial_x\cdot v)}{\sqrt{T}} \frac{\partial^k \rho}{\rho}, \tag{3.32}$$

where $c_{\sigma\nu\mu\phi}$ are constants and the sums are over $\phi \in \{\lambda, \eta, \kappa\}$, $0 \le \sigma \le k$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\le|\alpha|\le k}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{1\le|\alpha|\le k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The products $\Pi_\nu^{(k)}$ and $\Pi_\mu^{(k+1)}$ are defined as in the governing equation for $\gamma^{[k]}$ and the products $\Pi_\mu^{(k+1)}$ do not contain derivatives of order $k+1$ and are thus split between two or more derivative factors. Furthermore the term $\omega_\gamma^{[k-\frac{1}{2}]}$ is given by

$$\omega_\gamma^{[k-\frac{1}{2}]} = \sum_{\nu\mu} c_{\nu\mu} \Pi_\nu^{(k)} \Pi_\mu^{(k)} + g^2 h^{2k} \frac{\partial^k T}{T} \frac{\partial^k \rho}{\rho} - g^2 h^{2k} \frac{|\partial^{k-1}(\partial_x\cdot v)|^2}{T}, \tag{3.33}$$

where $c_{\nu\mu}$ are constants and at least one of the two products $\Pi_\nu^{(k)}$ or $\Pi_\mu^{(k)}$ is split between two or more derivative factors. Finally the flux $\varphi_\gamma^{[k-\frac{1}{2}]} = (\varphi_{\gamma 1}^{[k-\frac{1}{2}]}, \ldots, \varphi_{\gamma n}^{[k-\frac{1}{2}]})$ is in the form

$$\varphi_{\gamma l}^{[k-\frac{1}{2}]} = \sum_{\nu\mu} c_{\nu\mu l} \, h \, \Pi_\nu^{(k-1)} \Pi_\mu^{(k)}.$$

Proof. The proof—lengthy and tedious—presents no serious difficulties and is similar to that of Proposition 3.7. □

3.5 Balance equation for $\tilde{\gamma}^{[k]}$ and $\tilde{\gamma}^{[k-\frac{1}{2}]}$

We establish balance equations for $\tilde{\gamma}^{[k]}$, and $\tilde{\gamma}^{[k-\frac{1}{2}]}$. In Section 5, we will use these equations to derive a priori estimates and to establish that $\widetilde{\Gamma}^{[k]}$ satisfies conditional entropic principles.

Proposition 3.9 *Let (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations (3.1)–(3.5) with regularity (3.6)(3.7) and let $1 \leq k \leq l$. Then the following balance equation holds in $\mathcal{D}'\big((0,\bar{t})\times\mathbb{R}^n\big)$ and $L^1\big((0,\bar{t}), W^{l-k-1,1}\big)$*

$$\partial_t\tilde{\gamma}^{[k]} + \partial_x\cdot(v\tilde{\gamma}^{[k]}) + \partial_x\cdot\varphi_{\tilde{\gamma}}^{[k]} + \pi_{\tilde{\gamma}}^{[k]} + \Sigma_{\tilde{\gamma}}^{[k]} + \omega_{\tilde{\gamma}}^{[k]} = 0, \tag{3.34}$$

where $\varphi_{\tilde{\gamma}}^{[k]} \in L^1\big((0,\bar{t}), W^{l-k,1}\big)$ is a flux and $\pi_{\tilde{\gamma}}^{[k]}$, $\Sigma_{\tilde{\gamma}}^{[k]}$, $\omega_{\tilde{\gamma}}^{[k]} \in L^1\big((0,\bar{t}), W^{l-k,1}\big)$ are source terms. The term $\pi_{\tilde{\gamma}}^{[k]}$ is given by

$$\pi_{\tilde{\gamma}}^{[k]} = 2g^2h^{2(k+1)}\Big(\frac{\lambda}{e^{\varkappa\tau}}|\partial^{k+1}\tau|^2 + \frac{\eta}{e^{\varkappa\tau}}|\partial^{k+1}w|^2 + \frac{\frac{1}{3}\eta+\kappa}{e^{\varkappa\tau}}|\partial^{k}(\partial_x\cdot w)|^2\Big), \tag{3.35}$$

where $g = \rho T^{\frac{1}{2}(1-\varkappa)}$ and $h = 1/\rho T^{\frac{1}{2}-\varkappa}$ so that $\pi_{\tilde{\gamma}}^{[k]}$ only contains the temperature and velocity $(k+1)^{\text{th}}$ derivatives squared as expected from the hyperbolic–parabolic structure of system of partial differential equations. The term $\Sigma_{\tilde{\gamma}}^{[k]}$ is in the form

$$\Sigma_{\tilde{\gamma}}^{[k]} = \sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi}e^{-\varkappa\tau}\partial_\tau^\sigma\phi\,\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)} + \frac{k(1-2\varkappa)\lambda}{e^{\varkappa\tau}c_v}g^2h^{2(k+1)}|\partial^k r|^2\Delta\tau, \tag{3.36}$$

where $c_{\sigma\nu\mu\phi}$ are constants and the sum extends over $\phi \in \{\lambda,\eta,\kappa\}$, $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{0\leq|\alpha|\leq k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{0\leq|\alpha|\leq k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = g\,h^{k+1}\prod_{0\leq|\alpha|\leq k+1}\big(\partial^\alpha r\big)^{\nu_\alpha}\,\big(\partial^\alpha w\big)^{\nu'_\alpha}\,\big(\partial^\alpha\tau\big)^{\nu''_\alpha}, \tag{3.37}$$

where w denotes—with a slight abuse of notation—any of its components $w_1, \dots, w_n$, and μ and ν must be such that

$$\sum_{1\leq|\alpha|\leq k+1}|\alpha|(\nu_\alpha+\nu'_\alpha+\nu''_\alpha) = k+1, \qquad \sum_{|\alpha|=k+1}\nu_\alpha = 0, \qquad \sum_{|\alpha|=0}(\nu_\alpha+\nu''_\alpha) = 0,$$

so that there is a total of $k+1$ derivations and there is no derivative of order $k+1$ of density. Note that powers of the renormalized velocity w may appear in $\Pi_\nu^{(k+1)}$ but not of τ or r. In addition, there is at most one derivative of order $k+1$ of temperature or velocity components in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$

$$\sum_{|\alpha|=k+1}(\nu'_\alpha+\nu''_\alpha+\mu'_\alpha+\mu''_\alpha) \leq 1,$$

so that one of the terms $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is split between two or more derivative factors. Furthermore the term $\omega_{\tilde{\gamma}}^{[k]}$ is given by

$$\begin{aligned}\omega_{\tilde{\gamma}}^{[k]} = {} & \sum_{\nu\mu}c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + g^2h^{2k+1}\partial^k\tau\,\partial^k(\partial_x\tau)\cdot w + g^2h^{2k+1}\partial^k r\,\partial^k(\partial_x\tau)\cdot w \\ & -\frac{1}{c_v}g^2h^{2k+1}\partial^k w\cdot w\,\partial^k(\partial_x\cdot w) - \frac{1}{2c_v}g^2h^{2k+1}\partial^k w\cdot w\,\partial^k(\partial_x\tau)\cdot w,\end{aligned} \tag{3.38}$$

where $c_{\nu\mu}$ are constants and we use similar notation for $\Pi_\nu^{(k)}$ as for $\Pi_\mu^{(k+1)}$ and the summation extends over

$$\sum_{1\leq|\alpha|\leq k}|\alpha|(\nu_\alpha+\nu'_\alpha+\nu''_\alpha) = k, \qquad \sum_{1\leq|\alpha|\leq k}|\alpha|(\mu_\alpha+\mu'_\alpha+\mu''_\alpha) = k+1,$$

so that in particular $\sum_{|\alpha|=k+1}(\mu_\alpha+\mu'_\alpha+\mu''_\alpha) = 0$ and there are always at least two factors in the product $\Pi_\mu^{(k+1)}$. Finally the flux $\varphi_{\tilde{\gamma}}^{[k]} = (\varphi_{\tilde{\gamma}1}^{[k]}, \dots, \varphi_{\tilde{\gamma}n}^{[k]})$ is in the form

$$\varphi_{\tilde{\gamma}l}^{[k]} = \sum_{\sigma\nu\mu\phi}c_{\sigma\nu\mu\phi l}e^{-\varkappa\tau}\partial_\tau^\sigma\phi\,h\,\Pi_\nu^{(k)}\Pi_\mu^{(k+1)} + \sum_{\nu\mu}c_{\nu\mu l}\,h\,\Pi_\nu^{(k)}\Pi_\mu^{(k)}.$$

Proof. The proof is similar to that of Proposition 3.7 and is omitted. □

Proposition 3.10 *Let (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations* (3.1)–(3.5) *with regularity* (3.6)(3.7) *and let $1 \leq k \leq l$. Then the following balance equation holds in $\mathcal{D}'\big((0,\bar{t})\times\mathbb{R}^n\big)$ and $L^1\big((0,\bar{t}), W^{l-k-1,1}\big)$*

$$\partial_t \tilde{\gamma}^{[k-\frac{1}{2}]} + \partial_x\cdot(v\tilde{\gamma}^{[k-\frac{1}{2}]}) + \partial_x\cdot\varphi_{\tilde{\gamma}}^{[k-\frac{1}{2}]} + \pi_{\tilde{\gamma}}^{[k-\frac{1}{2}]} + \Sigma_{\tilde{\gamma}}^{[k-\frac{1}{2}]} + \omega_{\tilde{\gamma}}^{[k-\frac{1}{2}]} = 0, \tag{3.39}$$

where $\varphi_{\tilde{\gamma}}^{[k-\frac{1}{2}]} \in L^1\big((0,\bar{t}), W^{l-k,1}\big)$ is a flux and $\pi_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$, $\Sigma_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$, $\omega_{\tilde{\gamma}}^{[k-\frac{1}{2}]} \in L^1\big((0,\bar{t}), W^{l-k,1}\big)$ are source terms. The term $\pi_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$ is given by

$$\pi_{\tilde{\gamma}}^{[k-\frac{1}{2}]} = g^2 h^{2k} |\partial^k r|^2, \tag{3.40}$$

where $g = e^{r+\frac{1}{2}(1-\varkappa)\tau}$, and $h = e^{-r-(\frac{1}{2}-\varkappa)\tau}$ so that $\pi_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$ will help to complete the missing gradient terms in $\pi_{\tilde{\gamma}}^{[k-1]}$. The term $\Sigma_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$ is in the form

$$\begin{aligned}\Sigma_{\tilde{\gamma}}^{[k-\frac{1}{2}]} = & \sum_{\nu\mu} c_{\sigma\nu\mu\phi} e^{-\varkappa\tau} \partial_\tau^\sigma \phi\, \Pi_\nu^{(k)} \Pi_\mu^{(k+1)} - \frac{\kappa + \frac{4}{3}\eta}{e^{\varkappa\tau}} g^2 h^{2k+1} \partial^k r\, \partial^k(\partial_x\cdot w) \\ & + \tfrac{1}{2} \frac{\lambda}{c_v e^{\varkappa\tau}} g^2 h^{2k+1} \partial^{k-1}\partial_x r\cdot w\, \partial^{k-1}\Delta\tau,\end{aligned} \tag{3.41}$$

where $c_{\sigma\nu\mu\phi}$ are constants and the sums are over $\phi \in \{\lambda, \eta, \kappa\}$, $0 \leq \sigma \leq k$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\leq|\alpha|\leq k}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{1\leq|\alpha|\leq k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The products $\Pi_\nu^{(k)}$ and $\Pi_\mu^{(k+1)}$ are defined as in the governing equation for $\tilde{\gamma}^{[k]}$ and there is no derivative of order $k+1$ in $\Pi_\mu^{(k+1)}$. Furthermore the term $\omega_{\tilde{\gamma}}^{[k-\frac{1}{2}]}$ is given by

$$\begin{aligned}\omega_{\tilde{\gamma}}^{[k-\frac{1}{2}]} = & \sum_{\nu\mu} c_{\nu\mu} \Pi_\nu^{(k)} \Pi_\mu^{(k)} + g^2 h^{2k} \partial^k \tau \partial^k r - g^2 h^{2k} |\partial^{k-1}(\partial_x\cdot w)|^2 \\ & - \tfrac{1}{2} g^2 h^{2k} \partial^{k-1}(\partial_x\cdot w)\, \partial^{k-1}(\partial_x \tau)\cdot w - \frac{1}{2c_v} g^2 h^{2k} \partial^{k-1}(\partial_x\cdot w)\, \partial^{k-1}(\partial_x r)\cdot w \\ & - \frac{1}{4c_v} g^2 h^{2k} \partial^{k-1}(\partial_x \tau)\cdot w\, \partial^{k-1}(\partial_x r)\cdot w,\end{aligned} \tag{3.42}$$

where $c_{\nu\mu}$ are constants and at least one of the products $\Pi_\nu^{(k)}$ or $\Pi_\mu^{(k)}$ is split between derivatives factors. Finally the flux $\varphi_{\tilde{\gamma}}^{[k-\frac{1}{2}]} = (\varphi_{\tilde{\gamma}1}^{[k-\frac{1}{2}]}, \ldots, \varphi_{\tilde{\gamma}n}^{[k-\frac{1}{2}]})$ is in the form

$$\varphi_{\tilde{\gamma}l}^{[k-\frac{1}{2}]} = \sum_{\nu\mu} c_{\nu\mu l}\, h\, \Pi_\nu^{(k-1)} \Pi_\mu^{(k)}.$$

4 Weighted inequalities

We investigate weighted inequalities in Sobolev and Lebesgue spaces[7, 12, 15, 21, 22, 35]. These inequalities are required for renormalized variables with powers of temperature and density as weights as well as for temperature dependent thermal conductivity and viscosities.

4.1 Differential identities

Let α_i, $1 \leq i \leq n$, be nonnegative integers and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ be the corresponding multiindex. We denote by ∂^α the differential operator $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and by $|\alpha|$ its order $|\alpha| = \alpha_1 + \cdots + \alpha_n$. The derivative of superpositions has been investigated in particular by Vol'pert and Hudjaev[43] and the following proposition is established by induction on $|\alpha|$.

Lemma 4.1 *Let f and g be smooth functions and $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multiindex. Then we have*

$$\partial^\alpha (fg) = \sum_{0\leq\beta\leq\alpha} c_{\alpha\beta}\, \partial^\beta f \partial^{(\alpha-\beta)} g, \tag{4.1}$$

where $c_{\alpha\beta} = \alpha!/\beta!(\alpha-\beta)!$ are nonnegative integer coefficients, $\beta! = \beta_1! \cdots \beta_n!$, and where we write $0 \leq \beta \leq \alpha$ when $0 \leq \beta_i \leq \alpha_i$, $1 \leq i \leq n$.

Let $l \geq 1$, f be a smooth scalar function of $\mathsf{u} \in \mathbb{R}^l$, $\mathsf{u}_1, \dots, \mathsf{u}_l$ be smooth scalar functions of $x \in \mathbb{R}^n$, and let α be a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| \geq 1$. The partial derivatives of the superposition $f \circ \mathsf{u} = f(\mathsf{u}_1, \dots \mathsf{u}_l)$ can be written in the form

$$\partial^\alpha (f\circ \mathsf{u}) = \sum_{\sigma\mu} c_{\sigma\mu}\, \partial^\sigma f \prod_{\substack{1\leq|\beta|\leq|\alpha| \\ 1\leq j\leq l}} \big(\partial^\beta \mathsf{u}_j\big)^{\mu_{\beta j}}, \tag{4.2}$$

where $c_{\sigma\mu}$ are nonnegative integer and the sum is over $\sigma \in \mathbb{N}^l$, $1 \leq |\sigma| \leq |\alpha|$, $\mu = (\mu_{\beta j})_{1\leq|\beta|\leq|\alpha|, 1\leq j\leq l}$ with $\mu_{\beta j} \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, $j \in \mathbb{N}$, such that

$$\sum_{1\leq|\beta|\leq|\alpha|} \mu_{\beta j} = \sigma_j, \qquad \sum_{\substack{1\leq|\beta|\leq|\alpha| \\ 1\leq j\leq l}} \beta\, \mu_{\beta j} = \alpha, \tag{4.3}$$

so that we have in particular $\sum_{\beta j} |\beta|\, \mu_{\beta j} = |\alpha|$.

The rescaled unknowns $r = \log \rho$, $w = v/\sqrt{T}$, and $\tau = \log T$, naturally appear in higher order entropy estimates. We will need the following differential identities[15] easily established by induction on $|\alpha|$ and the next lemma will be used for temperature as well as for density.

Lemma 4.2 *Let T be smooth and positive and α be a multiindex. Then we have*

$$\frac{\partial^\alpha T}{T} = \sum_\mu c_\mu \prod_{1\leq|\beta|\leq|\alpha|} \big(\partial^\beta \tau\big)^{\mu_\beta} = \partial^\alpha \tau + \sum_\mu c_\mu \prod_{1\leq|\beta|\leq|\alpha|-1} \big(\partial^\beta \tau\big)^{\mu_\beta}, \tag{4.4}$$

where $\mu = (\mu_\beta)_{1\leq|\beta|\leq|\alpha|}$ with $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, and c_μ are nonnegative integer coefficients. The sum is extended over the μ such that

$$\sum_{1\leq|\beta|\leq|\alpha|} \beta\, \mu_\beta = \alpha,$$

so that we have in particular $\sum_{1\leq|\beta|\leq|\alpha|} |\beta|\, \mu_\beta = |\alpha|$, and the only term with $|\beta| = |\alpha|$ corresponds to $\partial^\alpha \tau$. Conversely, we have

$$\partial^\alpha \tau = \sum_\mu c'_\mu \prod_{1\leq|\beta|\leq|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta} = \frac{\partial^\alpha T}{T} + \sum_\mu c'_\mu \prod_{1\leq|\beta|\leq|\alpha|-1} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta}, \tag{4.5}$$

where c'_μ are integer coefficients and the sum is extended over the same set of μ.

Lemma 4.3 *Let T and v be smooth, T be positive, i with $1 \leq i \leq n$, and α be a multiindex. Then we have*

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \sum_{\mu\tilde{\alpha}} c_{\mu\tilde{\alpha}} \prod_{1\leq|\beta|\leq|\alpha|} \big(\partial^\beta \tau\big)^{\mu_\beta}\, \partial^{\tilde{\alpha}} w_i, \tag{4.6}$$

where $\mu = (\mu_\beta)_{1\leq|\beta|\leq|\alpha|}$, $\mu_\beta \in \mathbb{N}$, $\beta \in \mathbb{N}^n$, $\tilde{\alpha} \in \mathbb{N}^n$, $c_{\mu\tilde{\alpha}}$ are nonnegative integer coefficients, and the sum is extended over the μ and $\tilde{\alpha}$, such that

$$0 \leq \tilde{\alpha} \leq \alpha, \qquad \sum_{1\leq|\beta|\leq|\alpha|} \beta\, \mu_\beta + \tilde{\alpha} = \alpha.$$

More precisely, isolating the only term $\partial^\alpha w_i$ corresponding to $\tilde{\alpha} = \alpha$ and all the terms corresponding to $\tilde{\alpha} = (0, \dots, 0)$, we have

$$\frac{\partial^\alpha v_i}{\sqrt{T}} = \partial^\alpha w_i + \sum_{\mu\tilde{\alpha}} c_{\mu\tilde{\alpha}} \prod_{1\leq|\beta|\leq|\alpha|} \big(\partial^\beta \tau\big)^{\mu_\beta}\, \partial^{\tilde{\alpha}} w_i + \sum_\mu c_{\mu 0} \prod_{1\leq|\beta|\leq|\alpha|} \big(\partial^\beta \tau\big)^{\mu_\beta}\, w_i, \tag{4.7}$$

where the $\tilde{\alpha}$ in the middle sum are such that $1 \le |\tilde{\alpha}| < |\alpha|$. Conversely, we have

$$\partial^\alpha w_i = \sum_{\mu\tilde{\alpha}} c'_{\mu\tilde{\alpha}} \prod_{1\le|\beta|\le|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta} \frac{\partial^{\tilde{\alpha}} v_i}{\sqrt{T}}, \tag{4.8}$$

and more precisely

$$\partial^\alpha w_i = \frac{\partial^\alpha v_i}{\sqrt{T}} + \sum_{\mu\tilde{\alpha}} c'_{\mu\tilde{\alpha}} \prod_{1\le|\beta|\le|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta} \frac{\partial^{\tilde{\alpha}} v_i}{\sqrt{T}} + \sum_{\mu} c'_{\mu 0} \prod_{1\le|\beta|\le|\alpha|} \Big(\frac{\partial^\beta T}{T}\Big)^{\mu_\beta} \frac{v_i}{\sqrt{T}}, \tag{4.9}$$

where $c'_{\mu\tilde{\alpha}}$ are integer coefficients and the sums are extended over the same sets.

4.2 Weighted operators

A natural condition associated with weights[7, 12, 35] has been shown to be the Muckenhoupt property A_p, where $1 \le p \le \infty$.

Definition 4.4 *Let $g \in L^1_{loc}(\mathbb{R}^n)$ be positive and let $1 < p < \infty$. The function g satisfies the Muckenhoupt condition A_p if*

$$[g]_{A_p} = \sup_Q \Big(\frac{1}{|Q|}\int_Q g\,dx\Big)\Big(\frac{1}{|Q|}\int_Q g^{-\frac{1}{p-1}}\,dx\Big)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q.

For detailed studies about the Muckenhoupt property we refer to the book of Garcia-Cuerva and Rubio de Francia[12]. We have in particular $A_p \cap A_q = A_{\min(p,q)}$ and the weights of A_p have their logarithms in BMO[12, 35]. A locally summable function f belongs to the space $BMO(\mathbb{R}^n)$ if

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|}\int_Q \big|f(x) - \bar{f}_Q\big|\,dx < \infty,$$

where the supremum is taken over all cubes Q and where $\bar{f}_Q = 1/|Q| \int_Q f(x)\,dx$ denotes the average of f over Q[34]. The function space BMO has been introduced by John and Nirenberg[23] and naturally arises when estimating the norms of the weighted operators $T^\theta R_i T^{-\theta}$ where $R_i = (-\Delta)^{-1/2}\partial_i$, $1 \le i \le n$, are Riesz transforms, or when using the Coifman and Meyer inequalities[36, 37]. The space BMO and its dual $\mathcal{H}^1$ have already been used in the context of the Navier-Stokes equations[32, 29, 31].

Theorem 4.5 *There exists constants $b(n)$ and $B(n)$ such that for any $\theta \in \mathbb{R}$, any $\mathsf{u} \in BMO$, and any $1 < p < \infty$, the condition*

$$|\theta|\|\mathsf{u}\|_{BMO} < \tfrac{1}{2} b(n)\min(1, p-1),$$

implies that $\exp(\theta u) \in A_p$ and

$$\big[\exp(\theta u)\big]_{A_p} \le \big(1 + B(n)\big)^p.$$

Moreover, the constants $b(n)$ and $B(n)$ only depend on n and are thus invariant by a change of scale in the coordinate system.

Proof. These estimates are proved in [15] and the scale invariance of $b(n)$ and $B(n)$ is straightforward since both the BMO seminorm and the A_p condition number $[g]_{A_p}$ are scale invariant. □

We now investigate the continuity of Calderón-Zygmund operators in weighted Lebesgue spaces. In the following theorem the quantitites c_0, c_1, c_2 are the constants naturally associated with the norm of a Calderón-Zygmund operator $\mathcal{G}$[35].

Theorem 4.6 *Let $\mathcal{G}$ be a Calderón-Zygmund operator, let $1 < p < \infty$, and let g^p be a weight in A_p. Then the operator $\mathcal{G}$ is bounded in $L^p\big(g^p dx\big)$, or equivalently, the operator $g\mathcal{G}g^{-1}$ is bounded in L^p, with norm lower than $\mathcal{C}(c_0, c_1, c_2, n, p, [g^p]_{A_p})$, where c_0, c_1, c_2 are the constants naturally associated with the norm of $\mathcal{G}$.*

Proof. We refer to the books of Garcia-Cuerva and Rubio de Francia[12] and of Yves Meyer[35]. □

4.3 Multilinear estimates

We investigate weighted multilinear estimates with weights in A_p classes[7, 12, 15, 21, 22, 35] and we denote by $\mathcal{C}_0^0(\mathbb{R}^n)$ the set of continuous function that vanish at infinity. The following multilinear estimates have been obtained in previous work[15] by using the Wiener algebra $A(\mathbb{R}^n)$ instead of $\mathcal{C}_0^0(\mathbb{R}^n)$ but the proofs are similar thanks to the density of $\mathcal{D}(\mathbb{R}^n)$ in $W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$. The proof of this theorem essentially relies on the Coifman-Meyer theory and on Theorem 4.6.

Theorem 4.7 *Let $k \geq 1$, $l \geq 1$ be integers, and α^j, $1 \leq j \leq l$, be multiindices such that $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $k = \sum_{1 \leq j \leq l} |\alpha^j|$. Let $1 < p < \infty$, $g^p \in A_p$ and $\mathsf{u}_1, \dots, \mathsf{u}_l$, be such that there exist constants $\mathsf{u}_{j,\infty}$ with $\mathsf{u}_j - \mathsf{u}_{j,\infty} \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, and such that $g\partial^k u_j \in L^p$, $1 \leq j \leq l$. There exists a constant $c = c(k, n, p, [g^p]_{A_p})$ only depending on $(k, n, p, [g^p]_{A_p})$, such that*

$$\big\| g \prod_{1 \leq j \leq l} \partial^{\alpha^j} \mathsf{u}_j \big\|_{L^p} \leq c \sum_{1 \leq i \leq l} \Big(\prod_{\substack{1 \leq j \leq l \\ j \neq i}} \|\mathsf{u}_j\|_{BMO} \Big) \, \big\| g \partial^k \mathsf{u}_i \big\|_{L^p}, \tag{4.10}$$

and thus

$$\big\| g \prod_{1 \leq j \leq l} \partial^{\alpha^j} \mathsf{u}_j \big\|_{L^p} \leq c \|\mathsf{u}\|_{BMO}^{l-1} \|g \partial^k \mathsf{u}\|_{L^p}, \tag{4.11}$$

where

$$\|\mathsf{u}\|_{BMO} = \sum_{1 \leq j \leq l} \|\mathsf{u}_j\|_{BMO}, \qquad \|g\partial^m \mathsf{u}\|_{L^p}^p = \sum_{1 \leq j \leq l} \|g \partial^m \mathsf{u}_j\|_{L^p}^p.$$

We now investigate multilinear estimates where a weighted L^∞ norm of the gradient is used to decrease the total number of derivations k in the upper bound. We denote by $\mathcal{C}_0^1(\mathbb{R}^n)$ the set of continuously differentiable functions that vanish at infinity with their gradients.

Theorem 4.8 *Let $k \geq 2$, $l \geq 2$ be integers, and α^j, $1 \leq j \leq l$, be multiindices such that $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $k = \sum_{1 \leq j \leq l} |\alpha^j|$. Let $1 < p < \infty$, g be positive, $g \in L^1_{\mathrm{loc}}$ with $\log g \in BMO$, and $\mathsf{u}_1, \dots, \mathsf{u}_l$, be such that there exist constants $\mathsf{u}_{j,\infty}$ with $\mathsf{u}_j - \mathsf{u}_{j,\infty} \in W^{k-1,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$. Let h be the weight $h = \exp(\theta_1 u_1 + \cdots + \theta_l u_l)$, where $|\theta_j| \leq \bar{\theta}$ and $\bar{\theta} > 0$. There exist constants $\delta = \delta(k, n, p, \bar{\theta})$ and $c = c(k, n, p, \bar{\theta})$, only depending on $(k, n, p, \bar{\theta})$, such that if $\|\log g\|_{BMO} + \sum_{1 \leq j \leq l} \|u_j\|_{BMO} < \delta$, then, whenever $gh^{k-1}\partial^{k-1} u_j \in L^p$ and $gh^{k-2}\partial^{k-2} u_j \in L^p$, $1 \leq j \leq l$, the following estimates hold*

$$\begin{aligned} \big\| g h^k \prod_{1 \leq j \leq l} \partial^{\alpha^j} \mathsf{u}_j \big\|_{L^p} &\leq c \, \|\mathsf{u}\|_{BMO}^{l-2} \|h \partial_x \mathsf{u}\|_{L^\infty} \big\| g h^{k-1} \partial^{k-1} \mathsf{u} \big\|_{L^p} \\ &\quad + c \, \mathbf{1}_{k \geq 3} \|\mathsf{u}\|_{BMO}^{(l-3)^+} \|h \partial_x \mathsf{u}\|_{L^\infty}^2 \big\| g h^{k-2} \partial^{k-2} \mathsf{u} \big\|_{L^p}, \end{aligned} \tag{4.12}$$

where

$$\|h\partial_x \mathsf{u}\|_{L^\infty} = \sum_{1 \leq j \leq l} \|h \partial_x \mathsf{u}_j\|_{L^\infty}, \qquad \|g h^m \partial^m \mathsf{u}\|_{L^p}^p = \sum_{1 \leq j \leq l} \|g h^m \partial^m \mathsf{u}_j\|_{L^p}^p,$$

and where $\mathbf{1}_{k \geq 3} = 1$ if $k \geq 3$ and $\mathbf{1}_{k \geq 3} = 0$ if $k \leq 2$ so that in the special situation $2 \leq k \leq 3$, the second term in the right hand side of (4.12) is absent.

Proof. If there exists one multiindex α^{j_0} such that $|\alpha^{j_0}| = 1$ we can directly write that

$$\big\| g h^k \prod_{1 \leq j \leq l} \partial^{\alpha^j} \mathsf{u}_j \big\|_{L^p} \leq \|h \partial_x \mathsf{u}_{j_0}\|_{L^\infty} \, \big\| g h^{k-1} \prod_{\substack{1 \leq j \leq l \\ j \neq j_0}} \partial^{\alpha^j} \mathsf{u}_j \big\|_{L^p}, \tag{4.13}$$

and use the multilinear estimates of Theorem 4.7. The weight gh^{k-1} is in the A_p class and $[g^p h^{p(k-1)}]_{A_p}$ is bounded by a constant only depending on n and p from Theorem 4.5 for δ small enough since $\|\log(gh^{k-1})\|_{BMO} \leq (1 + k\bar{\theta})\delta$ provided we select $\delta \leq \frac{1}{2} b(n) \min(1, p-1)/(1 + k\bar{\theta})$. This covers in particular the situation where $2 \leq k \leq 3$ since it is assumed that $l \geq 2$ so that there is at least one first order derivative factor $\partial^{\alpha^{j_0}} \mathsf{u}_{j_0}$ with $|\alpha^{j_0}| = 1$ in this case.

Keeping in mind that $l \geq 2$, we can now assume that $|\alpha^1| \geq 2$ and $|\alpha^2| \geq 2$, so that $k \geq 4$, and write $\alpha^1 = \tilde{\alpha}^1 + e_{i_1}$, $\alpha^2 = \tilde{\alpha}^2 + e_{i_2}$, where $|\tilde{\alpha}^1| = k - 1$, $|\tilde{\alpha}^2| = k - 1$, and $i_1, i_2 \in \{1, \dots, n\}$. We have denoted by e_i, $1 \leq i \leq n$, the canonical basis of $\mathbb{R}^n$ with $e_i = (\delta_{i1}, \dots, \delta_{in})$, where δ_{ij} is the Kronecker symbol, so that $\partial^{e_i} = \partial_i$. We introduce the auxiliary functions $\mathsf{v}_1 = h\partial_{i_1}\mathsf{u}_1$ and $\mathsf{v}_2 = h\partial_{i_2}\mathsf{u}_2$ and write that

$$gh^k \prod_{1\le j\le l} \partial^{\alpha^j}\mathsf{u}_j = gh^k \partial^{\tilde\alpha_1}\big(\frac{\mathsf{v}_1}{h}\big)\,\partial^{\tilde\alpha_2}\big(\frac{\mathsf{v}_2}{h}\big) \prod_{3\le j\le l}\partial^{\alpha^j}\mathsf{u}_j.$$

We next expand the derivatives by using Lemma 4.1

$$\partial^{\tilde\alpha_1}\big(\frac{\mathsf{v}_1}{h}\big) = \frac{1}{h}\sum_{\tilde\beta_1\mu} c_{\tilde\beta_1\mu}\partial^{\tilde\beta_1}\mathsf{v}_1 \prod_{\substack{1\le|\beta|\le|\alpha|\\1\le j\le l}} (\partial^\beta \mathsf{u}_j)^{\mu_{\beta j}}, \tag{4.14}$$

where $c_{\tilde\beta_1\mu}$ are nonnegative integer coefficients, and the sum is over $0 \le \tilde\beta_1 \le \tilde\alpha_1$ and $\mu = (\mu_{\beta j})_{1\le|\beta|\le|\tilde\alpha_1|, 1\le j\le l}$ with $\mu_{\beta j}\in\mathbb{N}$, $\beta\in\mathbb{N}^n$, $1\le j\le l$, and $\sum_{\beta j}\beta\,\mu_{\beta j} = \tilde\alpha_1 - \tilde\beta_1$. We can thus write that

$$gh^k \prod_{1\le j\le l}\partial^{\alpha^j}\mathsf{u}_j = \sum_{\tilde\beta_1\tilde\beta_2\hat\alpha} c_{\tilde\beta_1\tilde\beta_2\hat\alpha}\, gh^{k-2}\partial^{\tilde\beta_1}\mathsf{v}_1\,\partial^{\tilde\beta_2}\mathsf{v}_2 \prod_{3\le j\le l}\partial^{\bar\alpha^j}\mathsf{u}_j \prod_{1\le j\le \hat l}\partial^{\hat\alpha^j}\hat{\mathsf{u}}_j, \tag{4.15}$$

where the derivatives factors arising from the derivation of $1/h$ in (4.14) are rewritten in the form $\prod_{1\le j\le\hat l}\partial^{\hat\alpha^j}\hat{\mathsf{u}}_j$, where $(\hat{\mathsf{u}}_1,\ldots,\hat{\mathsf{u}}_{\hat l})$ are proper replicates of $\mathsf{u}_1,\ldots,\mathsf{u}_l$. We can then use the inequality (4.10) of Theorem 4.7 to estimate the L^p norm of each term in the sum (4.15). Inequality (4.10) is used with the weight gh^{k-2} and with the variables $(\mathsf{v}_1,\mathsf{v}_2,\mathsf{u}_3,\ldots,\mathsf{u}_j,\hat{\mathsf{u}}_1,\ldots,\hat{\mathsf{u}}_{\hat l})$. The weight gh^{k-2} is in the A_p class for δ small enough and $[g^p h^{p(k-2)}]_{A_p}$ is bounded by a constant only depending on n and p from Theorem 4.5 provided that $\delta \le \frac{1}{2}b(n)\min(1,p-1)/(1+k\bar\theta)$. We can thus estimate the L^p norm of $gh^k\prod_{1\le j\le l}\partial^{\alpha^j}\mathsf{u}_j$, up to multiplicative constants depending on $(k,n,p,\bar\theta)$, in terms of

$$\|\mathsf{v}_2\|_{BMO}\prod_{3\le j\le l}\|\mathsf{u}_j\|_{BMO}\prod_{1\le j\le\hat l}\|\hat{\mathsf{u}}_j\|_{BMO}\;\; \big\|gh^{k-2}\partial^{k-2}\mathsf{v}_1\big\|_{L^p},$$

$$\|\mathsf{v}_1\|_{BMO}\prod_{3\le j\le l}\|\mathsf{u}_j\|_{BMO}\prod_{1\le j\le\hat l}\|\hat{\mathsf{u}}_j\|_{BMO}\;\; \big\|gh^{k-2}\partial^{k-2}\mathsf{v}_2\big\|_{L^p},$$

$$\|\mathsf{v}_1\|_{BMO}\|\mathsf{v}_2\|_{BMO}\prod_{\substack{3\le j\le l\\ j\ne i}}\|\mathsf{u}_j\|_{BMO}\prod_{1\le j\le\hat l}\|\hat{\mathsf{u}}_j\|_{BMO}\;\; \big\|gh^{k-2}\partial^{k-2}\mathsf{u}_i\big\|_{L^p}, \qquad 3\le i\le l,$$

and

$$\|\mathsf{v}_1\|_{BMO}\|\mathsf{v}_2\|_{BMO}\prod_{3\le j\le l}\|\mathsf{u}_j\|_{BMO}\prod_{\substack{1\le j\le\hat l\\ j\ne i}}\|\hat{\mathsf{u}}_j\|_{BMO}\;\; \big\|gh^{k-2}\partial^{k-2}\hat{\mathsf{u}}_i\big\|_{L^p}, \qquad 1\le i\le \hat l.$$

Expanding then the derivatives $\partial^{k-2}\mathsf{v}_j = \partial^{k-2}(h\partial_{i_j}\mathsf{u}_j)$, $j=1,2$, it is easily checked that $\|gh^{k-2}\partial^{k-2}\mathsf{v}_j\|_{L^p}$ is majorized by a multiplicative constant multiplied by $\sum_{1\le i\le l}\|gh^{k-1}\partial^{k-1}\mathsf{u}_i\|_{L^p}$ and the proof is complete since one may choose δ such that $0<\delta\le 1$. □

Remark 4.9 *The space of smooth functions with compact support $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{k,2}(\mathbb{R}^n)\cap BMO(\mathbb{R}^n)$—for the norm $\|\cdot\|_{W^{k,2}} + \|\cdot\|_{BMO}$ of course—if and only if $k\ge n/2$. Indeed, for $k<n/2$, $\mathcal{D}(\mathbb{R}^n)$ is not even dense in $W^{k,2}(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ and counterexemples are classically found in the form of series of needles[15]. On the other hand, for $k=n/2$, we have $W^{k,2}(\mathbb{R}^n)\cap BMO(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$, whereas for $k>n/2$, $W^{k,2}(\mathbb{R}^n)$ is included in $\mathcal{C}^0_0(\mathbb{R}^n)$. We have introduced the natural simplifying assumption $\mathsf{u}_j - \mathsf{u}_{j,\infty}\in W^{k,2}(\mathbb{R}^n)\cap\mathcal{C}^0_0(\mathbb{R}^n)$ since it will be sufficient for our applications and since for $k<n/2$, $\mathcal{D}(\mathbb{R}^n)$ is dense in $W^{k,2}(\mathbb{R}^n)\cap\mathcal{C}^0_0(\mathbb{R}^n)$ and $\mathcal{C}^0_0(\mathbb{R}^n)\subset L^\infty(\mathbb{R}^n)\subset BMO(\mathbb{R}^n)$. Similarly, when using the gradient norms $\|h\partial_x\mathsf{u}_j\|_{L^\infty}$, $1\le j\le l$, we have introduced the natural simplifying assumption that $\mathsf{u}_j - \mathsf{u}_{j,\infty}\in W^{k,2}(\mathbb{R}^n)\cap\mathcal{C}^1_0(\mathbb{R}^n)$ since it will be sufficient for our applications.*

4.4 Weighted products of derivatives

We first investigate products of derivatives of the rescaled unknowns τ and w with powers of temperature and density as natural weights[15]. Since in our applications w and τ will be parabolic variables, the total number of derivations k is left unchanged in the estimates.

Theorem 4.10 *Let $k \geq 1$ be an integer, $\bar\theta > 0$ be positive, $1 < p < \infty$, τ be such that $\tau - \tau_\infty \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$ for some constant τ_∞ and let $r \in BMO$. There exist positive constants $\delta(n,p,\bar\theta)$ and $c(k,n,p)$, only depending on $(n,p,\bar\theta)$ and (k,n,p), respectively, such that if $\|r\|_{BMO} + \|\tau\|_{BMO} < \delta$, then for any a, b with $|a|+|b| \leq \bar\theta$, any integer $l \geq 1$, and any multiindices α^j, $1 \leq j \leq l$, with $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $\sum_{1\leq j\leq l} |\alpha^j| = k$, whenever $e^{a\tau+br}\partial^k\tau \in L^p(\mathbb{R}^n)$, the following inequality holds*

$$\Big\| e^{a\tau+br} \prod_{1\leq j\leq l} \partial^{\alpha^j}\tau \Big\|_{L^p} \leq \; c\|\tau\|_{BMO}^{l-1} \, \big\| e^{a\tau+br}\partial^k\tau \big\|_{L^p}. \tag{4.16}$$

Further assuming that $w \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, $e^{a\tau+br}\partial^k w \in L^p(\mathbb{R}^n)$, and $0 \leq \bar l \leq l$, then

$$\begin{aligned}\Big\| e^{a\tau+br} \prod_{1\leq j\leq \bar l} \partial^{\alpha^j} w \prod_{\bar l+1\leq j\leq l} \partial^{\alpha^j}\tau \Big\|_{L^p} \leq \; & c\Big(\|w\|_{BMO} + \|\tau\|_{BMO} \Big)^{l-1} \\ & \times \Big(\big\| e^{a\tau+br}\partial^k w \big\|_{L^p} + \big\| e^{a\tau+br}\partial^k \tau \big\|_{L^p} \Big),\end{aligned} \tag{4.17}$$

where we have naturally defined $\big\| e^{a\tau+br}\,\partial^k w \big\|_{L^p}^p = \sum_{1\leq i\leq n} \big\| e^{a\tau+br}\,\partial^k w_i \big\|_{L^p}^p$ and in the left hand member of (4.17), with a slight abuse of notation, we have denoted by w any of its components $w_1, \dots, w_n$.

We now investigate products of derivatives of the rescaled unknowns r, τ and w. Since in our applications r will be a hyperbolic variable, the total number of derivations appearing in the estimates needs to be decreased by using a weighted L^∞ norm of the gradients.

Theorem 4.11 *Let $k \geq 2$ be an integer, $\bar\theta > 0$ be positive, $1 < p < \infty$, τ, r, w be such that $\tau - \tau_\infty, r - r_\infty, w \in W^{k-1,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$ for some constants τ_∞ and r_∞. Let a, b, $\bar a$, and $\bar b$ be constants with $|a| + |b| \leq \bar\theta$, $|\bar a| + |\bar b| \leq \bar\theta$, and let $g = \exp(a\tau + br)$ and $h = \exp(\bar a\tau + \bar b r)$. Let $l \geq 2$, let α^j, $1 \leq j \leq l$, be multiindices with $|\alpha^j| \geq 1$, $1 \leq j \leq l$, and $\sum_{1\leq j\leq l} |\alpha^j| = k$. There exist positive constants $\delta(k,n,p,\bar\theta)$ and $c(k,n,p,\bar\theta)$, only depending on $(k,n,p,\bar\theta)$, such that if $\|r\|_{BMO} + \|\tau\|_{BMO} < \delta$, then whenever $gh^{k-1}\partial^{k-1}r$, $gh^{k-1}\partial^{k-1}w$, $gh^{k-1}\partial^{k-1}\tau$, $gh^{k-2}\partial^{k-2}r$, $gh^{k-2}\partial^{k-2}w$, $gh^{k-2}\partial^{k-2}\tau \in L^p(\mathbb{R}^n)$, and $1 \leq \bar l \leq \tilde l \leq l$, we have the estimates*

$$\begin{aligned}\Big\| gh^k \prod_{1\leq j\leq \bar l} \partial^{\alpha^j} r \prod_{\bar l+1\leq j\leq \tilde l} \partial^{\alpha^j} w \prod_{\tilde l+1\leq j\leq l} \partial^{\alpha^j}\tau \Big\|_{L^p} \leq \; & c\|\widetilde{\mathrm{z}}\|_{BMO}^{l-2} \|h\partial_x\widetilde{\mathrm{z}}\|_{L^\infty} \|gh^{k-1}\partial^{k-1}\widetilde{\mathrm{z}}\|_{L^p} \\ & + c\mathbf{1}_{k\geq 3}\|\widetilde{\mathrm{z}}\|_{BMO}^{(l-3)^+} \|h\partial_x\widetilde{\mathrm{z}}\|_{L^\infty}^2 \|gh^{k-2}\partial^{k-2}\widetilde{\mathrm{z}}\|_{L^p},\end{aligned} \tag{4.18}$$

where we have denoted $\widetilde{\mathrm{z}} = (r, w, \tau)$ and

$$\|\widetilde{\mathrm{z}}\|_{BMO} = \|r\|_{BMO} + \|w\|_{BMO} + \|\tau\|_{BMO}, \tag{4.19}$$

$$\|h\partial_x\widetilde{\mathrm{z}}\|_{L^\infty} = \|h\partial_x r\|_{L^\infty} + \|h\partial_x w\|_{L^\infty} + \|h\partial_x \tau\|_{L^\infty}, \tag{4.20}$$

$$\|gh^m\partial^m\widetilde{\mathrm{z}}\|_{L^p}^p = \|gh^m\partial^m r\|_{L^p}^p + \|gh^m\partial^m w\|_{L^p}^p + \|gh^m\partial^m \tau\|_{L^p}^p \tag{4.21}$$

for any $m \in \mathbb{N}^*$ and in the left hand member of (4.18), with a slight abuse of notation, we have denoted by w any of its components $w_1, \dots, w_n$. In particular, in the situation where $2 \leq k \leq 3$, the second term in the right hand side of in (4.18) is absent.

Proof. Theorems 4.10 and 4.11 are direct consequences of the multilinear estimates of Theorems 4.7 and 4.8. □

4.5 Weighted products of renormalized derivatives

We now estimate products of derivatives of density, temperature and velocity components rescaled by the proper renormalizing factors.

Theorem 4.12 *Let $k \geq 1$ be an integer, $\bar\theta > 0$ be positive , $1 < p < \infty$, T be such that $T \geq T_{\min} > 0$ and $T - T_\infty \in W^{k,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$ for some positive constant T_∞ and ρ be positive such that $r = \log\rho \in BMO$. There exist positive constants $\delta(n,p,\bar\theta)$ and $c(k,n,p)$, only depending on $(n,p,\bar\theta)$ and (k,n,p), respectively, such that if $\|\log\rho\|_{BMO} + \|\log T\|_{BMO} < \delta$, then for any real a and b such that*

$|a|+|b|\leq\bar{\theta}$, any integer $l\geq 1$, and any multiindices α^j, $1\leq j\leq l$, with $|\alpha^j|\geq 1$, $1\leq j\leq l$, and $\sum_{1\leq j\leq l}|\alpha^j|=k$, whenever $T^a\rho^b(\partial^k T)/T\in L^p(\mathbb{R}^n)$, we have the estimates

$$\Big\|T^a\rho^b\prod_{1\leq j\leq l}\frac{\partial^{\alpha^j}T}{T}\Big\|_{L^p}\leq\ c\big\|\log T\big\|_{BMO}^{l-1}\ \big\|T^a\rho^b\frac{\partial^k T}{T}\big\|_{L^p}. \tag{4.22}$$

Assuming $v\in W^{k,2}(\mathbb{R}^n)\cap\mathcal{C}_0^0(\mathbb{R}^n)$, $\|\log\rho\|_{BMO}+\|v/\sqrt{T}\|_{L^\infty}+\|\log T\|_{BMO}<\delta$, whenever $T^a\rho^b(\partial^k v)/\sqrt{T}\in L^p(\mathbb{R}^n)$, we have for $0\leq\bar{l}\leq l$

$$\Big\|T^a\rho^b\prod_{1\leq j\leq\bar{l}}\frac{\partial^{\alpha^j}v}{\sqrt{T}}\prod_{\bar{l}+1\leq j\leq l}\frac{\partial^{\alpha^j}T}{T}\Big\|_{L^p}\leq\ c\Big(\|\frac{v}{\sqrt{T}}\|_{L^\infty}+\big\|\log T\big\|_{BMO}\Big)^{l-1}$$
$$\times\Big(\big\|T^a\rho^b\frac{\partial^k v}{\sqrt{T}}\big\|_{L^p}+\big\|T^a\rho^b\frac{\partial^k T}{T}\big\|_{L^p}\Big), \tag{4.23}$$

where, in the left hand member, with a slight abuse of notation, we have denoted by v any of its components $v_1,\dots,v_n$.

Theorem 4.13 *Let $k\geq 2$ be an integer, $\bar{\theta}>0$ be positive , $1<p<\infty$, ρ, v, T, be such that $\rho\geq\rho_{\min}$, $T\geq T_{\min}$, and $\rho-\rho_\infty, v, T-T_\infty\in W^{k-1,2}(\mathbb{R}^n)\cap\mathcal{C}_0^1(\mathbb{R}^n)$ for positive constants ρ_∞, $\rho_{\min}$, T_∞ and $T_{\min}$. Let a, b $\bar{a}$, and $\bar{b}$ be constants with $|a|+|b|\leq\bar{\theta}$, $|\bar{a}|+|\bar{b}|\leq\bar{\theta}$, and let $g=T^a\rho^b$, $h=T^{\bar{a}}\rho^{\bar{b}}$. Let $l\geq 2$, α^j, $1\leq j\leq l$, be multiindices with $|\alpha^j|\geq 1$, $1\leq j\leq l$, and $\sum_{1\leq j\leq l}|\alpha^j|=k$. There exist positive constants $\delta(k,n,p,\bar{\theta})$ and $c(k,n,p,\bar{\theta})$, only depending on $(k,n,p,\bar{\theta})$, such that if $\|\log\rho\|_{BMO}+\|v/\sqrt{T}\|_{L^\infty}+\|\log T\|_{BMO}<\delta(k,n,p,\bar{\theta})$, then whenever $gh^{k-1}(\partial^{k-1}\rho)/\rho$, $gh^{k-1}(\partial^{k-1}v)/\sqrt{T}$, $gh^{k-1}(\partial^{k-1}T)/T$, $gh^{k-2}(\partial^{k-2}\rho)/\rho$, $gh^{k-2}(\partial^{k-2}v)/\sqrt{T}$, $gh^{k-2}(\partial^{k-2}T)/T\in L^p(\mathbb{R}^n)$, we have for $0\leq\bar{l}\leq\tilde{l}\leq l$*

$$\Big\|gh^k\prod_{1\leq j\leq\bar{l}}\frac{\partial^{\alpha^j}\rho}{\rho}\prod_{\bar{l}+1\leq j\leq\tilde{l}}\frac{\partial^{\alpha^j}v}{\sqrt{T}}\prod_{\tilde{l}+1\leq j\leq l}\frac{\partial^{\alpha^j}T}{T}\Big\|_{L^p}\leq c\|\widetilde{z}\|_{BMO}^{\prime\,l-2}\|h\partial_x\widetilde{z}\|'_{L^\infty}\|gh^{k-1}\partial^{k-1}\widetilde{z}\|'_{L^p}$$
$$+\,c\mathbf{1}_{k\geq 3}\|\widetilde{z}\|_{BMO}^{\prime\,(l-3)^+}\|h\partial_x\widetilde{z}\|_{L^\infty}^{\prime 2}\|gh^{k-2}\partial^{k-2}\widetilde{z}\|'_{L^p}, \tag{4.24}$$

where, in the left hand member, with a slight abuse of notation, we have denoted by v any of its components $v_1,\dots,v_n$, and where $\widetilde{z}=(r,w,\tau)$ and

$$\|\widetilde{z}\|'_{BMO}=\|\log\rho\|_{BMO}+\|v/\sqrt{T}\|_{L^\infty}+\|\log T\|_{BMO}, \tag{4.25}$$

$$\|h\partial_x\widetilde{z}\|'_{L^\infty}=\|h\frac{\partial_x\rho}{\rho}\|_{L^\infty}+\|h\frac{\partial_x v}{\sqrt{T}}\|_{L^\infty}+\|h\frac{\partial_x T}{T}\|_{L^\infty}, \tag{4.26}$$

$$\|gh^m\partial^m\widetilde{z}\|_{L^p}^{\prime p}=\|gh^m\frac{\partial^m\rho}{\rho}\|_{L^p}^p+\|gh^m\frac{\partial^m v}{\sqrt{T}}\|_{L^p}^p+\|gh^m\frac{\partial^m T}{T}\|_{L^p}^p, \tag{4.27}$$

for any $m\in\mathbb{N}^*$. In particular, in the situation where $2\leq k\leq 3$, the second term in the right hand side of (4.24) is absent. Note that there is a L^∞ norm for the rescaled velocity w in $\|\widetilde{z}\|'_{BMO}$.

Proof. The proof of Theorems 4.12 and 4.13 essentially relies on Theorems 4.10 and 4.11 and on the differential identities established in Lemmas 4.2 and 4.3. Considering temperature as a typical example, the differential identities and Theorem 4.10 yield estimates in the form

$$\Big\|T^a\rho^b\prod_{1\leq j\leq l}\frac{\partial^{\alpha^j}T}{T}\Big\|_{L^p}\leq c\big\|\log T\big\|_{BMO}^{l-1}\ \big\|T^a\rho^b\partial^k\tau\big\|_{L^p}, \tag{4.28}$$

and similarly that

$$\Big\|T^a\rho^b\big(\frac{\partial^k T}{T}-\partial^k\tau\big)\Big\|_{L^p}\leq c\big\|\log T\big\|_{BMO}\ \big\|T^a\rho^b\partial^k\tau\big\|_{L^p},$$

where $c=c(k,n,p)$. Therefore for $c(k,n,p)\|\log T\|_{BMO}<1/2$ we have

$$\tfrac{1}{2}\,\big\|T^a\rho^b\partial^k\tau\big\|_{L^p}\ \leq\ \big\|T^a\rho^b\frac{\partial^k T}{T}\big\|_{L^p}\ \leq\ \tfrac{3}{2}\,\big\|T^a\rho^b\partial^k\tau\big\|_{L^p}, \tag{4.29}$$

and reinserting (4.29) in (4.28) completes the proof of (4.22). The same procedure can be applied to get estimates of $\big\|T^a\rho^b\partial^k\rho/\rho\big\|_{L^p}$ and $\big\|T^a\rho^b\partial^k v/\sqrt{T}\big\|_{L^p}$ and then to obtain (4.23) and (4.24). □

Remark 4.14 *Assuming that* $T - T_\infty \in W^{2,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$, $T \geq T_{\min} > 0$ *and* $\| \log T\|_{BMO}$ *is small enough, we obtain from Theorem 4.13 that*

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^6}{T^{5+a}}\, dx \leq c \|\log T\|_{BMO}^2 \Big\| \frac{\partial_x T}{T} \Big\|_{L^\infty}^2 \int_{\mathbb{R}^n} \frac{|\partial^2 T|^2}{T^{1+a}}\, dx. \tag{4.30}$$

In contrast, when $T - T_\infty \in W^{3,2}(\mathbb{R}^n) \cap \mathcal{C}_0^0(\mathbb{R}^n)$, $T \geq T_{\min} > 0$, *and* $\| \log T\|_{BMO}$ *is small enough, we obtain from Theorem 4.12 that*

$$\int_{\mathbb{R}^n} \frac{|\partial_x T|^6}{T^{5+a}}\, dx \leq c \|\log T\|_{BMO}^4 \int_{\mathbb{R}^n} \frac{|\partial^3 T|^2}{T^{1+a}}\, dx. \tag{4.31}$$

5 Higher order entropy estimates

In this section we investigate higher order entropy estimates for compressible flows spanning the whole space. We establish entropic inequalities when the quantities $\| \log \rho\|_{BMO}$, $\|v/\sqrt{T}\|_{L^\infty}$, $\| \log T\|_{BMO}$, $\|h\partial_x \rho/\rho\|_{L^\infty}$, $\|h\partial_x v/\sqrt{T}\|_{L^\infty}$, $\|h\partial_x T/T\|_{L^\infty}$, and $\|h^2\partial_x^2 T/T\|_{L^\infty}$ are small enough, where $h = 1/\rho T^{\frac{1}{2}-\varkappa}$ is a weight asssociated with the dependence of the local mean free path $l = \eta/\rho\sqrt{r_{\mathrm{g}} T}$ on density and temperature. In the following, all constants associated with a priori estimates and entropic inequalities may depend on the system parameters $\underline{\mathfrak{a}}$, $\overline{\mathfrak{a}}$, $\overline{\mathfrak{a}}_\sigma$, $\sigma \geq 1$, $\varkappa$, and c_v. However, these dependencies are made implicit in order to avoid notational complexities and only the dependence on k and n is made explicit.

5.1 Preliminaries

The balance equations of higher order correctors can be integrated over $\mathbb{R}^n$ and $[0, t]$ where $0 \leq t \leq \bar{t}$ thanks to the regularity properties of the solution. Considering the $\gamma^{[k]}$ balance equation (3.25) as a typical example, we have the following result.

Lemma 5.1 *Let* (ρ, v, T) *be a smooth solution of the compressible Navier-Stokes equations* (3.1)–(3.5) *with regularity* (3.6)(3.7) *and let* $1 \leq k \leq l$. *Then the following equation holds in* $\mathcal{D}'(0, \bar{t}\,)$ *and* $L^1(0, \bar{t}\,)$

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \int_{\mathbb{R}^n} \big(\pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]}\big)\, dx = 0, \tag{5.1}$$

and the following equation holds in $C^0[0, \bar{t}\,]$

$$\int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \int_0^t \int_{\mathbb{R}^n} \big(\pi_\gamma^{[k]} + \Sigma_\gamma^{[k]} + \omega_\gamma^{[k]}\big)\, dx = \int_{\mathbb{R}^n} \gamma_0^{[k]}\, dx, \tag{5.2}$$

where $\gamma_0^{[k]}$ *denotes the functional* $\gamma^{[k]}$ *evaluated at initial conditions.*

Proof. This lemma results from standard manipulations using distributional derivatives and test functions in the form of tensor products $\varphi(t)\psi(x)$. □

As a consequence of Lemma 5.1, integrating the balance equation (3.25) for $\gamma^{[k]}$ with $1 \leq k \leq l$, we deduce that

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k]}\, dx + \int_{\mathbb{R}^n} \pi_\gamma^{[k]}\, dx \leq \int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}|\, dx + \int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx, \tag{5.3}$$

so that we have to investigate the integrals $\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k]}|\, dx$ and $\int_{\mathbb{R}^n} |\omega_\gamma^{[k]}|\, dx$. Similarly, we obtain by integrating the balance equation (3.30) for $\gamma^{[k-\frac{1}{2}]}$ that

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[k-\frac{1}{2}]}\, dx + \int_{\mathbb{R}^n} \pi_\gamma^{[k-\frac{1}{2}]}\, dx \leq \int_{\mathbb{R}^n} |\Sigma_\gamma^{[k-\frac{1}{2}]}|\, dx + \int_{\mathbb{R}^n} |\omega_\gamma^{[k-\frac{1}{2}]}|\, dx, \tag{5.4}$$

and we have to investigate the integrals $\int_{\mathbb{R}^n} |\Sigma_\gamma^{[k-\frac{1}{2}]}|\, dx$ and $\int_{\mathbb{R}^n} |\omega_\gamma^{[k-\frac{1}{2}]}|\, dx$. We will simultaneouly estimate the analogous integrals $\int_{\mathbb{R}^n} |\Sigma_{\tilde{\gamma}}^{[k]}|\, dx$ and $\int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k]}|\, dx$ associated with the balance equation of

$\tilde{\gamma}^{[k]}$ as well as the integrals $\int_{\mathbb{R}^n} |\Sigma_{\tilde{\gamma}}^{[k-\frac{1}{2}]}|\, dx$ and $\int_{\mathbb{R}^n} |\omega_{\tilde{\gamma}}^{[k-\frac{1}{2}]}|\, dx$ associated with the balance equations for $\tilde{\gamma}^{[k-\frac{1}{2}]}$.

It will be convenient to denote by χ_γ the quantity

$$\begin{aligned}\chi_\gamma = \|\log\rho\|_{BMO} + \|\frac{v}{\sqrt{T}}\|_{L^\infty} + \|\log T\|_{BMO} \\ + \|h\frac{\partial_x\rho}{\rho}\|_{L^\infty} + \|h\frac{\partial_x v}{\sqrt{T}}\|_{L^\infty} + \|h\frac{\partial_x T}{T}\|_{L^\infty} + \|h^2\frac{\partial_x^2 T}{T}\|_{L^\infty},\end{aligned} \tag{5.5}$$

and similarly by $\chi_{\tilde{\gamma}}$ the quantity

$$\begin{aligned}\chi_{\tilde{\gamma}} = \|r\|_{BMO} + \|w\|_{L^\infty} + \|\tau\|_{BMO} \\ + \|hr\|_{L^\infty} + \|hw\|_{L^\infty} + \|h\tau\|_{L^\infty} + \|h^2\partial^2\tau\|_{L^\infty}.\end{aligned} \tag{5.6}$$

It can easily be established that $\chi_\gamma \le \chi_{\tilde{\gamma}}(1+\chi_{\tilde{\gamma}})$ and $\chi_{\tilde{\gamma}} \le \chi_\gamma(1+\chi_\gamma)$ so that $\chi_\gamma \le 1$ implies that $\frac{1}{3}\chi_\gamma \le \chi_{\tilde{\gamma}} \le 2\chi_\gamma$, and $\chi_{\tilde{\gamma}} \le 1$ implies that $\frac{1}{3}\chi_{\tilde{\gamma}} \le \chi_\gamma \le 2\chi_{\tilde{\gamma}}$, and assuming that either χ_γ or $\chi_{\tilde{\gamma}}$ is small is equivalent. We will establish that entropic inequalities hold for $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ when χ_γ or $\chi_{\tilde{\gamma}}$ are small enough. These quantities χ_γ and $\chi_{\tilde{\gamma}}$ are invariant under the change of scales (3.8) described in Remark 3.1. They can also be interpreted as involving the natural variables $\log\rho$, $v/\sqrt{r_{\rm g}T}$, and $\log T$, appearing in Maxwellian distributions[5] and the natural scale h associated with the local mean free path $\eta/\rho\sqrt{r_{\rm g}T}$. Since we have formally $v/\sqrt{r_{\rm g}T} = \mathcal{O}(\mathrm{Ma})$, $\log(T/T_\infty) = \mathcal{O}(\mathrm{Ma})$, and $\log(\rho/\rho_\infty) = \mathcal{O}(\mathrm{Ma})$, where Ma denotes the Mach number, the constraint that χ_γ or $\chi_{\tilde{\gamma}}$ remain small may be interpreted as a small Mach number constraint, which is consistent with Enskog expansion[20].

5.2 A priori estimates

We first investigate the integrals $\int_{\mathbb{R}^n} |\Sigma_\xi^{[k]}|\, dx$ and $\int_{\mathbb{R}^n} |\omega_\xi^{[k]}|\, dx$, where ξ denotes any of the symbols γ or $\tilde{\gamma}$, by using the weighted inequalities established in Section 4.

Proposition 5.2 *Let (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations* (3.1)–(3.5) *with regularity* (3.6)(3.7)*, let $1 \le k \le l$, and let ξ denote any of the symbols γ or $\tilde{\gamma}$. There exist positive constants $\delta(k,n)$ and $c_k = c(k,n)$ such that for $\chi_\xi < \delta$ we have*

$$\int_{\mathbb{R}^n} |\Sigma_\xi^{[k]}|\, dx \le c_k\, \chi_\xi \int_{\mathbb{R}^n} \big(\pi_\xi^{[k]} + \pi_\xi^{[k-\frac{1}{2}]} + \pi_\xi^{[k-1]} + \mathbf{1}_{k\ge 2}(\pi_\xi^{[k-\frac{3}{2}]} + \pi_\xi^{[k-2]})\big)\, dx. \tag{5.7}$$

$$\int_{\mathbb{R}^n} |\omega_\xi^{[k]}|\, dx \le c_k\, \chi_\xi \int_{\mathbb{R}^n} \big(\pi_\xi^{[k]} + \pi_\xi^{[k-\frac{1}{2}]} + \pi_\xi^{[k-1]} + \mathbf{1}_{k\ge 2}(\pi_\xi^{[k-\frac{3}{2}]} + \pi_\xi^{[k-2]})\big)\, dx. \tag{5.8}$$

Proof. We only give the proof for $\xi = \tilde{\gamma}$ since the proof for $\xi = \gamma$ is similar. We have from (3.36)

$$\Sigma_{\tilde{\gamma}}^{[k]} = \sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi} e^{-\varkappa\tau} \partial_\tau^\sigma\phi\, \Pi_\nu^{(k+1)} \Pi_\mu^{(k+1)} + \frac{k(1-2\varkappa)\lambda}{e^{\varkappa\tau} c_v} g^2 h^{2(k+1)} |\partial^k r|^2 \Delta\tau,$$

and the integral associated with the last term is directly majorized by

$$\int_{\mathbb{R}^n} \frac{\lambda}{e^{\varkappa\tau}} g^2 h^{2(k+1)} |\partial^k r|^2 |\Delta\tau|\, dx \le c\|h^2\partial^2\tau\|_{L^\infty} \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k-\frac{1}{2}]}\, dx,$$

where c is a constant since $\lambda e^{-\varkappa\tau}$ is bounded. Considering then the terms of $\Sigma_{\tilde{\gamma}}^{[k]}$ appearing in the sum we observe that the quantities $e^{-\varkappa\tau}\partial_\tau^\sigma\phi$ are bounded since $\partial_\tau^\sigma\phi = \sum_{1\le m\le\sigma} c_{\sigma m} T^m \partial_T^m \phi$, where $c_{\sigma m}$ are constants and where $\phi \in \{\kappa, \eta, \lambda\}$, so that we only have to estimate the L^2 norms of the products $\Pi_\nu^{(k+1)}$.

When $\Pi_\nu^{(k+1)}$ only contains derivatives of w and τ—in particular if there is a derivative of order $k+1$—we obtain from Theorem 4.10 applied to (w,τ) with k replaced by $k+1$, that when $\chi_{\tilde{\gamma}}$ is small enough

$$\|\Pi_\nu^{(k+1)}\|_{L^2} \le c\left(\|\log T\|_{BMO} + \|\frac{v}{\sqrt{T}}\|_{L^\infty}\right)^{N_\nu - 1} \left\{\int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[k]}\, dx\right\}^{\frac{1}{2}}, \tag{5.9}$$

where $N_\nu = \sum_{1\le|\alpha|\le k+1}(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = \sum_{1\le|\alpha|\le k+1}(\nu'_\alpha + \nu''_\alpha)$. However, if the product $\Pi_\nu^{(k+1)}$ is split—in particular if there is a derivative of density—we obtain from Theorem 4.11 applied to (r, w, τ) with k replaced by $k+1$, that when $\chi_{\tilde\gamma}$ is small enough

$$\begin{aligned}\|\Pi_\nu^{(k+1)}\|_{L^2} \le & c\,\|\widetilde{\mathrm{z}}\|_{BMO}^{N_\nu-2}\|h\partial_x\widetilde{\mathrm{z}}\|_{L^\infty}\,\|gh^k\partial^k\widetilde{\mathrm{z}}\|_{L^2}\\ &+ c\mathbf{1}_{k\ge2}\|\widetilde{\mathrm{z}}\|_{BMO}^{(N_\nu-3)^+}\,\|h\partial_x\widetilde{\mathrm{z}}\|_{L^\infty}^2\,\|gh^{k-1}\partial^{k-1}\widetilde{\mathrm{z}}\|_{L^2},\end{aligned}$$

keeping the notation of Theorem 4.11 for $\|h\partial_x\widetilde{\mathrm{z}}\|_{L^\infty}$ and $\|gh^m\partial^m\widetilde{\mathrm{z}}\|_{L^2}$. Therefore, we obtain that

$$\|\Pi_\nu^{(k+1)}\|_{L^2} \le c\,\chi_{\tilde\gamma}^{N_\nu-1}\left\{\int_{\mathbb{R}^n}\Big(\pi_{\tilde\gamma}^{[k-\frac12]} + \pi_{\tilde\gamma}^{[k-1]} + \mathbf{1}_{k\ge2}(\pi_{\tilde\gamma}^{[k-\frac32]} + \pi_{\tilde\gamma}^{[k-2]})\Big)\,dx\right\}^{\frac12}, \tag{5.10}$$

where $c = c(k,n)$ thanks to $\chi_{\tilde\gamma} \le 1$ and

$$\|gh^l\partial^l\widetilde{\mathrm{z}}\|_{L^2}^2 \le b\int_{\mathbb{R}^n}(\pi_{\tilde\gamma}^{[l-\frac12]} + \pi_{\tilde\gamma}^{[l-1]})\,dx, \quad 1\le l\le k,$$

where b is independent of l and n. Since one of the two products $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is split, we can combine the inequalities (5.9) and (5.10) in the form

$$\|\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}\|_{L^1} \le c\,\chi_{\tilde\gamma}\int_{\mathbb{R}^n}\Big(\pi_{\tilde\gamma}^{[k]} + \pi_{\tilde\gamma}^{[k-1]} + \pi_{\tilde\gamma}^{[k-\frac12]} + \mathbf{1}_{k\ge2}(\pi_{\tilde\gamma}^{[k-\frac32]} + \pi_{\tilde\gamma}^{[k-2]})\Big)\,dx,$$

where c depends on k and n. On the other hand, in the expression of $\omega_{\tilde\gamma}^{[k]}$, the products $\Pi_\mu^{(k+1)}$ are always split between several derivative factors, so that the inequality (5.8) is established in a similar way. The proof in the situation $\xi = \gamma$ is similar with Theorems 4.10 and 4.11 replaced by Theorems 4.12 and 4.13. □

Proposition 5.3 *Let (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations* (3.1)–(3.5) *with regularity* (3.6)(3.7)*, let $1\le k\le l$, and let ξ denote any of the symbols γ or $\tilde\gamma$. There exist positive constants $\delta(k,n)$ and $c_k = c(k,n)$ such that for $\chi_\xi < \delta$ we have*

$$\begin{aligned}\int_{\mathbb{R}^n}|\Sigma_\xi^{[k-\frac12]}|\,dx \le c_k\,\chi_\xi\int_{\mathbb{R}^n}&\big(\pi_\xi^{[k]} + \pi_\xi^{[k-\frac12]} + \pi_\xi^{[k-1]} + \mathbf{1}_{k\ge2}(\pi_\xi^{[k-\frac32]} + \pi_\xi^{[k-2]})\big)\,dx\\ &+ \mathsf{c}_0\left\{\int_{\mathbb{R}^n}\pi_\xi^{[k]}\,dx\right\}^{\frac12}\left\{\int_{\mathbb{R}^n}\pi_\xi^{[k-\frac12]}\,dx\right\}^{\frac12},\end{aligned} \tag{5.11}$$

$$\begin{aligned}\int_{\mathbb{R}^n}|\omega_\xi^{[k-\frac12]}|\,dx \le c_k\,\chi_\xi\int_{\mathbb{R}^n}&\big(\pi_\xi^{[k]} + \pi_\xi^{[k-\frac12]} + \pi_\xi^{[k-1]} + \mathbf{1}_{k\ge2}(\pi_\xi^{[k-\frac32]} + \pi_\xi^{[k-2]})\big)\,dx\\ &+ \mathsf{c}_0'\int_{\mathbb{R}^n}\pi_\xi^{[k-1]}\,dx + \mathsf{c}_0'\left\{\int_{\mathbb{R}^n}\pi_\xi^{[k-\frac12]}\,dx\right\}^{\frac12}\left\{\int_{\mathbb{R}^n}\pi_\xi^{[k-1]}\,dx\right\}^{\frac12},\end{aligned} \tag{5.12}$$

where c_0 and c_0' are constants independent of k and n.

Proof. Considering first the case $\xi = \gamma$ and the expression (3.41) for $\Sigma_\gamma^{[k-\frac12]}$, all terms in the sum are estimated as in the proof of Proposition 5.2. More specifically, the L^2 norm of $\Pi_\nu^{(k)}$ is estimated with Theorem 4.12 applied to ρ, v, and T, whereas the L^2 norm of the split product $\Pi_\mu^{(k+1)}$ is estimated with Theorem 4.13 applied to ρ, v, ad T with k replaced by $k+1$. Furthermore, the remaining extra terms are directly estimated in terms of $\pi_\gamma^{[k]}$, $\pi_\gamma^{[k-\frac12]}$ and $\pi_\gamma^{[k-1]}$. The same argument is valid for $\omega_\gamma^{[k-\frac12]}$ using the expression (3.33) as well as in the case $\xi = \tilde\gamma$ using (3.42) and (3.42). □

5.3 Zeroth order entropic inequalities

We now recast the classical zeroth order entropic inequality into a convenient form that will be used to investigate entropic principles associated with $\Gamma^{[k]}$.

Proposition 5.4 *Let $\gamma^{[0]}$ be given by* (3.17)*. Then $\gamma^{[0]} \geq 0$ and the following balance equation holds*

$$\partial_t \gamma^{[0]}/\mathsf{C}_0 + \partial_x\cdot\Big(\rho v(s_\infty - s) + \rho v c_p \frac{T-T_\infty}{T_\infty}\Big) + \partial_x\cdot\Big(\frac{q}{T_\infty} - \frac{q}{T} + \frac{\Pi\cdot v}{T}\Big) + \Big(\frac{\lambda|\partial_x T|^2}{T^2} + \frac{\eta|d|^2}{2T} + \frac{\kappa(\partial_x\cdot v)^2}{T}\Big)\, dx = 0. \tag{5.13}$$

Morover there exists positive constants B_0 and $\delta_0 > 0$ such that for $\mathsf{C}_0 \geq \mathsf{B}_0$ and $\chi_\gamma < \delta_0$ we have

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \int_{\mathbb{R}^n} \pi_\gamma^{[0]}\, dx \leq 0, \tag{5.14}$$

where we define from (3.26)

$$\pi_\gamma^{[0]} = 2g^2h^2\Big(\frac{\lambda}{T^\varkappa}\,\frac{|\partial_x T|^2}{T^2} + \frac{\eta}{T^\varkappa}\frac{|\partial_x v|^2}{T} + \frac{\frac{1}{3}\eta+\kappa}{T^\varkappa}\frac{(\partial_x\cdot v)^2}{T}\Big).$$

Proof. It is easily established that both the temperature and density parts of $\gamma^{[0]}$ are nonnegative so that $\gamma^{[0]} \geq 0$. Multiplying the total mass equation by $(\partial_\rho S^{(0)})_\infty = e_\infty/T_\infty + r_{\rm g} - s_\infty$, the total energy equation by $(\partial_{\mathcal{E}^{\rm tot}} S^{(0)})_\infty = 1/T_\infty$, and substracting this linear combination from the fluid entropy governing equation yields (5.13). Integrating this balance equation (5.13), keeping in mind the regularity assumptions such that fluxes and sources are in $L^1\big((0,\bar{t}\,), L^1(\mathbb{R}^n)\big)$, we obtain that

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \mathsf{C}_0 \int_{\mathbb{R}^n}\Big(\frac{\lambda|\partial_x T|^2}{T^2} + \frac{\eta|d|^2}{2T} + \frac{\kappa(\partial_x\cdot v)^2}{T}\Big)\, dx = 0.$$

From the properties of the transport coefficients we obtain

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \mathsf{C}_0(\underline{\mathfrak{a}}/2) \int_{\mathbb{R}^n} T^\varkappa\Big(\frac{|\partial_x T|^2}{T^2} + \frac{|\partial_x v + (\partial_x v)^t|^2}{T}\Big)\, dx \leq 0.$$

On the other hand, for any $v \in W^{1,2}$ and any index pair (i,j) we have[42]

$$2\partial_j v_i = (\partial_j v_i + \partial_i v_j) - \sum_{1\leq l\leq n} R_l R_j(\partial_l v_i + \partial_i v_l) + \sum_{1\leq l\leq n} R_l R_i(\partial_l v_j + \partial_j v_l) \tag{5.15}$$

where $R_i = (-\Delta)^{-1/2}\partial_i$ are the Riesz transforms, $1 \leq i \leq n$, and from the continuity of Calderòn-Zygmund operators in weighted Legesgue spaces established in Theorem 4.6 we deduce that there exits a constant $\bar{c}(n,\varkappa)$ such that

$$\int_{\mathbb{R}^n} \frac{|\partial_x v|^2}{T^{1-\varkappa}}\, dx \leq \bar{c} \int_{\mathbb{R}^n} \frac{|\partial_x v + (\partial_x v)^t|^2}{T^{1-\varkappa}}\, dx,$$

for $\|\log T\|_{BMO} < \delta(n,\varkappa)$ small enough. By combining these estimates and by using that $T^\varkappa = g^2h^2$ we obtain

$$\partial_t \int_{\mathbb{R}^n} \gamma^{[0]}\, dx + \mathsf{C}_0(\underline{\mathfrak{a}}/2\overline{\mathfrak{a}})\frac{1}{1+\bar{c}}\frac{1}{1+\frac{4}{3}n} \int_{\mathbb{R}^n} \pi_\gamma^{[0]}\, dx \leq 0,$$

and selecting $\mathsf{C}_0 \geq 2(1+\bar{c})(3+4n)\overline{\mathfrak{a}}/\underline{\mathfrak{a}}$ completes the proof. $\square$

We also recast the classical zeroth order entropic inequality into a convenient form that will be needed to investigate entropic principles associated with $\widetilde{\Gamma}^{[k]}$.

Proposition 5.5 *Let $\tilde{\gamma}^{[0]} = \gamma^{[0]}$ by given be* (3.17)*. Then $\tilde{\gamma}^{[0]} \geq 0$ and the balance equation* (5.13) *holds. Moreover there exists positive constants B_0 and $\delta_0 > 0$ such that for $\mathsf{C}_0 \geq \mathsf{B}_0$ and $\chi_{\tilde{\gamma}} < \delta_0$*

$$\partial_t \int_{\mathbb{R}^n} \tilde{\gamma}^{[0]}\, dx + \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx \leq 0, \tag{5.16}$$

where we define from (3.35)

$$\pi_{\tilde{\gamma}}^{[0]} = 2g^2h^2\Big(\frac{\lambda}{e^{\varkappa\tau}}\,|\partial_x\tau|^2 + \frac{\eta}{e^{\varkappa\tau}}|\partial_x w|^2 + \frac{\frac{1}{3}\eta+\kappa}{e^{\varkappa\tau}}(\partial_x\cdot w)^2\Big).$$

Proof. This is a consequence of the proof of Proposition 5.4 and of the differential relations

$$\frac{\partial_i v}{\sqrt{T}} = \partial_i w + \tfrac{1}{2} w \partial_i \tau, \qquad 1 \leq i \leq n,$$

which yield that $\int_{\mathbb{R}^n} \pi_\gamma^{[0]}\, dx$ is minorized by $(1 - c\chi_{\tilde{\gamma}}) \int_{\mathbb{R}^n} \pi_{\tilde{\gamma}}^{[0]}\, dx$. □

5.4 Higher order entropic inequalities

Our goal in this section is to obtain entropic inequalities for the $(2k)^{\text{th}}$ order kinetic entropy estimators

$$\Gamma^{[k]} = \gamma^{[0]} + \sum_{1 \leq i \leq k} (\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]}) \qquad k \geq 0, \tag{5.17}$$

and

$$\widetilde{\Gamma}^{[k]} = \tilde{\gamma}^{[0]} + \sum_{1 \leq i \leq k} (\tilde{\gamma}^{[i]} + a\tilde{\gamma}^{[i-\frac{1}{2}]}) \qquad k \geq 0. \tag{5.18}$$

The quantities $\gamma^{[i-\frac{1}{2}]}$ and $\tilde{\gamma}^{[i-\frac{1}{2}]}$, $1 \leq i \leq k$, are multiplied by a small rescaling factor a in (5.17) and (5.18) so as to not modify the majorizing properties of the correctors $\gamma^{[i]}$ and $\tilde{\gamma}^{[i]}$, $i \geq 0$.

Lemma 5.6 *Let (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations* (3.1)–(3.5) *with regularity* (3.6)(3.7)*, assume that $T \geq T_{\min}$. There exists $\mathsf{B}_0(T_\infty, T_{\min})$ such that for $\mathsf{C}_0 \geq \mathsf{B}_0$, $0 < a \leq 1$, and $0 \leq k \leq l$*

$$\tfrac{1}{2}(\gamma^{[0]} + \cdots + \gamma^{[k]}) \leq \Gamma^{[k]} \leq \tfrac{3}{2}(\gamma^{[0]} + \cdots + \gamma^{[k]}), \qquad 0 \leq k \leq l, \tag{5.19}$$

$$\tfrac{1}{2}(\tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]}) \leq \widetilde{\Gamma}^{[k]} \leq \tfrac{3}{2}(\tilde{\gamma}^{[0]} + \cdots + \tilde{\gamma}^{[k]}), \qquad 0 \leq k \leq l. \tag{5.20}$$

Moreover, assuming that $T \geq T_{\min}$ and $\rho \leq \rho_{\max}$, there exists exists $\mathsf{B}_0(T_\infty, T_{\min}, \rho_\infty, \rho_{\max})$ such that for $\mathsf{C}_0 \geq \mathsf{B}_0$,

$$\rho\big(|r - r_\infty|^2 + |w|^2 + c_v|\tau - \tau_\infty|^2\big) \leq \gamma^{[0]}. \tag{5.21}$$

Proof. Using the Cauchy-Schwartz inequality, it is straightforward to check that for any $1 \leq i \leq k \leq l$

$$\begin{aligned} |\gamma^{[i-\frac{1}{2}]}| \leq & \Big\{ \rho h^{2(i-1)} \Big| \frac{\partial^{i-1} v}{\sqrt{T}} \Big|^2 \Big\}^{\frac{1}{2}} \Big\{ \rho h^{2i} \Big| \frac{\partial^i \rho}{\rho} \Big|^2 \Big\}^{\frac{1}{2}} \\ \leq & \frac{1}{2} \Big(\rho h^{2(i-1)} \Big| \frac{\partial^{i-1} v}{\sqrt{T}} \Big|^2 + \rho h^{2i} \Big| \frac{\partial^i \rho}{\rho} \Big|^2 \Big). \end{aligned}$$

Therefore, half of the density part of $\gamma^{[i]}$ and of the velocity part of $\gamma^{[i-1]}$ compensate for $|\gamma^{[i-\frac{1}{2}]}|$ provided we ensure that $\gamma^{[0]} \geq \rho|v/\sqrt{T}|^2$ but this is a consequence of $\mathsf{C}_0 \geq 2T_\infty/T_{\min}$. The same method also applies for the modified estimators $\tilde{\gamma}^{[i-\frac{1}{2}]}$, $1 \leq i \leq k$, and this yields Inequalities (5.19) and (5.20) upon summing over $1 \leq i \leq k$. Inequality (5.21) is a consequence of

$$\frac{T_{\min}}{2T_\infty} |w|^2 \leq \frac{|v|^2}{2T_\infty},$$

$$\frac{T_{\min}}{2T_\infty} |\tau - \tau_\infty|^2 \leq \exp(\tau - \tau_\infty) - 1 - (\tau - \tau_\infty),$$

valid for $\tau_{\min} \leq \tau$, where $\tau_{\min} = \log T_{\min}$, $\tau_\infty = \log T_\infty$ and $T_{\min} \leq T_\infty$, and of

$$\frac{\rho_\infty}{2\rho_{\max}} |r - r_\infty|^2 \leq \exp(r_\infty - r) - 1 - (r_\infty - r),$$

valid for $r \leq r_{\max}$, where $r_{\max} = \log \rho_{\max}$, $r_\infty = \log \rho_\infty$ and $r_\infty \leq r_{\max}$ letting $\mathsf{B}_0 = \max(1, \frac{2T_\infty}{T_{\min}}, \frac{\rho_{\max}}{2\rho_\infty})$ and $\mathsf{C}_0 \geq \mathsf{B}_0$. □

Theorem 5.7 *Let (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations* (3.1)–(3.5) *with regularity* (3.6)(3.7) *and let $1 \leq k \leq l$. There exist positive constants $\bar{a}$ and $\delta_{\mathrm{N}}(k,n)$ such that for $a \leq \bar{a}$ and $\chi_\gamma < \delta_{\mathrm{N}} a$ we have*

$$\partial_t \int_{\mathbb{R}^n} \Gamma^{[k]}\, dx + \frac{1}{5} \int_{\mathbb{R}^n} \big(\pi_\gamma^{[0]} + \sum_{1\leq i\leq k} (\pi_\gamma^{[i]} + a\pi_\gamma^{[i-\frac{1}{2}]}) \big)\, dx \leq 0, \tag{5.22}$$

and for $a \leq \bar{a}$ and $\chi_{\tilde{\gamma}} < \delta_{\mathrm{N}} a$ we have

$$\partial_t \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, dx + \frac{1}{5} \int_{\mathbb{R}^n} \big(\pi_{\tilde{\gamma}}^{[0]} + \sum_{1\leq i\leq k} (\pi_{\tilde{\gamma}}^{[i]} + a\pi_{\tilde{\gamma}}^{[i-\frac{1}{2}]}) \big)\, dx \leq 0. \tag{5.23}$$

Proof. We only consider the case $\xi = \gamma$ since the proof is similar for the modified estimators $\xi = \tilde{\gamma}$. From the differential inequality (5.3) for $\gamma^{[i]}$, $1 \leq i \leq k \leq l$, and the results of Proposition 5.2, we obtain that

$$\begin{aligned}\partial_t \int_{\mathbb{R}^n} \gamma^{[i]}\, dx + (1 - 2c_i\, \chi_\gamma) \int_{\mathbb{R}^n} \pi_\gamma^{[i]}\, dx &\leq 2c_i\, \chi_\gamma \int_{\mathbb{R}^n} \big(\pi_\gamma^{[i-\frac{1}{2}]} + \pi_\gamma^{[i-1]}\big)\, dx \\ &+ \mathbf{1}_{i\geq 2} 2c_i\, \chi_\gamma \int_{\mathbb{R}^n} (\pi_\gamma^{[i-\frac{3}{2}]} + \pi_\gamma^{[i-2]})\, dx.\end{aligned} \tag{5.24}$$

Similarly, from the differential inequality (5.4), and the results of Proposition 5.3 we obtain that

$$\begin{aligned}\partial_t \int_{\mathbb{R}^n} \gamma^{[i-\frac{1}{2}]}\, dx + (1 - 2\epsilon_0 - 2c_i\, \chi_\gamma) \int_{\mathbb{R}^n} \pi_\gamma^{[i-\frac{1}{2}]}\, dx \leq (\frac{\mathsf{c}_0^2}{4\epsilon_0} + 2c_i\, \chi_\gamma) \int_{\mathbb{R}^n} \pi_\gamma^{[i]}\, dx \\ + (\mathsf{c}_0' + \frac{\mathsf{c}_0'^2}{4\epsilon_0} + 2c_i\, \chi_\gamma) \int_{\mathbb{R}^n} \pi_\gamma^{[i-1]}\, dx + \mathbf{1}_{i\geq 2} 2c_i\, \chi_\gamma \int_{\mathbb{R}^n} (\pi_\gamma^{[i-\frac{3}{2}]} + \pi_\gamma^{[i-2]})\, dx.\end{aligned} \tag{5.25}$$

Forming (5.24)+a(5.25), we obtain after some algebra

$$\begin{aligned}\partial_t \int_{\mathbb{R}^n} (\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]})\, dx + \big(1 - 2c_i\chi_\gamma - a(\frac{\mathsf{c}_0^2}{4\epsilon_0} + 2c_i\, \chi_\gamma)\big) \int_{\mathbb{R}^n} \pi_\gamma^{[i]}\, dx \\ + \Big(a(1 - 2\epsilon_0 - 2c_i\chi_\gamma) - 2c_i\chi_\gamma\Big) \int_{\mathbb{R}^n} \pi_\gamma^{[i-\frac{1}{2}]}\, dx \leq \\ + \Big(a(\mathsf{c}_0' + \frac{\mathsf{c}_0'^2}{4\epsilon_0} + 2c_i\, \chi_\gamma) + 2c_i\, \chi_\gamma\Big) \int_{\mathbb{R}^n} \pi_\gamma^{[i-1]}\, dx \\ + \mathbf{1}_{i\geq 2} 2(1+a)c_i\, \chi_\gamma \int_{\mathbb{R}^n} (\pi_\gamma^{[i-\frac{3}{2}]} + \pi_\gamma^{[i-2]})\, dx.\end{aligned} \tag{5.26}$$

Assuming then that

$$\begin{aligned}0 < a \leq 1, \qquad 2\epsilon_0 = \frac{1}{10}, \qquad 2\Big(\max_{1\leq i\leq k} c_i\Big)\chi_\gamma \leq \frac{a}{10}, \\ a\frac{\mathsf{c}_0^2}{4\epsilon_0} \leq \frac{1}{10}, \qquad a(\mathsf{c}_0' + \frac{\mathsf{c}_0'^2}{4\epsilon_0}) \leq \frac{1}{10},\end{aligned}$$

that is, $a \leq \bar{a}$ and $\chi_\gamma < \delta_{\mathrm{N}} a$ with

$$\bar{a} = \min\Big(1, \frac{4\epsilon_0}{10\mathsf{c}_0^2}, \frac{4\epsilon_0}{10(\mathsf{c}_0'^2 + 4\epsilon_0\mathsf{c}_0')}\Big), \qquad \delta_{\mathrm{N}} = \frac{1}{20 \max\limits_{1\leq i\leq k} c_i},$$

we obtain that

$$\begin{aligned}\partial_t \int_{\mathbb{R}^n} (\gamma^{[i]} + a\gamma^{[i-\frac{1}{2}]})\, dx + \frac{7}{10} \int_{\mathbb{R}^n} (\pi_\gamma^{[i]} + a\pi_\gamma^{[i-\frac{1}{2}]})\, dx \leq \\ + \frac{3}{10} \int_{\mathbb{R}^n} \pi_\gamma^{[i-1]}\, dx + \mathbf{1}_{i\geq 2} \frac{2}{10}\, \chi_\gamma \int_{\mathbb{R}^n} (a\pi_\gamma^{[i-\frac{3}{2}]} + \pi_\gamma^{[i-2]})\, dx.\end{aligned} \tag{5.27}$$

Summing for $1 \leq i \leq k$, and adding to the zeroth order inequality (5.14) we finally obtain (5.22) and the proof of (5.23) is similar. □

Corollary 5.8 *Let (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations* (3.1)–(3.5) *with regularity* (3.6)(3.7) *and let $1 \le k \le l$. There exist positive constants $\underline{b}$ and $\delta'_{\rm N}(k,n)$ such that for the fixed value $a = \bar{a}$ of Theorem 5.7 when $\chi_\gamma < \delta'_{\rm N}$ we have*

$$\partial_t \int_{\mathbb{R}^n} \Gamma^{[k]}\, dx + \underline{b} \int_{\mathbb{R}^n} \rho T^{1-\varkappa} \big(\gamma^{[1]} + \cdots + \gamma^{[k]}\big)\, dx \le 0, \tag{5.28}$$

and when $\chi_{\tilde{\gamma}} < \delta'_{\rm N}$ we have

$$\partial_t \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[k]}\, dx + \underline{b} \int_{\mathbb{R}^n} \rho T^{1-\varkappa} \big(\tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[k]}\big)\, dx \le 0. \tag{5.29}$$

Proof. This is a consequence of Theorem 5.7 in the special situation $a = \bar{a}$ letting $\delta'_{\rm N} = \delta_{\rm N}\bar{a}$, $\underline{b} = \bar{a}\min(1, \underline{\mathfrak{a}})/5(1 + c_v)$, and using $\rho T^{1-\varkappa} = g^2/\rho$. □

Theorem 5.7 and Corollary 5.8 show that the $(2k)^{\rm th}$ order kinetic entropy estimator $\Gamma^{[k]}$ and $\widetilde{\Gamma}^{[k]}$ obey entropic principles. Upon integrating these inequalities (5.28) and (5.29), a priori estimates are obtained for the solutions of the compressible Navier-Stokes equations. These entropic inequalities and the related a priori estimates are also invariant—up to a multiplicative factor—by the change of scales (3.8) described in Remark 3.1 and naturally associated to the Navier-Stokes equations. Since we have formally $v/\sqrt{r_{\rm g} T} = \mathcal{O}({\rm Ma})$, $\log(T/T_\infty) = \mathcal{O}({\rm Ma})$, and $\log(\rho/\rho_\infty) = \mathcal{O}({\rm Ma})$, where Ma denotes the Mach number, the constraint that χ_γ or $\chi_{\tilde{\gamma}}$ remain small may be interpreted as a small Mach number constraint, which is consistent with Enskog expansion[20]. These estimates also provide a thermodynamic interpretation of the corresponding weighted Sobolev norms involving either renormalized derivatives for $\Gamma^{[k]}$, or derivatives of the renormalized variable $\widetilde{\rm z}$—which is also a normal variable[19, 27]—for $\widetilde{\Gamma}^{[k]}$, and involving as well the dependence on density and temperature of the local mean free path through the factor h. This factor h ensure in particular that the operator $h\partial_x$ is scale invariant.

6 Global solutions

In this section, as an example of application of higher order entropy estimates, we investigate global existence of smooth solutions when the initial values $\log(\rho_0/\rho_\infty)$, $v_0/\sqrt{T_0}$, and $\log(T_0/T_\infty)$ are small enough in appropriate weighted spaces.

6.1 Local existence

We denote by z the combined unknown ${\rm z} = (\rho, v, T)$ and accordingly by ${\rm z}_\infty$ the equilibrium point ${\rm z}_\infty = (\rho_\infty, 0, T_\infty)$ with $\rho_\infty > 0$, $v_\infty = 0$ and $T_\infty > 0$. We denote by $\mathcal{O}_{\rm z} = (0,\infty) \times \mathbb{R}^n \times (0,\infty)$ the natural domain for the variable z.

Theorem 6.1 *Let $n \ge 1$ and $l \ge [n/2]+3$ be integers and let $b > 0$ be given. Let $\mathcal{O}_0$ be an open bounded convex set such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}_{\rm z}$, d_1 with $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\rm z})$, and define $\mathcal{O}_1 = \{\, {\rm z} \in \mathcal{O}_{\rm z};\ d({\rm z}, \overline{\mathcal{O}}_0) < d_1 \,\}$. There exists $\bar{t} > 0$ small enough, which only depends on $\mathcal{O}_0$, d_1, and b, such that for any ${\rm z}_0$ with $\|{\rm z}_0 - {\rm z}_\infty\|_{W^{l,2}} < b$ and ${\rm z}_0 \in \overline{\mathcal{O}}_0$, there exists a unique local solution ${\rm z} = (\rho, v, T)$ to the system* (3.1)–(3.3) *with initial condition*

$$\big(\rho(0,x), v(0,x), T(0,x)\big) = \big(\rho_0(x), v_0(x), T_0(x)\big), \tag{6.1}$$

such that

$$\big(\rho(t,x), v(t,x), T(t,x)\big) \in \mathcal{O}_1, \tag{6.2}$$

and

$$\rho - \rho_\infty \in C^0\big([0,\bar{t}], W^{l,2}(\mathbb{R}^n)\big) \cap C^1\big([0,\bar{t}], W^{l-1,2}(\mathbb{R}^n)\big), \tag{6.3}$$

$$v, T - T_\infty \in C^0\big([0,\bar{t}], W^{l,2}(\mathbb{R}^n)\big) \cap C^1\big([0,\bar{t}], W^{l-2,2}(\mathbb{R}^n)\big) \cap L^2\big((0,\bar{t}), W^{l+1,2}(\mathbb{R}^n)\big). \tag{6.4}$$

In addition, there exists $C > 0$ which only depends on $\mathcal{O}_0$, d_1, and b, such that

$$\sup_{0\le s\le \bar{t}} \Big\{ \|\rho(s)-\rho_\infty\|^2_{W^{l,2}} + \|v(s)\|^2_{W^{l,2}} + \|T(s)-T_\infty\|^2_{W^{l,2}} \Big\}$$
$$+ \int_0^{\bar{t}} \Big\{ \|\rho(s)-\rho_\infty\|^2_{W^{l,2}} + \|v(s)\|^2_{W^{l+1,2}} + \|T(s)-T_\infty\|^2_{W^{l+1,2}} \Big\}\, ds$$
$$\le C\Big(\|\rho_0-\rho_\infty\|^2_{W^{l,2}} + \|v_0\|^2_{W^{l,2}} + \|T_0-T_\infty\|^2_{W^{l,2}} \Big). \tag{6.5}$$

Proof. We refer the reader to Kawashima[25, 26] for a general proof concerning hyperbolic-parabolic symmetric systems in normal form. This proof is also adapted to the parameter dependent case in Giovangigli and Graille[18]. □

6.2 Properties of the solutions

We establish in this section that the solutions constructed in Theorem 6.1 are as smooth as expected from initial data.

Theorem 6.2 *The solutions obtained in Theorem 6.1 inherit the regularity of* z_0*, that is, for any* $k \ge l$ *such that* $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{k,2}$*, we have*

$$\rho - \rho_\infty \in C^0\big([0,\bar{t}], W^{k,2}\big) \cap C^1\big([0,\bar{t}], W^{k-1,2}\big), \tag{6.6}$$

$$v,\, T - T_\infty \in C^0\big([0,\bar{t}], W^{k,2}\big) \cap C^1\big([0,\bar{t}], W^{k-2,2}\big) \cap L^2\big((0,\bar{t}), W^{k+1,2}\big). \tag{6.7}$$

In particular, z *is smooth when* $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{k,2}(\mathbb{R}^n)$ *for any* $k \in \mathbb{N}$.

Proof. Let $k \ge l$ be such that $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{k,2}$ and denote by $e^{[k]}$ the quantity $e^{[k]} = |\partial^k \rho|^2 + |\partial^k v|^2 + |\partial^k T|^2$. We have to estimate $e^{[k]}$ in order to establish (6.6)(6.7).

Assume first that the regularity properties (6.6)(6.7) hold. A balance equation for $e^{[k]}$ can easily be derived—and is simpler than that of $\gamma^{[k]}$ of $\tilde{\gamma}^{[k]}$—and written in the form

$$\partial_t e^{[k]} + \partial_x\!\cdot\!(v e^{[k]}) + \partial_x\!\cdot\!\varphi_e^{[k]} + \pi_e^{[k]} + \Sigma_e^{[k]} + \omega_e^{[k]} = 0. \tag{6.8}$$

This equation holds in $\mathcal{D}'\big((0,\bar{t})\times\mathbb{R}^n\big)$ and $L^1\big((0,\bar{t}), W^{-1,1}\big)$, $e^{[k]} \in C^0\big([0,\bar{t}], L^1\big)$, and $\varphi_e^{[k]}$, $\pi_e^{[k]}$, $\Sigma_e^{[k]}$, $\omega_e^{[k]} \in L^1\big((0,\bar{t}), L^1(\mathbb{R}^n)\big)$. The term $\pi_e^{[k]}$ is given by

$$\pi_e^{[k]} = \frac{2\lambda}{\rho c_v}\, |\partial^{k+1} T|^2 + \frac{2\eta}{\rho} |\partial^{k+1} v|^2 + \frac{2(\frac{1}{3}\eta + \kappa)}{\rho} |\partial^k (\partial_x\!\cdot\! v)|^2, \tag{6.9}$$

and the term $\Sigma_e^{[k]}$ is in the form

$$\Sigma_e^{[k]} = \sum_{\sigma\nu\mu} c_{\sigma\nu\mu\phi} T^{a_{\nu\mu\phi}} \rho^{b_{\nu\mu\phi}} \partial_T^\sigma \phi\, \widehat{\Pi}_\nu^{(k+1)} \widehat{\Pi}_\mu^{(k+1)}, \tag{6.10}$$

where the sums are over $0 \le \sigma \le k$, $\phi \in \{\lambda, \eta, \kappa\}$, $\nu = (\nu_\alpha, \nu'_\alpha, \nu''_\alpha)_{1\le|\alpha|\le k+1}$, $\mu = (\mu_\alpha, \mu'_\alpha, \mu''_\alpha)_{1\le|\alpha|\le k+1}$, $\nu_\alpha, \nu'_\alpha, \nu''_\alpha, \mu_\alpha, \mu'_\alpha, \mu''_\alpha \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$. The quantities $a_{\nu\mu\phi}$ and $b_{\nu\mu\phi}$ are integers depending on ν, μ and ϕ. The products $\widehat{\Pi}_\nu^{(k+1)}$ are defined by

$$\widehat{\Pi}_\nu^{(k+1)} = \prod_{1\le|\alpha|\le k+1} \big(\partial^\alpha \rho\big)^{\nu_\alpha} \big(\partial^\alpha v\big)^{\nu'_\alpha} \big(\partial^\alpha T\big)^{\nu''_\alpha}, \tag{6.11}$$

where v denotes any of its components $v_1, \dots, v_n$, and ν must be such that

$$\sum_{1\le|\alpha|\le k+1} |\alpha| (\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k+1, \qquad \sum_{|\alpha|=k+1} \nu_\alpha = 0,$$

so that there is a total of $k+1$ derivations and there is no derivative of order $k+1$ of density. In addition, we have $\sum_{|\alpha|=k+1} (\nu'_\alpha + \nu''_\alpha + \mu'_\alpha + \mu''_\alpha) \le 1$, so that there is at most one derivative of $(k+1)^{\text{th}}$

order in the product $\widehat{\Pi}_\nu^{(k+1)}\widehat{\Pi}_\mu^{(k+1)}$. Furthermore the term $\omega_e^{[k]}$ is given by

$$\omega_e^{[k]} = \sum_{\nu\mu} c_{\nu\mu} T^{a_{\nu\mu}} \rho^{b_{\nu\mu}} \widehat{\Pi}_\nu^{(k)}\widehat{\Pi}_\mu^{(k+1)} - \frac{2T}{c_v}\partial^k T\,\partial^k(\partial_x\cdot v) \tag{6.12}$$

$$+ \frac{2T}{\rho}\partial^k\rho\,\partial^k(\partial_x\cdot v) - 2\rho\partial^k\rho\,\partial^k(\partial_x\cdot v), \tag{6.13}$$

where we use similar notation for $\widehat{\Pi}_\nu^{(k)}$ as for $\widehat{\Pi}_\mu^{(k+1)}$ and the summation extends over

$$\sum_{1\le|\alpha|\le k} |\alpha|(\nu_\alpha + \nu'_\alpha + \nu''_\alpha) = k, \qquad \sum_{1\le|\alpha|\le k} |\alpha|(\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = k+1,$$

so that in particular $\sum_{|\alpha|=k+1}(\mu_\alpha + \mu'_\alpha + \mu''_\alpha) = 0$ and there are always at least two derivative factors in the product $\widehat{\Pi}_\mu^{(k+1)}$. Finally the flux $\varphi_e^{[k]} = (\varphi_{e1}^{[k]},\dots,\varphi_{en}^{[k]})$ is in the form

$$\varphi_{el}^{[k]} = \sum_{\sigma\nu\mu l} c_{\sigma\nu\mu\phi l} T^{a_{\nu\mu\phi}} \rho^{b_{\nu\mu\phi}} \partial_T^\sigma\phi\, \widehat{\Pi}_\nu^{(k)}\widehat{\Pi}_\mu^{(k+1)}. \tag{6.14}$$

After integrating Equation (6.8) over $\mathbb{R}^n$ and using uniform lower bounds on $\lambda/\rho c_v$ and η/ρ thanks to (6.2), we obtain that these exists a $\delta > 0$ with

$$\partial_t \int_{\mathbb{R}^n} e^{[j]}\,dx + 2\delta \int_{\mathbb{R}^n} \big(|\partial^{j+1}T|^2 + |\partial^{j+1}v|^2\big)\,dx \le c \int_{\mathbb{R}^n} \big(|\Sigma_e^{[k]}| + |\omega_e^{[k]}|\big)\,dx, \quad 1\le j\le k.$$

Now regrouping all derivatives of order $k+1$ appearing in $\Sigma_e^{[k]}$ in the left member, using $xy \le \epsilon x^2 + 4y^2/\epsilon$, we only have to estimate the L^2 norm of multiple products with $k+1$ derivations $\widehat{\Pi}_\nu^{(k+1)}$ with at least two derivative factors or of multiple products with only k derivations $\widehat{\Pi}_\nu^{(k)}$. From Theorem 4.8 and since $\|\mathrm{z}-\mathrm{z}_\infty\|_{L^\infty}$, and $\|\partial\mathrm{z}\|_{L^\infty}$, are finite thanks to $l > n/2+1$, whenever the product $\widehat{\Pi}_\nu^{(k+1)}$ is split, we have estimates in the form

$$\|\widehat{\Pi}_\nu^{(k+1)}\|_{L^2}^2 \le c\big(1 + \|\partial\mathrm{z}\|_{L^\infty}\big)^{2(k-1)} \int_{\mathbb{R}^n} \big(e^{[1]} + \cdots + e^{[k]}\big)\,dx,$$

where c only depends on $\|\mathrm{z}\|_{L^\infty}$. The products $\widehat{\Pi}_\nu^{(k)}$ are also estimated thanks to Theorem 4.7. Combining these estimates, we obtain after some algebra that

$$\partial_t \int_{\mathbb{R}^n} e^{[j]}\,dx + \delta \int_{\mathbb{R}^n} (|\partial^{j+1}T|^2 + |\partial^{j+1}v|^2)\,dx \le c \int_{\mathbb{R}^n} \big(e^{[1]} + \cdots + e^{[k]}\big)\,dx, \quad 1\le j\le k,$$

where δ and c depend on L^∞ estimates of z and $\partial\mathrm{z}$. Upon summing these inequalities and using Gronwall lemma we deduce that $\int_{\mathbb{R}^n} e^{[k]}\,dx$ remain uniformly bounded over the whole time interval under consideration $[0,\bar{t}\,]$ and we thus have a uniform upper bound B for the sobolev norm $\|\mathrm{z}-\mathrm{z}_\infty\|_{W^{k,2}} \le B$. This also implies that $\int_0^{\bar{t}}\int_{\mathbb{R}^n}|\partial^{j+1}T|^2\,dxdt$ and $\int_0^{\bar{t}}\int_{\mathbb{R}^n}|\partial^{j+1}v|^2\,dxdt$ are finite.

Now from the local existence theorem, there exists a positive time $0 < t' \le \bar{t}$ constructed with the parameters $\mathcal{O}_0$, d_1 and $2B$, where a solution with regularity (6.6)(6.7) exists and coincide with z. The preceding estimates then show that the local existence theorem can be used repeatedly over $[0,\bar{t}\,]$ since we have the uniform bound $\|\mathrm{z}-\mathrm{z}_\infty\|_{W^{k,2}} \le B$ over this interval so that finally (6.6)(6.7) hold over $[0,\bar{t}\,]$. Moreover, when $\mathrm{z}_0 - \mathrm{z}_\infty$ is in $W^{k,2}$ for any $k\ge 0$, $\mathrm{z}-\mathrm{z}_\infty$ is in $C^0([0,\bar{t}\,],W^{k,2})$ for any k, and we recover the regularity with respect to time from the governing equations so that z is smooth. □

In the next propositions, we reformulate for convenience the local existence theorem in terms of the combined unknown $\widetilde{\mathrm{z}} = (r,w,\tau)$ associated with the renormalized variables r, w and τ.

Lemma 6.3 *Denote by $\mathcal{F} : (0,\infty)\times\mathbb{R}^n\times(0,\infty) \to \mathbb{R}^{n+2}$ the application defined by $\mathcal{F}(\mathrm{z}) = \widetilde{\mathrm{z}}$, that is, $\mathcal{F}(\rho,v,T) = (r,w,\tau) = (\log\rho, v/\sqrt{T}, \log T)$. Then $\mathcal{F}$ is a C^∞ diffeomorphism and its jacobian matrix reads*

$$\partial_{\mathrm{z}}\mathcal{F} = \begin{pmatrix} \frac{1}{\rho} & 0 & 0 \\ 0 & \frac{I}{\sqrt{T}} & -\frac{1}{2}\frac{v}{T^{\frac{3}{2}}} \\ 0 & 0 & \frac{1}{T} \end{pmatrix}.$$

Moreover, for any $\mathrm{M}_r > 0$, $\mathrm{M}_w > 0$, $\mathrm{M}_\tau > 0$, defining $\widetilde{\mathcal{O}} = (-\mathrm{M}_r,\mathrm{M}_r)\times(-\mathrm{M}_w,\mathrm{M}_w)^n\times(-\mathrm{M}_\tau,\mathrm{M}_\tau)$, the corresponding open set $\mathcal{O} = \mathcal{F}^{-1}(\widetilde{\mathcal{O}})$ is convex.

Proof. The proof is similar to that of the incompressible case[15]. □

Proposition 6.4 *Let* $\mathrm{M}_r > 0$, $\mathrm{M}_w > 0$, $\mathrm{M}_\tau > 0$, *define*

$$\widetilde{\mathcal{O}}_0 = (-\mathrm{M}_r, \mathrm{M}_r) \times (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau),$$

and $\mathcal{O}_0 = \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$. *Let* $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathrm{z}})$, $\mathcal{O}_1 = \{\, \mathrm{z} \in \mathcal{O}_{\mathrm{z}};\ d(\mathrm{z}, \overline{\mathcal{O}}_0) < d_1 \,\}$, *and select an arbitrary* $b > 0$. *From Theorem 6.1 we have a local solution built with the paramaters* $\mathcal{O}_0$, d_1, *and* b. *This solution is then such that*

$$r - r_\infty \in C^0\big([0,\bar t], W^{l,2}\big) \cap C^1\big([0,\bar t], W^{l-1,2}\big), \tag{6.15}$$

$$w,\, \tau - \tau_\infty \in C^0\big([0,\bar t], W^{l,2}\big) \cap C^1\big([0,\bar t], W^{l-2,2}\big) \cap L^2\big((0,\bar t), W^{l+1,2}\big), \tag{6.16}$$

and there exists $C > 0$ *which only depend on* $\mathcal{O}_0$, d_1, *and* b, *such that*

$$\begin{aligned} &\sup_{0\le s\le \bar t}\Big\{ \|r(s) - r_\infty\|_{W^{l,2}}^2 + \|w(s)\|_{W^{l,2}}^2 + \|\tau(s) - \tau_\infty\|_{W^{l,2}}^2 \Big\} \\ &\quad + \int_0^{\bar t} \Big\{ \|r(s) - r_\infty\|_{W^{l,2}}^2 + \|w(s)\|_{W^{l+1,2}}^2 + \|\tau(s) - \tau_\infty\|_{W^{l+1,2}}^2 \Big\}\, ds \\ &\qquad \le C\Big(\|r_0 - r_\infty\|_{W^{l,2}}^2 + \|w_0\|_{W^{l,2}}^2 + \|\tau_0 - \tau_\infty\|_{W^{l,2}}^2 \Big). \end{aligned} \tag{6.17}$$

Moreover, the kinetic estimators are such that $\Gamma^{[l]}, \widetilde{\Gamma}^{[l]} \in C\big([0,\bar t], L^1(\mathbb{R}^n)\big)$.

Proof. The set $\mathcal{O}_0 = \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$ is convex and from Theorem 6.1, there exists a local solution built with $\mathcal{O}_0$, d_1 and b. We then have estimates in the form

$$\underline{c}_{\mathrm{z}} \|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{l,2}} \le \|\mathrm{z} - \mathrm{z}_\infty\|_{W^{l,2}} \le \overline{c}_{\mathrm{z}} \|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{l,2}}, \tag{6.18}$$

where $\underline{c}_{\mathrm{z}}$ and $\overline{c}_{\mathrm{z}}$ only depend on $\mathcal{O}_1$ and l thanks to the classical estimates

$$\|\psi(\phi) - \psi(0)\|_{W^{k,2}} \le C_0 \|\psi\|_{C^k(\overline{\mathcal{O}}_\phi)} (1 + \|\phi\|_{L^\infty})^{k-1} \|\phi\|_{W^{k,2}}, \tag{6.19}$$

where $\mathcal{O}_\phi$ is a convex open set with $\phi(x) \in \mathcal{O}_\phi$, $x \in \mathbb{R}^n$. Similarly, the regularity properties are direct consequences of the estimates

$$\|\psi(\phi) - \psi(\hat\phi)\|_{W^{k,2}} \le C_0 \|\psi\|_{C^{k+1}(\overline{\mathcal{O}}_\phi)} (1 + \|\phi\|_{W^{k,2}} + \|\hat\phi\|_{W^{k,2}})^k \|\phi - \hat\phi\|_{W^{k,2}}, \tag{6.20}$$

where $\mathcal{O}_\phi$ is a convex open set with $\phi(x) \in \mathcal{O}_\phi$, $\hat\phi(x) \in \mathcal{O}_\phi$, $x \in \mathbb{R}^n$, and k is such that $k \ge [n/2] + 1$. The properties $\Gamma^{[l]}, \widetilde{\Gamma}^{[l]} \in C\big([0,\bar t], L^1(\mathbb{R}^n)\big)$ are then straighforward to establish. □

6.3 Global existence

In this section, we investigate global existence of solutions for which the quantity $\chi_{\tilde\gamma} = \|r\|_{BMO} + \|w\|_{L^\infty} + \|\tau\|_{BMO} + \|hr\|_{L^\infty} + \|hw\|_{L^\infty} + \|h\tau\|_{L^\infty} + \|h^2\partial^2\tau\|_{L^\infty}$ remains small. We investigate solutions with given bounds $\rho_{\min} < \rho < \rho_{\max}$ and $T_{\min} < T < T_{\max}$, where $\rho_{\min} < \rho_\infty < \rho_{\max}$ and $T_{\min} < T_\infty < T_{\max}$, and assume that C_0 has been chosen large enough as in Lemma 5.6. We will also use the results of Corollary 5.8 and assume that the fixed value $a = \bar a$ has been selected for the parameter a in this section.

Theorem 6.5 *Let* $n \ge 1$ *and* $l \ge [n/2] + 3$ *be integers. Assume that the coefficients* λ, κ, *and* η *satisfy* (3.11)(3.12). *There exists* $\delta_\Gamma(l, n, T_{\min}, T_{\max}, \rho_{\min}, \rho_{\max}) > 0$ *such that for* ρ_0, v_0, *and* T_0 *satisfying* $T_{\min} < \inf_{\mathbb{R}^n} T_0$, $\sup_{\mathbb{R}^n} T_0 < T_{\max}$, $\rho_{\min} < \inf_{\mathbb{R}^n}\rho_0$, $\sup_{\mathbb{R}^n}\rho_0 < \rho_{\max}$, $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{l,2}$, *and*

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \le \delta_\Gamma, \tag{6.21}$$

where $\widetilde{\Gamma}_0^{[l]}$ *denotes the functional* $\widetilde{\Gamma}^{[l]}$ *evaluated at initial conditions, there exists a unique global solution* $\mathrm{z} = (\rho, v, T)$ *with initial conditions*

$$\big(\rho(0,x), v(0,x), T(0,x)\big) = \big(\rho_0(x), v_0(x), T_0(x)\big), \tag{6.22}$$

such that

$$\rho - \rho_\infty, r - r_\infty \in C^0\big([0,\infty), W^{l,2}\big) \cap C^1\big([0,\infty), W^{l-1,2}\big), \tag{6.23}$$

$$v, w, T - T_\infty, \tau - \tau_\infty \in C^0\big([0,\infty), W^{l,2}\big) \cap C^1\big([0,\infty), W^{l-2,2}\big), \tag{6.24}$$

$$\partial_x \rho, \partial_x r \in L^2\big((0,\infty), W^{l-1,2}\big) \qquad \partial_x T, \partial_x \tau, \partial_x v, \partial_x w \in L^2\big((0,\infty), W^{l,2}\big), \tag{6.25}$$

and we have the estimates

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx + \underline{b} \int_0^t \int_{\mathbb{R}^n} \rho T^{1-\varkappa} \big(\tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[l]}\big)\, dx dt \le \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx. \tag{6.26}$$

Furthermore, this solution is smooth and we have

$$\lim_{t\to\infty} \|\mathrm{z}(t,\cdot) - \mathrm{z}_\infty\|_{L^\infty} = 0. \tag{6.27}$$

Proof. We investigate solutions such that $\rho_{\min} < \rho < \rho_{\max}$ and $T_{\min} < T < T_{\max}$. For such solutions, thanks to classical estimates in the form

$$\|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{C_0^2} \le c_0 \|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{l_0+2,2}},$$

where $l_0 = [n/2] + 1$ we have the inequalities

$$\|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{L^\infty} + \chi_{\tilde{\gamma}} \le c_\chi \|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{l_0+2,2}},$$

and

$$c_\Gamma \|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{l,2}}^2 \le \int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx,$$

thanks to Lemma 5.6 where c_χ and c_Γ depend on $T_{\min}$, $T_{\max}$, $\rho_{\min}$, $\rho_{\max}$, and l. In order to obtain a value of δ_Γ small enough, so that the higher order entropic estimates of Theorem 5.7 hold, we will ensure that $\delta_\Gamma \le c_\Gamma \delta_{\mathrm{N}}'^2 / 4c_\chi^2$ where δ_{N}' is defined in Corollary 5.8 and this value will indeed insure that $\chi_{\tilde{\gamma}} \le \delta_{\mathrm{N}}'/2$. Corresponding to this value of δ_Γ, we have estimates in the forms $\|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{L^\infty} \le c_\chi (\delta_\Gamma / c_\Gamma)^{1/2}$ and $\|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{l,2}} \le (\delta_\Gamma / c_\Gamma)^{1/2}$. We now select $\mathrm{M}_r > 0$, $\mathrm{M}_w > 0$, and $\mathrm{M}_\tau > 0$, such that

$$\log(\rho_{\min}/\rho_\infty) < -\mathrm{M}_r < \mathrm{M}_r < \log(\rho_{\max}/\rho_\infty),$$

$$\log(T_{\min}/T_\infty) < -\mathrm{M}_\tau < \mathrm{M}_\tau < \log(T_{\max}/T_\infty),$$

and define

$$\widetilde{\mathcal{O}}_0 = (-\mathrm{M}_r, \mathrm{M}_r) \times (-\mathrm{M}_w, \mathrm{M}_w)^n \times (-\mathrm{M}_\tau, \mathrm{M}_\tau),$$

and for $\delta > 0$

$$\widetilde{\mathcal{O}}_\delta = \{\ z \in \mathbb{R}^{n+2};\ \|z - \widetilde{\mathrm{z}}_\infty\| \le c_\chi (\delta / c_\Gamma)^{1/2}\ \}.$$

For δ_0 small enough we have

$$\widetilde{\mathcal{O}}_{2\delta_0} = \{\ z \in \mathbb{R}^{n+2};\ \|z - \widetilde{\mathrm{z}}_\infty\| \le \sqrt{2} c_\chi (\delta_0 / c_\Gamma)^{1/2}\ \} \ \subset \widetilde{\mathcal{O}}_0,$$

and we now set

$$\delta_\Gamma = \min\Big(\frac{c_\Gamma \delta_{\mathrm{N}}'^2}{4 c_\chi^2}, \delta_0 \Big).$$

The open set $\mathcal{O}_0 = \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_0)$ is convex and let $0 < d_1 < d(\overline{\mathcal{O}}_0, \partial \mathcal{O}_{\mathrm{z}})$, and define $\mathcal{O}_1 = \{\, z \in \mathcal{O}_{\mathrm{z}};\ d(z, \mathcal{O}_0) < d_1 \,\}$ and $\widetilde{\mathcal{O}}_1 = \mathcal{F}(\mathcal{O}_1)$. Now for functions taking their values in $\mathcal{O}_1$ we have inequalities in the form $\|\mathrm{z} - \mathrm{z}_\infty\|_{W^{k,2}} \le \overline{c}_{\mathrm{z}} \|\widetilde{\mathrm{z}} - \widetilde{\mathrm{z}}_\infty\|_{W^{k,2}}$ where $\overline{c}_{\mathrm{z}}$ only depends on k and $\mathcal{O}_1$. We thus obtain the a priori estimate $\|\mathrm{z} - \mathrm{z}_\infty\|_{W^{l,2}} \le \overline{c}_{\mathrm{z}} (\delta_\Gamma / c_\Gamma)^{1/2}$. We now set $b = \overline{c}_{\mathrm{z}} (\delta_\Gamma / c_\Gamma)^{1/2} + 1$ and from Theorem 6.1 and Proposition 6.4 we have local solutions over a time interval $[0, \bar{t}\,]$ built with the paramaters $\mathcal{O}_0$, d_1, and b.

Let now ρ_0, v_0, and T_0 satisfy $T_{\min} < \inf_{\mathbb{R}^n} T_0$, $\sup_{\mathbb{R}^n} T_0 < T_{\max}$, $\rho_{\min} < \inf_{\mathbb{R}^n}\rho$, $\sup_{\mathbb{R}^n}\rho < \rho_{\max}$, $\mathrm{z}_0 - \mathrm{z}_\infty \in W^{l,2}$, and $\int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \leq \delta_\Gamma$. Then by construction $\mathrm{z}_0 \in \mathcal{O}_0$ and $\|\mathrm{z}_0 - \mathrm{z}_\infty\|_{W^{l,2}} < b$, and we have a local solution over the time interval $[0, \bar{t}\,]$. Letting

$$\begin{aligned}\chi_{\tilde{\gamma}}(t) =& \|r(t,\cdot)\|_{BMO} + \|w(t,\cdot)\|_{L^\infty} + \|\tau(t,\cdot)\|_{BMO} + \|h(t,\cdot)\partial_x r(t,\cdot)\|_{L^\infty} \\ &+ \|h(t,\cdot)\partial_x w(t,\cdot)\|_{L^\infty} + \|h(t,\cdot)\partial_x \tau(t,\cdot)\|_{L^\infty} + \|h^2(t,\cdot)\partial_x^2 \tau(t,\cdot)\|_{L^\infty},\end{aligned}$$

we also have by construction $\chi_{\tilde{\gamma}}(0) \leq \delta'_{\mathrm{N}}/2$ and we claim that for any $t \in [0, \bar{t}\,]$ we also have $\chi_{\tilde{\gamma}}(t) \leq \delta'_{\mathrm{N}}/2$. We introduce the set

$$\mathcal{E} = \{\ s \in\ (0, \bar{t}\,]; \forall t \in [0, s],\ \chi_{\tilde{\gamma}}(t) \leq (2/3)\delta'_{\mathrm{N}}, \quad \mathrm{z}(t) \in \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_{2\delta_0})\ \},$$

which is not empty since $t \to \chi_{\tilde{\gamma}}(t)$ is continuous, $\chi_{\tilde{\gamma}}(0) \leq \delta_{\mathrm{N}}/2$, and $\widetilde{\mathrm{z}}(0) \in \widetilde{\mathcal{O}}_{\delta_0}$ so that $\mathrm{z}(0) \in \mathcal{F}^{-1}(\widetilde{\mathcal{O}}_{\delta_0})$. Denoting $\mathsf{e} = \sup \mathcal{E}$ we have $\chi_{\tilde{\gamma}}(t) \leq (2/3)\delta'_{\mathrm{N}}$ over $[0, \mathsf{e}]$ so that the entropic estimates of Theorem 5.7 hold and we have

$$\int_{\mathbb{R}^n} \widetilde{\Gamma}^{[l]}\, dx \leq \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx \leq \delta_\Gamma, \qquad 0 \leq t \leq \mathsf{e}.$$

This now implies that $\chi_{\tilde{\gamma}}(t) \leq \delta'_{\mathrm{N}}/2$ and that $\rho_{\min} < \rho < \rho_{\max}$ and $T_{\min} < T < T_{\max}$ uniformly over $[0, \mathsf{e}]$ so that $\mathsf{e} = \bar{t}$. From the above a priori estimates, we also obtain that for $t \in [0, \bar{t}\,]$ we have $\|\widetilde{\mathrm{z}}(t) - \widetilde{\mathrm{z}}_\infty\|_{L^\infty} \leq c_\chi(\delta_\Gamma/c_\Gamma)^{1/2}$, so that $\mathrm{z}(t) \in \mathcal{O}_0$, and $\|\mathrm{z}(t) - \mathrm{z}_\infty\|_{W^{l,2}} \leq b - 1 < b$, in particular at $t = \bar{t}$. We may now use again the local existence theorem over $[\bar{t}, 2\bar{t}\,]$ and an easy induction shows that the solution is a global solution.

The asymptotic stability is obtained by letting $\Phi(t) = \int_{\mathbb{R}^n} (\tilde{\gamma}^{[1]} + \cdots + \tilde{\gamma}^{[l-1]})\, dx$ and establishing that

$$\int_0^\infty |\Phi(t)|\, dt + \int_0^\infty |\partial_t \Phi(t)|\, dt \leq C \int_{\mathbb{R}^n} \widetilde{\Gamma}_0^{[l]}\, dx.$$

This shows that $\lim_{t\to\infty} \|\partial_x \widetilde{\mathrm{z}}(t,\cdot)\|_{W^{l-2,2}} = 0$, and using the interpolation inequality

$$\|\phi\|_{C^0} \leq C_0\ \|\partial_x^{l-1}\phi\|_{L^2}^{a}\ \|\phi\|_{L^2}^{1-a},$$

where $n/a = 2(l-1)$ we conclude that $\lim_{t\to\infty} \|\widetilde{\mathrm{z}}(t,\cdot) - \widetilde{\mathrm{z}}_\infty\|_{C^0} = 0$, and next that $\lim_{t\to\infty} \|\mathrm{z}(t,\cdot) - \mathrm{z}_\infty\|_{C^0} = 0$. □

Asymptotic stability of constant equilibrium states is usually obtained for $\mathrm{z}_0 - \mathrm{z}_\infty$ small enough in appropriate spaces. Assuming that $\log(\rho_0/\rho_\infty)$, $v_0/\sqrt{T_0}$, and $\log(T_0/T_\infty)$ are small enough seems more natural since these quantities are scale invariant and since the Knudsen and Mach numbers are of the same order of magnitude. The corresponding a priori estimates have a natural thermodynamic interpretation with higher order entropies. A complete analysis of the asymptotic expansions for small Mach and Knudsen numbers, however, is out of the scope of the present paper[16, 17].

7 Conclusion

We have investigated higher order kinetic entropy estimators for compressible fluid models in the natural situation where the volume viscosity, the shear viscosity and the thermal conductivity coefficients depend on temperature. We have establish that entropic inequalities hold for such estimators provided that the quantities $\|\log\rho\|_{BMO}$, $\|v/\sqrt{T}\|_{L^\infty}$, $\|\log T\|_{BMO}$, $\|h\partial_x\rho/\rho\|_{L^\infty}$, $\|h\partial_x v/\sqrt{T}\|_{L^\infty}$, $\|h\partial_x T/T\|_{L^\infty}$, and $\|h^2\partial_x^2 T/T\|_{L^\infty}$, are small enough. As an example of application, we have established a global existence theorem provided that the initial values $\log(\rho_0/\rho_\infty)$, $\log(T_0/T_\infty)$ and $v_0/\sqrt{T_0}$ are small enough in appropriate weighted spaces.

A Derivation of the $\gamma^{[k]}$ balance equation

We derive the balance equation for the entropic correctors $\gamma^{[k]}$. The proof is lengthy and tedious but presents no serious difficulties.

To obtain more concise analytic expressions it is convenient to define $a_k = 1 + k(1 - 2\varkappa)$ and $b_k = -1 + 2k$ in such a way that

$$\gamma^{[k]} = \frac{1}{T^{a_k-1}\rho^{b_k}}\Big(\frac{|\partial^k\rho|^2}{\rho^2} + \frac{|\partial^k v|^2}{T} + c_v\frac{|\partial^k T|^2}{T^2}\Big).$$

In order to obtain a balance equation for $\gamma^{[k]}$ for smooth solutions we form its time differential $\partial_t\gamma^{[k]}$

$$\begin{aligned}\partial_t\gamma^{[k]} &+ \Big(\frac{(a_k-1)|\partial^k\rho|^2}{T^{a_k}\rho^{b_k+2}} + \frac{a_k|\partial^k v|^2}{T^{a_k+1}\rho^{b_k}} + \frac{c_v(a_k+1)|\partial^k T|^2}{T^{a_k+2}\rho^{b_k}}\Big)\partial_t T \\ &+ \Big(\frac{(b_k+2)|\partial^k\rho|^2}{T^{a_k-1}\rho^{b_k+3}} + \frac{b_k|\partial^k v|^2}{T^{a_k}\rho^{b_k+1}} + \frac{c_v b_k|\partial^k T|^2}{T^{a_k+1}\rho^{b_k+1}}\Big)\partial_t\rho - 2c_v\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha T\partial^\alpha\partial_t T}{T^{a_k+1}\rho^{b_k}} \\ &- 2\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha\rho\partial^\alpha\partial_t\rho}{T^{a_k-1}\rho^{b_k+2}} - 2\sum_{\substack{1\le i\le n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i\partial^\alpha\partial_t v_i}{T^{a_k}\rho^{b_k}} = 0, \qquad \text{(A.1)}\end{aligned}$$

and we use the governing equations in order to express $\partial_t T$, $\partial_t\rho$ and $\partial_t v$ in terms of spatial gradients

$$\partial_t\rho = -\rho\partial_x\!\cdot v - v\!\cdot\partial_x\rho, \qquad \text{(A.2)}$$

$$\partial_t v_i = \frac{1}{\rho}\sum_{1\le j\le n}\partial_j\big(\eta\partial_j v_i + \eta\partial_i v_j + (\kappa - \tfrac{2}{3}\eta)\partial_x\!\cdot v\delta_{ij}\big) - \frac{1}{\rho}\partial_i(\rho T) - v\!\cdot\partial_x v_i, \qquad \text{(A.3)}$$

$$\partial_t T = \frac{1}{\rho c_v}\sum_{1\le j\le n}\partial_j(\lambda\partial_j T) + \frac{\eta}{2\rho c_v}|d|^2 + \frac{\kappa}{\rho c_v}(\partial_x\!\cdot v)^2 - \frac{T}{c_v}\partial_x\!\cdot v - v\!\cdot\partial_x T. \qquad \text{(A.4)}$$

We denote respectively by $\mathcal{T}^T$, $\mathcal{T}^\rho$, $\mathcal{T}^{\partial T}$, $\mathcal{T}^{\partial\rho}$, and $\mathcal{T}^{\partial v}$, the five sums appearing in the governing equation for $\partial_t\gamma^{[k]}$, keeping in mind that the time derivative terms $\partial_t\rho$, $\partial_t v$ and $\partial_t T$ have been replaced by their expressions (A.2)–(A.4). We first examine separately higher order derivative contributions associated with each sum $\mathcal{T}^T$, $\mathcal{T}^\rho$, $\mathcal{T}^{\partial T}$, $\mathcal{T}^{\partial\rho}$, and $\mathcal{T}^{\partial v}$. The lower order derivative terms of convective origin are examined all together at the end.

The term in $\mathcal{T}^T$ associated with $|\partial^k\rho|^2\lambda\Delta T$, which is not of the admissible form, is isolated in $\Sigma_\gamma^{[k]}$ whereas all terms associated with $|\partial^k\rho|^2|\partial_x T|^2$, $|\partial^k\rho|^2|d|^2$, and $|\partial^k\rho|^2|\partial_x\!\cdot v|^2$ are of the admissible form, that is, in the form

$$\sum_{\sigma\nu\mu\phi} c_{\sigma\nu\mu\phi}T^{\sigma-\varkappa}\partial_T^\sigma\phi\;\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$$

where $c_{\sigma\nu\mu\phi}$ are constants and the products $\Pi_\nu^{(k+1)}$ and $\Pi_\mu^{(k+1)}$ are defined by

$$\Pi_\nu^{(k+1)} = g\;h^{k+1}\prod_{1\le|\alpha|\le k+1}\Big(\frac{\partial^\alpha\rho}{\rho}\Big)^{\nu_\alpha}\;\Big(\frac{\partial^\alpha v}{\sqrt{T}}\Big)^{\nu'_\alpha}\;\Big(\frac{\partial^\alpha T}{T}\Big)^{\nu''_\alpha}.$$

The sums are over $\phi\in\{\lambda,\eta,\kappa\}$, $0\le\sigma\le k$, $\nu = (\nu_\alpha,\nu'_\alpha,\nu''_\alpha)_{1\le|\alpha|\le k+1}$, $\mu = (\mu_\alpha,\mu'_\alpha,\mu''_\alpha)_{1\le|\alpha|\le k+1}$, $\nu_\alpha,\nu'_\alpha,\nu''_\alpha,\mu_\alpha,\mu'_\alpha,\mu''_\alpha\in\mathbb{N}$, $\alpha\in\mathbb{N}^n$. and μ and ν must be such that $\sum_{1\le|\alpha|\le k+1}|\alpha|(\nu_\alpha+\nu'_\alpha+\nu''_\alpha) = k+1$, $\sum_{1\le|\alpha|\le k+1}(\mu_\alpha+\mu'_\alpha+\mu''_\alpha) = k+1$, $\sum_{|\alpha|=k+1}(\nu_\alpha+\mu_\alpha) = 0$, $\sum_{|\alpha|=k+1}(\nu'_\alpha+\nu''_\alpha+\mu'_\alpha+\mu''_\alpha)\le 1$, so that there is no derivative of order $k+1$ of density and at most one derivative of order $k+1$ of temperature or velocity components in the product $\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$. In particular, one of the terms $\Pi_\nu^{(k+1)}$ or $\Pi_\mu^{(k+1)}$ is always split between two or more derivative factors.

Similarly, all terms of $\mathcal{T}^T$ in the form $|\partial^k T|^2|d|^2$, $|\partial^k T|^2|\partial_x\!\cdot v|^2$ $|\partial^k v|^2|d|^2$, and $|\partial^k v|^2|\partial_x\!\cdot v|^2$ are of the admissible form. On the other hand, the terms associated with $|\partial^k T|^2\partial_x\!\cdot(\lambda\partial_x T)$ and $|\partial^k T|^2\partial_x\!\cdot(\lambda\partial_x T)$ are integrated by parts. They yield flux contributions and source terms in the form

$$-\sum_{1\le l\le n}\partial_l\Big(\frac{(a_k+1)|\partial^k T|^2}{T^{2+a_k}\rho^{b_k}} + \frac{a_k|\partial^k v|^2}{c_v T^{1+a_k}\rho^{b_k}}\Big)\lambda\partial_l T,$$

which are easily rewritten as sums of terms like $c_{\sigma\nu\mu}T^{\sigma-\varkappa}\partial_T^\sigma\lambda\;\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\text{th}}$ order. All other contributions from $\mathcal{T}^T$ as well as all contributions from $\mathcal{T}^\rho$ and $\mathcal{T}^{\partial\rho}$ are of lower order type.

We now consider the term $\mathcal{T}^{\partial T}$ with each contribution at a time. The most important contribution in $\mathcal{T}^{\partial T}$ is that associated with

$$-2c_v \sum_{\substack{1\leq l\leq n\\|\alpha|=k}} \frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{a_k+1}\rho^{b_k}}\partial^\alpha\Big(\frac{1}{\rho c_v}\partial_l(\lambda\partial_l T)\Big).$$

We then write

$$\frac{1}{\rho}\partial_l(\lambda\partial_l T) = \partial_l(\frac{1}{\rho}\lambda\partial_l T) + \frac{\lambda\partial_l T\partial_l\rho}{\rho^2}$$

and the contributions associated with $\partial_l(\lambda\partial_l T/\rho c_v)$ are integrated by parts. This yields source terms in the form

$$+2 \sum_{\substack{1\leq l\leq n\\|\alpha|=k}} \frac{k!}{\alpha!}\partial_l\Big(\frac{\partial^\alpha T}{T^{1+a_k}\rho^{b_k}}\Big)\partial^\alpha\Big(\frac{\lambda\partial_l T}{\rho}\Big).$$

After expanding the derivatives, using the differential identities of Section 4.1, the above sum can be written

$$2 \sum_{\substack{1\leq l\leq n\\|\alpha|=k}} \frac{k!}{\alpha!}\Big(\frac{\partial^\alpha\partial_l T}{T^{1+a_k}\rho^{b_k}} - (1+a_k)\frac{\partial^\alpha T\partial_l T}{T^{2+a_k}\rho^{b_k}} - b_k\frac{\partial^\alpha T\partial_l\rho}{T^{1+a_k}\rho^{1+b_k}}\Big)$$

$$\times\frac{1}{\rho}\Big(\lambda\partial^\alpha\partial_l T + \sum_{\tilde\alpha\nu\mu} c_{\alpha\tilde\alpha\nu\mu}T^\sigma\partial_T^\sigma\lambda \prod_\beta\big(\frac{\partial^\beta T}{T}\big)^{\nu_\beta}\prod_\beta\big(\frac{\partial^\beta\rho}{\rho}\big)^{\mu_\beta}\,\partial^{\alpha-\tilde\alpha}\partial_l T\Big),$$

where the summations and products extend over $1\leq l\leq n$, $|\alpha|=k$, $0\leq\tilde\alpha\leq\alpha$, $\tilde\alpha\neq 0$, $1\leq\sigma\leq|\tilde\alpha|$, $\sum_\beta\beta(\nu_\beta+\mu_\beta)=\tilde\alpha$, $1\leq|\beta|\leq|\tilde\alpha|$, and $\sum_\beta\nu_\beta=\sigma$. We can now extract for $\pi_\gamma^{[k]}$ the term in the form $\lambda(\partial^\alpha\partial_l T)^2$ which can be written

$$2 \sum_{\substack{1\leq l\leq n\\|\alpha|=k}} \frac{k!}{\alpha!}\frac{(\partial^\alpha\partial_l T)^2}{T^{1+a_k}\rho^{b_k}} = 2\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\frac{(\partial^\alpha T)^2}{T^{1+a_k}\rho^{b_k}},$$

thanks to the properties of multinomial coefficients[8, 41]. All other terms are of admissible form for $\Sigma_\gamma^{[k]}$, i.e., in the form $c_{\sigma\nu\mu}T^{\sigma-\varkappa}\partial_T^\sigma\lambda\,\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\rm th}$ order since $\sum_\beta|\beta|\nu_\beta+1+|\alpha-\tilde\alpha|=k+1$. More specifically, we can factorize T^{-a_k} in the first factors, $T^{1+\varkappa}$ in the parenthesis, and all the terms involving derivatives of $\partial_T^\sigma\lambda$ are multiplied and divided by T^σ thanks to $\sum_\beta\nu_\beta=\sigma$.

The contributions associated with $\lambda\partial_l T\partial_l\rho/\rho^2$ are integrated by parts thanks to a decomposition in the form $\alpha=\tilde\alpha+e_{i_\alpha}$ where $|\tilde\alpha|=k-1$, as well as the contributions in $\mathcal{T}^{\partial T}$ associated with $\eta|d|^2+\kappa(\partial_x\!\cdot v)^2$, and only yield admissible source terms. More specifically, we decompose each multiindex α with $|\alpha|=k$ into $\alpha=\tilde\alpha+e_{i_\alpha}$ where $|\tilde\alpha|=k-1$, i_α is chosen arbitrarily with $\alpha_{i_\alpha}\neq 0$, and $e_1,\dots,e_n$ denotes the canonical basis of $\mathbb{N}^n$, so that we have $\partial^\alpha=\partial^{\tilde\alpha}\partial_{i_\alpha}$. We can then integrate these terms by parts and obtain sources in the form

$$\sum_{\substack{1\leq i,j\leq n\\|\alpha|=k}} \partial_{i_\alpha}\Big(\frac{\partial^\alpha T}{T^{1+a_k}\rho^{b_k}}\Big)\partial^{\tilde\alpha}\big(\eta d_{ij}^2\big).$$

Upon expanding the derivatives with the help of the differential identities established in the Section 4.1, all these terms are of admissible form for $\Sigma_\gamma^{[k]}$.

We now consider the sum $\mathcal{T}^{\partial v}$ and its most important contribution is that corresponding to $\partial^\alpha\partial_x\!\cdot(\eta d+\kappa\partial_x\!\cdot vI)$ which reads

$$-2\sum_{\substack{1\leq i,l\leq n\\|\alpha|=k}} \frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}\rho^{b_k}}\partial^\alpha\Big(\frac{1}{\rho}\partial_l(\eta\partial_l v_i+\eta\partial_i v_l+(\kappa-\tfrac{2}{3}\eta)\,\partial_x\!\cdot v\,\delta_{il})\Big),$$

where δ_{il} is the Kronecker symbol. We first consider the contribution associated with $\eta\partial_l v_i$ using the identity

$$\frac{1}{\rho}\partial_l(\eta\partial_l v_i) = \partial_l\Big(\frac{1}{\rho}\eta\partial_l v_i\Big) + \frac{\partial_l\rho\partial_i v_l}{\rho^2}$$

and focus on the contributions of the terms $\partial_l(\eta\partial_l v_i/\rho)$. The contributions associated with $\partial_l\rho\partial_i v_l$ are of admissible form for $\Sigma_\gamma^{[k]}$ after one integration by parts using $\alpha = \tilde{\alpha} + e_{i_\alpha}$ and the corresponding details are omitted. After integration by parts we obtain sources in the form

$$2 \sum_{\substack{1\leq i,l\leq n\\ |\alpha|=k}} \frac{k!}{\alpha!} \partial_l\Big(\frac{\partial^\alpha v_i}{T^{a_k}\rho^{b_k}}\Big)\partial^\alpha\Big(\frac{\eta\partial_l v_i}{\rho}\Big).$$

Expanding the derivatives, the sum is rewritten

$$\sum_{\substack{1\leq i,l\leq n\\ |\alpha|=k}} \frac{k!}{\alpha!}\Big(\frac{\partial^\alpha\partial_l v_i}{T^{a_k}\rho^{b_k}} - a_k\frac{\partial^\alpha v_i\partial_l T}{T^{a_k+1}\rho^{b_k}} - b_k\frac{\partial^\alpha v_i\partial_l\rho}{T^{a_k+1}\rho^{b_k+1}}\Big)$$

$$\times\frac{1}{\rho}\Big(\eta\partial^\alpha\partial_l v_i + \sum_{\tilde{\alpha}\nu\mu} c_{\alpha\tilde{\alpha}\nu\mu}T^\sigma\partial_T^\sigma\eta\prod_\beta\big(\frac{\partial^\beta T}{T}\big)^{\nu_\beta}\prod_\beta\big(\frac{\partial^\beta\rho}{\rho}\big)^{\mu_\beta}\partial^{\alpha-\tilde{\alpha}}\partial_l v_i\Big),$$

where the summations and products extend over $1\leq i,l\leq n$, $|\alpha|=k$, $0\leq\tilde{\alpha}\leq\alpha$, $\tilde{\alpha}\neq 0$, $1\leq\sigma\leq|\tilde{\alpha}|$, $\sum_\beta\beta(\nu_\beta+\mu_\beta)=\tilde{\alpha}$, $1\leq|\beta|\leq|\tilde{\alpha}|$, and $\sum_\beta\nu_\beta=\sigma$. We can extract the term in the form $\eta(\partial^\alpha\partial_l v_i)^2$ for $\pi_\gamma^{[k]}$ which is rewritten as

$$2\sum_{\substack{1\leq i,l\leq n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{(\partial^\alpha\partial_l v_i)^2}{T^{a_k}\rho^{b_k}} = 2\sum_{\substack{1\leq i\leq n\\ |\alpha|=k+1}}\frac{(k+1)!}{\alpha!}\frac{(\partial^\alpha v_i)^2}{T^{a_k}\rho^{b_k}},$$

thanks to the properties of multinomial coefficients. All the other terms are of admissible form for $\Sigma_\gamma^{[k]}$, that is, in the form $c_{\sigma\nu\mu}T^{\sigma-\varkappa}\partial_T^\sigma\eta\,\Pi_\nu^{(k+1)}\Pi_\mu^{(k+1)}$ with at most one derivative of $(k+1)^{\text{th}}$ order.

The contributions associated with $\eta\partial_i v_l$ is treated in an analogous way with the identity

$$\partial_l(\eta\partial_i v_l) = \partial_T\eta\partial_l T\partial_i v_l + \partial_i(\eta\partial_l v_l) - \partial_T\eta\partial_i T\partial_l v_i,$$

and yields a source term for $\pi_\gamma^{[k]}$ in the form

$$2\eta\sum_{\substack{1\leq l\leq n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{(\partial^\alpha\partial_l v_l)^2}{T^{a_k}\rho^{b_k}}.$$

Finally, the terms $(\kappa-\frac{2}{3}\eta)\partial_x\cdot v\,\delta_{il}$ can be treated in a similar way and yields a source term for $\pi_\gamma^{[k]}$ in the form

$$2(\kappa-\tfrac{2}{3}\eta)\sum_{\substack{1\leq l\leq n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{(\partial^\alpha\partial_l v_l)^2}{T^{a_k}\rho^{b_k}},$$

as well as contributions of the admissible form.

Lower order convective terms first yield the contributions

$$-\Big(\frac{(a_k-1)|\partial^k\rho|^2}{T^{a_k}\rho^{b_k+2}} + \frac{a_k|\partial^k v|^2}{T^{a_k+1}\rho^{b_k}} + \frac{c_v(a_k+1)|\partial^k T|^2}{T^{a_k+2}\rho^{b_k}}\Big)v\cdot\partial_x T$$

$$-\Big(\frac{(b_k+2)|\partial^k\rho|^2}{T^{a_k-1}\rho^{b_k+3}} + \frac{b_k|\partial^k v|^2}{T^{a_k}\rho^{b_k+1}} + \frac{c_v b_k|\partial^k T|^2}{T^{a_k+1}\rho^{b_k+1}}\Big)v\cdot\partial_x\rho$$

$$+2c_v\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha T\partial^\alpha(v\cdot\partial_x T)}{T^{a_k+1}\rho^{b_k}} + 2\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha\rho\partial^\alpha(v\cdot\partial_x\rho)}{T^{a_k-1}\rho^{b_k+2}} - 2\sum_{\substack{1\leq i\leq n\\ |\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i\partial^\alpha(v\cdot\partial_x v_i)}{T^{a_k}\rho^{b_k}},$$

and all terms proportional to v are easily recast in the form $v\cdot\partial_x\gamma^{[k]}$, so that the only remaining contributions are the sources

$$2c_v \sum_{\substack{|\alpha|=k\\1\leq l\leq n}} \sum_{\substack{0\leq\beta\leq\alpha\\1\leq|\beta|}} c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{a_k+1}\rho^{b_k}}\partial^\beta v_l\partial^{(\alpha-\beta)}\partial_l T$$

$$2 \sum_{\substack{|\alpha|=k\\1\leq l\leq n}} \sum_{\substack{0\leq\beta\leq\alpha\\1\leq|\beta|}} c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha \rho}{T^{a_k-1}\rho^{b_k+2}}\partial^\beta v_l\partial^{(\alpha-\beta)}\partial_l \rho$$

$$2 \sum_{\substack{|\alpha|=k\\1\leq i,l\leq n}} \sum_{\substack{0\leq\beta\leq\alpha\\1\leq|\beta|}} c_{\alpha\beta}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}\rho^{b_k}}\partial^\beta v_l\partial^{(\alpha-\beta)}\partial_l \rho$$

which are easily rewritten in the form $c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}$.

The remaining first order terms are then in the form

$$\begin{aligned}
&-\Big(\frac{(a_k-1)|\partial^k\rho|^2}{T^{a_k}\rho^{b_k+2}}+\frac{a_k|\partial^k v|^2}{T^{a_k+1}\rho^{b_k}}+\frac{c_v(a_k+1)|\partial^k T|^2}{T^{a_k+2}\rho^{b_k}}\Big)\frac{T\partial_x\!\cdot v}{c_v}\\
&-\Big(\frac{(b_k+2)|\partial^k\rho|^2}{T^{a_k-1}\rho^{b_k+3}}+\frac{b_k|\partial^k v|^2}{T^{a_k}\rho^{b_k+1}}+\frac{c_v b_k|\partial^k T|^2}{T^{a_k+1}\rho^{b_k+1}}\Big)\rho\partial_x\!\cdot v\\
&+2\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha T}{T^{a_k+1}\rho^{b_k}}\partial^\alpha(T\partial_x\!\cdot v)+2\sum_{|\alpha|=k}\frac{k!}{\alpha!}\frac{\partial^\alpha \rho}{T^{a_k-1}\rho^{b_k+2}}\partial^\alpha(\rho\partial_x\!\cdot v)\\
&+2\sum_{\substack{1\leq i\leq n\\|\alpha|=k}}\frac{k!}{\alpha!}\frac{\partial^\alpha v_i}{T^{a_k}\rho^{b_k}}\partial^\alpha\Big(T\frac{\partial_i\rho}{\rho}+\partial_i T\Big).
\end{aligned}$$

The two first sum are easily recast in the admissible form $c_{\nu\mu}\Pi_\nu^{(k)}\Pi_\mu^{(k+1)}$. In the last three sums, it is then important to separate admissible terms form unsplit ones, that is, to separate terms with three or more derivatives—which are then of the admissible form—from quadratic terms. The third and fourth terms yields the special source terms

$$+2\frac{\partial^k T\partial^k(\partial_x\!\cdot v)}{T^{a_k}\rho^{b_k}}+2\frac{\partial^k \rho\partial^k(\partial_x\!\cdot v)}{T^{a_k-1}\rho^{b_k+1}}. \tag{A.5}$$

In the last sum, the contributions associated with $\partial_i T$ are integrated by parts and yield admissible terms plus the special term

$$-2\frac{\partial^k T\partial^k(\partial_x\!\cdot v)}{T^{a_k}\rho^{b_k}},$$

which compensates with the first term of (A.5). Finally, the special contributions associated with $T\partial_i\rho/\rho = T\partial_i\log\rho$ are integrated by parts and yields the source term

$$-2\frac{\partial^k \rho\partial^k(\partial_x\!\cdot v)}{T^{a_k-1}\rho^{b_k+1}},$$

which compensate with the second term of (A.5). This compensations of quadratic terms involving hyperbolic variables are the consequence of the symmetric structure of the system of partial differential equations.

Let now (ρ, v, T) be a smooth solution of the compressible Navier-Stokes equations (3.1)–(3.5) with regularity (3.6)(3.7), and assume that $T \geq T_{\min}$ and that $\rho \geq \rho_{\min}$. The preceding derivation of the $\gamma^{[k]}$ balance equation can then be justified for $0 \leq k \leq l$ by using mollifiers and classical properties of commutators[25, 26, 43].

Moreover, from classical interpolation inequalities the following lemmas ensure that $\varphi_\gamma^{[k]}$, $\pi_\gamma^{[k]}$, $\Sigma_\gamma^{[k]}$, $\omega_\gamma^{[k]} \in L^1\big((0,\bar t\,), W^{l-k,1}\big)$.

Lemma A.1 *Let $i \geq 1$, α^j, $1 \leq j \leq i$, be multiindices such that $|\alpha^j| \geq 1$, $1 \leq j \leq i$, and let $k = \sum_{1\leq j\leq i}|\alpha^j|$. Let $\mathsf{u}_1, \dots, \mathsf{u}_i$, be such that there exist constants $\mathsf{u}_{j,\infty}$ with $\mathsf{u}_j - \mathsf{u}_{j,\infty} \in W^{m,2}(\mathbb{R}^n)\cap\mathcal{C}_0^0(\mathbb{R}^n)$ and assume that $1 \leq k \leq m$. There exists a constant $c = c(m,n)$ only depending on (m,n), such that*

$$\|\prod_{1\leq j\leq l}\partial^{\alpha^j}\mathsf{u}_j\|_{W^{m-k,2}} \leq c\|\mathsf{u}-\mathsf{u}_\infty\|_{L^\infty}^{i-1}\Big(\|\partial^k\mathsf{u}\|_{L^2}+\cdots+\|\partial^m\mathsf{u}\|_{L^2}\Big), \tag{A.6}$$

where

$$\|\mathsf{u}-\mathsf{u}_\infty\|_{L^\infty} = \sum_{1\le j\le i} \|\mathsf{u}_j - \mathsf{u}_{j,\infty}\|_{L^\infty}$$

$$\|\partial^m \mathsf{u}\|_{L^2}^2 = \sum_{1\le j\le i} \|\partial^m \mathsf{u}_j\|_{L^2}^2.$$

and the derivatives of $\prod_{1\le j\le l} \partial^{\alpha^j}\mathsf{u}_j$ *can be evaluated by using Leibnitz' formula.*

Lemma A.2 *Let* $i \ge 2$, α^j, $1 \le j \le i$, *be multiindices such that* $|\alpha^j| \ge 1$, $1 \le j \le i$, *and let* $k = \sum_{1\le j\le i} |\alpha^j|$. *Let* $\mathsf{u}_1, \dots, \mathsf{u}_i$, *be such that there exist constants* $\mathsf{u}_{j,\infty}$ *with* $\mathsf{u}_j - \mathsf{u}_{j,\infty} \in W^{m,2}(\mathbb{R}^n) \cap \mathcal{C}_0^1(\mathbb{R}^n)$ *and assume that* $2 \le k \le m+1$. *There exists a constant* $c = c(m,n)$ *only depending on* (m,n), *such that*

$$\| \prod_{1\le j\le l} \partial^{\alpha^j}\mathsf{u}_j \|_{W^{m+1-k,2}} \le c\Big(\|\mathsf{u}-\mathsf{u}_\infty\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\Big)^{i-1} \Big(\|\partial^1 \mathsf{u}\|_{L^2} + \cdots + \|\partial^m \mathsf{u}\|_{L^2}\Big), \tag{A.7}$$

where

$$\|\partial_x \mathsf{u}\|_{L^\infty} = \sum_{1\le j\le i} \|\partial_x \mathsf{u}_j\|_{L^\infty},$$

and the derivatives of $\prod_{1\le j\le l} \partial^{\alpha^j}\mathsf{u}_j$ *can be evaluated by using Leibnitz' formula.*

These Lemmas can be established by using classical interpolation inequalities[43] or by using Theorems 4.7 and 4.8 with a weight unity.

Lemma A.3 *Let* $m \ge 0$, *and* a, $b \in W^{m,2}(\mathbb{R}^n)$. *Then* $ab \in W^{m,1}$ *and there exists a constant* $c(m,n)$ *only depending on* (m,n) *such that*

$$\big\|ab\big\|_{W^{m,1}} \le c\,\|a\|_{W^{m,2}}\,\|b\|_{W^{m,2}}, \tag{A.8}$$

and the derivatives of ab *can be evaluated by using Leibnitz' formula.*

This Lemma is a direct consequence of Hölder inequality.

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