Asymptotic stability and relaxation for fast chemistry fluids

Vincent Giovangigli[1] and Wen An Yong[2]

[1]CMAP–CNRS, École Polytechnique, 91128 Palaiseau, FRANCE

[2]ZCAM, Tsinghua University, Beijing, 100084, CHINA and
Beijing Computational Science Research Center, Beijing 100084, CHINA

Abstract

Global existence for multicomponent reactive fluids with fast chemistry is investigated. The system of partial differential equations derived from the kinetic theory is symmetrizable hyperbolic-parabolic with stiff chemical sources. New a priori estimates are obtained uniformly with respect to chemistry relaxation times and lead to asymptotic stability results for well prepared initial conditions. Convergence towards the chemical equilibrium fluid model when chemistry times go to zero is established as well as error estimates.

1. Introduction

Chemical equilibrium fluids are reduced models which are of interest in various scientific and engineering applications such as reentry of space vehicles into Earth's atmosphere [1, 42], engine rocket nozzle flows [52], or chemical reactors [31]. These equilibrium models are valid when the chemical characteristic times are smaller than the flow characteristic times and lead to an important reduction of the number of unknown variables with species densities replaced by atom densities. This is a strong motivation for investigating the fast chemistry limit of multicomponent flow models. In this work, global existence results for fast chemistry fluids are established as well as convergence towards the chemical equilibrium fluid model as chemical characteristic times go to zero.

The system of partial differential equations modeling fluids out of chemical equilibrium as derived from the kinetic theory of gases is first presented [18]. The balance equations express the conservation of species mass, momentum and energy and involve convective, dissipative as well as chemical source terms. The dissipative transport fluxes have a complex structure derived from the kinetic theory and couple all species equations as well as the energy equation through the Soret and Dufour cross effets. The chemistry terms are written for an arbitrary complex chemical reaction mechanism with rates deduced from the kinetic theory as well as from statistical themodynamics. Thermodynamic properties obtained from the kinetic theory of dilute gases coincide with that of ideal gas

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mixtures and chemical equilibrium states are characterized in terms of Gibbs potentials. The reaction mechanism is assumed to be sufficiently detailed in such a way that its chemical equilibrium states coincide with natural equilibrium states obtained when all possible reactions are considered. The slow variables of the out of equilibrium system—that are the relevant variables of the limiting chemical equilibrium fluid—are then associated with atomic mass densities, momentum and energy. The governing equations finally form a second order quasilinear system of partial differential equations in terms of the conservative variable $\mathsf{u} \in \mathbb{R}^{\mathsf{n}}$ that is of dimension $\mathsf{n} = n + d + 1$ where n denotes the number of reactive species and d the space dimension.

Symmetrized forms for the system of partial differential equations are then discussed [30, 16, 48, 34, 36, 9, 21, 22, 18, 10, 58, 13, 56, 37, 4]. Existence of symmetrized forms is related to the existence of a mathematical entropy σ compatible with convective terms, dissipative terms and chemistry. The natural entropic symmetrized form for fluids out of chemical equilibrium is evaluated and appears as a symmetric second order system in terms of the entropic variable $\mathsf{v} = (\partial_{\mathsf{u}}\sigma)^t$. These entropic forms or entropic variables have been a key tool in the study of cross diffusion effects [21, 22, 18, 33, 8]. The source term is shown to be of quasilinear form as is typical in a relaxation framework and often encountered in mathematical physics [57]. Normal forms, that is, symmetric hyperbolic-parabolic composite forms of the system of partial differential equations [34, 36] are further investigated using normal variables w and the mathematical framework needed to investigate the fast chemistry limit is completed by introducing the small parameter ϵ associated with fast chemistry relaxation. Strict dissipativity of the system in normal form is investigated and it is established that there exists a compensating matrix K compatible with the fast manifold. More specifically, denoting by $\overline{\mathcal{E}}$ the slow equilibrium manifold with respect to the normal variable and by π the orthogonal projector onto the fast manifold $\overline{\mathcal{E}}^{\perp}$, it is established that there exist a compensating matrix such that $K\pi = 0$. This is a natural assumption since, on the one hand, hyperbolic-parabolic coupling aspects are associated with total mass, momentum and energy conservation equations, and, on the other hand, chemical reactions neither create mass, momentum nor energy. The governing equations at chemical equilibrium, the thermodynamics of chemical equilibrium, the corresponding symmetrized forms as well as strict dissipativity at equilibrium are also investigated.

The system of partial differential equations in the normal variable $\mathsf{w} \in \mathbb{R}^{\mathsf{n}}$ out of chemical equilibrium is found in the form

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t \mathsf{w} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w})\partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_j \mathsf{w}\big) + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\mathsf{w} = \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}(\mathsf{w})\,\partial_i \mathsf{w}\,\partial_j \mathsf{w}, \tag{1.1}$$

where ∂_t denotes the time derivative operator, ∂_i the space derivative operator in the ith direction, $\mathcal{D} = \{1,\dots,d\}$ the spatial directions, $\mathsf{w} \in \mathbb{R}^{\mathsf{n}}$ the normal variable decomposed into $\mathsf{w} = (\mathsf{w}_{\mathrm{I}}, \mathsf{w}_{\mathrm{II}})^t$, $\mathsf{w}_{\mathrm{I}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{I}}}$ the hyperbolic components with $\mathsf{n}_{\mathrm{I}} = 1$, $\mathsf{w}_{\mathrm{II}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{II}}}$ the parabolic components with $\mathsf{n}_{\mathrm{II}} = n + d$, and $\epsilon \in (0, 1]$

the positive relaxation parameter. The matrix $\overline{\mathsf{A}}_0 \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$ is symmetric positive definite and block-diagonal, $\overline{\mathsf{A}}_i \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$, $i \in \mathcal{D}$ are symmetric, $\overline{\mathsf{B}}_{ij} \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$ and $\overline{\mathsf{B}}_{ij}^t = \overline{\mathsf{B}}_{ji}$, $i, j \in \mathcal{D}$, $\overline{\mathsf{B}}_{ij}$ has nonzero components only into the right lower $\overline{\mathsf{B}}_{ij}^{\mathrm{II},\mathrm{II}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{II}},\mathsf{n}_{\mathrm{II}}}$ blocks, $\overline{\mathsf{B}}^{\mathrm{II},\mathrm{II}} = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{II},\mathrm{II}}(\mathsf{w})\xi_i\xi_j$ is positive definite for $\boldsymbol{\xi} \in \Sigma^{d-1}$ where Σ^{d-1} is the sphere in d dimension, $\overline{\mathsf{L}} \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$ is positive semi-definite with a fixed nullspace $\overline{\mathcal{E}}$, and $\overline{\mathrm{M}}_{ij} \in \mathbb{R}^{\mathsf{n},\mathsf{n},\mathsf{n}}$, $i, j \in \mathcal{D}$, are third order tensors. The nullspace $\overline{\mathcal{E}} \subset \mathbb{R}^{\mathsf{n}}$ of the linearized source term $\overline{\mathsf{L}}$ represents the slow manifold and is of dimension $\mathsf{n}_{\mathrm{e}} = n_{\mathrm{a}} + d + 1$ where n_{a} denotes the number of atoms. The orthogonal complement $\overline{\mathcal{E}}^{\perp} \subset \mathbb{R}^{\mathsf{n}}$ of dimension $n - n_{\mathrm{a}}$ is the fast manifold and $\pi\mathsf{w} \in \mathbb{R}^{\mathsf{n}}$ the fast variable. The quadratic residual $\overline{\mathrm{Q}} = \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}(\mathsf{w})\,\partial_i\mathsf{w}\,\partial_j\mathsf{w}$ may also be written $\overline{\mathrm{Q}} = -\sum_{i,j\in\mathcal{D}} \partial_i(\partial_{\mathsf{w}}\mathsf{v})^t\,(\partial_{\mathsf{v}}\mathsf{w})^t\,\overline{\mathsf{B}}_{ij}\partial_j\mathsf{w}$ where v denotes the entropic variable and only involves the parabolic components $\overline{\mathrm{M}}_{ij}(\mathsf{w})\partial_i\mathsf{w}\partial_j\mathsf{w} = \big(0,\, \overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II},\mathrm{II}}(\mathsf{w})\partial_i\mathsf{w}_{\mathrm{II}}\partial_j\mathsf{w}_{\mathrm{II}}\big)^t$. Various forms of the quadratic residual $\overline{\mathrm{Q}}$ will be discussed in more details in the following. The system coefficients $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i$, $i \in \mathcal{D}$, $\overline{\mathsf{B}}_{ij}$, $i, j \in \mathcal{D}$, and $\overline{\mathrm{M}}_{ij}$, $i \in \mathcal{D}$, are assumed to be sufficiently smooth.

As a first mathematical step in order to solve the Cauchy problem in $\mathbb{R}^d$, a local existence theorem for (1.1) is established—generalizing previous work on the relaxation of internal energy [29]—without assuming that the matrix $\overline{\mathsf{A}}_0$ in front of the time derivative operator ∂_t leaves invariant the fast manifold $\overline{\mathcal{E}}^{\perp}$, or equivalently without assuming the commutation relation $\overline{\mathsf{A}}_0\pi = \pi\,\overline{\mathsf{A}}_0$ where π denotes the orthogonal projector on the fast manifold $\overline{\mathcal{E}}^{\perp}$. The required a priori estimates are more intricate to obtain than that in the 'commutative case' where a simplified analysis is feasible using the commutation of π with the differential operator $\overline{\mathsf{A}}_0[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\overline{\mathsf{L}}]$ [29]. For a suitable positive time $\bar{\tau}$, and denoting by $\mathsf{w}^\star$ an equilibrium state, $\mathsf{w} - \mathsf{w}^\star$ is estimated in $C^0\big([0,\bar{\tau}], H^l\big)$, $\pi\mathsf{w}/\sqrt{\epsilon}$ in $L^2\big((0,\bar{\tau}), H^l\big)$, and $\partial_t\mathsf{w}$ and $\pi\mathsf{w}/\epsilon$ in $L^2\big((0,\bar{\tau}), H^{l-1}\big)$ for $l \geq [d/2] + 2$ where $H^l = H^l(\mathbb{R}^d)$ denotes the usual Sobolev space in $\mathbb{R}^d$. These new uniform estimates lead to local existence results on a time interval independent of the relaxation parameter $\epsilon \in (0, 1]$ for well prepared initial conditions, that is, assuming that w_0 is close to the equilibrium manifold. Stronger estimates of $\partial_t^2\mathsf{w}$ and $\pi\partial_t\mathsf{w}/\epsilon$ in $L^2\big((0,\bar{\tau}), H^{l-2}\big)$, assuming that the initial time derivative $\partial_t\mathsf{w}_0$ is close to the equilibrium manifold, are not required in this work and would only be needed for a two term Chapman-Enskog expansion [29].

A priori estimates on time intervals of arbitrary length are then investigated by using the strict dissipativity of the system of partial equations (1.1), extending Kawashima theory to the stiff case. The differences with the estimates established by Kawashima [34] are the inclusion of extra terms associated with the rescaled fast variables $\pi\mathsf{w}/\sqrt{\epsilon}$ and $\pi\mathsf{w}/\epsilon$ and the coupling with the estimates for time derivative $\partial_t\mathsf{w}$. These time derivative estimates indeed require to use an energy method coupled to that of the solution as well as that of the fast variable $\pi\mathsf{w}$. Combining these estimates with the local existence theorem leads to global existence results for well prepared initial conditions that may be summarized in the following form where $|\cdot|_l$ denotes the $H^l(\mathbb{R}^d)$ norm :

Theorem 1.1. *Let* $d\geq 1$ *and* $l\geq [d/2] + 2$ *be integers and* $\mathsf{w}^\star$ *an equilibrium* *state. There exists* $\bar{b} > 0$ *small enough such that if* w_0 *satisfies* $\mathsf{w}_0 - \mathsf{w}^\star \in H^l$ *and* $|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2 < \bar{b}^2$ *there exists a unique global solution to the Cauchy problem with initial condition* $\mathsf{w}(0,x) = \mathsf{w}_0(x)$ *and*

$$\begin{aligned}
&\mathsf{w}_{\mathrm{I}} - \mathsf{w}_{\mathrm{I}}^\star \in C^0\big([0,\infty), H^l\big) \cap C^1\big([0,\infty), H^{l-1}\big), \qquad &&\partial_x \mathsf{w}_{\mathrm{I}} \in L^2\big((0,\infty), H^{l-1}\big),\\
&\mathsf{w}_{\mathrm{II}} - \mathsf{w}_{\mathrm{II}}^\star \in C^0\big([0,\infty), H^l\big) \cap C^1\big([0,\infty), H^{l-2}\big), \qquad &&\partial_x \mathsf{w}_{\mathrm{II}} \in L^2\big((0,\infty), H^{l}\big).
\end{aligned}$$

Furthermore, there exists a constant $\bar{c}$ *independent of* ϵ *such that* w *satisfies the estimate*

$$\begin{aligned}
|\mathsf{w}(t) - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}(t)|_{l-1}^2 + \int_0^t |\partial_x \mathsf{w}_{\mathrm{I}}|_{l-1}^2\, d\tau + \int_0^t |\partial_x \mathsf{w}_{\mathrm{II}}|_{l}^2\, d\tau + \frac{1}{\epsilon}\int_0^t |\pi\mathsf{w}(\tau)|_l^2\, d\tau\\
+ \frac{1}{\epsilon^2}\int_0^t |\pi\mathsf{w}(\tau)|_{l-1}^2\, d\tau + \int_0^t |\partial_t \mathsf{w}(\tau)|_{l-1}^2\, d\tau \le \bar{c}^2\Big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2\Big), \quad (1.2)
\end{aligned}$$

and $\sup_{x\in\mathbb{R}^d} |\mathsf{w}(t,x) - \mathsf{w}^\star|$ *goes to zero as* $t \to \infty$.

Key points are notably the strict dissipativity of the second order terms, the compatibility condition between compensating matrices and the fast manifold $K\pi = 0$, and the use of the appropriate norm (1.2) involving the fast variable $\pi\mathsf{w}$ and the time derivative. Applying these results to the system of equations modeling gas out of chemical equilibrium, global existence of solutions is obtained around constant states in all space dimensions uniformly with respect to the chemical relaxation time $\epsilon \in (0,1]$ extending previous work associated with finite rate chemistry [22, 18].

The singular limit $\epsilon \to 0$ for the system of partial differential equations modeling fluids out of chemical equilibrium is finally investigated. Various other relaxation models have also been investigated in the literature in different physical and mathematical contexts and notably in an hyperbolic setting [9, 43, 59, 10, 37, 54, 55, 58, 60, 61, 53, 44, 45] as well as for reaction-diffusion systems [3, 5, 14]. The limiting chemical equilibrium fluid describes the conservation of atom mass, momentum and energy and global existence of solutions has already been established [34, 18]. Such limiting equations for chemical equilibrium fluids may also be deduced from the kinetic theory [12]. The limiting equations may be symmetrized with an entropic variable v_{e} as well as a normal variable w_{e} associated with w. Denoting by $\overline{\Pi}_{\mathsf{e}}$ the embedding from the lower dimensional space of equilibrium normal variables w_{e} onto the slow manifold $\overline{\mathcal{E}}$, then a priori estimates out of chemical equilibrium are combined with stability results at equilibrium in order to establish the following convergence results :

Theorem 1.2. *Let* $d \ge 1$ *and* $l \ge [d/2] + 4$ *be integers and let* $\bar{b}$ *from Theorem 1.1. For any* w_0 *with* $\mathsf{w}_0 - \mathsf{w}^\star \in H^l$, $\pi\mathsf{w}_0 = 0$ *and* $|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 < \bar{b}^2$ *there exists a unique solution* w *of the out of equilibrium system such that the estimates* (1.2) *hold and there exists a unique global solution* w_{e} *of the limiting equilibrium system starting from the equilibrium projection of* w_0*. Then the out of equilibrium solution converges toward the chemical equilibrium solution* *pointwise* $\lim_{\epsilon\to 0} \mathsf{w}(t,x) = \overline{\Pi}_{\rm e}\mathsf{w}_{\rm e}(t,x)$ *and for any time* $\bar\tau$ *there exists a constant* c *depending on* $\bar\tau$ *with the error estimate*

$$\sup_{\tau\in[0,\bar\tau]} |\mathsf{w} - \overline{\Pi}_{\rm e}\mathsf{w}_{\rm e}|_{l-2} \leq \mathsf{c}\,\epsilon.$$

To the best of the authors' knowledge, this is the first rigorous justification of the *fast chemistry fluid limit* as well as the first error estimate. The dissipation matrices at equilibrium are also given by $\overline{\mathsf{B}}^{\rm e}_{ij} = \overline{\Pi}^t_{\rm e}\overline{\mathsf{B}}_{ij}\overline{\Pi}_{\rm e}$ in such a way that cross effects also appear for the limiting fluid at equilibrium.

The nonequilibrium model is summarized in Section 2, symmetrization and strict dissipativity is discussed in Section 3. Equations at equilibrium are discussed in Section 4. New a priori estimates and global existence results are established in Section 5 and convergence towards the equilibrium reduced model is established in Section 6.

2. Governing equations

The system of equations modeling multicomponent reactive fluids as derived from the kinetic theory is presented [7, 15, 18]. The mathematical assumptions on the system coefficients are summarized and the system of partial differential equations is rewritten in quasilinear form.

2.1. Conservation equations

In multicomponent flows, the conservation of species mass, momentum, and energy may be written in the form [7, 15, 18]

$$\partial_t\rho_k + \boldsymbol{\nabla}\cdot(\rho_k\boldsymbol{v}) + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{J}}_k = m_k\omega_k, \qquad k\in\mathfrak{S}, \tag{2.1}$$

$$\partial_t(\rho\boldsymbol{v}) + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v} + p\boldsymbol{I}) + \boldsymbol{\nabla}\cdot\boldsymbol{\Pi} = 0, \tag{2.2}$$

$$\partial_t(\mathcal{E} + \tfrac12\rho|\boldsymbol{v}|^2) + \boldsymbol{\nabla}\cdot\big((\mathcal{E} + \tfrac12\rho|\boldsymbol{v}|^2 + p)\boldsymbol{v}\big) + \boldsymbol{\nabla}\cdot(\boldsymbol{Q} + \boldsymbol{\Pi}\cdot\boldsymbol{v}) = 0, \tag{2.3}$$

where ∂_t denotes the time derivative, $\boldsymbol{\nabla}$ the space derivative operator, ρ_k the mass density of the kth species, $\boldsymbol{v}$ the mass average flow velocity, $\boldsymbol{\mathcal{J}}_k$ the diffusion flux of the kth species, m_k the molar mass of the kth species, ω_k the molar production rate of the kth species, $\mathfrak{S} = \{1,\dots,n\}$ the set of species indices, $n \geqslant 1$ the number of species, $\rho = \sum_{k\in\mathfrak{S}}\rho_k$ the total mass density, p the pressure, $\boldsymbol{\Pi}$ the viscous tensor, $\mathcal{E}$ the internal energy per unit volume and $\boldsymbol{Q}$ the heat flux. It is assumed here for the sake of simplicity that there are no forces acting on the species. The spatial dimension is denoted by d and the components of $\boldsymbol{v}$ and $\boldsymbol{\nabla}$ are written as $\boldsymbol{v} = (v_1,\dots,v_d)^t$ and $\boldsymbol{\nabla} = (\partial_1,\dots,\partial_d)^t$ where v_i denotes the velocity in the ith spatial direction, ∂_i the derivation in the ith spatial direction and bold symbols are used for vectors in $\mathbb{R}^d$ or tensors in $\mathbb{R}^{d,d}$.

These equations have to be completed by relations expressing the thermodynamic properties like p and $\mathcal{E}$, the chemical production rates ω_k, $k\in\mathfrak{S}$, and the transport fluxes $\boldsymbol{\Pi}\in\mathbb{R}^{d,d}$, $\boldsymbol{\mathcal{J}}_k\in\mathbb{R}^d$, $k\in\mathfrak{S}$, and $\boldsymbol{Q}\in\mathbb{R}^d$.

2.2. Thermodynamics

The thermodynamics deduced from the kinetic theory of dilute gases corresponds to that of ideal mixtures with temperature dependent specific heats. Nonideal thermodynamics—that are notably important for supercritical fluids—could also be taken into account in the model but lay out of the scope of the present work [25, 26].

Denoting by T the absolute temperature, the state variables $\rho_1, \dots, \rho_n$ and T are used for convenience and $\varrho \in \mathbb{R}^n$ stands for the vector of species partial densities $\varrho = (\rho_1, \dots, \rho_n)^t$. The internal energy per unit volume $\mathcal{E}$ and the pressure p may be written as

$$\mathcal{E}(\varrho, T) = \sum_{k\in\mathfrak{S}} \rho_k e_k(T), \qquad p(\varrho, T) = \sum_{k\in\mathfrak{S}} \rho_k r_k T \tag{2.4}$$

where e_k is the internal energy per unit mass of the kth species, $r_k = R/m_k$ the specific gas constant of the kth species and R the universal gas constant. The internal energy e_k of the kth species per unit mass is given by

$$e_k(T) = e_k^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} c_{\mathrm{v}k}(\tau)\, d\tau, \qquad k \in \mathfrak{S}, \tag{2.5}$$

where e_k^{st} is the standard formation energy of the kth species at the standard temperature T^{st} and $c_{\mathrm{v}k}$ the constant volume specific heat of the kth species. The formation energy at zero temperature of the kth species is defined by $e_k^0 = e_k(0) = e_k^{\mathrm{st}} - \int_0^{T^{\mathrm{st}}} c_{\mathrm{v}k}(\tau)\, d\tau$. The (physical) entropy per unit volume $\mathcal{S}$ may be written in the form

$$\mathcal{S}(\varrho, T) = \sum_{k\in\mathfrak{S}} \rho_k s_k(\rho_k, T), \tag{2.6}$$

where s_k is the entropy per unit mass of the kth species. This entropy s_k is given by

$$s_k(\rho_k, T) = s_k^{\mathrm{st}} + \int_{T^{\mathrm{st}}}^{T} \frac{c_{\mathrm{v}k}(T')}{T'}\, dT' - \frac{R}{m_k} \log\Big(\frac{\rho_k r_k T^{\mathrm{st}}}{p^{\mathrm{st}}}\Big), \qquad k \in \mathfrak{S}, \tag{2.7}$$

where s_k^{st} is the formation entropy of the kth species at the standard temperature T^{st} and standard pressure p^{st}. Similarly, one can introduce the mixture enthalpy per unit volume $\mathcal{H} = \sum_{k\in\mathfrak{S}} \rho_k h_k(T)$ with $h_k(T) = e_k(T) + r_k T$, $k \in \mathfrak{S}$, the mixture Gibbs function per unit volume $\mathcal{G} = \sum_{k\in\mathfrak{S}} \rho_k g_k(\rho_k, T)$, with $g_k(\rho_k, T) = h_k(T) - T s_k(\rho_k, T)$, $k \in \mathfrak{S}$, as well as the species chemical potentials per unit mass

$$\mu_k(\rho_k, T) = \frac{g_k}{RT}, \qquad k \in \mathfrak{S}. \tag{2.8}$$

Using (2.5)(2.7) these chemical potential μ_k, $k \in \mathfrak{S}$, may also be written

$$m_k \mu_k = m_k \mu_k^{\mathrm{u}}(T) + \log(\rho_k/m_k), \qquad k \in \mathfrak{S}, \tag{2.9}$$

where $\mu_k^{\mathrm{u}}(T)$ denotes the value of μ_k when $\rho_k = m_k$. The species mass fractions y_k, $k \in \mathfrak{S}$, partial pressures p_k, $k \in \mathfrak{S}$, mole per unit volume n_k, $k \in \mathfrak{S}$, and mole fractions x_k, $k \in \mathfrak{S}$, are further defined by $\mathsf{y}_k = \rho_k/\rho$, $p_k = \rho_k r_k T$, $n_k = \rho_k/m_k = p_k/RT$, and $\mathsf{x}_k = p_k/p$, for $k \in \mathfrak{S}$, respectively. The mole fractions may be written $\mathsf{x}_k = m\mathsf{y}_k/m_k$ where m is the mean molar weight with $(\sum_{k\in\mathfrak{S}} \mathsf{y}_k)/m = \sum_{k\in\mathfrak{S}} \mathsf{y}_k/m_k$, the mole and mass fraction vectors are defined by $\mathsf{x} = (\mathsf{x}_1, \ldots, \mathsf{x}_n)^t$ and $\mathsf{y} = (\mathsf{y}_1, \ldots, \mathsf{y}_n)^t$, the vector of chemical potentials by $\mu = (\mu_1, \ldots, \mu_n)^t$, the vector of molar masses by $\mathfrak{m} = (m_1, \ldots, m_n)^t$, the vector $\mathbb{1}$ is defined by $\mathbb{1} = (1, \ldots, 1)^t$ and x, y, μ, $\mathfrak{m}$, $\mathbb{1} \in \mathbb{R}^n$, with $\varrho = \rho\mathsf{y}$.

2.3. Chemical kinetics

The reaction mechanism is composed by an arbitrary set of elementary chemical reactions between the species and may be written

$$\sum_{k\in\mathfrak{S}} \nu_{ki}^{\mathrm{f}}\, \mathfrak{M}_k \rightleftarrows \sum_{k\in\mathfrak{S}} \nu_{ki}^{\mathrm{b}}\, \mathfrak{M}_k, \qquad i \in \mathfrak{R}, \tag{2.10}$$

where $\mathfrak{M}_k$ is the chemical symbol of the kth species, ν_{ki}^{f} and ν_{ki}^{b} the forward and backward stoichiometric coefficients of the kth species in the ith reaction, $\mathfrak{R} = \{1, \ldots, n_{\mathrm{r}}\}$ the set of reaction indices, and $n_{\mathrm{r}} \geqslant 1$ the number of chemical reactions. The overall stoichiometric coefficients are defined by $\nu_{ki} = \nu_{ki}^{\mathrm{b}} - \nu_{ki}^{\mathrm{f}}$, and the reaction vectors $\nu_i^{\mathrm{f}}, \nu_i^{\mathrm{b}}, \nu_i \in \mathbb{R}^n$, by $\nu_i^{\mathrm{f}} = (\nu_{1i}^{\mathrm{f}}, \ldots, \nu_{ni}^{\mathrm{f}})^t$, $\nu_i^{\mathrm{b}} = (\nu_{1i}^{\mathrm{b}}, \ldots, \nu_{ni}^{\mathrm{b}})^t$, and $\nu_i = (\nu_{1i}, \ldots, \nu_{ni})^t$. Note that elementary reactions are natural molecular events and are always reversible [18].

The species of the mixture are assumed to be constituted by atoms, and $\mathfrak{a}_{il}$ denotes the number of lth atom in the ith species, $\mathfrak{A} = \{1, \ldots, n_{\mathrm{a}}\}$ the set of atom indices, $1 \le n_{\mathrm{a}} \le n$ the number of atoms—or elements—in the mixture and $\mathfrak{a}_l \in \mathbb{R}^n$ the lth atom vector $\mathfrak{a}_l = (\mathfrak{a}_{1l}, \ldots, \mathfrak{a}_{nl})^t$. In order to investigate chemical equilibrium, one may choose the set of species large enough in such a way that the atomic species are present in the mixture and—without loss of generality—it may be assumed that the atomic species correspond to the first n_{a} species. In this situation, m_l denotes the mass of the lth atom for $l \in \mathfrak{A}$, $\mathfrak{a}_{kl} = \delta_{kl}$ for $k, l \in \mathfrak{A}$, and the atom vectors $\mathfrak{a}_l$, $l \in \mathfrak{A}$, are linearly independent. Note that the atoms vectors $\mathfrak{a}_l \in \mathbb{R}^n$ are only defined for $l \in \mathfrak{A}$ and have no meaning for $l \in \mathfrak{S}\backslash\mathfrak{A}$. The Euclidean scalar product is denoted by $\langle\ ,\ \rangle$ and the conservation of atoms in chemical reactions reads $\langle \nu_i, \mathfrak{a}_l\rangle = 0$ for $i \in \mathfrak{R}$ and $l \in \mathfrak{A}$. Letting $M \in \mathbb{R}^{n,n}$ be the mass matrix $M = \mathrm{diag}(m_1, \ldots, m_n)$, the mass reaction vectors $\widetilde{\nu}_i$, $\widetilde{\nu}_i^{\mathrm{f}}$, $\widetilde{\nu}_i^{\mathrm{b}} \in \mathbb{R}^n$, $i \in \mathfrak{R}$, and the atom vectors per unit mass $\widetilde{\mathfrak{a}}_l \in \mathbb{R}^n$, $l \in \mathfrak{A}$, are given by

$$\widetilde{\nu}_i = M\nu_i, \qquad \widetilde{\nu}_i^{\mathrm{f}} = M\nu_i^{\mathrm{f}}, \qquad \widetilde{\nu}_i^{\mathrm{b}} = M\nu_i^{\mathrm{b}}, \qquad \widetilde{\mathfrak{a}}_l = m_l M^{-1}\mathfrak{a}_l, \tag{2.11}$$

and these vectors are such that $\widetilde{\mathfrak{a}}_{kl} = \delta_{kl}$ for $k, l \in \mathfrak{A}$, and $\langle \widetilde{\nu}_i, \widetilde{\mathfrak{a}}_l\rangle = 0$, for $i \in \mathfrak{R}$ and $l \in \mathfrak{A}$. The mass density of the lth atom present in the species then reads $\widetilde{\rho}_l = \langle \widetilde{\mathfrak{a}}_l, \varrho\rangle$ and is distinct from the mass density of the lth atom as a species

ρ_l. More specifically, letting n_k be the number of mole of the kth species per unit volume and $\widetilde{n}_l$ be the total number of moles of the lth atom present *inside* all species per unit volume, then $\widetilde{n}_l = \sum_{k\in\mathfrak{S}} \mathfrak{a}_{kl} n_k$ differ from n_l and using $\rho_k = m_k n_k$, $k \in \mathfrak{S}$, and $\widetilde{\rho}_l = m_l \widetilde{n}_l$, $l \in \mathfrak{A}$, yields that $\widetilde{\rho}_l = \langle \widetilde{\mathfrak{a}}_l, \varrho \rangle$.

The vector space spanned by the reaction vectors in $\mathbb{R}^n$ is denoted by $\mathcal{R} = \mathrm{Span}\{\, \nu_i,\ i \in \mathfrak{R}\,\}$ and the vector space spanned by the atom vectors by $\mathcal{A} = \mathrm{Span}\{\, \mathfrak{a}_l,\ l \in \mathfrak{A}\,\}$. From atom conservation one generally has $\mathcal{R} \subset \mathcal{A}^\perp$ where $\perp$ denotes the perpendicular symbol and it is assumed in this paper that the reaction mechanism is sufficiently representative of natural elementary reactions in such a way that the reaction vectors ν_i, $i \in \mathfrak{R}$, are spanning the maximum space

$$\mathcal{R} = \mathcal{A}^\perp. \tag{2.12}$$

This condition further means that the chemical equilibrium states associated with the reaction mechanism $M\mu \in \mathcal{R}^\perp$ coincide with natural equilibrium states obtained when all possible reactions are considered $M\mu \in \mathcal{A}$. In this situation, the slow variables obtained from the species partial densities $\rho_1, \dots, \rho_n$ naturally reduce to the atom partial densities $\widetilde{\rho}_1, \dots, \widetilde{\rho}_{n_\mathrm{a}}$. The linear spaces $\mathcal{A}$ and $\mathcal{R}$ are then of dimension n_a and $n - n_\mathrm{a}$, respectively. Keeping in mind that the atomic species have been chosen as the first n_a species, one may introduce for species that are not atoms $k \in \mathfrak{S}\backslash\mathfrak{A} = \{n_\mathrm{a}+1, \dots, n\}$, the *formation reaction vectors*

$$\mathfrak{v}_k = (-\mathfrak{a}_{k1}, \dots, -\mathfrak{a}_{kn_\mathrm{a}}, 0, \dots, 0, 1, 0, \dots, 0)^t, \qquad k \in \mathfrak{S}\backslash\mathfrak{A}, \tag{2.13}$$

that may be written $\mathfrak{v}_k = \mathsf{e}_k - \sum_{l\in\mathfrak{A}} \mathfrak{a}_{kl} \mathsf{e}_l$, $k \in \mathfrak{S}\backslash\mathfrak{A}$, where e_i, $i \in \mathfrak{S}$, denotes the basis vectors of $\mathbb{R}^n$. It is then easily checked that any reaction vector may be decomposed as $\nu_i = \sum_{k\in\mathfrak{S}\backslash\mathfrak{A}} \nu_{ki} \mathfrak{v}_k$, in such a way that the formation vectors $\mathfrak{v}_k$, $k \in \mathfrak{S}\backslash\mathfrak{A}$, form a convenient set of basis vectors of the linear space $\mathcal{R}$. These formation vectors correspond to the formation chemical reactions $\sum_{l\in\mathfrak{A}} \mathfrak{a}_{kl}\, \mathfrak{M}_l \rightleftarrows \mathfrak{M}_k$, $k \in \mathfrak{S}\backslash\mathfrak{A}$ that are not necessarily in the reaction mechanism. The linear space $\mathcal{A}$ is thus of dimension n_a and spanned by $\mathfrak{a}_l$, $l \in \mathfrak{A}$, wheras the linear space $\mathcal{R}$ is of dimension $n - n_\mathrm{a}$ and spanned by $\mathfrak{v}_k$, $k \in \mathfrak{S}\backslash\mathfrak{A}$. The mass weighted formation vectors denoted by $\widetilde{\mathfrak{v}}_k = M\mathfrak{v}_k$, for $k \in \mathfrak{S}\backslash\mathfrak{A}$, also form a convenient basis of $M\mathcal{R}$ and are such that $\langle \widetilde{\mathfrak{v}}_k, \widetilde{\mathfrak{a}}_l \rangle = 0$, for $k \in \mathfrak{S}\backslash\mathfrak{A}$ and $l \in \mathfrak{A}$.

The molar production rates that are considered are the Maxwellian production rates obtained from the kinetic theory [18, 11] when chemical characteristic times are larger than the mean free times of the molecules and the characteristic times of internal energy relaxation. Denoting by $\omega = (\omega_1, \dots, \omega_n)^t \in \mathbb{R}^n$ the vector of molar production rates then $\omega = \sum_{i\in\mathfrak{R}} \nu_i \tau_i$ where τ_i is the rate of progress of the ith reaction and

$$M\omega = \sum_{i\in\mathfrak{R}} \widetilde{\nu}_i \tau_i. \tag{2.14}$$

The rate of progress of the ith reaction τ_i reads

$$\tau_i = \mathcal{K}_i^\mathrm{s} \big(\exp\langle \mu, \widetilde{\nu}_i^\mathrm{f} \rangle - \exp\langle \mu, \widetilde{\nu}_i^\mathrm{b} \rangle \big), \tag{2.15}$$

where $\mathcal{K}_i^{\mathrm{s}}$ is the kinetic constant of the ith reaction assumed to be positive. These rates (2.15) obtained from the kinetic theory of reactive gases [12, 18] are compatible with the law of mass action and have also been derived from statistical mechanics and statistical thermodynamics [41, 32, 38]. In the following lemma the quasilinear form of the mass production rates $M\omega$ is investigated [18, 57]. We denote by $\mathbb{R}^{k,k}$ the vector space of matrices of size $k \geq 1$ and for any $A \in \mathbb{R}^{k,k}$, $N(A)$ denotes its nullspace and $R(A)$ its range.

Lemma 2.1. *The chemical source term $M\omega \in \mathbb{R}^n$ may be written*

$$M\omega = -\Lambda\mu, \tag{2.16}$$

where the matrix $\Lambda \in \mathbb{R}^{n,n}$ is given by

$$\Lambda = \sum_{i\in\mathfrak{R}} \Lambda_i \widetilde{\nu}_i \otimes \widetilde{\nu}_i. \tag{2.17}$$

The coefficients Λ_i, $i \in \mathfrak{R}$, are positive and may be written $\Lambda_i = \mathcal{K}_i^{\mathrm{s}}\zeta_i$ where $\mathcal{K}_i^{\mathrm{s}}$ is the reaction constant of the ith reaction and ζ_i the nonequilibrium factor $\zeta_i = \int_0^1 \exp\bigl(\langle \widetilde{\nu}_i^{\mathrm{f}}, \mu\rangle + \xi\langle \widetilde{\nu}_i^{\mathrm{b}} - \widetilde{\nu}_i^{\mathrm{f}}, \mu\rangle\bigr)\,\mathrm{d}\xi$. The matrix Λ is symmetric positive semi-definite with nullspace $N(\Lambda) = M^{-1}\mathcal{A}$ and range $R(\Lambda) = M\mathcal{R}$ in $\mathbb{R}^n$.

Proof. The expressions (2.16)(2.17) of $M\omega$ and Λ are direct consequences of (2.14)(2.15) and of the identity $e^\beta - e^\alpha = (\beta - \alpha)\int_0^1 \exp\bigl(\alpha + \xi(\beta-\alpha)\bigr)\,\mathrm{d}\xi$. The coefficients Λ_i, $i \in \mathfrak{R}$, are positive since the reaction constants $\mathcal{K}_i^{\mathrm{s}}$ are assumed to be positive. From (2.17) it is deduced that Λ is symmetric and that $\langle \Lambda x, x\rangle = \sum_{i\in\mathfrak{R}} \Lambda_i \langle \widetilde{\nu}_i, x\rangle^2$ for $x \in \mathbb{R}^n$. This shows that Λ is positive semi-definite and that $\Lambda x = 0$ if and only if $\langle \widetilde{\nu}_i, x\rangle = 0$ for $i \in \mathfrak{R}$, that is, if and only if $x \in (M\mathcal{R})^\perp = M^{-1}\mathcal{A}$. □

2.4. Transport fluxes

The transport fluxes $\boldsymbol{\Pi} \in \mathbb{R}^{d,d}$, $\boldsymbol{\mathcal{J}}_k \in \mathbb{R}^d$, $k \in \mathfrak{S}$, and $\boldsymbol{Q} \in \mathbb{R}^d$ due to macroscopic variable gradients can be obtained from the Chapman-Enskog expansion [49, 50, 7, 15, 11, 18]. The viscous tensor $\boldsymbol{\Pi}$ may be written

$$\boldsymbol{\Pi} = -\kappa\, \boldsymbol{\nabla}\cdot\boldsymbol{v}\boldsymbol{I} - \eta\bigl(\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^t - \tfrac{2}{d'}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\boldsymbol{I}\bigr), \tag{2.18}$$

where κ denotes the volume viscosity, η the shear viscosity, $\boldsymbol{I}$ the d dimensional identity tensor, and d' the dimension of the velocity space in the underlying kinetic framework. It is assumed in the following that the dimension of the kinetic velocity space d' is such that $2 \leq d'$ and $1 \leq d \leq d'$. The assumption $1 \leq d \leq d'$ means that the spatial dimension of the model has been reduced from d' to d so that $\boldsymbol{\Pi}$ is the left upper block of the full d'-dimensional viscous tensor. The assumption $2 \leq d'$ is also natural since $d' = 3$ in our physical world and $\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^t - \frac{2}{d'}(\boldsymbol{\nabla \cdot v})\boldsymbol{I} = 0$ when $d' = 1$. The viscous tensor may be rewritten for convenience as

$$\boldsymbol{\Pi} = -\kappa' \boldsymbol{\nabla \cdot v}\, \boldsymbol{I} - \eta\big(\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^t - \tfrac{2}{d}(\boldsymbol{\nabla \cdot v})\, \boldsymbol{I}\big), \tag{2.19}$$

where the modified volume viscosity $\kappa' = \kappa + 2\eta(d' - d)/d'd$ is nonnegative, keeping in mind that $1 \leq d \leq d'$, and where the deviatoric tensor $\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^t - \frac{2}{d}(\boldsymbol{\nabla \cdot v})\, \boldsymbol{I}$ is traceless.

The species mass diffusive fluxes $\boldsymbol{\mathcal{J}}_k \in \mathbb{R}^d$, $k \in \mathfrak{S}$, and the heat flux $\boldsymbol{Q} \in \mathbb{R}^d$ may be written in the form [49, 50, 7, 15, 11, 18].

$$\boldsymbol{\mathcal{J}}_k = -\sum_{l \in \mathfrak{S}} C_{kl}(\boldsymbol{\nabla} \mathsf{x}_l + \mathsf{x}_l \boldsymbol{\nabla} \log p + \mathsf{x}_l \widetilde{\chi}_l \boldsymbol{\nabla} \log T), \qquad k \in \mathfrak{S}, \tag{2.20}$$

$$\boldsymbol{Q} = -\lambda \boldsymbol{\nabla} T + \sum_{k \in \mathfrak{S}} (RT\frac{\widetilde{\chi}_k}{m_k} + h_k)\boldsymbol{\mathcal{J}}_k, \tag{2.21}$$

where C_{kl}, $k, l \in \mathfrak{S}$, denotes the multicomponent flux diffusion coefficients, $\widetilde{\chi}_k$, $k \in \mathfrak{S}$, the rescaled thermal diffusion ratios and λ the thermal conductivity. When the mass fractions are nonzero, it is also possible to define the species diffusion velocities $\mathbf{v}_k \in \mathbb{R}^d$, $k \in \mathfrak{S}$, by letting

$$\mathbf{v}_k = \frac{\boldsymbol{\mathcal{J}}_k}{\rho_k} = -\sum_{l \in \mathfrak{S}} D_{kl}(\boldsymbol{\nabla} \mathsf{x}_l + \mathsf{x}_l \boldsymbol{\nabla} \log p + \mathsf{x}_l \widetilde{\chi}_l \boldsymbol{\nabla} \log T),$$

where $D_{kl} = C_{kl}/\rho_k$, $k, l \in \mathfrak{S}$, are the multicomponent diffusion coefficients. The transport coefficients κ, η, λ, $C = (C_{kl})_{k,l \in \mathfrak{S}} \in \mathbb{R}^{n,n}$, $D = (D_{kl})_{k,l \in \mathfrak{S}} \in \mathbb{R}^{n,n}$, or $\widetilde{\chi} = (\widetilde{\chi}_1, \ldots, \widetilde{\chi}_n)^t \in \mathbb{R}^n$, have important properties inherited from the kinetic framework [50, 7, 15, 11, 18]. They satisfy symmetry properties, mass conservation constraints, as well as positivity properties. These multicomponent transport coefficients κ, η, λ, $C = (C_{kl})_{k,l \in \mathfrak{S}}$, $D = (D_{kl})_{k,l \in \mathfrak{S}}$, or $\widetilde{\chi} = (\widetilde{\chi}_1, \ldots, \widetilde{\chi}_n)^t$, are also smooth functions of the state variables. Note that the matrices C and D are generally irreducible and the governing equations have thus a complex structure [17, 18]. Finally, there exist many alternative forms for multicomponent transport fluxes that are out of the scope of this work and the reader is referred to [50, 7, 15, 18].

2.5. Mathematical assumptions

In order to investigate the fast chemistry asymptotic analysis, the reaction constants are naturally rescaled as

$$\mathcal{K}_i^{\mathrm{s}} = \frac{\widehat{\mathcal{K}}_i^{\mathrm{s}}}{\epsilon}, \tag{2.22}$$

where $\epsilon \in (0, 1]$ denotes the chemistry relaxation parameter and $\widehat{\mathcal{K}}_i^{\mathrm{s}}$, $i \in \mathfrak{R}$, the rescaled reaction constants that remain finite as $\epsilon \to 0$. The parameter ϵ

typically represents a ratio between chemistry times and fluid mechanics times and this parameter converges to zero $\epsilon \to 0$ in the fast chemistry limit. Accordingly, we introduce the rescaled reaction rates $\tau_i = \widehat{\tau}_i/\epsilon$, $i \in \mathfrak{R}$, production rates $\omega_k = \widehat{\omega}_k/\epsilon$, $k \in \mathfrak{S}$, production vector $\widehat{\omega} = (\widehat{\omega}_1, \dots, \widehat{\omega}_n)^t \in \mathbb{R}^n$, linearized operator $\Lambda = \widehat{\Lambda}/\epsilon \in \mathbb{R}^{n,n}$, and constants $\Lambda_i = \widehat{\Lambda}_i/\epsilon$, $i \in \mathfrak{R}$, in such a way

$$\widehat{\omega}_k = \sum_{i\in\mathfrak{R}} \nu_{ki}\widehat{\tau}_i, \quad k \in \mathfrak{S}, \qquad \widehat{\tau}_i = \widehat{\mathcal{K}}_i^{\mathrm{s}}\big(\exp\langle\mu, \widetilde{\nu}_i^{\mathrm{f}}\rangle - \exp\langle\mu, \widetilde{\nu}_i^{\mathrm{b}}\rangle\big), \quad i \in \mathfrak{R}, \tag{2.23}$$

and

$$M\widehat{\omega} = -\widehat{\Lambda}\,\mu, \qquad \widehat{\Lambda} = \sum_{i\in\mathfrak{R}} \widehat{\Lambda}_i \widetilde{\nu}_i \otimes \widetilde{\nu}_i. \tag{2.24}$$

The following assumptions on thermodynamic properties, rescaled chemical production rates, and transport coefficients have been extracted from the kinetic theory of gases. The integer $\varkappa \geq 3$ denotes the regularity class of transport coefficients and thermodynamic functions. The assumption $\varkappa \geq 3$ is used in Sections 2, 3, and 4, whereas it is assumed that $\varkappa \geq l + 4$ where l denotes the regularity class for strong solutions to the Cauchy problem with $l \geq l_0 + 1$ in Section 5 and $l \geq l_0 + 3$ in Section 6 where $l_0 = [d/2] + 1$.

($\mathsf{H_1}$) *The molar masses m_k, $k \in \mathfrak{S}$, and the perfect gas constant R are positive. The formation energies e_k^{st}, $k \in \mathfrak{S}$, and entropies s_k^{st}, $k \in \mathfrak{S}$, are real constants. The specific heats $c_{\mathrm{v}k}$, $k \in \mathfrak{S}$, are $C^\varkappa$ functions of $T \in [0, \infty)$. There exist positive constants $\underline{c}_{\mathrm{v}}$ and $\overline{c}_{\mathrm{v}}$ such that $0 < \underline{c}_{\mathrm{v}} \leqslant c_{\mathrm{v}k}(T) \leqslant \overline{c}_{\mathrm{v}}$ for $T \geq 0$ and $k \in \mathfrak{S}$.*

($\mathsf{H_2}$) *The stoichiometric coefficients ν_{ki}^{f} and ν_{ki}^{b}, $k \in \mathfrak{S}$, $i \in \mathfrak{R}$, the atomic coefficients $\mathfrak{a}_{kl}$, $k \in \mathfrak{S}$, $l \in \mathfrak{A}$, are nonnegative integers. The atom vectors $\mathfrak{a}_l \in \mathbb{R}^n$, $l \in \mathfrak{A}$, and the reaction vectors $\nu_i \in \mathbb{R}^n$, $i \in \mathfrak{R}$, satisfy the atom conservation relations $\langle \nu_i, \mathfrak{a}_l\rangle = 0$, $i \in \mathfrak{R}$, $l \in \mathfrak{A}$, and the species molar masses $\mathfrak{m} \in \mathbb{R}^n$ are given by $\mathfrak{m} = \sum_{l\in\mathfrak{A}} m_l\, \mathfrak{a}_l$.*

($\mathsf{H_3}$) *The chemical reaction mechanisms is such that $\mathcal{R} = \mathcal{A}^\perp \subset \mathbb{R}^n$ and the atomic elements are the first n_{a} species. The rescaled reaction constants $\widehat{\mathcal{K}}_i^{\mathrm{s}}$ are $C^\varkappa$ positive functions of $T > 0$ for $i \in \mathfrak{R}$.*

($\mathsf{H_4}$) *The flux diffusion matrix $C = (C_{kl})_{k,l\in\mathfrak{S}} \in \mathbb{R}^{n,n}$ the rescaled thermal diffusion ratios vector $\widetilde{\chi} = (\widetilde{\chi}_1, \dots, \widetilde{\chi}_n)^t \in \mathbb{R}^n$ the volume viscosity κ, the shear viscosity η, and the thermal conductivity λ are $C^\varkappa$ functions of (ϱ, T) for $T > 0$ and $\rho_i > 0$, $i \in \mathfrak{S}$. These coefficients satisfy the mass conservation relations $N(C) = \mathrm{Span}\{\varrho\}$, $R(C) = \mathbb{1}^\perp$, and $\widetilde{\chi} \in \mathsf{x}^\perp$ in $\mathbb{R}^n$ The dimension of the underlying kinetic velocity space d' is such that $2 \leq d'$ and $d \leq d'$.*

($\mathsf{H_5}$) *The thermal conductivity λ and the shear viscosity η are positive. The volume viscosity κ is nonnegative. The diffusion matrix $D \in \mathbb{R}^{n,n}$ where $D_{kl} = C_{kl}/\rho_k$, $k, l \in \mathfrak{S}$, is symmetric positive semi-definite with nullspace $N(D) = \mathrm{Span}\{\varrho\}$,*

Remark 2.2. *The coefficients C, λ, η, $\widetilde{\chi}$ and κ have smooth extensions to the domain $\rho_i \geq 0$, $i \in \mathfrak{S}$, and $\rho > 0$. This is also the case for the non diagonal coefficients D_{ij} for $i \neq j$ whereas the coefficient $\rho_i D_{ii}$ has a finite positive limit when $\rho_i \to 0$ [17, 11].*

The properties (H_1)-(H_5) are assumed to hold whenever the equations modeling multicomponent reactive fluids are considered. The constraints on C and D insure in particular that $\sum_{k\in\mathfrak{S}} \boldsymbol{\mathcal{J}}_k = \sum_{k\in\mathfrak{S}} \rho_k \mathbf{v}_k = 0$ so that diffusion *is not artificially creating mass.* Adding the n species governing equations indeed yields the total mass conservation equation $\partial_t \rho + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0$. On the other hand, the positivity properties of transport coefficients—that are deduced from properties of multicomponent Boltzmann linearized collision operators—further insure that entropy production due to transport processes is nonnegative.

In the following proposition, we investigate the balance equation for entropy and evaluate the rate of entropy production [18]. In this Proposition, $\boldsymbol{d}_k = \boldsymbol{\nabla}\mathsf{x}_k + \mathsf{x}_k \boldsymbol{\nabla} \log p + \mathsf{x}_k \widetilde{\chi}_k \boldsymbol{\nabla} \log T$ denotes the diffusion driving force of the kth species, κ' the modified volume viscosity coefficient, $\mathbb{D}$ the differential symbol, and for any matrices A and B, $|A|^2 = \sum_{ij} a_{ij}^2$ denotes the Frobenius norm and $A{:}B = \sum_{ij} a_{ij} b_{ij}$ the corresponding scalar product.

Proposition 2.3. *The differential $\mathbb{D}\mathcal{S}$ of entropy is given by Gibb's relation*

$$T \, \mathbb{D}\mathcal{S} = \mathbb{D}\mathcal{E} - \sum_{k\in\mathfrak{S}} g_k \, \mathbb{D}\rho_k. \tag{2.25}$$

Moreover, $\mathcal{S}$ satisfies the balance equation

$$\partial_t \mathcal{S} + \boldsymbol{\nabla}\cdot(\boldsymbol{v}\mathcal{S}) + \boldsymbol{\nabla}\cdot\Big(\frac{\boldsymbol{Q}}{T} - \sum_{k\in\mathfrak{S}} \frac{g_k}{T} \boldsymbol{\mathcal{J}}_k\Big) = \frac{\eta}{2T} \big|\boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{\nabla}\boldsymbol{v}^t - \tfrac{2}{d}(\boldsymbol{\nabla}\cdot\boldsymbol{v})\,\boldsymbol{I}\big|^2$$

$$+ \frac{\kappa'}{T}(\boldsymbol{\nabla}\cdot\boldsymbol{v})^2 + \frac{\lambda}{T^2}|\boldsymbol{\nabla}T|^2 + \frac{\rho R}{m} \sum_{k,l\in\mathfrak{S}} D_{kl} \boldsymbol{d}_k\cdot\boldsymbol{d}_l - \sum_{k\in\mathfrak{S}} \frac{g_k m_k \omega_k}{T}. \tag{2.26}$$

Proof. From the expressions of $\mathcal{E}$ and $\mathcal{S}$ it is first obtained that $\partial_{\rho_k}\mathcal{S} = s_k - r_k$ in such a way that $\partial_{\rho_k}\mathcal{S} = e_k/T - g_k/T$, and next that $\partial_T \mathcal{S} = \sum_{k\in\mathfrak{S}} \rho_k c_{\mathrm{v}k}/T = \partial_T \mathcal{E}/T$, and this yields the volumetric Gibbs' relation (2.25) since we have $\mathbb{D}\mathcal{E} = \partial_T \mathcal{E}\, \mathbb{D}T + \sum_{k\in\mathfrak{S}} e_k \, \mathbb{D}\rho_k$.

Using then (2.25), the convective derivative $\partial_t \mathcal{S} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\mathcal{S}$ is expressed in terms of $\partial_t \mathcal{E} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\mathcal{E}$ and $\partial_t \rho_k + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho_k$, $k \in \mathfrak{S}$. The convective derivative of $\partial_t \rho_k + \boldsymbol{v}\cdot\boldsymbol{\nabla}\rho_k$ is given by the kth species conservation equation. On the other hand, $\partial_t \mathcal{E} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\mathcal{E}$ is deduced from the total energy conservation equation after subtracting the balance equation for kinetic energy obtained by multiplying the momentum equation by the velocity vector. After some algebra, this yields

$$\begin{aligned} \partial_t \mathcal{S} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\mathcal{S} = & \frac{1}{T}(-\boldsymbol{\nabla}\cdot\boldsymbol{Q} - \mathcal{E}\,\boldsymbol{\nabla}\cdot\boldsymbol{v} - p\,\boldsymbol{\nabla}\cdot\boldsymbol{v} - \boldsymbol{\Pi}{:}\boldsymbol{\nabla}\boldsymbol{v}) \\ & - \sum_{k\in\mathfrak{S}} \frac{g_k}{T}(m_k\omega_k - \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{J}}_k - \rho_k\,\boldsymbol{\nabla}\cdot\boldsymbol{v}), \end{aligned}$$

Noting that $\mathcal{G} = \sum_{k\in\mathfrak{S}} \rho_k g_k$ and $(\mathcal{E} + p - \mathcal{G})/T = \mathcal{S}$, and using

$$\frac{\boldsymbol{\nabla\cdot Q}}{T} = \boldsymbol{\nabla\cdot}\Big(\frac{\boldsymbol{Q}}{T}\Big) + \frac{\boldsymbol{Q\cdot\nabla} T}{T^2}, \qquad \frac{g_k \boldsymbol{\nabla\cdot\mathcal{J}}_k}{T} = \boldsymbol{\nabla\cdot}\Big(\frac{g_k\boldsymbol{\mathcal{J}}_k}{T}\Big) - \boldsymbol{\mathcal{J}}_k\boldsymbol{\cdot\nabla}\Big(\frac{g_k}{T}\Big),$$

we obtain that

$$\partial_t \mathcal{S} + \boldsymbol{\nabla\cdot}(\boldsymbol{v}\mathcal{S}) + \boldsymbol{\nabla\cdot}\Big(\frac{\boldsymbol{Q}}{T} - \sum_{k\in\mathfrak{S}} \frac{g_k}{T}\boldsymbol{\mathcal{J}}_k\Big) = -\frac{\boldsymbol{\Pi:\nabla v}}{T}$$
$$- \frac{\boldsymbol{Q\cdot\nabla} T}{T^2} - \sum_{k\in\mathfrak{S}} \boldsymbol{\mathcal{J}}_k\boldsymbol{\cdot\nabla}\Big(\frac{g_k}{T}\Big) - \sum_{k\in\mathfrak{S}} \frac{m_k\omega_k g_k}{T}.$$

From the expression (2.19) of the viscous tensor $\boldsymbol{\Pi}$, the tensor product $-\boldsymbol{\Pi:\nabla v}$ is then rewritten in the form $\kappa'(\boldsymbol{\nabla\cdot v})^2 + (\eta/2)\big|\boldsymbol{\nabla v} + \boldsymbol{\nabla v}^t - \frac{2}{d}(\boldsymbol{\nabla\cdot v})\,\boldsymbol{I}\big|^2$. Finally, using the thermodynamic identity

$$\boldsymbol{\nabla}\Big(\frac{g_k}{T}\Big) = -\frac{h_k}{T^2}\boldsymbol{\nabla} T + r_k \frac{\boldsymbol{\nabla}\mathsf{x}_k}{\mathsf{x}_k} + r_k\frac{\boldsymbol{\nabla} p}{p},$$

the expression (2.20) for the diffusive fluxes $\boldsymbol{\mathcal{J}}_k$, $k \in \mathfrak{S}$, the relations $\rho_k/m_k = (\rho/m)\mathsf{x}_k$ and $C_{kl} = \rho_k D_{kl}$, $k,l \in \mathfrak{S}$, and the expression (2.21) for the heat flux completes the proof. □

From assumptions (H_5) it is deduced that the entropy production terms in (2.26) associated with transport processes are nonnegative whereas entropy production due to chemical reactions $-\sum_{k\in\mathfrak{S}} g_k m_k \omega_k/T = -R\langle\mu, M\omega\rangle$ is investigated in the next section.

2.6. *Chemical equilibrium*

In the following lemma, entropy production due to chemistry is investigated as well as the notion of chemical equilibrium.

Proposition 2.4. *For any state $(\varrho, T)^t \in (0,\infty)^{n+1}$ the entropy production due to chemistry $-R\langle\mu, M\omega\rangle$ is nonnegative and may be written*

$$-R\langle\mu, M\omega\rangle = \sum_{i\in\mathfrak{R}} R\mathcal{K}_i^{\mathrm{s}}\big(\langle\mu,\widetilde{\nu}_i^{\mathrm{f}}\rangle - \langle\mu,\widetilde{\nu}_i^{\mathrm{b}}\rangle\big)\,\big(\exp\langle\mu,\widetilde{\nu}_i^{\mathrm{f}}\rangle - \exp\langle\mu,\widetilde{\nu}_i^{\mathrm{b}}\rangle\big). \quad (2.27)$$

Furthermore the following statements are equivalent :

(i) *The entropy production due to chemistry vanishes $\langle\mu, M\omega\rangle = 0$.*

(ii) *The reaction rates of progress vanish $\tau_j = 0$, $\quad j \in \mathfrak{R}$.*

(iii) *The species production rates vanish $\omega = 0$.*

(iv) *The vector μ belongs to $M^{-1}\mathcal{A} = (M\mathcal{R})^\perp$.*

Any point $(\varrho, T)^t \in (0,\infty)^{n+1}$ that satisfies these properties is termed a chemical equilibrium point.

Proof. It is first deduced from (2.14) that $-\langle \mu, M\omega\rangle = \sum_{i\in\mathfrak{R}} \tau_i \langle \mu, (\widetilde{\nu}_i^{\rm f} - \widetilde{\nu}_i^{\rm b})\rangle$ so that from (2.15)

$$-\langle \mu, M\omega\rangle = \sum_{i\in\mathfrak{R}} \mathcal{K}_i^{\rm s}\big(\langle \mu, \widetilde{\nu}_i^{\rm f}\rangle - \langle \mu, \widetilde{\nu}_i^{\rm b}\rangle\big)\,\big(\exp\langle \mu, \widetilde{\nu}_i^{\rm f}\rangle - \exp\langle \mu, \widetilde{\nu}_i^{\rm b}\rangle\big),$$

and this yields (2.27). Note that each chemical reaction yields a nonnegative entropy production as in the underlying kinetic model [18, 12].

From the expression of entropy production (2.27), it is then obtained that $\langle \mu, M\omega\rangle = 0$ implies $\langle \mu, \widetilde{\nu}_j\rangle = 0$, $j \in \mathfrak{R}$, and so $\tau_j = 0$, $j\in\mathfrak{R}$, and $\omega = 0$ and it is established that (i) implies (ii). The fact that (ii) implies (iii) is a consequence of $\omega = \sum_{j\in\mathfrak{R}} \tau_j \nu_j$. One also deduces from the expression of entropy production $-R\langle \mu, M\omega\rangle$ that (iii) implies (i) so that the three statements (i), (ii), and (iii) are equivalent. Finally, it is easily established that (iv) is equivalent to $\langle \mu, \widetilde{\nu}_j\rangle = 0$, $j\in\mathfrak{R}$, so that (ii) and (iv) are also equivalent since $\mathcal{A} = \mathcal{R}^\perp$. $\square$

Only positive equilibrium states $\varrho > 0$ which are in the interior $(0,\infty)^n$ of the natural densities simplex $[0,\infty)^n$ are considered in Proposition 2.4. Spurious points with zero mass fractions where the source terms ω_k, $k\in\mathfrak{S}$, also vanish—termed 'boundary equilibrium points'—are of a different nature [18]. Detailed chemical reaction mechanisms often exclude such boundary equilibrium points—unless some atom is missing in the mixture—because of three body recombination reactions [18]. The expression of entropy production due to chemical reactions (2.27) may also be seen as a macroscopic consequence of Boltzmann H theorem involving only reactive collisions [12, 18].

Proposition 2.5. *For any $T_{\rm e} > 0$ and any $\widetilde{\varrho} = (\widetilde{\rho}_1, \ldots, \widetilde{\rho}_{n_{\rm a}})^t > 0$ there exists a unique equilibrium point $\varrho_{\rm e}$ such that $\langle \varrho_{\rm e}, \widetilde{\mathfrak{a}}_l\rangle = \widetilde{\rho}_l$, for $l\in\mathfrak{A}$. In other words, there exits a unique $\varrho_{\rm e} > 0$ such that $\mu(\varrho_{\rm e}, T_{\rm e}) \in M^{-1}\mathcal{A}$ and $\langle \varrho_{\rm e}, \widetilde{\mathfrak{a}}_l\rangle = \widetilde{\rho}_l$, for $l\in\mathfrak{A}$ and the equilibrium state $\varrho_{\rm e} = (\rho_{\rm e1}, \ldots, \rho_{\rm en})^t$ is a $C^\varkappa$ function of $(\widetilde{\varrho}, T_{\rm e})$.*

Proof. The proof is only sketched and we refer to [46, 39, 18] for more details. For $\widetilde{\varrho} > 0$, letting $\varrho_{\rm sp} = \sum_{l\in\mathfrak{A}} \widetilde{\rho}_l \mathsf{e}_l + \delta \sum_{k\in\mathfrak{S}\setminus\mathfrak{A}} \widetilde{\mathfrak{v}}_k$, then $\langle \varrho_{\rm sp}, \widetilde{\mathfrak{a}}_l\rangle = \widetilde{\rho}_l$ and $\varrho_{\rm sp} > 0$ for $\delta > 0$ small enough. The equilibrium point is then investigated in the simplex

$$\mathfrak{I} = (\varrho_{\rm sp} + M\mathcal{R}) \cap (0,\infty)^n.$$

The Helmholtz free energy $\mathcal{F} = \mathcal{E} - T_{\rm e}\mathcal{S}$ is continuous on the closure $\overline{\mathfrak{I}} = (\varrho_{\rm sp} + M\mathcal{R}) \cap [0,\infty)^n$ of $\mathfrak{I}$ and is strictly convex on $\mathfrak{I}$. Moreover $\mathcal{F}$ cannot achieve its maximum at the boundaries where the appropriate derivatives have the wrong sign [18]. The maximum is thus achieved in the interior $\varrho_{\rm e} > 0$ and this point is shown to be an equilibrium point using $\partial_\varrho \mathcal{F} = RT_{\rm e}\mu$ and that $M\mathcal{R}$ is in the tangent space of the simplex so that $\langle \mu_{\rm e}, M\nu_i\rangle = 0$ for any $i\in\mathfrak{R}$. The uniqueness of the equilibrium point on $\mathfrak{I}$ is a consequence of the strict convexity of $\mathcal{F}$. Finally, the equilibrium partial densities $\varrho_{\rm e}$ are $C^\varkappa$ functions of $(\widetilde{\varrho}, T_{\rm e})$ as a direct application of the implicit function theorem [46, 39, 18, 26]. $\square$

The energy per unit volume at equilibrium $\mathcal{E}_{\rm e}$ is defined by $\mathcal{E}_{\rm e}(\widetilde{\varrho}, T_{\rm e}) = \mathcal{E}\big(\varrho_{\rm e}(\widetilde{\varrho}, T_{\rm e}), T_{\rm e}\big)$ and the specific heat at equilibrium $c_{\rm ev}$ at constant volume by $\rho_{\rm e} c_{\rm ev} = \partial_{T_{\rm e}} \mathcal{E}_{\rm e}(\widetilde{\varrho}, T_{\rm e})$. The entropy per unit volume of the mixture is given by $\mathcal{S}_{\rm e}(\widetilde{\varrho}, T_{\rm e}) = \mathcal{S}\big(\varrho_{\rm e}(\widetilde{\varrho}, T_{\rm e}), T_{\rm e}\big)$ and the Gibbs function per unit volume by $\mathcal{G}_{\rm e} = \mathcal{E}_{\rm e} + p_{\rm e} - T_{\rm e}\mathcal{S}_{\rm e}$. The pressure $p_{\rm e}$ is given by $p_{\rm e} = RT_{\rm e} \sum_{k\in\mathfrak{S}} \rho_{{\rm e}k}/m_k$ and the total density $\rho_{\rm e}$ can be written $\rho_{\rm e} = \sum_{k\in\mathfrak{S}} \rho_{{\rm e}k} = \sum_{l\in\mathfrak{A}} \widetilde{\rho}_l$. Differential expressions as well as concavity properties of the entropy per unit volume $\mathcal{S}_{\rm e}$ in terms of the state variables $(\widetilde{\varrho}, T_{\rm e})$ and $(\widetilde{\varrho}, \mathcal{E}_{\rm e})$ are now investigated in the following new lemmas.

Lemma 2.6. *The energy and entropy per unit volume at chemical equilibrium* $\mathcal{E}_{\rm e}$ *and* $\mathcal{S}_{\rm e}$ *are* $C^\varkappa$ *functions of the variables* $(\widetilde{\varrho}, T_{\rm e})$ *over the domain* $T_{\rm e} > 0$, $\widetilde{\varrho} > 0$ *and* $c_{\rm ev} = \partial_{T_{\rm e}} \mathcal{E}_{\rm e}/\rho_{\rm e}$ *is positive. Moreover, letting* $s_{{\rm e}k} = s_k\big(\rho_{{\rm e}k}(\widetilde{\varrho}, T_{\rm e}), T_{\rm e}\big)$, $k \in \mathfrak{S}$, *then*

$$\partial_{T_{\rm e}} \mathcal{S}_{\rm e} = \frac{\rho_{\rm e} c_{\rm ev}}{T_{\rm e}}, \qquad \partial_{\widetilde{\rho}_l} \mathcal{S}_{\rm e} = \sum_{k\in\mathfrak{S}} (s_{{\rm e}k} - r_k) \partial_{\widetilde{\rho}_l} \rho_{{\rm e}k}, \quad l \in \mathfrak{A}. \tag{2.28}$$

Proof. The regularity of $\mathcal{E}_{\rm e}$ and $\mathcal{S}_{\rm e}$ is a consequence of that of $\mathcal{E}$, $\mathcal{S}$, and $\varrho_{\rm e}$. A direct calculation using $\mathcal{E}_{\rm e} = \sum_{k\in\mathfrak{S}} \rho_{{\rm e}k}(\widetilde{\varrho}, T_{\rm e}) e_k(T_{\rm e})$ then yields

$$\mathbb{D}\mathcal{E}_{\rm e} = \rho_{\rm e} c_{\rm ev}\, \mathbb{D} T_{\rm e} + \sum_{l\in\mathfrak{A}} \sum_{k\in\mathfrak{S}} \partial_{\widetilde{\rho}_l} \rho_{{\rm e}k} e_k \,\mathbb{D}\widetilde{\rho}_l, \tag{2.29}$$

where $\mathbb{D}$ denotes the total derivative and

$$\rho_{\rm e} c_{\rm ev} = \partial_{T_{\rm e}} \mathcal{E}_{\rm e} = \sum_{k\in\mathfrak{S}} \partial_{T_{\rm e}} \rho_{{\rm e}k}(\widetilde{\varrho}, T_{\rm e})\, e_k(T_{\rm e}) + \sum_{k\in\mathfrak{S}} \rho_{{\rm e}k}(\widetilde{\varrho}, T_{\rm e})\, c_{{\rm v}k}(T_{\rm e}). \tag{2.30}$$

The second term in the right hand side of (2.30) is positive since mass densities and specific heats are positive and it is thus sufficient to establish that the first sum is nonnegative. Differentiating with respect to temperature $T_{\rm e}$ the constraint $\langle \widetilde{\mathfrak{a}}_l, \varrho_{\rm e}\rangle = \widetilde{\rho}_l$ yields $\langle \widetilde{\mathfrak{a}}_l, \partial_{T_{\rm e}} \varrho_{\rm e}\rangle = 0$ for $l \in \mathfrak{A}$, so that $\partial_{T_{\rm e}} \varrho_{\rm e} \in (M^{-1}\mathcal{A})^\perp = M\mathcal{R}$. Similarly, differentiating the relations $\langle \widetilde{\mathfrak{v}}_k, \mu_{\rm e}\rangle = 0$, $k \in \mathfrak{S}\backslash\mathfrak{A}$, with respect to $T_{\rm e}$ yields $\partial_{T_{\rm e}} \mu_{\rm e} \in M^{-1}\mathcal{A} = (M\mathcal{R})^\perp$ and using (2.8) with (2.5)(2.7) it is next obtained that

$$-RT_{\rm e}^2 \partial_{T_{\rm e}} \mu_{\rm e} = \big(e_1(T_{\rm e}) - r_1 T_{\rm e}^2 \partial_{T_{\rm e}} \log \rho_{{\rm e}1}, \ldots, e_n(T_{\rm e}) - r_n T_{\rm e}^2 \partial_{T_{\rm e}} \log \rho_{{\rm e}n}\big)^t \in M^{-1}\mathcal{A}.$$

Multiplying scalarly this relation by $\partial_{T_{\rm e}} \varrho_{\rm e} \in M\mathcal{R} = (M^{-1}\mathcal{A})^\perp$ yields

$$\sum_{k\in\mathfrak{S}} \partial_{T_{\rm e}} \rho_{{\rm e}k}(\widetilde{\varrho}, T_{\rm e})\, e_k(T_{\rm e}) = \sum_{k\in\mathfrak{S}} r_k T_{\rm e}^2 \rho_{{\rm e}k}(\widetilde{\varrho}, T_{\rm e}) \big(\partial_{T_{\rm e}} \log \rho_{{\rm e}k}(\widetilde{\varrho}, T_{\rm e})\big)^2,$$

and this proves that $c_{\rm ev} > 0$.

A direct calculation using $\mathcal{S}_{\rm e} = \sum_{k\in\mathfrak{S}} \rho_{{\rm e}k}(\widetilde{\varrho}, T_{\rm e}) s_k\big(\rho_{{\rm e}k}(\widetilde{\varrho}, T_{\rm e}), T_{\rm e}\big)$ next yields

$$\mathbb{D}\mathcal{S}_{\rm e} = \partial_{T_{\rm e}} \mathcal{S}_{\rm e}\, \mathbb{D} T_{\rm e} + \sum_{l\in\mathfrak{A}} \sum_{k\in\mathfrak{S}} (s_{{\rm e}k} - r_k) \partial_{\widetilde{\rho}_l} \rho_{{\rm e}k} \,\mathbb{D}\widetilde{\rho}_l, \tag{2.31}$$

where

$$\partial_{T_\mathrm{e}}\mathcal{S}_\mathrm{e} = \frac{1}{T_\mathrm{e}}\sum_{k\in\mathfrak{S}}\rho_{\mathrm{e}k}c_{\mathrm{v}k} + \sum_{k\in\mathfrak{S}}(s_{\mathrm{e}k}-r_k)\partial_{T_\mathrm{e}}\rho_{\mathrm{e}k}.$$

However, since $s_{\mathrm{e}k} - r_k = (e_{\mathrm{e}k} - g_{\mathrm{e}k})/T_\mathrm{e}$, $\mu_{\mathrm{e}k} = g_{\mathrm{e}k}/RT_\mathrm{e}$, $\mu_\mathrm{e} \in M^{-1}\mathcal{A}$, and $\partial_{T_\mathrm{e}}\varrho_{\mathrm{e}k} \in M\mathcal{R} = (M^{-1}\mathcal{A})^\perp$, we deduce the identity $\sum_{k\in\mathfrak{S}}(s_{\mathrm{e}k}-r_k)\partial_{T_\mathrm{e}}\rho_{\mathrm{e}k} = \sum_{k\in\mathfrak{S}} e_{\mathrm{e}k}\partial_{T_\mathrm{e}}\rho_{\mathrm{e}k}/T_\mathrm{e}$ and finally that $\partial_{T_\mathrm{e}}\mathcal{S}_\mathrm{e} = \rho_\mathrm{e}c_\mathrm{ev}/T_\mathrm{e} = \partial_{T_\mathrm{e}}\mathcal{E}_\mathrm{e}/T_\mathrm{e}$ and the proof of (2.28) is complete. □

Lemma 2.7. *The map $(\widetilde{\varrho}, T_\mathrm{e}) \mapsto (\widetilde{\varrho}, \mathcal{E}_\mathrm{e})$ is a $C^\varkappa$ diffeomorphism from the domain $T_\mathrm{e} > 0$, $\widetilde{\varrho} > 0$ onto an open set $\mathcal{O}_{(\widetilde{\varrho},\mathcal{E}_\mathrm{e})}$. Over this domain $\mathcal{O}_{(\widetilde{\varrho},\mathcal{E}_\mathrm{e})}$, denoting by $\overline{\partial}$ the derivation with respect to the variable $(\widetilde{\varrho}, \mathcal{E}_\mathrm{e})$ then*

$$\overline{\partial}_{\mathcal{E}_\mathrm{e}}\mathcal{S}_\mathrm{e} = \frac{1}{T_\mathrm{e}}, \qquad \overline{\partial}_{\widetilde{\rho}_l}\mathcal{S}_\mathrm{e} = -\frac{g_{\mathrm{e}l}}{T_\mathrm{e}}, \quad l \in \mathfrak{A}, \tag{2.32}$$

where $g_{\mathrm{e}l} = g_l\big(\rho_{\mathrm{e}l}(\widetilde{\varrho}, T_\mathrm{e}), T_\mathrm{e}\big)$ so that $T_\mathrm{e}\,\mathbb{D}\mathcal{S}_\mathrm{e} = \mathbb{D}\mathcal{E}_\mathrm{e} - \sum_{l\in\mathfrak{A}} g_{\mathrm{e}l}\,\mathbb{D}\widetilde{\rho}_l$, where $\mathbb{D}$ denotes the total derivative. Moreover, letting $\mathfrak{s}_l = \sum_{k\in\mathfrak{S}} e_{\mathrm{e}k}\partial_{\widetilde{\rho}_l}\rho_{\mathrm{e}k} = e_{\mathrm{e}l} - r_l T_\mathrm{e}^2\partial_{T_\mathrm{e}}\log\rho_{\mathrm{e}l}$ for $l \in \mathfrak{A}$, the following relations hold

$$\overline{\partial}^2_{\mathcal{E}_\mathrm{e},\mathcal{E}_\mathrm{e}}\mathcal{S}_\mathrm{e} = -\frac{1}{\rho_\mathrm{e}c_\mathrm{ev}T_\mathrm{e}^2}, \qquad \overline{\partial}^2_{\mathcal{E}_\mathrm{e},\widetilde{\rho}_l}\mathcal{S}_\mathrm{e} = \frac{\mathfrak{s}_l}{\rho_\mathrm{e}c_\mathrm{ev}T_\mathrm{e}^2}, \quad l \in \mathfrak{A},$$

$$\overline{\partial}^2_{\widetilde{\rho}_l,\widetilde{\rho}_{l'}}\mathcal{S}_\mathrm{e} = -\frac{\mathfrak{s}_l\mathfrak{s}_{l'}}{\rho_\mathrm{e}c_\mathrm{ev}T_\mathrm{e}^2} - \sum_{k\in\mathfrak{S}}\frac{1}{m_k}\frac{\partial_{\widetilde{\rho}_l}\rho_{\mathrm{e}k}\partial_{\widetilde{\rho}_{l'}}\rho_{\mathrm{e}k}}{\rho_{\mathrm{e}k}},$$

and $\mathcal{S}_\mathrm{e}$ is a strictly concave function of $(\widetilde{\varrho}, \mathcal{E}_\mathrm{e})$.

Proof. The fact that $(\widetilde{\varrho}, T_\mathrm{e}) \mapsto (\widetilde{\varrho}, \mathcal{E}_\mathrm{e})$ is a $C^\varkappa$ diffeomorphism from the domain $T_\mathrm{e} > 0$, $\widetilde{\varrho} > 0$ onto an open set is a direct consequence of $c_\mathrm{ev} > 0$ and of the inverse function theorem. Using (2.29)(2.31) it is obtained after some algebra that

$$T_\mathrm{e}\,\mathbb{D}\mathcal{S}_\mathrm{e} = \mathbb{D}\mathcal{E}_\mathrm{e} - \sum_{l\in\mathfrak{A}}\sum_{k\in\mathfrak{S}} g_{\mathrm{e}k}\partial_{\widetilde{\rho}_l}\rho_{\mathrm{e}k}\,\mathbb{D}\widetilde{\rho}_l.$$

Differentiating the relations $\langle\widetilde{\mathfrak{a}}_{l'}, \varrho_\mathrm{e}\rangle = \widetilde{\rho}_{l'}$, for $l' \in \mathfrak{A}$, with respect to $\widetilde{\rho}_l$, and keeping in mind that $\mathfrak{a}_{kl} = \widetilde{\mathfrak{a}}_{kl} = \delta_{kl}$ for $k, l \in \mathfrak{A}$, it is obtained that $\partial_{\widetilde{\rho}_l}\varrho_\mathrm{e} - \mathsf{e}_l \in (M^{-1}\mathcal{A})^\perp = M\mathcal{R}$ for $l \in \mathfrak{A}$, where e_k, $k \in \mathfrak{S}$, are the basis vectors of $\mathbb{R}^n$. This now implies that $\sum_{k\in\mathfrak{S}} g_{\mathrm{e}k}\partial_{\widetilde{\rho}_l}\rho_{\mathrm{e}k} = g_{\mathrm{e}l}$ and finally one obtains the relation

$$T_\mathrm{e}\,\mathbb{D}\mathcal{S}_\mathrm{e} = \mathbb{D}\mathcal{E}_\mathrm{e} - \sum_{l\in\mathfrak{A}} g_{\mathrm{e}l}\,\mathbb{D}\widetilde{\rho}_l,$$

that is close to that out of equilibrium. This shows that $\overline{\partial}_{\mathcal{E}_\mathrm{e}}\mathcal{S}_\mathrm{e} = 1/T_\mathrm{e}$ and $\overline{\partial}_{\widetilde{\rho}_l}\mathcal{S}_\mathrm{e} = -g_{\mathrm{e}l}/T_\mathrm{e}$ and from (2.29) one also has $\overline{\partial}_{\mathcal{E}_\mathrm{e}}T_\mathrm{e} = 1/\rho_\mathrm{e}c_\mathrm{ev}$ and $\overline{\partial}_{\widetilde{\rho}_l}T_\mathrm{e} = -\sum_{k\in\mathfrak{S}} e_k(T_\mathrm{e})\partial_{\widetilde{\rho}_l}\rho_{\mathrm{e}k}/\rho_\mathrm{e}c_\mathrm{ev}$, for $l \in \mathfrak{A}$.

Differentiating $\overline{\partial}_{\mathcal{E}_e}\mathcal{S}_e = 1/T_e$ with respect to $\mathcal{E}_e$ and $\widetilde{\rho}_l$ then yields that $\overline{\partial}^2_{\mathcal{E}_e,\mathcal{E}_e}\mathcal{S}_e = -1/T_e^2\rho_e c_{ev}$ and $\overline{\partial}^2_{\mathcal{E}_e,\widetilde{\rho}_l}\mathcal{S}_e = \sum_{k\in\mathfrak{S}} e_{ek}\partial_{\widetilde{\rho}_l}\rho_{ek}/T_e^2\rho_e c_{ev}$. The second expression of $\mathfrak{s}_l$ and thus of $\overline{\partial}^2_{\mathcal{E}_e,\widetilde{\rho}_l}\mathcal{S}_e$ is obtained by first noting that $\langle \partial_{T_e}\mu_e, \partial_{\widetilde{\rho}_l}\varrho_e - \mathsf{e}_l\rangle = 0$ in such a way that

$$\Big\langle -\frac{(e_{e1},\ldots,e_{en})^t}{RT_e^2} - M^{-1}\partial_{T_e}\log\varrho_e,\ \partial_{\widetilde{\rho}_l}\varrho_e - \mathsf{e}_l \Big\rangle = 0.$$

Differentiating $\langle \mu_e, \widetilde{\mathfrak{v}}_k\rangle = 0$, $k \in \mathfrak{S}\backslash\mathfrak{A}$, with respect to $\widetilde{\rho}_l$ also yields $\partial_{\widetilde{\rho}_l}\mu_e \in M^{-1}\mathcal{A}$, and using $\partial_{\widetilde{\rho}_l}\mu_{ek} = \partial_{\widetilde{\rho}_l}\log\varrho_{ek}/m_k$ yields

$$\langle M^{-1}\partial_{T_e}\log\varrho_e, \partial_{\widetilde{\rho}_l}\varrho_e\rangle = \langle \partial_{T_e}\varrho_e, M^{-1}\partial_{\widetilde{\rho}_l}\log\varrho_e\rangle = 0,$$

and this now implies that $\mathfrak{s}_l = \sum_{k\in\mathfrak{S}} e_k(T_e)\partial_{\widetilde{\rho}_l}\rho_{ek} = e_{el} - r_l T_e^2\partial_{T_e}\log\rho_{el}$. Finally, deriving the relation $\overline{\partial}_{\widetilde{\rho}_l}\mathcal{S}_e = -g_{el}/T_e$ with respect to $\widetilde{\rho}_{l'}$ yields the expression of $\overline{\partial}^2_{\widetilde{\rho}_l,\widetilde{\rho}_{l'}}\mathcal{S}_e$.

In order to establish the concavity of $\mathcal{S}_e$, letting $x = (x_1,\ldots,x_{n_a},x_{\mathcal{E}})^t$, it is noted that

$$-\langle\overline{\partial}\mathcal{S}_e x, x\rangle = \frac{1}{\rho_e c_{ev} T_e^2}\Big(x_{\mathcal{E}} - \sum_{l\in\mathfrak{A}}\mathfrak{s}_l x_l\Big)^2 + \sum_{k\in\mathfrak{S}}\frac{1}{m_k}\Big(\sum_{l\in\mathfrak{A}} x_l\partial_{\widetilde{\rho}_l}\rho_{ek}\Big)^2,$$

and this expression is nonnegative. If it is zero, then $\sum_{l\in\mathfrak{A}} x_l\partial_{\widetilde{\rho}_l}\rho_{ek} = 0$ for any $k\in\mathfrak{S}$ and this implies that $\sum_{l\in\mathfrak{A}} x_l\mathsf{e}_l = \sum_{l\in\mathfrak{A}} x_l(\mathsf{e}_l - \partial_{\widetilde{\rho}_l}\varrho_e) \in (M^{-1}\mathcal{A})^\perp$. This implies that $\langle\widetilde{\mathfrak{a}}_l, \sum_{l\in\mathfrak{A}} x_l\mathsf{e}_l\rangle = x_l = 0$, $l\in\mathfrak{A}$, and next $x_{\mathcal{E}} = 0$ and finally $x = 0$ so that $\mathcal{S}_e$ is strictly concave. □

The open set $\mathcal{O}_{(\widetilde{\varrho},\mathcal{E}_e)} \subset \mathbb{R}^{n_a+1}$ may be investigated under stronger assumptions associated with stable atomic elements and heats of formation at zero temperature [18] but the precise expression of this open set is not required in this work. Since $\mathcal{S}_e$ is a strictly concave function, thermodynamic stability naturally holds at chemical equilibrium [25].

2.7. Ozone decomposition

In order to illustrate the general formalism valid for arbitrary complex reaction mechanisms described in previous sections, we consider here a simple example of detailed chemical reaction mechanisms. More specifically, the detailed reaction mechanism of ozone combustion [6, 51] is investigated. This mechanism involves the three reactive species

$$\{\ \mathrm{O},\ \mathrm{O_2},\ \mathrm{O_3}\ \}, \tag{2.33}$$

and only one atom O. The species indexing set is thus $\mathfrak{S} = \{1,2,3\}$ with atomic oxygen O being species 1, oxygen O_2 species 2, and ozone O_3 species 3 with

$n = 3$. On the other hand, O is atom number 1 with $\mathfrak{A} = \{1\}$ and $n_{\mathrm{a}} = 1$. The atom decomposition vector $\mathfrak{a}_1$ is then

$$\mathfrak{a}_1 = (1, 2, 3)^t, \tag{2.34}$$

and the species mass are $m_1 = m_{\mathrm{O}}$, $m_2 = 2m_{\mathrm{O}}$, and $m_3 = 3m_{\mathrm{O}}$ where m_{O} is the mass per mole of atomic oxygen.

The ozone decomposition mechanism is typically written in chemistry as [6, 51]

$$\mathrm{O_3} + \mathfrak{M} \rightleftharpoons \mathrm{O} + \mathrm{O_2} + \mathfrak{M}, \tag{2.35}$$

$$\mathrm{O_3} + \mathrm{O} \rightleftharpoons 2\mathrm{O_2}, \tag{2.36}$$

$$\mathrm{O_2} + \mathfrak{M} \rightleftharpoons 2\mathrm{O} + \mathfrak{M}, \tag{2.37}$$

where $\mathfrak{M}$ denotes any of the three species O, $\mathrm{O_2}$, or $\mathrm{O_3}$. Reaction (2.35), as written in chemistry, is thus a notational shortcut for the three reactions

$$\mathrm{O_3} + \mathrm{O} \rightleftharpoons \mathrm{O} + \mathrm{O_2} + \mathrm{O},$$

$$\mathrm{O_3} + \mathrm{O_2} \rightleftharpoons \mathrm{O} + \mathrm{O_2} + \mathrm{O_2},$$

$$\mathrm{O_3} + \mathrm{O_3} \rightleftharpoons \mathrm{O} + \mathrm{O_2} + \mathrm{O_3},$$

that are numbered as reactions $1, 2, 3$. Reaction (2.36) is then numbered as reaction number 4, whereas (2.37) is again a notational shortcut for the three reactions

$$\mathrm{O_2} + \mathrm{O} \rightleftharpoons 2\mathrm{O} + \mathrm{O},$$

$$\mathrm{O_2} + \mathrm{O_2} \rightleftharpoons 2\mathrm{O} + \mathrm{O_2},$$

$$\mathrm{O_2} + \mathrm{O_3} \rightleftharpoons 2\mathrm{O} + \mathrm{O_3},$$

that are numbered as reactions $5, 6$, and 7. The detailed reaction mechanism of ozone combustion therefore involves $n = 3$ species, $n_{\mathrm{a}} = 1$ atom, $n_{\mathrm{r}} = 7$ chemical reactions with $\mathfrak{S} = \{1, 2, 3\}$, $\mathfrak{A} = \{1\}$ and $\mathfrak{R} = \{1, 2, 3, 4, 5, 6, 7\}$. The corresponding reaction vectors can be written

$$\nu_1 = \nu_2 = \nu_3 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \qquad \nu_4 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \qquad \nu_5 = \nu_6 = \nu_7 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix},$$

and it is easily checked that the O atom is conserved in each chemical reaction so that $\langle \mathfrak{a}_1, \nu_i \rangle = 0$ for $i \in \{1, 2, 3, 4, 5, 6, 7\}$. The mass atom vector $\widetilde{\mathfrak{a}}_1 = M^{-1}\mathfrak{a}_1$ and mass reaction vectors $\widetilde{\nu}_i = M\nu_i$, $i \in \{1, 2, 3, 4, 5, 6, 7\}$, are easily evaluated as

$$\widetilde{\mathfrak{a}}_1 = (1, 1, 1)^t,$$

and

$$\widetilde{\nu}_1 = \widetilde{\nu}_2 = \widetilde{\nu}_3 = m_{\mathrm{O}} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \qquad \widetilde{\nu}_4 = m_{\mathrm{O}} \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}$$

$$\widetilde{\nu}_5 = \widetilde{\nu}_6 = \widetilde{\nu}_7 = m_{\mathrm{O}} \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix},$$

and it is easily checked that $\langle \widetilde{\mathfrak{a}}_1, \widetilde{\nu}_i \rangle = 0$ for $i \in \{1,2,3,4,5,6,7\}$. In particular, the equilibrium condition for the ozone mixture reads

$$\mu \in \mathbb{R}\widetilde{\mathfrak{a}}_1 = \mathrm{Span}(\widetilde{\mathfrak{a}}_1).$$

The reaction vector space $\mathcal{R}$ is then of dimension 2 and given by

$$\mathcal{R} = \mathrm{Span}\{\nu_1, \nu_4\} = \mathrm{Span}\{\nu_1, \nu_5\} = \mathrm{Span}\{\nu_4, \nu_5\},$$

with the linear relation $-\nu_1 + \nu_4 + \nu_5 = 0$ whereas the atom vector space $\mathcal{A}$ reads

$$\mathcal{A} = \mathrm{Span}\{\mathfrak{a}_1\} = \mathbb{R}\mathfrak{a}_1 = \mathcal{R}^{\perp}.$$

The species formation vectors $\mathfrak{v}_k$ defined for $k \in \mathfrak{S}\backslash\mathfrak{A} = \{2,3\}$ are given by

$$\mathfrak{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \qquad \mathfrak{v}_3 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix},$$

and we observe that $\mathcal{R} = \mathrm{Span}\{\mathfrak{v}_2, \mathfrak{v}_3\}$ with $\mathfrak{v}_2 = -\nu_5$ and $\mathfrak{v}_3 = -\nu_4 - 2\nu_5$. The formation reactions may be written $2\mathrm{O} \rightleftharpoons \mathrm{O}_2$ and $3\mathrm{O} \rightleftharpoons \mathrm{O}_3$ and are not part of the mechanism (2.35)-(2.37) even though we have $\mathcal{R} = \mathrm{Span}\{\mathfrak{v}_2, \mathfrak{v}_3\}$.

Denoting by $n_k = \rho_k/m_k$ the number of mole of the kth species per unit volume where $k \in \{1,2,3\}$ then $\widetilde{n}_1 = n_1 + 2n_2 + 3n_3$ is the total number of moles of the O atom present *inside* all species per unit volume. Since there is only one atom, the partial atom mass density $\widetilde{\rho}_1 = m_1 \widetilde{n}_1$ also coincide with the total density $\widetilde{\rho}_1 = \rho$. At chemical equilibrium, for a fixed $T = T_{\mathrm{e}}$ and $\widetilde{\rho}_1 = m_1 \widetilde{n}_1$, there exists a unique positive equilibrium state $\varrho_{\mathrm{e}} = (\rho_{\mathrm{e}1}, \rho_{\mathrm{e}2}, \rho_{\mathrm{e}3})^t$ such that $M\mu(\varrho_{\mathrm{e}}, T_{\mathrm{e}}) \in \mathbb{R}\mathfrak{a}_1$. In order to investigate such equilibrium points, we may use (2.9) where $\mu_k^{\mathrm{u}}(T)$ is the value of μ_k at unit molar concentration $\rho_k = m_k$ or $n_k = 1$. Letting then

$$K_2(T) = \frac{m_2}{m_1^2} \exp\bigl(2m_1\mu_1^{\mathrm{u}} - m_2\mu_2^{\mathrm{u}}\bigr), \qquad K_3(T) = \frac{m_3}{m_1^3} \exp\bigl(3m_1\mu_1^{\mathrm{u}} - m_3\mu_3^{\mathrm{u}}\bigr),$$

we have at chemical equilibrium $2m_1\mu_1 = m_2\mu_2$ and $3m_1\mu_1 = m_3\mu_3$ in such a way that

$$\rho_{\mathrm{e}2} = K_2 \rho_{\mathrm{e}1}^2, \qquad \rho_{\mathrm{e}3} = K_3 \rho_{\mathrm{e}1}^3.$$

Then $\rho_{\mathrm{e}1}$ is the unique positive solution of the atom conservation relation

$$\widetilde{\rho}_1 = \rho_{\mathrm{e}1} + K_2 \rho_{\mathrm{e}1}^2 + K_3 \rho_{\mathrm{e}1}^3,$$

easily shown to be a smooth function of $\widetilde{\rho}_1 = \rho$ and T so that the equilibrium state ϱ_{e} is a smooth function $\widetilde{\rho}_1$ and T in agreement with the general theory.

Finally the molar production rates are obtained in the form

$$\begin{aligned}
\omega_1 &= \tau_1 + \tau_2 + \tau_3 - \tau_4 + 2\tau_5 + 2\tau_6 + 2\tau_7, \\
\omega_2 &= \tau_1 + \tau_2 + \tau_3 + 2\tau_4 - \tau_5 - \tau_6 - \tau_7, \\
\omega_3 &= -\tau_1 - \tau_2 - \tau_3 - \tau_4,
\end{aligned}$$

where τ_i denotes the rate of progress of reaction $i \in \{1,2,3,4,5,6,7\}$. These rate of progress may also be rewritten as given by the law of mass action. More specifically, defining the forward and reverse reaction constants as

$$\mathcal{K}_i^{\mathrm{f}} = \mathcal{K}_i^{\mathrm{s}} \exp\langle \mu^{\mathrm{u}}, \widetilde{\nu}_i^{\mathrm{f}} \rangle, \qquad \mathcal{K}_i^{\mathrm{b}} = \mathcal{K}_i^{\mathrm{s}} \exp\langle \mu^{\mathrm{u}}, \widetilde{\nu}_i^{\mathrm{b}} \rangle,$$

and using (2.9) and (2.15), the rate of progress of the ith reaction may be written

$$\tau_i = \mathcal{K}_i^{\mathrm{f}} \prod_{k\in\mathfrak{S}} \Big(\frac{\rho_k}{m_k}\Big)^{\nu_{ki}^{\mathrm{f}}} - \mathcal{K}_i^{\mathrm{b}} \prod_{k\in\mathfrak{S}} \Big(\frac{\rho_k}{m_k}\Big)^{\nu_{ki}^{\mathrm{b}}} = \mathcal{K}_i^{\mathrm{f}} \prod_{k\in\mathfrak{S}} n_k^{\nu_{ki}^{\mathrm{f}}} - \mathcal{K}_i^{\mathrm{b}} \prod_{k\in\mathfrak{S}} n_k^{\nu_{ki}^{\mathrm{b}}}.$$

We have in particular $\tau_1 = \mathcal{K}_1^{\mathrm{f}} n_1 n_3 - \mathcal{K}_1^{\mathrm{b}} n_1^2 n_2$, $\tau_2 = \mathcal{K}_2^{\mathrm{f}} n_2 n_3 - \mathcal{K}_2^{\mathrm{b}} n_1 n_2^2$, $\tau_3 = \mathcal{K}_3^{\mathrm{f}} n_3^2 - \mathcal{K}_3^{\mathrm{b}} n_1 n_2 n_3$, $\tau_4 = \mathcal{K}_4^{\mathrm{f}} n_1 n_3 - \mathcal{K}_4^{\mathrm{b}} n_2^2$, $\tau_5 = \mathcal{K}_5^{\mathrm{f}} n_1 n_2 - \mathcal{K}_5^{\mathrm{b}} n_1^3$, $\tau_6 = \mathcal{K}_6^{\mathrm{f}} n_2^2 - \mathcal{K}_6^{\mathrm{b}} n_1^2 n_2$, and $\tau_7 = \mathcal{K}_7^{\mathrm{f}} n_2 n_3 - \mathcal{K}_7^{\mathrm{b}} n_1^2 n_3$. The ratio of forward and reverse constants is also related to thermodynamics $\mathcal{K}_i^{\mathrm{b}}/\mathcal{K}_i^{\mathrm{f}} = \exp\big(-\langle \mu^{\mathrm{u}}, \widetilde{\nu}_i \rangle\big)$ as given by the kinetic theory of gases and statistical mechanics. Such reciprocity relations between macroscopic forward and reverse reaction constants, that may be seen as Onsager relations for chemistry, arise from reciprocity relations between molecular reactive transition probabilities [11, 12, 18].

2.8. Vector notation

The equations governing multicomponent flows (2.1)–(2.3) can conveniently be rewritten in vector form

$$\partial_t \mathsf{u} + \sum_{i\in\mathcal{D}} \partial_i \mathsf{F}_i + \sum_{i\in\mathcal{D}} \partial_i \mathsf{F}_i^{\mathrm{diss}} = \frac{1}{\epsilon}\Omega, \tag{2.38}$$

where u is the conservative variable, ∂_i the spatial derivative operator in the ith spatial direction, $\mathcal{D} = \{1,\dots,d\}$ the indexing set of spatial directions, d the spatial dimension, F_i the convective flux in the ith direction, $\mathsf{F}_i^{\mathrm{diss}}$ the dissipative flux in the ith direction, Ω the rescaled source term, and $\epsilon \in (0,1]$ the relaxation parameter. Letting $\mathsf{n} = n + d + 1$, the conservative variable $\mathsf{u} \in \mathbb{R}^{\mathsf{n}}$ is found to be

$$\mathsf{u} = \big(\rho_1, \dots, \rho_n, \rho\boldsymbol{v}, \mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2\big)^t, \tag{2.39}$$

and the natural variable $\mathsf{z} \in \mathbb{R}^{\mathsf{n}}$ is defined by $\mathsf{z} = \big(\rho_1, \dots, \rho_n, \boldsymbol{v}, T\big)^t$. For convenience, the velocity components in $\mathbb{R}^d$ of vectors in $\mathbb{R}^{\mathsf{n}} = \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}$ are written as vectors of $\mathbb{R}^d$ and bold symbols are used for vector or tensor quantities in the physical space $\mathbb{R}^d$. The map $\mathsf{z} \mapsto \mathsf{u}$ is a $C^{\varkappa}$ diffeomorphism from the open set $\mathcal{O}_{\mathsf{z}} = (0,\infty)^n \times \mathbb{R}^d \times (0,\infty)$ onto a convex open set $\mathcal{O}_{\mathsf{u}}$ of $\mathbb{R}^{\mathsf{n}}$ [22, 18].

Proposition 2.8. *The map* $\mathsf{z} \longmapsto \mathsf{u}$ *is a* $C^{\varkappa}$ *diffeomorphism from the open set* $\mathcal{O}_{\mathsf{z}}$ *onto the open convex set* $\mathcal{O}_{\mathsf{u}}$ *given by*

$$\mathcal{O}_{\mathsf{u}} = \{\ u \in \mathbb{R}^{\mathsf{n}}; u_i > 0,\ 1 \le i \le n, \quad u_{\mathsf{n}} - \phi(u_1, \dots, u_{n+d}) > 0\ \},$$

where $\phi(u_1, \dots, u_{n+d}) = \frac{1}{2}(u_{n+1}^2 + \cdots + u_{n+d}^2) / \sum_{i \in \mathfrak{S}} u_i + \sum_{i \in \mathfrak{S}} u_i e_i^0$ *and* e_i^0 *is the energy of formation of the* i^{th} *species at zero temperature.*

The convective and diffusive fluxes $\mathsf{F}_i \in \mathbb{R}^{\mathsf{n}}$ and $\mathsf{F}_i^{\text{diss}} \in \mathbb{R}^{\mathsf{n}}$ in the ith direction are given by

$$\begin{aligned} \mathsf{F}_i &= \big(\rho_1 v_i, \dots,\ \rho_n v_i,\ \rho v_i \boldsymbol{v} + p \boldsymbol{e}_i,\ (\mathcal{E} + \tfrac{1}{2}\rho|\boldsymbol{v}|^2 + p) v_i\big)^t, \\ \mathsf{F}_i^{\text{diss}} &= \big(\mathcal{J}_{1i}, \dots,\ \mathcal{J}_{ni},\ \boldsymbol{\Pi}_i,\ Q_i + \boldsymbol{\Pi}_i \cdot \boldsymbol{v}\big)^t, \end{aligned}$$

where $\boldsymbol{e}_i \in \mathbb{R}^d$ denotes the ith basis vector in the physical space $\mathbb{R}^d$, $\mathcal{J}_{ki}$ the diffusion flux of the kth species in the ith direction, Q_i the heat flux in the ith direction, $\boldsymbol{\Pi} = (\Pi_{ij})_{i,j \in \mathcal{D}}$ the viscous tensor, and $\boldsymbol{\Pi}_i \in \mathbb{R}^d$ the vector $\boldsymbol{\Pi}_i = (\Pi_{1i}, \dots, \Pi_{di})^t$, so that $\boldsymbol{\mathcal{J}}_k = (\mathcal{J}_{k1}, \dots, \mathcal{J}_{kd})^t$ and $\boldsymbol{Q} = (Q_1, \dots, Q_d)^t$.

The dissipative fluxes may further be expressed as $\mathsf{F}_i^{\text{diss}} = -\sum_{j \in \mathcal{D}} \widehat{\mathsf{B}}_{ij}(\mathsf{z}) \partial_j \mathsf{z}$ where $\widehat{\mathsf{B}}_{ij}$, $i, j \in \mathcal{D}$, are—uniquely defined—matrices of $\mathbb{R}^{\mathsf{n},\mathsf{n}}$ since all transport fluxes are linear expressions in terms of the gradients of the natural variable z. Since $\mathsf{z} \mapsto \mathsf{u}$ is a smooth diffeomorphism, defining $\mathsf{B}_{ij}(\mathsf{u}) = \widehat{\mathsf{B}}_{ij}(\mathsf{z}) \partial_{\mathsf{u}} \mathsf{z}$, for $i, j \in \mathcal{D}$, it is obtained that $\mathsf{F}_i^{\text{diss}} = -\sum_{j \in \mathcal{D}} \mathsf{B}_{ij}(\mathsf{u}) \partial_j \mathsf{u}$, $i \in \mathcal{D}$, where the dissipation matrix $\mathsf{B}_{ij} \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$ relates the dissipative flux in the ith direction $\mathsf{F}_i^{\text{diss}}$ to the gradient of the conservative variable in the jth direction $\partial_j \mathsf{u}$. Further denoting by $\mathsf{A}_i = \partial_{\mathsf{u}} \mathsf{F}_i \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$, $i \in \mathcal{D}$, the convective flux Jacobian matrices, and $\Omega \in \mathbb{R}^{\mathsf{n}}$ the rescaled source term

$$\Omega = \big(m_1 \widehat{\omega}_1, \dots, m_n \widehat{\omega}_n, \mathbf{0}, 0\big)^t,$$

one may write the system of partial differential equation in a quasilinear form whose structure is addressed in the next section.

3. Hyperbolic-Parabolic structure

Symmetrization with respect to entropic and normal variables is first summarized for abstract systems in quasilinear form as well as strict dissipativity. Symmetrized forms are evaluated and strict dissipativity is next discussed for the system of partial differential equations modeling multicomponent reactive fluids.

3.1. Entropic variables

Consider a second order quasilinear system of conservation laws in the general form

$$\partial_t \mathsf{u} + \sum_{i\in\mathcal{D}} \mathsf{A}_i(\mathsf{u})\partial_i \mathsf{u} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}(\mathsf{u})\partial_j \mathsf{u}\big) - \frac{1}{\epsilon}\Omega(\mathsf{u}) = 0, \tag{3.1}$$

where $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$, $\mathcal{O}_{\mathsf{u}}$ is an open convex set of $\mathbb{R}^{\mathsf{n}}$, and $\mathsf{n} \geq 1$. The convective jacobians are defined by $\mathsf{A}_i = \partial_{\mathsf{u}}\mathsf{F}_i \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$, $i \in \mathcal{D}$, and it is assumed that the fluxes $\mathsf{F}_i \in \mathbb{R}^{\mathsf{n}}$, $i \in \mathcal{D}$, the dissipation matrices $\mathsf{B}_{ij} \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$, $i,j \in \mathcal{D}$, and the source term $\Omega \in \mathbb{R}^{\mathsf{n}}$, are $C^{\varkappa}$ over $\mathcal{O}_{\mathsf{u}}$ where $\varkappa \geq 3$.

A mathematical entropy for the system of partial differential equations (3.1) must be compatible with the convective terms, the dissipative terms as well as the source term. The definition presented in [26, 27] is simplified to the situation where the set $\mathcal{O}_{\mathsf{u}}$ is convex. In the following definition, properties $(\mathsf{E}_1)(\mathsf{E}_2)$ concerning the convective terms have been adapted from [30, 16], properties $(\mathsf{E}_3)(\mathsf{E}_4)$ associated with the dissipative terms have been adapted from [34, 47, 35, 36], properties (E_5)-(E_7) concerning the source terms have been adapted from [9, 37], and Σ^{d-1} denotes the sphere in d dimension.

Definition 3.1. *Consider a $C^{\varkappa}$ function $\mathsf{u} \mapsto \sigma(\mathsf{u})$ defined over the open convex domain $\mathcal{O}_{\mathsf{u}}$. The function σ is said to be an entropy function for the system* (3.1) *if the following properties hold.*

(E_1) *The Hessian matrix $\partial^2_{\mathsf{u}}\sigma(\mathsf{u}) = \partial_{\mathsf{u}}(\partial_{\mathsf{u}}\sigma)^t(\mathsf{u})$ is positive definite over $\mathcal{O}_{\mathsf{u}}$.*

(E_2) *There exist $C^{\varkappa}$ functions $\mathsf{u} \mapsto \mathsf{q}_i(\mathsf{u})$ such that $\partial_{\mathsf{u}}\sigma(\mathsf{u})\,\mathsf{A}_i(\mathsf{u}) = \partial_{\mathsf{u}}\mathsf{q}_i(\mathsf{u})$ for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$ and $i \in \mathcal{D}$.*

(E_3) *The relations $\big(\mathsf{B}_{ij}(\mathsf{u})\,\big(\partial^2_{\mathsf{u}}\sigma(\mathsf{u})\big)^{-1}\big)^t = \mathsf{B}_{ji}(\mathsf{u})\big(\partial^2_{\mathsf{u}}\sigma(\mathsf{u})\big)^{-1}$ hold for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$ and $i,j \in \mathcal{D}$.*

(E_4) *The matrix $\sum_{i,j\in\mathcal{D}} \mathsf{B}_{ij}(\mathsf{u})\big(\partial^2_{\mathsf{u}}\sigma(\mathsf{u})\big)^{-1}\xi_i\xi_j$ is positive semi-definite for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$.*

(E_5) *There exists a fixed vector space $\mathcal{E} \subset \mathbb{R}^{\mathsf{n}}$ such that $\Omega(\mathsf{u}) \in \mathcal{E}^{\perp}$ for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$ and $\Omega(\mathsf{u}) = 0$ if and only if $\big(\partial_{\mathsf{u}}\sigma(\mathsf{u})\big)^t \in \mathcal{E}$ and if and only if $\partial_{\mathsf{u}}\sigma(\mathsf{u})\ \Omega(\mathsf{u}) = 0$.*

(E_6) *If $\Omega(\mathsf{u}) = 0$, then the matrix $\partial_{\mathsf{u}}\Omega(\mathsf{u})\,\big(\partial^2_{\mathsf{u}}\sigma(\mathsf{u})\big)^{-1}$ is symmetric and its nullspace is given by $N\big(\partial_{\mathsf{u}}\Omega(\mathsf{u})\,(\partial^2_{\mathsf{u}}\sigma(\mathsf{u}))^{-1}\big) = \mathcal{E}$.*

(E_7) *The inequality $\partial_{\mathsf{u}}\sigma(\mathsf{u})\ \Omega(\mathsf{u}) \leq 0$ holds for $\mathsf{u} \in \mathcal{O}_{\mathsf{u}}$.*

Existence of an entropy is closely associated with symmetrization properties [30, 16, 34, 47, 35, 36, 9, 37, 26, 27, 28]. The difficulties associated with nonideal fluids where only *local* symmetrization are feasible and where $\mathcal{O}_{\mathsf{u}}$ may not be convex [26] are avoided here. Note also that more general source terms with no symmetry properties at equilibrium have been considered by Chen, Levermore and Liu [9] and Yong [56].

Definition 3.2. *Consider a $C^{\varkappa-1}$ diffeomorphism $\mathsf{u} \mapsto \mathsf{v} \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$ from $\mathcal{O}_\mathsf{u}$ onto an open set $\mathcal{O}_\mathsf{v}$ and the system in the v variable*

$$\widetilde{\mathsf{A}}_0(\mathsf{v})\partial_t \mathsf{v} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}_i(\mathsf{v})\partial_i \mathsf{v} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}_{ij}(\mathsf{v})\partial_j \mathsf{v}\big) - \frac{1}{\epsilon}\widetilde{\Omega}(\mathsf{v}) = 0, \tag{3.2}$$

where $\widetilde{\mathsf{A}}_0 = \partial_\mathsf{v}\mathsf{u}$, $\widetilde{\mathsf{A}}_i = \mathsf{A}_i\partial_\mathsf{v}\mathsf{u} = \partial_\mathsf{v}\mathsf{F}_i$, $\widetilde{\mathsf{B}}_{ij} = \mathsf{B}_{ij}\partial_\mathsf{v}\mathsf{u}$, and $\widetilde{\Omega} = \Omega$, have at least regularity $\varkappa-2$. The system is said of the symmetric form if properties (S_1)-(S_7) *hold.*

(S_1) *The matrix $\widetilde{\mathsf{A}}_0(\mathsf{v})$ is symmetric positive definite for $\mathsf{v} \in \mathcal{O}_\mathsf{v}$.*

(S_2) *The matrices $\widetilde{\mathsf{A}}_i(\mathsf{v})$, $i \in \mathcal{D}$, are symmetric for $\mathsf{v} \in \mathcal{O}_\mathsf{v}$.*

(S_3) *The relations $\widetilde{\mathsf{B}}_{ij}^t(\mathsf{v}) = \widetilde{\mathsf{B}}_{ji}(\mathsf{v})$ hold for $\mathsf{v} \in \mathcal{O}_\mathsf{v}$ and $i, j \in \mathcal{D}$.*

(S_4) *The matrix $\widetilde{\mathsf{B}}(\mathsf{v}, \boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}} \widetilde{\mathsf{B}}_{ij}(\mathsf{v})\xi_i\xi_j$ is positive semi-definite for $\mathsf{v} \in \mathcal{O}_\mathsf{v}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$.*

(S_5) *There exists a fixed vector space $\mathcal{E} \subset \mathbb{R}^\mathsf{n}$ such that $\widetilde{\Omega}(\mathsf{v}) \in \mathcal{E}^\perp$ for $\mathsf{v} \in \mathcal{O}_\mathsf{v}$ and $\widetilde{\Omega}(\mathsf{v}) = 0$ if and only if $\mathsf{v} \in \mathcal{E}$ and if and only if $\langle \mathsf{v}, \widetilde{\Omega}(\mathsf{v})\rangle = 0$.*

(S_6) *If $\widetilde{\Omega}(\mathsf{v}) = 0$, then $\partial_\mathsf{v}\widetilde{\Omega}(\mathsf{v})$ is symmetric and $N\big(\partial_\mathsf{v}\widetilde{\Omega}(\mathsf{v})\big) = \mathcal{E}$.*

(S_7) *The inequality $\langle \mathsf{v}, \widetilde{\Omega}(\mathsf{v})\rangle \leq 0$ holds for $\mathsf{v} \in \mathcal{O}_\mathsf{v}$.*

The manifold $\mathcal{E} \subset \mathbb{R}^\mathsf{n}$ is naturally termed the equilibrium manifold or the slow manifold, since $\widetilde{\Omega}(\mathsf{v}) = 0$ when $\mathsf{v} \in \mathcal{E}$, and $\mathcal{E}^\perp$ is termed the fast manifold. The equivalence between symmetrization (S_1)-(S_7) and entropy (E_1)-(E_7) for hyperbolic-parabolic systems of conservation laws is obtained with $\mathsf{v} = (\partial_\mathsf{u}\sigma)^t$ [26, 28].

Theorem 3.3. *Assume that the system* (3.1) *admits a $C^\varkappa$ entropy function σ defined over an open convex domain $\mathcal{O}_\mathsf{u}$. Then the system can be symmetrized with the entropic variable $\mathsf{v} = (\partial_\mathsf{u}\sigma)^t$. Conversely, assume that the system can be symmetrized with the $C^{\varkappa-1}$ diffeomorphism $\mathsf{u} \mapsto \mathsf{v}$. Then there exists a $C^\varkappa$ entropy over the open convex set $\mathcal{O}_\mathsf{u}$ such that $\mathsf{v} = (\partial_\mathsf{u}\sigma)^t$.*

Sketch of the proof. The equivalence of (S_1)-(S_2) and (E_1)-(E_2) is classical and is essentially obtained with Poincaré's lemma. Then (S_3)-(S_7) and (E_3)-(E_7) are identical statements with $\mathsf{v} = (\partial_\mathsf{u}\sigma)^t$. $\square$

3.2. Normal variables

In order to split between hyperbolic and parabolic variables, it is next necessary to rewrite the system in normal form, that is, into a symmetric hyperbolic–parabolic composite form [34, 36, 21, 22]. The properties of normal variables $\mathsf{w} \in \mathbb{R}^{\mathsf{n}}$ are specified in the following definition where $\mathcal{O}_{\mathsf{w}}$ denotes the open set where w lives.

Definition 3.4. *Consider a symmetrized system as in Definition 3.2 and let* $\mathsf{v} \mapsto \mathsf{w}$ *be a* $C^{\varkappa-1}$ *diffeomorphism from the open set* $\mathcal{O}_{\mathsf{v}}$ *onto an open set* $\mathcal{O}_{\mathsf{w}}$*. Letting* $\mathsf{v} = \mathsf{v}(\mathsf{w})$ *in the symmetrized system* (3.2) *and multiplying on the left side by* $(\partial_{\mathsf{w}}\mathsf{v})^t$ *a new system in the variable* w *is obtained*

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t \mathsf{w} \ +\sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w})\partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_j \mathsf{w}\big) - \frac{1}{\epsilon}\overline{\Omega}(\mathsf{w}) = \overline{\mathrm{Q}}(\mathsf{w},\partial_x \mathsf{w}), \quad (3.3)$$

where $\overline{\mathsf{A}}_0 = (\partial_{\mathsf{w}}\mathsf{v})^t\ \widetilde{\mathsf{A}}_0\ (\partial_{\mathsf{w}}\mathsf{v})$, $\overline{\mathsf{B}}_{ij} = (\partial_{\mathsf{w}}\mathsf{v})^t\ \widetilde{\mathsf{B}}_{ij}\ (\partial_{\mathsf{w}}\mathsf{v})$, $\overline{\mathsf{A}}_i = (\partial_{\mathsf{w}}\mathsf{v})^t\ \widetilde{\mathsf{A}}_i\ (\partial_{\mathsf{w}}\mathsf{v})$, $\overline{\Omega} = (\partial_{\mathsf{w}}\mathsf{v})^t\widetilde{\Omega}$*, have at least regularity* $\varkappa-2$ *and* $\overline{\mathrm{Q}} = -\sum_{i,j\in\mathcal{D}} \partial_i(\partial_{\mathsf{w}}\mathsf{v})^t\ (\partial_{\mathsf{v}}\mathsf{w})^t\ \overline{\mathsf{B}}_{ij}\partial_j \mathsf{w}$ *is quadratic in the gradients. The quadratic term may also be written* $\overline{\mathrm{Q}} = \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}(\mathsf{w})\,\partial_i \mathsf{w}\,\partial_j \mathsf{w}$ *where* $\overline{\mathrm{M}}_{ij}(\mathsf{w})$, $i,j\in\mathcal{D}$*, are third order tensors that are functions of* $\mathsf{w}\in\mathcal{O}_{\mathsf{w}}$ *and have at least regularity* $\varkappa-3$*. In other words, the* k*th component* $\overline{\mathrm{Q}}_k$ *is in the form* $\overline{\mathrm{Q}}_k = \sum_{1\le l,l'\le \mathsf{n}} (\overline{\mathrm{M}}_{ij})_{kll'}\,\partial_i \mathsf{w}_l \partial_j \mathsf{w}_{l'}$ *with* $(\overline{\mathrm{M}}_{ij})_{kll'} = -\sum_{1\le r,s\le \mathsf{n}} \partial_{\mathsf{w}_l}(\partial_{\mathsf{w}_k}\mathsf{v}_r)\partial_{\mathsf{v}_r}\mathsf{w}_s(\overline{\mathsf{B}}_{ij})_{sl'}$ *This system satisfies in particular properties* $(\overline{\mathsf{S}}_1)$-$(\overline{\mathsf{S}}_4)$*, that is, properties* (S_1)-(S_4) *rewritten in terms of overbar matrices. This system* (3.3) *is said to be of the normal form if there exists a partition of* $\{1,\ldots,\mathsf{n}\}$ *into* $\mathrm{I} = \{1,\ldots,\mathsf{n}_{\mathrm{I}}\}$ *and* $\mathrm{II} = \{\mathsf{n}_{\mathrm{I}}+1,\ldots,\mathsf{n}_{\mathrm{I}}+\mathsf{n}_{\mathrm{II}}\}$ *with* $\mathsf{n} = \mathsf{n}_{\mathrm{I}} + \mathsf{n}_{\mathrm{II}}$ *such that the following properties hold.*

(N_1) *The matrices* $\overline{\mathsf{A}}_0$ *and* $\overline{\mathsf{B}}_{ij}$ *have the block structure*

$$\overline{\mathsf{A}}_0 = \begin{bmatrix} \overline{\mathsf{A}}_0^{\mathrm{I,I}} & 0_{\mathsf{n}_{\mathrm{I}},\mathsf{n}_{\mathrm{II}}} \\ 0_{\mathsf{n}_{\mathrm{II}},\mathsf{n}_{\mathrm{I}}} & \overline{\mathsf{A}}_0^{\mathrm{II,II}} \end{bmatrix}, \qquad \overline{\mathsf{B}}_{ij} = \begin{bmatrix} 0_{\mathsf{n}_{\mathrm{I}},\mathsf{n}_{\mathrm{I}}} & 0_{\mathsf{n}_{\mathrm{I}},\mathsf{n}_{\mathrm{II}}} \\ 0_{\mathsf{n}_{\mathrm{II}},\mathsf{n}_{\mathrm{I}}} & \overline{\mathsf{B}}_{ij}^{\mathrm{II,II}} \end{bmatrix}.$$

(N_2) *The matrix* $\overline{\mathsf{B}}^{\mathrm{II,II}}(\mathsf{w},\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{II,II}}(\mathsf{w})\xi_i\xi_j$ *is positive definite for* $\mathsf{w}\in\mathcal{O}_{\mathsf{w}}$ *and* $\boldsymbol{\xi}\in\Sigma^{d-1}$.

(N_3) *The quadratic residual is in the form*

$$\overline{\mathrm{Q}}(\mathsf{w},\partial_x\mathsf{w}) = \big(\overline{\mathrm{Q}}_{\mathrm{I}}(\mathsf{w},\partial_x\mathsf{w}_{\mathrm{II}}),\overline{\mathrm{Q}}_{\mathrm{II}}(\mathsf{w},\partial_x\mathsf{w})\big)^t.$$

The vector and matrix block structure induced by the partitioning of $\mathbb{R}^{\mathsf{n}}$ *into* $\mathbb{R}^{\mathsf{n}} = \mathbb{R}^{\mathsf{n}_{\mathrm{I}}}\times\mathbb{R}^{\mathsf{n}_{\mathrm{II}}}$ *has been used in* (N_1)-(N_3) *so that* $\mathsf{w} = (\mathsf{w}_{\mathrm{I}},\mathsf{w}_{\mathrm{II}})^t$ *for instance and* $0_{i,j}$ *denotes the zero matrix with* i *lines and* j *columns.*

The main interest of normal forms is that the resulting subsystem of partial differential equations governing the variable w_{I} is symmetric hyperbolic whereas the subsystem governing the variable w_{II} is symmetric strongly parabolic [34, 36]. Incidentally, for the symmetric subsystem governing w_{II}, Petrovsky parabolicity is *equivalent* to strong parabolicity, i.e., to the Legendre-Hadamard condition [26]. A sufficient condition for system (3.2) to be recast into a normal form is that the nullspace naturally associated with the dissipation matrix $\widetilde{\mathsf{B}}$ is a fixed subspace of $\mathbb{R}^{\mathsf{n}}$. This is Condition (N) introduced by Kawashima and Shizuta [36] which has been strengthened in [22].

(N) *The nullspace $N(\widetilde{\mathsf{B}})$ of the matrix $\widetilde{\mathsf{B}}(\mathsf{v},\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}} \widetilde{\mathsf{B}}_{ij}(\mathsf{v})\xi_i\xi_j$ does not depend on $\mathsf{v}\in\mathcal{O}_{\mathsf{v}}$ and $\boldsymbol{\xi}\in\Sigma^{d-1}$ and $\widetilde{\mathsf{B}}_{ij}(\mathsf{v})N(\widetilde{\mathsf{B}}) = 0$, for $i,j\in\mathcal{D}$.*

Letting $\mathsf{n}_{\mathrm{I}} = \dim\big(N(\widetilde{\mathsf{B}})\big)$ and $\mathsf{n}_{\mathrm{II}} = \mathsf{n}-\mathsf{n}_{\mathrm{I}}$, let P be an arbitrary constant nonsingular matrix of dimension n such that its first n_{I} columns span the nullspace $N(\widetilde{\mathsf{B}})$. In order to characterize more easily normal forms for symmetric systems of conservation laws satisfying (N) one may introduce the auxiliary variables $\mathsf{u}' = \mathsf{P}^t\mathsf{u}$ and $\mathsf{v}' = \mathsf{P}^{-1}\mathsf{v}$ [22, 18]. The dissipation matrices corresponding to these auxiliary variables have nonzero coefficients only in the lower right block of size $\mathsf{n}_{\mathrm{II}} = \mathsf{n}-\mathsf{n}_{\mathrm{I}}$. Normal symmetric forms are then equivalently obtained from the v' symmetric equation [22, 18].

Theorem 3.5. *Consider a system of conservation laws* (3.2) *that is symmetric in the sense of Definition 3.2 and assume that the nullspace invariance property* (N) *holds. Denoting by $\mathsf{u}' = \mathsf{P}^t\mathsf{u}$ and $\mathsf{v}' = \mathsf{P}^{-1}\mathsf{v}$, the auxiliary variable, all normal forms of the system* (3.2) *are given by changes of variable in the form $\mathsf{w} = \big(\mathcal{F}_{\mathrm{I}}(\mathsf{u}'_{\mathrm{I}}), \mathcal{F}_{\mathrm{II}}(\mathsf{v}'_{\mathrm{II}})\big)^t$ where $\mathcal{F}_{\mathrm{I}}$ and $\mathcal{F}_{\mathrm{II}}$ are any diffeomorphisms of $\mathbb{R}^{\mathsf{n}_{\mathrm{I}}}$ and $\mathbb{R}^{\mathsf{n}_{\mathrm{II}}}$, respectively, and $\overline{\mathsf{Q}} = -\sum_{i,j\in\mathcal{D}} \partial_i(\partial_{\mathsf{w}}\mathsf{v})^t\,(\partial_{\mathsf{v}}\mathsf{w})^t\,\overline{\mathsf{B}}_{ij}\partial_j\mathsf{w}$ is in the form*

$$\overline{\mathsf{Q}} = \Big(0,\, \overline{\mathsf{Q}}_{\mathrm{II}}(\mathsf{w},\partial_x\mathsf{w}_{\mathrm{II}})\Big)^t = \Big(0,\, \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}^{\mathrm{II,II,II}}(\mathsf{w})\,\partial_i\mathsf{w}_{\mathrm{II}}\,\partial_j\mathsf{w}_{\mathrm{II}}\Big)^t, \tag{3.4}$$

where $\overline{\mathrm{M}}_{ij}^{\mathrm{II,II,II}}(\mathsf{w})$ are third order tensors depending on w with regularity at least $\varkappa-3$. Finally, when $\mathcal{F}_{\mathrm{II}}$ is linear, the quadratic residual $\overline{\mathsf{Q}}$ is zero.

It is further assumed in the following that the rescaled source term $\overline{\Omega}$ associated with the normal variable w is in quasilinear form

$$\overline{\Omega} = -\overline{\mathsf{L}}(\mathsf{w})\mathsf{w}, \tag{3.5}$$

where $\overline{\mathsf{L}}(\mathsf{w})$ is a symmetric positive semi-definite matrix of size n with a fixed nullspace $N(\overline{\mathsf{L}}) = \overline{\mathcal{E}}$ of dimension $\mathsf{n}_{\mathrm{e}} = \dim(\overline{\mathcal{E}})$ with $1\le \mathsf{n}_{\mathrm{e}} < \mathsf{n}$ and the map $\mathsf{w}\mapsto\overline{\mathsf{L}}(\mathsf{w})$ is assumed to be $C^{\varkappa}$ over $\mathcal{O}_{\mathsf{w}}$. Under these assumptions, it is easy to establish properties ($\overline{\mathsf{S}}_5$)-($\overline{\mathsf{S}}_7$), that are analogous to (S_5)-(S_7), for the source term in normal form $\overline{\Omega}$. We use the terminology *quasilinear* since the matrix $\overline{\mathsf{L}}$ has several invariant properties being symmetric positive semi-definite with fixed range and nullspace. Source terms $\widetilde{\Omega}$ associated with the natural entropic symmetrized form (3.2) that are in quasilinear form with respect to v are often encountered in mathematical physics [57]. As a consequence, the assumption (3.5) concerning $\overline{\Omega}$ is also natural in terms of *properly chosen* normal variables w [28]. This is notably the case when the normal variable is such that $\pi\mathsf{w} = \pi(\partial_{\mathsf{v}}\mathsf{w})\mathsf{v}$ as investigated in [28]. More importantly, it is the case for chemistry source terms as established in Section 2.3 and investigated in more details in Theorem 3.10 and Theorem 3.12 as well as for energy relaxation terms in multitemperature flows [28, 29].

3.3. Strict parabolic dissipativity

Consider an abstract system of conservation laws (3.1) with an entropy and a constant state $\mathsf{u}^\star \in \mathbb{R}^{\mathsf{n}}$ such that $\Omega(\mathsf{u}^\star) = 0$. The nullspace invariance property (N) is assumed to hold and the system may be rewritten into a normal form. Let $\mathsf{v}^\star \in \mathbb{R}^{\mathsf{n}}$ and $\mathsf{w}^\star \in \mathbb{R}^{\mathsf{n}}$ denote the corresponding constant states in the v and w variables respectively. If one linearizes the system (3.3) around the constant stationary state $\mathsf{w}^\star$, the following linear system in the variable $\delta\mathsf{w} = \mathsf{w} - \mathsf{w}^\star$ is obtained

$$\overline{\mathsf{A}}_0(\mathsf{w}^\star)\partial_t\delta\mathsf{w} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w}^\star)\partial_i\delta\mathsf{w} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}(\mathsf{w}^\star)\partial_i\partial_j\delta\mathsf{w} + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w}^\star)\delta\mathsf{w} = 0. \quad (3.6)$$

Investigating smooth global solutions around constant equilibrium states requires such linearized normal form to be strictly dissipative [34, 47, 35, 36, 22, 18, 37, 24, 26]. By Fourier transform, the spectral problem associated with the constant coefficient linear system (3.6) reads

$$\lambda\overline{\mathsf{A}}_0(\mathsf{w}^\star)\phi + \Big(\mathrm{i}\zeta\overline{\mathsf{A}}(\mathsf{w}^\star,\boldsymbol{\xi}) + \zeta^2\overline{\mathsf{B}}(\mathsf{w}^\star,\boldsymbol{\xi}) + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w}^\star)\Big)\phi = 0, \quad (3.7)$$

where $\zeta \in \mathbb{R}$, $\boldsymbol{\xi} \in \Sigma^{d-1}$, $\phi \in \mathbb{C}^{\mathsf{n}}$, $\overline{\mathsf{A}}(\mathsf{w}^\star,\boldsymbol{\xi}) = \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w}^\star)\xi_i$, $\overline{\mathsf{B}}(\mathsf{w}^\star,\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}(\mathsf{w}^\star)\xi_i\xi_j$, and $\mathrm{i}^2 = -1$. Strict dissipativity for systems (3.6) with *fixed* ϵ has notably been investigated by Kawashima [34] and Shizuta and Kawashima [47]. Let $\mathcal{S}(\zeta,\boldsymbol{\xi},\epsilon)$ denote the set of complex numbers λ such that there exists $\phi \in \mathbb{C}^{\mathsf{n}}$, $\phi \neq 0$, satisfying (3.7). When investigating global solutions, it will be assumed that the system is strictly *parabolic* dissipative, i.e., that the system without sources $\overline{\mathsf{L}} = 0$, is strictly dissipative. In other words, it is assumed that for $\zeta \neq 0$ the eigenvalues of $\lambda \in \mathcal{S}(\zeta,\boldsymbol{\xi},\infty)$ have a negative real part. The following equivalent forms of the 'Kawashima condition' have been established by Shizuta and Kawashima [47] for (K_1)-(K_4) and Beauchard and Zuazua [2] for (K_5).

Theorem 3.6. *Assume that the matrix $\overline{\mathsf{A}}_0(\mathsf{w}^\star)$ is symmetric positive definite, that the matrices $\overline{\mathsf{A}}_i(\mathsf{w}^\star)$, $i \in \mathcal{D}$, are symmetric, that the reciprocity relations $\overline{\mathsf{B}}_{ij}(\mathsf{w}^\star)^t = \overline{\mathsf{B}}_{ji}(\mathsf{w}^\star)$, $i,j \in \mathcal{D}$ hold, that $\overline{\mathsf{B}}(\mathsf{w}^\star,\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}(\mathsf{w}^\star)\xi_i\xi_j$ is positive semi-definite for $\boldsymbol{\xi} \in \Sigma^{d-1}$, and denote $\overline{\mathsf{A}}(\mathsf{w}^\star,\boldsymbol{\xi}) = \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w}^\star)\xi_i$. The system* *of partial differential equations* (3.3) *is said to be strictly parabolic dissipative at* $\mathsf{w}^\star$ *when any of the following equivalent properties holds.*

(K_1) *There exists a* C^∞ *map* $K : \Sigma^{d-1} \mapsto \mathbb{R}^{d,d}$ *such that for any* $\boldsymbol{\xi} \in \Sigma^{d-1}$ *the product* $K(\boldsymbol{\xi})\overline{\mathsf{A}}_0(\mathsf{w}^\star)$ *is skew-symmetric,* $K(-\boldsymbol{\xi}) = -K(\boldsymbol{\xi})$*, and* $K(\boldsymbol{\xi})\,\overline{\mathsf{A}}(\mathsf{w}^\star, \boldsymbol{\xi}) + \overline{\mathsf{B}}(\mathsf{w}^\star, \boldsymbol{\xi})$ *is positive definite.*

(K_2) *For any* $\zeta \in \mathbb{R}$*,* $\zeta \neq 0$*, and any* $\boldsymbol{\xi} \in \Sigma^{d-1}$*, the eigenvalues* $\lambda \in \mathcal{S}(\zeta, \boldsymbol{\xi}, \infty)$ *have a negative real part* $\Re(\lambda) < 0$*.*

(K_3) *Let* $\phi \in \mathbb{R}^{\mathsf{n}}\backslash\{0\}$ *such that* $\overline{\mathsf{B}}(\mathsf{w}^\star, \boldsymbol{\xi})\phi = 0$ *for some* $\boldsymbol{\xi} \in \Sigma^{d-1}$*. Then* $\theta\overline{\mathsf{A}}_0(\mathsf{w}^\star)\phi + \overline{\mathsf{A}}(\mathsf{w}^\star, \boldsymbol{\xi})\phi \neq 0$ *for any* $\theta \in \mathbb{R}$*.*

(K_4) *There exists* $\delta > 0$ *such that for any* $\zeta \in \mathbb{R}$*,* $\boldsymbol{\xi} \in \Sigma^{d-1}$*, the eigenvalue* $\lambda \in \mathcal{S}(\zeta, \boldsymbol{\xi}, \infty)$ *have their real part majorized by* $\Re(\lambda) \leq -\delta|\zeta|^2/(1+|\zeta|^2)$*.*

(K_5) *Letting* $\widehat{\mathsf{B}}^\star = \big(\overline{\mathsf{A}}_0(\mathsf{w}^\star)\big)^{-1}\overline{\mathsf{B}}(\mathsf{w}^\star, \boldsymbol{\xi})$ *and* $\widehat{\mathsf{A}}^\star = \big(\overline{\mathsf{A}}_0(\mathsf{w}^\star)\big)^{-1}\overline{\mathsf{A}}(\mathsf{w}^\star, \boldsymbol{\xi})$ *the Kalman condition is satisfied* $\operatorname{rank}\big[\,\widehat{\mathsf{B}}^\star, \widehat{\mathsf{A}}^\star\,\widehat{\mathsf{B}}^\star, \dots, (\widehat{\mathsf{A}}^\star)^{\mathsf{n}-1}\,\widehat{\mathsf{B}}^\star\,\big] = \mathsf{n}$*.*

A physical interpretation of the 'Kawashima condition' (K_1)-(K_5) is that all waves $\phi\,\exp(\theta t + x \cdot \boldsymbol{\xi})$ associated with the hyperbolic operator $\overline{\mathsf{A}}_0(\mathsf{w}^\star)\partial_t + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w}^\star)\partial_i$ lead to dissipation, i.e., entropy production, since there are not in the nullspace of $\overline{\mathsf{B}}$, as shown by (K_3). Note that only the symmetric part of the product $K(\boldsymbol{\xi})\,\overline{\mathsf{A}}(\mathsf{w}^\star, \boldsymbol{\xi})$ plays a role in (K_1). The traditional Kalman condition involving the $\mathsf{n}^2 \times \mathsf{n}$ matrix with first block $\widehat{\mathsf{B}}^\star$, second block $\widehat{\mathsf{B}}^\star\,\widehat{\mathsf{A}}^\star$, and kth block $\widehat{\mathsf{B}}^\star\,(\widehat{\mathsf{A}}^\star)^{k-1}$ has been rewritten in the form (K_5) with the $\mathsf{n} \times \mathsf{n}^2$ matrix $\big[\,\widehat{\mathsf{B}}^\star, \widehat{\mathsf{A}}^\star\,\widehat{\mathsf{B}}^\star, \dots, (\widehat{\mathsf{A}}^\star)^{\mathsf{n}-1}\,\widehat{\mathsf{B}}^\star\,\big]$ thanks to the symmetry of $\overline{\mathsf{A}}_0(\mathsf{w}^\star)$ and $\overline{\mathsf{B}}(\mathsf{w}^\star, \boldsymbol{\xi})$. It is not known in general if the matrix $K(\boldsymbol{\xi})$ may be written $\sum_{j\in\mathcal{D}} K_j\xi_j$, but it is generally possible to obtain compensating matrices in this form in practical applications. One may now deduce some properties of the eigenvalues $\mathcal{S}(\zeta, \boldsymbol{\xi}, \epsilon)$ for the complete system (3.7).

Proposition 3.7. *Assume that the system* (3.6) *is strictly parabolic dissipative at* $\mathsf{w}^\star$*. Further assume that there exists a compensating matrix* K *compatible with the fast manifold in such a way that*

$$K(\boldsymbol{\xi})\,\pi = 0, \qquad \boldsymbol{\xi} \in \Sigma^{d-1}. \tag{3.8}$$

Then the system is also strictly dissipative for any $\epsilon > 0$ *and there exists* $\delta > 0$ *such that for any* $\zeta \in \mathbb{R}$*, any* $\boldsymbol{\xi} \in \Sigma^{d-1}$*, and any* $\epsilon > 0$*, the eigenvalues* $\lambda \in \mathcal{S}(\zeta, \boldsymbol{\xi}, \epsilon)$ *of the linearized normal form after Fourier transform* (3.7) *are such that* $\Re(\lambda) \leq -\delta|\zeta|^2/(1 + |\zeta|^2)$*.*

Proof. The proof is identical to the proofs presented in [34, 47] using the compatibility between the compensating matrix and the fast manifold $K\pi = 0$. $\square$

Remark 3.8. *When the compensating matrices are not compatible with the fast manifold and* (3.8) *does not hold, one may obtain estimates in the form* $\Re(\lambda) \leq -\delta|\zeta|^2/\big(\frac{1}{\epsilon} + 1 + |\zeta|^2\big)$ *that degenerate as* $\epsilon \to 0$*.*

3.4. Fast normal variable

In order to establish a priori estimates for the fast variable $\pi\mathsf{w}$ it is first required to have an appropriate governing partial differential equation. In the situation where $\overline{\mathsf{A}}_0$ and π commute, such a partial differential equation may be obtained by applying the projector π to the governing equations in normal form [29]. Such an equation is derived here in the general situation where the matrix $\overline{\mathsf{A}}_0$ and the projector π do not necessarily commute.

Let $\overline{\mathsf{a}}_1, \ldots, \overline{\mathsf{a}}_{\mathsf{n}_\mathrm{e}}$ be a basis of the slow manifold or equilibrium space $\overline{\mathcal{E}}$ and $\overline{\mathsf{a}}_{\mathsf{n}_\mathrm{e}+1}, \ldots, \overline{\mathsf{a}}_{\mathsf{n}}$ a basis of the fast manifold $\overline{\mathcal{E}}^{\perp}$, where $1 \leq \mathsf{n}_\mathrm{e} < \mathsf{n}$ denotes the dimension of $\overline{\mathcal{E}}$. Let the linear operators $\overline{\Pi}_\mathsf{e} = \mathbb{R}^{\mathsf{n}_\mathsf{e}} \mapsto \mathbb{R}^{\mathsf{n}}$ and $\overline{\Pi}_\mathsf{r} = \mathbb{R}^{\mathsf{n}-\mathsf{n}_\mathrm{e}} \mapsto \mathbb{R}^{\mathsf{n}}$ be given by their matrices in the canonical bases

$$\overline{\Pi}_\mathsf{e} = [\overline{\mathsf{a}}_1, \ldots, \overline{\mathsf{a}}_{\mathsf{n}_\mathrm{e}}], \qquad \overline{\Pi}_\mathsf{r} = [\overline{\mathsf{a}}_{\mathsf{n}_\mathrm{e}+1}, \ldots, \overline{\mathsf{a}}_{\mathsf{n}}].$$

The metric matrices $\overline{\mathcal{J}}_\mathsf{e}$ and $\overline{\mathcal{J}}_\mathsf{r}$ of order n_e and $\mathsf{n}-\mathsf{n}_\mathrm{e}$, respectively are defined by $\overline{\mathcal{J}}^{-1}_{\mathsf{e}\,i,j} = \langle \overline{\mathsf{a}}_i, \overline{\mathsf{a}}_j \rangle$, $1 \leq i,j \leq \mathsf{n}_\mathrm{e}$, and $\overline{\mathcal{J}}^{-1}_{\mathsf{r}\,i,j} = \langle \overline{\mathsf{a}}_i, \overline{\mathsf{a}}_j \rangle$, $\mathsf{n}_\mathrm{e}+1 \leq i,j \leq \mathsf{n}$. Each vector $\mathfrak{r} \in \mathbb{R}^{\mathsf{n}}$ admits a unique decomposition $\mathfrak{r} = \mathfrak{r}_{\overline{\mathcal{E}}} + \mathfrak{r}_{\overline{\mathcal{E}}^\perp}$ where $\mathfrak{r}_{\overline{\mathcal{E}}} \in \overline{\mathcal{E}}$ and $\mathfrak{r}_{\overline{\mathcal{E}}^\perp} \in \overline{\mathcal{E}}^{\perp}$ and after a little algebra, it is easily shown that the vector $\overline{\mathcal{J}}_\mathsf{e}\overline{\Pi}_\mathsf{e}^t\mathfrak{r}$ represents the coordinates of $\mathfrak{r}_{\overline{\mathcal{E}}}$ with respect to $\overline{\mathsf{a}}_1, \ldots, \overline{\mathsf{a}}_{\mathsf{n}_\mathrm{e}}$ whereas the vector $\overline{\mathcal{J}}_\mathsf{r}\overline{\Pi}_\mathsf{r}^t\mathfrak{r}$ represents the coordinates of $\mathfrak{r}_{\overline{\mathcal{E}}^\perp}$ with respect to $\overline{\mathsf{a}}_{\mathsf{n}_\mathrm{e}+1}, \ldots, \overline{\mathsf{a}}_{\mathsf{n}}$, in such a way that

$$\mathfrak{r}_{\overline{\mathcal{E}}} = \overline{\Pi}_\mathsf{e}\overline{\mathcal{J}}_\mathsf{e}\overline{\Pi}_\mathsf{e}^t\,\mathfrak{r}, \qquad \mathfrak{r}_{\overline{\mathcal{E}}^\perp} = \overline{\Pi}_\mathsf{r}\overline{\mathcal{J}}_\mathsf{r}\overline{\Pi}_\mathsf{r}^t\,\mathfrak{r}, \qquad \mathbb{I}_\mathsf{n} = \overline{\Pi}_\mathsf{e}\overline{\mathcal{J}}_\mathsf{e}\overline{\Pi}_\mathsf{e}^t + \overline{\Pi}_\mathsf{r}\overline{\mathcal{J}}_\mathsf{r}\overline{\Pi}_\mathsf{r}^t,$$

where $\mathbb{I}_\mathsf{n}$ denotes the identity tensor in $\mathbb{R}^{\mathsf{n}}$, and $\pi = \overline{\Pi}_\mathsf{r}\overline{\mathcal{J}}_\mathsf{r}\overline{\Pi}_\mathsf{r}^t$, $\overline{\mathcal{J}}_\mathsf{e}\overline{\Pi}_\mathsf{e}^t\overline{\Pi}_\mathsf{e} = \mathbb{I}_{\mathsf{n}_\mathrm{e}}$ and $\overline{\mathcal{J}}_\mathsf{r}\overline{\Pi}_\mathsf{r}^t\overline{\Pi}_\mathsf{r} = \mathbb{I}_{\mathsf{n}-\mathsf{n}_\mathrm{e}}$.

Proposition 3.9. *The fast normal variable $\pi\mathsf{w}$ satisfies the following partial differential equation*

$$\begin{aligned}\overline{\mathsf{A}}^\pi_0(\mathsf{w})\partial_t(\pi\mathsf{w}) \; + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}^\pi_i(\mathsf{w})\partial_i\mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^\pi_{ij}(\mathsf{w})\partial_i\partial_j\mathsf{w} \qquad\qquad& \\ + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\pi\mathsf{w} = \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}^\pi_{ij}(\mathsf{w})\,\partial_i\mathsf{w}\,\partial_j\mathsf{w},& \qquad (3.9)\end{aligned}$$

where $\overline{\mathsf{A}}^\pi_0 = \pi\pi_\mathsf{A}\overline{\mathsf{A}}_0\pi + (\mathbb{I}-\pi)\overline{\mathsf{A}}_0(\mathbb{I}-\pi)$, $\overline{\mathsf{A}}^\pi_i = \pi\pi_\mathsf{A}\overline{\mathsf{A}}_i$, $\overline{\mathsf{B}}^\pi_{ij} = \pi\pi_\mathsf{A}\overline{\mathsf{B}}_{ij}$, $\overline{\mathrm{M}}^\pi_{ij} = \pi\pi_\mathsf{A}(\overline{\mathrm{M}}_{ij} + \partial_\mathsf{w}\overline{\mathsf{B}}_{ij})$, and $\pi_\mathsf{A} = \mathbb{I} - \overline{\mathsf{A}}_0\overline{\Pi}_\mathsf{e}(\overline{\Pi}_\mathsf{e}^t\overline{\mathsf{A}}_0\overline{\Pi}_\mathsf{e})^{-1}\overline{\Pi}_\mathsf{e}^t$. The matrix $\overline{\mathsf{A}}^\pi_0$ is symmetric positive definite and the regularity class of $\overline{\mathsf{A}}^\pi_0$, $\overline{\mathsf{A}}^\pi_i$, $\overline{\mathsf{B}}^\pi_{ij}$, is at least $\varkappa-2$ and that of $\overline{\mathrm{M}}^\pi_{ij}$ is at least $\varkappa-3$, as for $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i$, $\overline{\mathsf{B}}_{ij}$, and $\overline{\mathrm{M}}_{ij}$, respectively.

Proof. Letting $\mathsf{w}_\mathsf{r} = \overline{\mathcal{J}}_\mathsf{r}\overline{\Pi}_\mathsf{r}^t\mathsf{w}$ and $\mathsf{w}_\mathsf{e} = \overline{\mathcal{J}}_\mathsf{e}\overline{\Pi}_\mathsf{e}^t\mathsf{w}$, then $\mathsf{w} = \overline{\Pi}_\mathsf{r}\mathsf{w}_\mathsf{r} + \overline{\Pi}_\mathsf{e}\mathsf{w}_\mathsf{e}$ and $\pi\mathsf{w} = \overline{\Pi}_\mathsf{r}\mathsf{w}_\mathsf{r}$. Applying both operators $\overline{\Pi}_\mathsf{r}^t$ and $\overline{\Pi}_\mathsf{e}^t$ to (3.3), with $\overline{\mathrm{Q}}$ in the form (3.4), using $\mathsf{w} = \overline{\Pi}_\mathsf{r}\mathsf{w}_\mathsf{r} + \overline{\Pi}_\mathsf{e}\mathsf{w}_\mathsf{e}$, and eliminating the time derivative $\partial_t\mathsf{w}_\mathsf{e}$ with linear combinations, the following governing equations for w_r is first obtained

$$\overline{\Pi}_\mathsf{r}^t \pi_\mathsf{A} \overline{\mathsf{A}}_0 \overline{\Pi}_\mathsf{r} \partial_t \mathsf{w}_\mathsf{r} + \sum_{i\in\mathcal{D}} \overline{\Pi}_\mathsf{r}^t \pi_\mathsf{A} \overline{\mathsf{A}}_i \partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\Pi}_\mathsf{r}^t \pi_\mathsf{A} \overline{\mathsf{B}}_{ij} \partial_i \partial_j \mathsf{w} + \frac{1}{\epsilon} \overline{\Pi}_\mathsf{r}^t \overline{\mathsf{L}}\, \overline{\Pi}_\mathsf{r} \mathsf{w}_\mathsf{r} =$$

$$\sum_{i,j\in\mathcal{D}} \overline{\Pi}_\mathsf{r}^t \pi_\mathsf{A} (\overline{\mathrm{M}}_{ij} + \partial_\mathsf{w} \overline{\mathsf{B}}_{ij}) \partial_i \mathsf{w}\, \partial_j \mathsf{w},$$

where $\pi_\mathsf{A} = \mathbb{I} - \overline{\mathsf{A}}_0 \overline{\Pi}_\mathsf{e} (\overline{\Pi}_\mathsf{e}^t \overline{\mathsf{A}}_0 \overline{\Pi}_\mathsf{e})^{-1} \overline{\Pi}_\mathsf{e}^t$ denotes the projector onto $\overline{\mathcal{E}}^\perp$ parallel to $\overline{\mathsf{A}}_0 \overline{\mathcal{E}}$ with $\pi_\mathsf{A}^2 = \pi_\mathsf{A}$, $N(\pi_\mathsf{A}) = \overline{\mathsf{A}}_0 \overline{\mathcal{E}}$, $R(\pi_\mathsf{A}) = \overline{\mathcal{E}}^\perp$. Multiplying this equation by $\overline{\Pi}_\mathsf{r} \overline{\mathcal{J}}_\mathsf{r}$, using $\mathsf{w}_\mathsf{r} = \overline{\mathcal{J}}_\mathsf{r} \overline{\Pi}_\mathsf{r}^t \pi \mathsf{w}$, $\pi = \overline{\Pi}_\mathsf{r} \overline{\mathcal{J}}_\mathsf{r} \overline{\Pi}_\mathsf{r}^t$, $\overline{\mathsf{L}} = \pi \overline{\mathsf{L}} \pi$, and adding for convenience the matrix $(\mathbb{I} - \pi) \overline{\mathsf{A}}_0 (\mathbb{I} - \pi)$ to the resulting matrix in front of the time derivative then yields (3.9). One may then establish that $\pi \pi_\mathsf{A} = \pi_\mathsf{A}$, $\pi_\mathsf{A} \pi = \pi$, and that $\pi_\mathsf{A} \overline{\mathsf{A}}_0 = \overline{\mathsf{A}}_0 - \overline{\mathsf{A}}_0 \overline{\Pi}_\mathsf{e} (\overline{\Pi}_\mathsf{e}^t \overline{\mathsf{A}}_0 \overline{\Pi}_\mathsf{e})^{-1} \overline{\Pi}_\mathsf{e}^t \overline{\mathsf{A}}_0 = \overline{\mathsf{A}}_0 \pi_\mathsf{A}^t$. The matrix $\pi_\mathsf{A} \overline{\mathsf{A}}_0$ is then symmetric positive semi-definite since $\pi_\mathsf{A} \overline{\mathsf{A}}_0 = \pi_\mathsf{A}^2 \overline{\mathsf{A}}_0 = \pi_\mathsf{A} \overline{\mathsf{A}}_0 \pi_\mathsf{A}^t$ with nullspace $N(\pi_\mathsf{A} \overline{\mathsf{A}}_0) = N(\pi_\mathsf{A}^t) = R(\pi_\mathsf{A})^\perp = \overline{\mathcal{E}}$. This now implies that $\pi \pi_\mathsf{A} \overline{\mathsf{A}}_0 \pi$ is symmetric positive semi-definite with the same nullspace since $\pi = \pi^t$, $\pi \pi_\mathsf{A} = \pi_\mathsf{A}$, and $\pi_\mathsf{A}^t \pi = \pi_\mathsf{A}^t \pi^t = (\pi \pi_\mathsf{A})^t = \pi_\mathsf{A}^t$. The matrix $\overline{\mathsf{A}}_0^\pi$ is thus positive definite since $\pi \pi_\mathsf{A} \overline{\mathsf{A}}_0 \pi$ has nullspace $\overline{\mathcal{E}}$ and $(\mathbb{I} - \pi) \overline{\mathsf{A}}_0 (\mathbb{I} - \pi)$ has nullspace $\overline{\mathcal{E}}^\perp$. The regularity properties of the system coefficients are then direct consequences of their expressions. $\square$

3.5. Entropic form for multicomponent flows

In this section the natural entropic symmetrized form for the system of partial differential equations modeling multicomponent reative fluids is evaluated [21, 22, 18]. The mathematical entropy $\sigma = -\mathcal{S}/R$ is used where the $1/R$ factor is introduced for convenience. For this particular system of partial differential equations with $\mathsf{n} = n+d+1$, the velocity components of all quantities in $\mathbb{R}^{n+d+1}$ are denoted as vectors of $\mathbb{R}^d$ and the corresponding partitioning is also used for matrices.

There is also a uniqueness theorem for mathematical entropies that are independent of transport coefficients [27]. This result strengthen the representation theorem of normal variable as well as the importance of the following natural entropic symmetrized form.

Theorem 3.10. *Assume that* (H_1)-(H_5) *hold. Then the function* $\sigma = -\mathcal{S}/R$ *is a mathematical entropy for the system* (2.1)–(2.3) *governing multicomponent fluids with the entropic variable*

$$\mathsf{v} = (\partial_\mathsf{u} \sigma)^t = \frac{1}{RT} \Big(g_1 - \tfrac{1}{2}|\boldsymbol{v}|^2, \ldots, g_n - \tfrac{1}{2}|\boldsymbol{v}|^2, \boldsymbol{v}, -1 \Big)^t. \tag{3.10}$$

The map $\mathsf{u} \mapsto \mathsf{v}$ *is a* $C^{\varkappa-1}$ *diffeomorphism from* $\mathcal{O}_\mathsf{u}$ *onto* $\mathcal{O}_\mathsf{v} = \mathbb{R}^n \times \mathbb{R}^d \times (-\infty, 0)$. *The system written in term of the entropic variable* v *reads*

$$\widetilde{\mathsf{A}}_0(\mathsf{v}) \partial_t \mathsf{v} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}_i(\mathsf{v}) \partial_i \mathsf{v} - \sum_{i,j\in\mathcal{D}} \partial_i \big(\widetilde{\mathsf{B}}_{ij}(\mathsf{v}) \partial_j \mathsf{v} \big) + \frac{1}{\epsilon} \widetilde{\mathsf{L}}(\mathsf{v})\, \mathsf{v} = 0, \tag{3.11}$$

with $\widetilde{\mathsf{A}}_0 = \partial_{\mathsf{v}}\mathsf{u}$, $\widetilde{\mathsf{A}}_i = \mathsf{A}_i\partial_{\mathsf{v}}\mathsf{u}$, $\widetilde{\mathsf{B}}_{ij} = \mathsf{B}_{ij}\partial_{\mathsf{v}}\mathsf{u}$, *and* $\widetilde{\Omega} = \Omega = -\widetilde{\mathsf{L}}\mathsf{v}$, *and is of the symmetric form. The matrix* $\widetilde{\mathsf{A}}_0$ *is given by*

$$\widetilde{\mathsf{A}}_0 = \begin{bmatrix} \Gamma & & Sym \\ \boldsymbol{v}\otimes\Gamma\mathbb{1} & \langle\Gamma\mathbb{1},\mathbb{1}\rangle\boldsymbol{v}\otimes\boldsymbol{v} + \rho RT\boldsymbol{I} & \\ (\Gamma e^{\mathrm{tl}})^t & \langle\Gamma e^{\mathrm{tl}},\mathbb{1}\rangle\boldsymbol{v}^t + \rho RT\boldsymbol{v}^t & \Upsilon \end{bmatrix}, \tag{3.12}$$

where Γ *is the diagonal matrix* $\Gamma = \mathrm{diag}(m_1\rho_1,\dots,m_n\rho_n)$, e^{tl} *the vector* $e^{\mathrm{tl}} = (e_1^{\mathrm{tl}},\dots,e_n^{\mathrm{tl}})^t$ *where* $e_i^{\mathrm{tl}} = e_i + \frac{1}{2}|\boldsymbol{v}|^2$, *and* $\Upsilon = \langle\Gamma e^{\mathrm{tl}}, e^{\mathrm{tl}}\rangle + \rho RT|\boldsymbol{v}|^2 + \rho RT^2 c_{\mathrm{v}}$. *Since* $\widetilde{\mathsf{A}}_0$ *is symmetric, only its left lower triangular part is given with "Sym" written in the upper triangular part. Denoting by* $\boldsymbol{\xi} = (\xi_1,\dots,\xi_d)^t$ *an arbitrary vector of* $\mathbb{R}^d$ *and letting* $\widetilde{\mathsf{A}} = \sum_{i\in\mathcal{D}}\xi_i\widetilde{\mathsf{A}}_i$ *and* $h^{\mathrm{tl}} = h + \frac{1}{2}|\boldsymbol{v}|^2$, *then*

$$\widetilde{\mathsf{A}} = \boldsymbol{v}\cdot\boldsymbol{\xi}\,\widetilde{\mathsf{A}}_0 + RT\begin{bmatrix} 0 & & Sym \\ \boldsymbol{\xi}\otimes\varrho & \rho(\boldsymbol{\xi}\otimes\boldsymbol{v} + \boldsymbol{v}\otimes\boldsymbol{\xi}) & \\ \boldsymbol{v}\cdot\boldsymbol{\xi}\,\varrho^t & \boldsymbol{v}\cdot\boldsymbol{\xi}\,\rho\boldsymbol{v}^t + \rho h^{\mathrm{tl}}\boldsymbol{\xi}^t & 2\rho h^{\mathrm{tl}}\boldsymbol{v}\cdot\boldsymbol{\xi} \end{bmatrix}. \tag{3.13}$$

The decomposition $\widetilde{\mathsf{B}}_{ij} = \widetilde{\mathsf{B}}^{D\lambda}\delta_{ij} + \kappa RT\,\widetilde{\mathsf{B}}^{\kappa}_{ij} + \eta RT\,\widetilde{\mathsf{B}}^{\eta}_{ij}$ *holds with*

$$\widetilde{\mathsf{B}}^{D\lambda} = \frac{RT}{p}\begin{bmatrix} \mathcal{D} & & Sym \\ 0_{d,n} & 0_{d,d} & \\ (\mathcal{D}\hbar)^t & 0_{1,d} & \lambda pT + \langle\mathcal{D}\hbar,\hbar\rangle \end{bmatrix}, \tag{3.14}$$

where $\mathcal{D}$ *is the matrix of size n with components* $\rho_k\rho_l D_{kl}$ *and* $\hbar$ *is the vector of size n with components* $\hbar_i = h_i + \frac{RT}{m_i}\widetilde{\chi}_i$. *Moreover, denoting by* $\boldsymbol{\xi} = (\xi_1,\dots,\xi_d)^t$ *and* $\boldsymbol{\zeta} = (\zeta_1,\dots,\zeta_d)^t$ *arbitrary vectors of* $\mathbb{R}^d$, *the matrices* $\widetilde{\mathsf{B}}^{\kappa}_{ij}$ *and* $\widetilde{\mathsf{B}}^{\eta}_{ij}$, $i,j\in\mathcal{D}$, *are given by*

$$\sum_{i,j\in\mathcal{D}}\xi_i\zeta_j\widetilde{\mathsf{B}}^{\kappa}_{ij} = \begin{bmatrix} 0_{n,n} & 0_{n,d} & 0_{n,1} \\ 0_{d,n} & \boldsymbol{\xi}\otimes\boldsymbol{\zeta} & \boldsymbol{v}\cdot\boldsymbol{\zeta}\,\boldsymbol{\xi} \\ 0_{1,n} & \boldsymbol{v}\cdot\boldsymbol{\xi}\,\boldsymbol{\zeta}^t & \boldsymbol{v}\cdot\boldsymbol{\xi}\,\boldsymbol{v}\cdot\boldsymbol{\zeta} \end{bmatrix}, \tag{3.15}$$

$$\sum_{i,j\in\mathcal{D}}\xi_i\zeta_j\widetilde{\mathsf{B}}^{\eta}_{ij} = \begin{bmatrix} 0_{n,n} & 0_{n,d} & 0_{n,1} \\ 0_{d,n} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\boldsymbol{I} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \frac{2}{d'}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\xi}\,\boldsymbol{\zeta} - \frac{2}{d'}\boldsymbol{v}\cdot\boldsymbol{\zeta}\,\boldsymbol{\xi} \\ 0_{1,n} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{v}^t + \boldsymbol{v}\cdot\boldsymbol{\zeta}\,\boldsymbol{\xi}^t - \frac{2}{d'}\boldsymbol{v}\cdot\boldsymbol{\xi}\,\boldsymbol{\zeta}^t & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\,\boldsymbol{v}\cdot\boldsymbol{v} + \frac{d'-2}{d'}\boldsymbol{v}\cdot\boldsymbol{\xi}\,\boldsymbol{v}\cdot\boldsymbol{\zeta} \end{bmatrix}. \tag{3.16}$$

The equilibrium manifold or slow manifold $\mathcal{E}$ *is of dimension* $\mathsf{n}_{\mathrm{e}} = n_{\mathrm{a}} + d + 1$ *and given by*

$$\mathcal{E} = M^{-1}\mathcal{A}\times\mathbb{R}^d\times\mathbb{R}. \tag{3.17}$$

The linear space $\mathcal{E}$ *is spanned by the vectors* $\mathsf{a}_l = (\widetilde{\mathfrak{a}}_l,\mathbf{0},0)^t$, $l\in\mathfrak{A}$, *and the vectors* $\mathsf{a}_{n_{\mathrm{a}}+l} = \mathsf{f}_{n+l}$, *for* $1\le l\le d+1$, *where* f_i, $1\le i\le\mathsf{n}$, *denotes the basis vectors of* $\mathbb{R}^{\mathsf{n}}$. *The fast manifold* $\mathcal{E}^{\perp}$ *is spanned by the vectors* $\mathsf{a}_{k+d+1} = (\widetilde{\mathfrak{v}}_k,\mathbf{0},0)^t$, $k\in\mathfrak{S}\backslash\mathfrak{A}$, *where* $\widetilde{\mathfrak{v}}_k = M\mathfrak{v}_k$ *is the mass weighted formation reaction* *vector of the kth species for* $k \in \mathfrak{S}\backslash\mathfrak{A} = \{ n_{\rm a}+1,\dots,n \}$*. The source term* $\widetilde{\Omega}$ *is in quasilinear form with* $\widetilde{\Omega}(\mathsf{v}) = -\widetilde{\mathsf{L}}(\mathsf{v})\mathsf{v}$ *and*

$$\widetilde{\mathsf{L}} = \sum_{i\in\mathfrak{R}} \widehat{\Lambda}_i \mathfrak{p}_i \otimes \mathfrak{p}_i, \tag{3.18}$$

with $\mathfrak{p}_i = (\widetilde{\nu}_i, \mathbf{0}, 0)^t$*,* $i \in \mathfrak{R}$*, where* $\widehat{\Lambda}_i$ *is the rescaled constant as described in* (2.24) *and the nullspace of* $\widetilde{\mathsf{L}}$ *is the equilibrium manifold* $N(\widetilde{\mathsf{L}}) = \mathcal{E}$*.*

Sketch of the proof. The variable v and the matrices $\widetilde{\mathsf{A}}_0$, $\widetilde{\mathsf{A}}_i$, $i \in \mathcal{D}$, and $\widetilde{\mathsf{B}}_{ij}$, $i,j \in \mathcal{D}$, are evaluated by using systematically the natural variable z. More specifically, the following expressions are easily derived $\mathsf{v}^t = \partial_\mathsf{z}\sigma(\partial_\mathsf{z}\mathsf{u})^{-1}$, $\widetilde{\mathsf{A}}_0 = \partial_\mathsf{z}\mathsf{u}(\partial_\mathsf{z}\mathsf{v})^{-1}$, $\widetilde{\mathsf{A}} = \partial_\mathsf{z}\mathsf{F}_i(\partial_\mathsf{z}\mathsf{v})^{-1}$, and $\widetilde{\mathsf{B}}_{ij} = \widehat{\mathsf{B}}_{ij}(\partial_\mathsf{z}\mathsf{v})^{-1}$, where F_i denotes the convective flux in the ith direction and $\widehat{\mathsf{B}}_{ij}$, $i,j \in \mathcal{D}$, the unique matrices such that $\mathsf{F}_i^{\rm diss} = -\sum_{j\in\mathcal{D}} \widehat{\mathsf{B}}_{ij}\partial_j \mathsf{z}$, $i \in \mathcal{D}$. All derivatives with respect to z are easily evaluated and subsequently—after lengthy calculations—the variable v and the afore-mentioned matrices.

The symmetry properties then follow and using the expressions of $\widetilde{\mathsf{A}}_0$ and $\widetilde{\mathsf{B}}_{ij}$, $i,j \in \mathcal{D}$, it is next established that $\widetilde{\mathsf{A}}_0$ is positive definite and $\widetilde{\mathsf{B}}$ is positive semi-definite [21, 22, 18, 26]. Finally, the properties of the source term $\widetilde{\Omega}(\mathsf{v}) = \Omega(\mathsf{u})$ are directly obtained from those of $M\omega$ analyzed in Section 2 $\square$

3.6. Normal form for multicomponent flows

The symmetric system (3.11) may be rewritten in normal form, that is, in the form of a symmetric hyperbolic-parabolic composite system, where hyperbolic and parabolic variables are split [48, 34, 47, 35, 36, 21, 22, 23, 57]. It has been established in previous work that the nullspace invariance property holds for multicomponent flows [22, 18].

Lemma 3.11. *The nullspace of the matrix* $\widetilde{\mathsf{B}}(\mathsf{v},\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}} \widetilde{\mathsf{B}}_{ij}(\mathsf{v})\xi_i\xi_j$ *is independent of* $\mathsf{v} \in \mathcal{O}_\mathsf{v}$ *and* $\boldsymbol{\xi} \in \Sigma^{d-1}$ *and given by* $N(\widetilde{\mathsf{B}}) = \mathrm{Span}\{(\mathbb{I}, \mathbf{0}, 0)^t\}$ *and* $\widetilde{\mathsf{B}}_{ij}(\mathsf{v})N(\widetilde{\mathsf{B}}) = 0$*,* $i,j \in \mathcal{D}$*, for* $\mathsf{v} \in \mathcal{O}_\mathsf{v}$*.*

From Theorem 3.5, all normal forms of the system (3.2) are obtained with variables w in the form

$$\mathsf{w} = \Big(\mathcal{F}_{\rm I}\big(\,\rho\,\big), \mathcal{F}_{\rm II}\Big(\frac{g_2-g_1}{RT},\dots,\frac{g_n-g_1}{RT}, \frac{\boldsymbol{v}}{T}, \frac{-1}{RT}\Big)\Big)^t, \tag{3.19}$$

where $\mathcal{F}_{\rm I}$ and $\mathcal{F}_{\rm II}$ are diffeomorphism of $\mathbb{R}$ and $\mathbb{R}^{n+d}$. Incidentally, the number of hyperbolic variables for fluid flows may sometimes differ from unity as for instance with ambipolar plasmas in three dimensions where it is seven and includes density, electric field, and magnetic field [20]. In the following theorem, the normal form corresponding to a normal variable especially convenient for investigating chemical equilibrium fluids is evaluated.

Theorem 3.12. *Assume that* (H_1)-(H_5) *hold and consider the normal variable*

$$\mathsf{w} = \Big(\rho,\, \frac{g_2-g_1}{RT},\ldots,\, \frac{g_n-g_1}{RT},\, \boldsymbol{v}, T\Big)^t = \big(\rho,\, \mu_2-\mu_1,\ldots,\, \mu_n-\mu_1,\, \boldsymbol{v}, T\big)^t, \quad (3.20)$$

and the diffeomorphism $\mathsf{v} \longmapsto \mathsf{w}$ *from* $\mathcal{O}_{\mathsf{v}}$ *onto the open set* $\mathcal{O}_{\mathsf{w}} = (0,\infty)\times\mathbb{R}^{n-1}\times\mathbb{R}^d\times(0,\infty)$. *Then the system of partial differential equations in normal form may be written*

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\mathsf{w} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\mathsf{w} - \sum_{i,j\in\mathcal{D}}\partial_i\big(\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_j\mathsf{w}\big) + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\mathsf{w} = \overline{\mathsf{Q}}(\mathsf{w},\partial_x\mathsf{w}), \quad (3.21)$$

where the matrix $\overline{\mathsf{A}}_0$ *is given by*

$$\overline{\mathsf{A}}_0 = \begin{bmatrix} \overline{\mathsf{A}}_0^{\mathrm{I,I}} & \mathit{Sym} \\ 0_{n+d,1} & \overline{\mathsf{A}}_0^{\mathrm{II,II}} \end{bmatrix},$$

with

$$\overline{\mathsf{A}}_0^{\mathrm{I,I}} = \frac{1}{\langle\Gamma\mathbb{I},\mathbb{I}\rangle} \qquad \overline{\mathsf{A}}_0^{\mathrm{II,II}} = \begin{bmatrix} \overline{\mathsf{A}}^{\mathrm{II,II}} & & \mathit{Sym} \\ 0_{d,n-1} & \frac{\rho}{RT}\boldsymbol{I} & \\ \mathsf{a}^t & 0 & \overline{\Upsilon} \end{bmatrix}.$$

The quadratic residual $\overline{\mathsf{Q}} = -\sum_{i,j\in\mathcal{D}}\partial_i(\partial_{\mathsf{w}}\mathsf{v})^t\,(\partial_{\mathsf{v}}\mathsf{w})^t\,\overline{\mathsf{B}}_{ij}\partial_j\mathsf{w}$ *may be written* $\overline{\mathsf{Q}} = (0,\ldots,0,\overline{\mathsf{Q}}_{\boldsymbol{v}},\overline{\mathsf{Q}}_T)^t$ *with*

$$\overline{\mathsf{Q}}_{\boldsymbol{v}} = -\sum_{i\in\mathcal{D}}\frac{\partial_i T}{RT^2}\boldsymbol{\Pi}_i, \qquad \overline{\mathsf{Q}}_T = -\sum_{i,j\in\mathcal{D}}\frac{1}{RT^2}\partial_i v_j\boldsymbol{\Pi}_{ij} - 2\sum_{i\in\mathcal{D}}\frac{\partial_i T}{RT^3}Q_i. \quad (3.22)$$

The matrix $\overline{\mathsf{A}}^{\mathrm{II,II}}$ *is the square matrix of dimension* $n-1$ *with coefficients*

$$\overline{\mathsf{A}}^{\mathrm{II,II}}_{kl} = \Gamma_{kl} - \frac{(\Gamma\mathbb{I})_k(\Gamma\mathbb{I})_l}{\langle\Gamma\mathbb{I},\mathbb{I}\rangle}, \quad 2\le k,l\le n,$$

a *is the vector of dimension* $n-1$ *with* $\mathsf{a}_l RT^2 = (\Gamma e^{\mathrm{tl}})_l - (\Gamma\mathbb{I})_l\langle\Gamma e^{\mathrm{tl}},\mathbb{I}\rangle/\langle\Gamma\mathbb{I},\mathbb{I}\rangle$, $2\le l\le n$, *and* $\overline{\Upsilon}$ *is given by*

$$\overline{\Upsilon} = \frac{\langle\Gamma e^{\mathrm{tl}}, e^{\mathrm{tl}}\rangle + \rho RT^2 c_{\mathrm{v}}}{R^2T^4} - \frac{\langle\Gamma e^{\mathrm{tl}},\mathbb{I}\rangle^2}{R^2T^4\langle\Gamma\mathbb{I},\mathbb{I}\rangle},$$

where $\mathbb{I} = (1,\ldots,1)^t$, $e^{\mathrm{tl}} = (e^{\mathrm{tl}}_1,\ldots,e^{\mathrm{tl}}_n)^t$, *and* $\Gamma = \mathrm{diag}(m_1\rho_1,\ldots,m_n\rho_n)$. *Denoting by* $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_d)^t$ *an arbitrary vector of* $\mathbb{R}^d$, *the matrices* $\overline{\mathsf{A}}_i$, $i\in\mathcal{D}$, *are given by*

$$\sum_{i\in\mathcal{D}}\xi_i\overline{\mathsf{A}}_i = \overline{\mathsf{A}}_0\boldsymbol{v}\cdot\boldsymbol{\xi} + \frac{\rho}{\langle\Gamma\mathbb{I},\mathbb{I}\rangle}\begin{bmatrix} 0 & & & \mathit{Sym} \\ 0_{n-1,1} & 0_{n-1,n-1} & & \\ \boldsymbol{\xi} & \boldsymbol{\xi}\otimes y & 0_{d,d} & \\ 0 & 0_{n-1,1} & z\boldsymbol{\xi}^t & 0 \end{bmatrix},$$

where y is the vector of dimension $n-1$ with components $y_l = \langle \Gamma 1\!\!1, 1\!\!1\rangle \mathsf{y}_l - (\Gamma 1\!\!1)_l$, $2 \le l \le n$, and $z = \big(\langle \Gamma 1\!\!1, 1\!\!1\rangle h^{\mathrm{tl}} - \langle \Gamma e^{\mathrm{tl}}, 1\!\!1\rangle\big)/RT^2$.

The matrices $\overline{\mathsf{B}}_{ij}$ have the structure $\overline{\mathsf{B}}_{ij} = \delta_{ij}\overline{\mathsf{B}}^{D\lambda} + \frac{\kappa}{RT}\overline{\mathsf{B}}^{\kappa}_{ij} + \frac{\eta}{RT}\overline{\mathsf{B}}^{\eta}_{ij}$ and denoting by $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_d)^t$ and $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_d)^t$ arbitrary vectors of $\mathbb{R}^d$, then

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \overline{\mathsf{B}}^{\kappa}_{ij} = \begin{bmatrix} 0_{n,n} & 0_{n,d} & 0_{n,1} \\ 0_{d,n} & \boldsymbol{\xi}\otimes\boldsymbol{\zeta} & 0_{d,1} \\ 0_{1,n} & 0_{1,d} & 0 \end{bmatrix}, \tag{3.23}$$

$$\sum_{i,j\in\mathcal{D}} \xi_i\zeta_j \overline{\mathsf{B}}^{\eta}_{ij} = \begin{bmatrix} 0_{n,n} & 0_{n,d} & 0_{n,1} \\ 0_{d,n} & \boldsymbol{\xi}\cdot\boldsymbol{\zeta}\boldsymbol{I} + \boldsymbol{\zeta}\otimes\boldsymbol{\xi} - \frac{2}{d'}\boldsymbol{\xi}\otimes\boldsymbol{\zeta} & 0_{d,1} \\ 0_{1,n} & 0_{1,d} & 0 \end{bmatrix}, \tag{3.24}$$

$$\overline{\mathsf{B}}^{D\lambda} = \frac{RT}{p}\begin{bmatrix} 0 & & & Sym \\ 0 & \mathcal{D}_{n-1,n-1} & & \\ 0 & 0_{d,n} & 0_{d,d} & \\ 0 & \frac{(\mathcal{D}\hbar)^t_{n-1}}{RT^2} & 0_{1,d} & \frac{\lambda pT + \langle \mathcal{D}\hbar, \hbar\rangle}{R^2T^4} \end{bmatrix}, \tag{3.25}$$

where $\mathcal{D}_{n-1,n-1}$ is the matrix of size $n-1$ with coefficients $(\rho_k\rho_l D_{kl})_{2\le k,l\le n}$ and $(\mathcal{D}\hbar)_{n-1}$ are the $n-1$ last components of $\mathcal{D}\hbar$.

The equilibrium manifold with respect to the normal variable is the linear space $\overline{\mathcal{E}}$ spanned by the vectors $\overline{\mathsf{a}}_1 = \mathsf{f}_1$, $\overline{\mathsf{a}}_l = (\widetilde{\mathfrak{a}}_l - \widetilde{\mathfrak{a}}_{1l}1\!\!1, \mathbf{0}, 0)^t$, for $l \in \mathfrak{A}\backslash\{1\} = \{\,2,\ldots,n_{\mathrm{a}}\,\}$, and $\overline{\mathsf{a}}_{n_{\mathrm{a}}+l} = \mathsf{f}_{n+l}$, for $1 \le l \le d+1$, where f_i, $1 \le i \le \mathsf{n}$, denote the basis vectors of $\mathbb{R}^{\mathsf{n}}$. The fast manifold $\overline{\mathcal{E}}^{\perp}$ is spanned by the vectors $\overline{\mathsf{a}}_{k+d+1} = (\widetilde{\mathfrak{v}}_k - \widetilde{\mathfrak{v}}_{1k}\mathsf{e}_1, \mathbf{0}, 0)^t$, for $l \in \mathfrak{S}\backslash\mathfrak{A} = \{\,n_{\mathrm{a}}+1,\ldots,n\,\}$ where e_i, $1 \le i \le n$ denote the basis vectors of $\mathbb{R}^n$. The source term $\overline{\Omega}$ for the normal form given by

$$\overline{\Omega} = \Big(0, m_2\widehat{\omega}_2, \ldots, m_n\widehat{\omega}_n, \mathbf{0}, 0\Big)^t,$$

is in quasilinear form $\overline{\Omega}(\mathsf{w}) = -\overline{\mathsf{L}}(\mathsf{w})\,\mathsf{w}$ with

$$\overline{\mathsf{L}} = \sum_{i\in\mathfrak{R}} \widehat{\Lambda}_i \mathfrak{p}'_i\otimes\mathfrak{p}'_i, \tag{3.26}$$

where $\mathfrak{p}'_i = (\widetilde{\nu}'_i, \mathbf{0}, 0)^t$, $\widetilde{\nu}'_i = (0, m_2\nu_{2i}, \ldots, m_n\nu_{ni}, \mathbf{0}, 0)^t$, and $\widehat{\Lambda}_i$ is the rescaled constant (2.24). The nullspace of $\overline{\mathsf{L}}$ is the equilibrium manifold $N(\overline{\mathsf{L}}) = \overline{\mathcal{E}}$ and $\overline{\mathsf{L}} = \overline{\mathsf{L}}\pi = \pi\overline{\mathsf{L}}$ where π denoted the orthogonal projector onto $\overline{\mathcal{E}}^{\perp}$.

Sketch of the proof. The matrices $\overline{\mathsf{A}}_0$, $\overline{\mathsf{A}}_i$, $i \in \mathcal{D}$, and $\overline{\mathsf{B}}_{ij}$, $i,j \in \mathcal{D}$, are again evaluated by using the natural variable z. We may write for instance that $\overline{\mathsf{A}}_0 = (\partial_{\mathsf{w}}\mathsf{z})^t\,(\partial_{\mathsf{z}}\mathsf{v})^t\,\partial_{\mathsf{z}}\mathsf{u}\,\partial_{\mathsf{w}}\mathsf{z}$ with similar expressions for $\overline{\mathsf{A}}_i$, $i \in \mathcal{D}$, and $\overline{\mathsf{B}}_{ij}$, $i,j \in \mathcal{D}$, using $\partial_{\mathsf{z}}\mathsf{F}_i$ and $\widehat{\mathsf{B}}_{ij}$, $i,j \in \mathcal{D}$.

An interesting variant is to use the auxiliary variables $\mathsf{u}' = \mathsf{P}^t\mathsf{u}$ and $\mathsf{v}' = \mathsf{P}^{-1}\mathsf{v}$ where $\mathsf{P} \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$ is obtained from the identity matrix by replacing the first column vector f_1 by $(1\!\!1, \mathbf{0}, 0)^t = \sum_{i\in\mathfrak{S}} \mathsf{f}_i$ that spans $N(\widetilde{\mathsf{B}})$. The dissipation matrices corresponding to these auxiliary variables have nonzero coefficients only in the lower right block of size $n + d$. The normal forms is then obtained from the v' conservation equation with $\partial_{\mathsf{w}}\mathsf{v}'$ evaluated as $\partial_{\mathsf{w}}\mathsf{v}' = \partial_{\mathsf{z}}\mathsf{v}'(\partial_{\mathsf{z}}\mathsf{w})^{-1}$.

Finally the properties of the source term $\overline{\Omega} = (\partial_{\mathsf{w}}\mathsf{v})^t\widetilde{\Omega}$ are easily obtained from that of $\widetilde{\Omega}$. □

Strict parabolic dissipativity of the system in normal form is is now investigated. It is indeed possible to find a compensating matrix in the form $K(\boldsymbol{\xi}) = \sum_{j\in\mathcal{D}} K_j\xi_j$ with $\boldsymbol{\xi} \in \Sigma^{d-1}$ and such that $K(\boldsymbol{\xi})\pi = 0$ for any $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$.

Proposition 3.13. *Let $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$ be fixed, $\delta > 0$, and $K(\boldsymbol{\xi})$ be the matrix defined for $\boldsymbol{\xi} \in \Sigma^{d-1}$ by*

$$K(\boldsymbol{\xi}) = \sum_{j\in\mathcal{D}} \xi_j K_j = \delta \begin{bmatrix} 0 & 0_{1,n-1} & \boldsymbol{\xi}^t & 0 \\ 0_{n-1,1} & 0_{n-1,n-1} & 0_{n-1,d} & 0_{n-1,1} \\ -\boldsymbol{\xi} & 0_{d,n-1} & 0_{d,d} & 0_{d,1} \\ 0 & 0_{1,n-1} & 0_{1,d} & 0 \end{bmatrix} \big(\overline{\mathsf{A}}_0(\mathsf{w})\big)^{-1}.$$

Then for sufficiently small positive δ, the map $\boldsymbol{\xi} \to K(\boldsymbol{\xi})$ is a compensating function, that is, the product $K(\boldsymbol{\xi})\overline{\mathsf{A}}_0(\mathsf{w})$ is skew-symmetric, $K(-\boldsymbol{\xi}) = -K(\boldsymbol{\xi})$ and the matrix $K(\boldsymbol{\xi})\overline{\mathsf{A}}(\mathsf{w},\boldsymbol{\xi}) + \overline{\mathsf{B}}(\mathsf{w},\boldsymbol{\xi})$ is positive definite for $\boldsymbol{\xi} \in \Sigma^{d-1}$. Furthermore, the compatibility relation between the compensating matrix $K(\boldsymbol{\xi})$ and the fast manifold $K(\boldsymbol{\xi})\pi = 0$ holds.

Proof. It is obvious by construction that the products $K_j\overline{\mathsf{A}}_0(\mathsf{w})$, $j \in \mathcal{D}$, are skew-symmetric. On the other hand, a direct calculation yields that

$$K(\boldsymbol{\xi})\overline{\mathsf{A}}(\mathsf{w},\boldsymbol{\xi}) = \frac{\delta RT}{\langle \Gamma\mathbb{1}, \mathbb{1}\rangle} \begin{bmatrix} |\boldsymbol{\xi}|^2 & \mathcal{O}(|\boldsymbol{\xi}|^2) & \mathcal{O}(|\boldsymbol{\xi}|^2) & \mathcal{O}(|\boldsymbol{\xi}|^2) \\ 0_{n-1,1} & 0_{n-1,n-1} & 0_{n-1,d} & 0_{n-1,1} \\ \mathcal{O}(|\boldsymbol{\xi}|^2) & 0_{d,n-1} & \mathcal{O}(|\boldsymbol{\xi}|^2) & 0 \\ 0 & 0_{1,n-1} & 0_{1,d} & 0 \end{bmatrix}.$$

Using now the property that $\overline{\mathsf{B}}^{\mathrm{II,II}}(\mathsf{w},\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{II,II}}_{ij}(\mathsf{w})\xi_i\xi_j$ is positive definite for $\boldsymbol{\xi} \in \Sigma^{d-1}$, and since $RT/\langle\Gamma\mathbb{1},\mathbb{1}\rangle > 0$ and $|\boldsymbol{\xi}|^2 = 1$ for $\boldsymbol{\xi} \in \Sigma^{d-1}$, one obtains that $K(\boldsymbol{\xi})\overline{\mathsf{A}}(\mathsf{w},\boldsymbol{\xi}) + \overline{\mathsf{B}}(\mathsf{w},\boldsymbol{\xi})$ is positive definite for $\boldsymbol{\xi} \in \Sigma^{d-1}$ and δ sufficiently small. The relation $K(\boldsymbol{\xi})\pi = 0$ is next established by first noting that the equilibrium manifold $\overline{\mathcal{E}}$ is included in the linear space spanned by $\mathsf{f}_2, \dots, \mathsf{f}_n$. For any $2 \le i \le n$, it is then checked that the vector $\big(\overline{\mathsf{A}}_0(\mathsf{w})\big)^{-1}\mathsf{f}_i$ is in the space spanned by $\mathsf{f}_2, \dots, \mathsf{f}_n$ and f_{n} with $\mathsf{n} = n + d + 1$. All these base vectors are then in the nullspace of $K(\boldsymbol{\xi})\overline{\mathsf{A}}_0(\mathsf{w}) = \delta(\mathsf{f}_1 \otimes \mathsf{f}_\xi - \mathsf{f}_\xi \otimes \mathsf{f}_1)$ where $\mathsf{f}_\xi = \sum_{l\in\mathcal{D}} \xi_l \mathsf{f}_{n+l}$. □

3.7. Ozone normal variable

We illustrate the previous developments about normal forms associated with arbitrary complex mixtures and chemical reactions mechanisms by considering the special situation of ozone decomposition presented in Section 2.7.

In this situation, the normal variable $\mathsf{w} \in \mathbb{R}^{4+d}$ is found in the form

$$\mathsf{w} = (\rho, \mu_2 - \mu_1, \mu_3 - \mu_1, \boldsymbol{v}, T)^t,$$

and is of dimension $\mathsf{n} = 4 + d$. The total mass density ρ also coincides with the oxygen atom mass dentisy $\widetilde{\rho}_1 = \rho$ since there is only one atom. The equilibrium manifold with respect to the normal variable $\overline{\mathcal{E}}$ is of dimension $\mathsf{n}_\mathrm{e} = 2 + d$ and is spanned by the vectors

$$\overline{\mathsf{a}}_1 = \mathsf{f}_1, \qquad \overline{\mathsf{a}}_{1+l} = \mathsf{f}_{3+l}, \qquad 1 \leq l \leq d+1,$$

where f_i, $1 \leq i \leq 4 + d$, denote here the basis vectors of $\mathbb{R}^{4+d}$. Note that the set $\mathfrak{A}\backslash\{1\} = \emptyset$ is empty in our situation since there is only one atom. The fast manifold $\overline{\mathcal{E}}^\perp$ is of dimension $\mathsf{n}_\mathsf{r} = 2$ and spanned by the renormalized vectors

$$\overline{\mathsf{a}}_{3+d} = (0, 1, 0, \mathbf{0}, 0)^t = \mathsf{f}_2, \qquad \overline{\mathsf{a}}_{4+d} = (0, 0, 1, \mathbf{0}, 0)^t = \mathsf{f}_3,$$

so that

$$\overline{\Pi}_\mathrm{e} = [\mathsf{f}_1, \mathsf{f}_3 + 1, \dots, \mathsf{f}_{3+d}, \mathsf{f}_{4+d}], \qquad \overline{\Pi}_\mathsf{r} = [\mathsf{f}_2, \mathsf{f}_3],$$

and the metric matrices reduce to $\overline{\mathcal{J}}_\mathrm{e} = \mathbb{I}_{2+d}$ and $\overline{\mathcal{J}}_\mathsf{r} = \mathbb{I}_2$. The fast variable then takes the simple form

$$\pi\mathsf{w} = (0, \mu_2 - \mu_1, \mu_3 - \mu_1, \mathbf{0}, 0)^t, \qquad \overline{\mathcal{J}}_\mathsf{r}\overline{\Pi}_\mathsf{r}^t\mathsf{w} = \begin{pmatrix} \mu_2 - \mu_1 \\ \mu_3 - \mu_1 \end{pmatrix},$$

and of course vanish at equilibrium where μ becomes proportional to $\widetilde{\mathfrak{a}}_1 = (1, 1, 1)^t$. On the other hand, the slow variable takes the simple form

$$w_\mathrm{e} = \overline{\mathcal{J}}_\mathrm{e}\overline{\Pi}_\mathrm{e}^t\mathsf{w} = (\rho, \boldsymbol{v}, T)^t,$$

and the limiting normal variable reads $\mathsf{w}_\mathrm{e} = (\rho_\mathrm{e}, \boldsymbol{v}_\mathrm{e}, T_\mathrm{e})^t$. The limiting fluid obtained when $\epsilon \to 0$ is then a compressible fluid with the chemical equilibrium thermodynamics investigated in Section 2.6.

Finally, the matrix $\overline{\mathsf{L}}$ only has nonzero entries $\overline{\mathsf{L}}_{ij}$ for $2 \leq i, j \leq 3$ since $\overline{\mathsf{L}} = \pi\overline{\mathsf{L}}\pi$ and the reduced matrix $\overline{\mathsf{L}}^{\mathsf{r},\mathsf{r}} = \overline{\Pi}_\mathsf{r}^t\overline{\mathsf{L}}\,\overline{\Pi}_\mathsf{r}$ may be written

$$\overline{\mathsf{L}}^{\mathsf{r},\mathsf{r}} = \theta\begin{pmatrix} 2 \\ -3 \end{pmatrix}\otimes\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \theta'\begin{pmatrix} 4 \\ -3 \end{pmatrix}\otimes\begin{pmatrix} 4 \\ -3 \end{pmatrix} + \theta''\begin{pmatrix} -2 \\ 0 \end{pmatrix}\otimes\begin{pmatrix} -2 \\ 0 \end{pmatrix},$$

where $\theta, \theta', \theta''$ are positive so that $\overline{\mathsf{L}}^{\mathsf{r},\mathsf{r}}$ is positive definite and $\overline{\mathsf{L}}$ positive definite over $\mathrm{Span}\{\mathsf{f}_2, \mathsf{f}_3\}$.

4. Equations at equilibrium

Projectors associated with the slow and fast variables are first discussed as well as reduced equilibrium equations in a general setting. The special situation of chemical equilibrium flows is then considered.

4.1. Reduced systems at equilibrium

The proper projectors associated with the slow manifold $\mathcal{E}$ need to be introduced in order derive the equilibrium limit equations. Proceeding as in Section 3.4, let $\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{n}_\mathrm{e}}$ denotes a basis of the slow manifold or equilibrium space $\mathcal{E}$ and $\mathsf{a}_{\mathsf{n}_\mathrm{e}+1}, \dots, \mathsf{a}_\mathsf{n}$ a basis of the fast manifold $\mathcal{E}^\perp$. The linear operators $\Pi_\mathsf{e} = \mathbb{R}^{\mathsf{n}_\mathrm{e}} \longrightarrow \mathbb{R}^\mathsf{n}$ and $\Pi_\mathsf{r} = \mathbb{R}^{\mathsf{n}-\mathsf{n}_\mathrm{e}} \longrightarrow \mathbb{R}^\mathsf{n}$ are defined by their matrices in the canonical bases

$$\Pi_\mathsf{e} = [\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{n}_\mathrm{e}}], \qquad \Pi_\mathsf{r} = [\mathsf{a}_{\mathsf{n}_\mathrm{e}+1}, \dots, \mathsf{a}_\mathsf{n}].$$

The metric matrices $\mathcal{J}_\mathsf{e}$ and $\mathcal{J}_\mathsf{r}$ of order n_e and $\mathsf{n}-\mathsf{n}_\mathrm{e}$, respectively, are defined by $\mathcal{J}^{-1}_{\mathsf{e}\,i,j} = \langle \mathsf{a}_i, \mathsf{a}_j \rangle$, $1 \le i,j \le \mathsf{n}_\mathrm{e}$ and $\mathcal{J}^{-1}_{\mathsf{r}\,i,j} = \langle \mathsf{a}_i, \mathsf{a}_j \rangle$, $\mathsf{n}_\mathrm{e}+1 \le i,j \le \mathsf{n}$. Each vector $\mathfrak{r} \in \mathbb{R}^\mathsf{n}$ admits a unique decomposition $\mathfrak{r} = \mathfrak{r}_\mathcal{E} + \mathfrak{r}_{\mathcal{E}^\perp}$ where $\mathfrak{r}_\mathcal{E} = \Pi_\mathsf{e}\mathcal{J}_\mathsf{e}\Pi_\mathsf{e}^t\,\mathfrak{r} \in \mathcal{E}$ and $\mathfrak{r}_{\mathcal{E}^\perp} = \Pi_\mathsf{r}\mathcal{J}_\mathsf{r}\Pi_\mathsf{r}^t \in \mathcal{E}^\perp$ and $\mathbb{I}_\mathsf{n} = \Pi_\mathsf{e}\mathcal{J}_\mathsf{e}\Pi_\mathsf{e}^t + \Pi_\mathsf{r}\mathcal{J}_\mathsf{r}\Pi_\mathsf{r}^t$, $\mathcal{J}_\mathsf{e}\Pi_\mathsf{e}^t\Pi_\mathsf{e} = \mathbb{I}_{\mathsf{n}_\mathrm{e}}$, $\mathcal{J}_\mathsf{r}\Pi_\mathsf{r}^t\Pi_\mathsf{r} = \mathbb{I}_{\mathsf{n}-\mathsf{n}_\mathrm{e}}$ and $\tilde{\pi} = \Pi_\mathsf{r}\mathcal{J}_\mathsf{r}\Pi_\mathsf{r}^t$ where $\tilde{\pi}$ is the orthogonal projector onto $\mathcal{E}^\perp$.

It is assumed that there exists $\mathsf{u}^\star$ with $\Omega(\mathsf{u}^\star) = 0$ and the corresponding entropic variable is denoted by $\mathsf{v}^\star$. The state $\mathsf{v}^\star$ is in the equilibrium manifold $\mathcal{O}_\mathsf{v} \cap \mathcal{E}$ so that $\mathcal{O}_\mathsf{v} \cap \mathcal{E} \neq \emptyset$. It is then possible to parametrize the equilibrium manifold by its projection on the slow manifold denoted by u_e as shown in the following lemma [28].

Proposition 4.1. *There exists a convex domain* $\mathcal{O}_{\mathsf{u}_\mathrm{e}}$ *containing* $\mathsf{u}_\mathrm{e}^\star = \Pi_\mathsf{e}^t\mathsf{u}^\star$ *such that for any* $\mathsf{u}_\mathrm{e} \in \mathcal{O}_{\mathsf{u}_\mathrm{e}}$ *there exists a unique* $\mathsf{u}_\mathrm{eq} \in \mathcal{O}_\mathsf{u}$ *such that* $\Omega(\mathsf{u}_\mathrm{eq}) = 0$ *and* $\mathsf{u}_\mathrm{e} = \Pi_\mathsf{e}^t\,\mathsf{u}_\mathrm{eq}$. *The map* $\mathsf{u}_\mathrm{e} \mapsto \mathsf{u}_\mathrm{eq}$ *is a* $C^\varkappa$ *diffeomorphism from* $\mathcal{O}_{\mathsf{u}_\mathrm{e}}$ *onto an open set* $\mathcal{O}'_\mathsf{u} \subset \mathcal{O}_\mathsf{u}$ *containing* $\mathsf{u}^\star$ *and its differential satisfies*

$$\partial_\mathsf{u}\Omega\big(\mathsf{u}_\mathrm{eq}\big)\,\partial_{\mathsf{u}_\mathrm{e}}\mathsf{u}_\mathrm{eq} = 0, \qquad \Pi_\mathsf{e}^t\,\partial_{\mathsf{u}_\mathrm{e}}\mathsf{u}_\mathrm{eq} = \mathbb{I}_{\mathsf{n}_\mathrm{e}}. \tag{4.1}$$

Denoting by $\mathsf{v}_\mathrm{eq} = \mathsf{v}(\mathsf{u}_\mathrm{eq})$ *the symmetric variable associated with* u_eq*, then the map* $\mathsf{u}_\mathrm{e} \mapsto \mathsf{v}_\mathrm{eq}$ *is at least* $C^{\varkappa-1}$ *and* $\mathsf{v}_\mathrm{eq} \in \mathcal{E}$.

Proof. The proof is only sketched and the reader is referred to [28] for more details. The map

$$\Phi = (\mathsf{u}, \mathsf{u}_\mathrm{e})^t \mapsto \big(\Pi_\mathsf{r}^t\Omega(\mathsf{u}), \Pi_\mathsf{e}^t\mathsf{u} - \mathsf{u}_\mathrm{e}\big)^t,$$

from $\mathcal{O}_\mathsf{u} \times \Pi_\mathsf{e}^t\,\mathcal{O}_\mathsf{u}$ to $\mathbb{R}^\mathsf{n}$ is $C^\varkappa$, $\Phi(\mathsf{u}^\star, \mathsf{u}_\mathrm{e}^\star) = 0$, and its partial differential with respect to u is invertible at $(\mathsf{u}^\star, \mathsf{u}_\mathrm{e}^\star)^t$. Denoting $(\partial_\mathsf{u}\Omega)^\star = \partial_\mathsf{u}\Omega(\mathsf{u}^\star)$, the two conditions $\Pi_\mathsf{r}^t(\partial_\mathsf{u}\Omega)^\star\mathfrak{r} = 0$ and $\Pi_\mathsf{e}^t\mathfrak{r} = 0$ indeed imply that $\mathfrak{r} = 0$. This may be checked by writing $\mathfrak{r} = (\partial_\mathsf{v}\mathsf{u})^\star\mathfrak{r}'$ which yields $\Pi_\mathsf{r}^t(\partial_\mathsf{v}\Omega)^\star\mathfrak{r}' = 0$ and thus $\mathfrak{r}' \in \mathcal{E}$ since $\pi(\partial_\mathsf{v}\Omega)^\star\mathfrak{r}' = (\partial_\mathsf{v}\Omega)^\star\mathfrak{r}'$ and $N\big((\partial_\mathsf{v}\Omega)^\star\big) = \mathcal{E}$, and then $\langle \mathfrak{r}', \mathfrak{r} \rangle = \langle (\partial_\mathsf{v}\mathsf{u})^\star\mathfrak{r}', \mathfrak{r}' \rangle = 0$ since $\Pi_\mathsf{e}^t\mathfrak{r} = 0$ and finally $\mathfrak{r}' = 0 = \mathfrak{r}$ since $(\partial_\mathsf{v}\mathsf{u})^\star$ is positive definite. From the implicit function theorem, one may parametrize locally the equilibrium manifold in the form $\big(\mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e}), \mathsf{u}_\mathrm{e}\big)$ with $\mathsf{u}_\mathrm{e} \in \mathcal{O}_{\mathsf{u}_\mathrm{e}}$ and the corresponding map $\mathsf{u}_\mathrm{e} \mapsto \mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})$ is $C^\varkappa$. The proof is then complete observing that $\Omega = 0$ if and only if $\Pi_\mathsf{r}^t\Omega = 0$ and differentiating $\Omega\big(\mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})\big) = 0$ and $\Pi_\mathsf{e}^t\mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})\big) = \mathsf{u}_\mathrm{e}$ in order to obtain (4.1). $\square$

The limiting governing equations for the slow variable u_e as $\epsilon \to 0$ may then be obtained formally by applying the projection operator Π_e^t to the governing equations in conservative form, letting $\mathsf{u}_\mathrm{e} = \Pi_\mathsf{e}^t \mathsf{u}$, and superimposing the equilibrium condition $\mathsf{u} = \mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})$. These equations are obtained in the form

$$\partial_t \mathsf{u}_\mathrm{e} + \sum_{i\in\mathcal{D}} \mathsf{A}_i^\mathrm{e}(\mathsf{u}_\mathrm{e})\partial_i \mathsf{u}_\mathrm{e} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}^\mathrm{e}(\mathsf{u}_\mathrm{e})\partial_j \mathsf{u}_\mathrm{e}\big) = 0, \tag{4.2}$$

where $\mathsf{A}_i^\mathrm{e}(\mathsf{u}_\mathrm{e}) = \Pi_\mathsf{e}^t \mathsf{A}_i\big(\mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})\big)\,\partial_{\mathsf{u}_\mathrm{e}}\mathsf{u}_\mathrm{eq}$, $\mathsf{B}_{ij}^\mathrm{e}(\mathsf{u}_\mathrm{e}) = \Pi_\mathsf{e}^t \mathsf{B}_{ij}\big(\mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})\big)\,\partial_{\mathsf{u}_\mathrm{e}}\mathsf{u}_\mathrm{eq}$, and u_eq is the unique equilibrium point obtained in Proposition 4.1. The convective terms are in conservative form with $\mathsf{A}_i^\mathrm{e}(\mathsf{u}_\mathrm{e}) = \partial_{\mathsf{u}_\mathrm{e}}\mathsf{F}_i^\mathrm{e}(\mathsf{u}_\mathrm{e})$ and the convective fluxes at equilibrium read $\mathsf{F}_i^\mathrm{e}(\mathsf{u}_\mathrm{e}) = \Pi_\mathsf{e}^t \mathsf{F}_i\big(\mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})\big)$, $i \in \mathcal{D}$. From the properties of A_i, $i \in \mathcal{D}$ and B_{ij}, $i,j \in \mathcal{D}$, and since $\mathsf{u} \mapsto \mathsf{u}_\mathrm{eq}$ has regularity $C^\varkappa$, we deduce that A_i^e and $\mathsf{B}_{ij}^\mathrm{e}$ are $C^{\varkappa-1}$ over $\mathcal{O}_{\mathsf{u}_\mathrm{e}}$ and the natural initial condition for u_e is $\mathsf{u}_\mathrm{e0} = \Pi_\mathsf{e}^t \mathsf{u}_0$. Note that the equilibrium system (4.2) is equivalent to a *one term* Chapman-Enskog expansion of the conservative variable u in the fast chemistry limit [40, 9, 28, 29]. Our aim in this work is to establish that the difference $\mathsf{u}_\mathrm{e} - \Pi_\mathsf{e}^t \mathsf{u}$ between the equilibrium solution u_e that satisfies (4.2) and the projection $\Pi_\mathsf{e}^t \mathsf{u}$ of the out of equilibrium solution u that satisfies (3.1) is $\mathcal{O}(\epsilon)$. This will rigorously justify (4.2) and will also yield an error estimate.

More accurate equations at equilibrium may also be obtained by using a *two term* Chapman-Enskog expansion [40, 9, 28, 29] instead of a one term expansion (4.2). These higher order equations obtained with two term Chapman-Enskog expansions are required in particular when dissipative terms are of first order with respect to ϵ but lay out of the scope of the present work. Symmetrizability of the system of partial differential equation at equilibrium has been established in [28].

Proposition 4.2. *The $C^\varkappa$ map $\mathsf{u}_\mathrm{e} \mapsto \sigma_\mathrm{e}(\mathsf{u}_\mathrm{e})$ defined over the open convex domain $\mathcal{O}_{\mathsf{u}_\mathrm{e}}$ by*

$$\sigma_\mathrm{e}(\mathsf{u}_\mathrm{e}) = \sigma\big(\mathsf{u}_\mathrm{eq}(\mathsf{u}_\mathrm{e})\big), \qquad \mathsf{u}_\mathrm{e} \in \mathcal{O}_{\mathsf{u}_\mathrm{e}}, \tag{4.3}$$

is a mathematical entropy for the system of partial differential equations (4.2)*. Denoting by $\mathsf{v}_\mathrm{eq} = \mathsf{v}(\mathsf{u}_\mathrm{eq})$ the symmetrizing variable corresponding to u_eq, the corresponding entropic variable v_e is given by*

$$\mathsf{v}_\mathrm{e} = \mathcal{J}_\mathsf{e}\Pi_\mathsf{e}^t\,\mathsf{v}_\mathrm{eq},$$

and such that $\mathsf{v}_\mathrm{eq} = \Pi_\mathsf{e}\mathsf{v}_\mathrm{e}$. The map $\mathsf{u}_\mathrm{e} \mapsto \mathsf{v}_\mathrm{e}$ is a $C^{\varkappa-1}$ diffeomorphism for the open set $\mathcal{O}_{\mathsf{u}_\mathrm{e}}$ onto an open set $\mathcal{O}_{\mathsf{v}_\mathrm{e}}$ and the symmetrized equations read

$$\widetilde{\mathsf{A}}_0^\mathrm{e}\partial_t \mathsf{v}_\mathrm{e} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}_i^\mathrm{e}(\mathsf{v}_\mathrm{e})\partial_i \mathsf{v}_\mathrm{e} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}_{ij}^\mathrm{e}(\mathsf{v}_\mathrm{e})\partial_j \mathsf{v}_\mathrm{e}\big) = 0, \tag{4.4}$$

where $\widetilde{\mathsf{A}}_0^\mathrm{e} = \partial_{\mathsf{v}_\mathrm{e}}\mathsf{u}_\mathrm{e} = \Pi_\mathsf{e}^t\widetilde{\mathsf{A}}_0\Pi_\mathsf{e}$, $\widetilde{\mathsf{A}}_i^\mathrm{e} = \mathsf{A}_i^\mathrm{e}\partial_{\mathsf{v}_\mathrm{e}}\mathsf{u}_\mathrm{e} = \Pi_\mathsf{e}^t\widetilde{\mathsf{A}}_i\Pi_\mathsf{e}$, and $\widetilde{\mathsf{B}}_{ij}^\mathrm{e} = \mathsf{B}_{ij}^\mathrm{e}\partial_{\mathsf{v}_\mathrm{e}}\mathsf{u}_\mathrm{e} = \Pi_\mathsf{e}^t\widetilde{\mathsf{B}}_{ij}\Pi_\mathsf{e}$ have regularity at least $\varkappa - 2$.

In practical applications, *for suitable choices of normal variables* w, the resulting equilibrium manifold $\overline{\mathcal{E}}$ in terms of the normal variable—that may generally be obtained from $\overline{\mathcal{E}}^{\perp} = (\partial_{\mathsf{w}}\mathsf{v})^t\mathcal{E}^{\perp}$ or $\overline{\mathcal{E}} = (\partial_{\mathsf{w}}\mathsf{v})^{-1}\mathcal{E}$—is also a *fixed linear space*. Appropriate projections of w then often coincide with a normal variable w_{e} of the reduced system. This will be indeed the case in the special situation of chemical equilibrium flows investigated in the next section. It is also often found in practical applications that $N(\widetilde{\mathsf{B}}) \subset \mathcal{E}$ so that the source terms generally only concern the parabolic variables.

4.2. Application to multicomponent flows

In the particular situation of multicomponent flows, the slow manifold $\mathcal{E} = M^{-1}\mathcal{A} \times \mathbb{R}^d \times \mathbb{R}$ is of dimension $\mathsf{n}_{\mathrm{e}} = n_{\mathrm{a}} + d + 1$ and is spanned by the vectors $\mathsf{a}_l = (\widetilde{\mathfrak{a}}_l, \mathbf{0}, 0)^t$, $l \in \mathfrak{A}$, and $\mathsf{a}_{n_{\mathrm{a}}+l} = \mathsf{f}_{n+l}$, for $1 \le l \le d+1$. The vectors $\widetilde{\mathfrak{a}}_l = m_l M^{-1}\mathfrak{a}_l$, $l \in \mathfrak{A}$, are associated with atom or elemental decomposition per unit mass and f_i, $1 \le i \le \mathsf{n}$, denotes the basis vectors of $\mathbb{R}^{\mathsf{n}}$. On the other hand, the fast manifold $\mathcal{E}^{\perp}$ is spanned by the vectors $\mathsf{a}_{k+d+1} = (\widetilde{\mathfrak{v}}_k, \mathbf{0}, 0)^t$, $k \in \mathfrak{S}\backslash\mathfrak{A} = \{\, n_{\mathrm{a}} + 1, \dots, n \,\}$, where $\widetilde{\mathfrak{v}}_k$, $k \in \mathfrak{S}\backslash\mathfrak{A}$, denotes the mass weighted formation reaction vectors. The corresponding linear operators $\Pi_{\mathsf{e}} = [\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{n}_{\mathrm{e}}}]$ and $\Pi_{\mathsf{r}} = [\mathsf{a}_{\mathsf{n}_{\mathrm{e}}+1}, \dots, \mathsf{a}_{\mathsf{n}}]$ are introduced as in Section 4.1 as well as the matrices $\mathcal{J}_{\mathsf{e}}$ and $\mathcal{J}_{\mathsf{r}}$ so that $\mathbb{I}_{\mathsf{n}} = \Pi_{\mathsf{e}}\mathcal{J}_{\mathsf{e}}\Pi_{\mathsf{e}}^t + \Pi_{\mathsf{r}}\mathcal{J}_{\mathsf{r}}\Pi_{\mathsf{r}}^t$, $\tilde{\pi} = \Pi_{\mathsf{r}}\mathcal{J}_{\mathsf{r}}\Pi_{\mathsf{r}}^t$, $\mathcal{J}_{\mathsf{e}}\Pi_{\mathsf{e}}^t\Pi_{\mathsf{e}} = \mathbb{I}_{\mathsf{n}_{\mathrm{e}}}$, and $\mathcal{J}_{\mathsf{r}}\Pi_{\mathsf{r}}^t\Pi_{\mathsf{r}} = \mathbb{I}_{\mathsf{n}-\mathsf{n}_{\mathrm{e}}}$.

The slow conservative variable then reads [12, 18]

$$\mathsf{u}_{\mathsf{e}} = (\Pi_{\mathsf{e}})^t\mathsf{u} = \Big(\widetilde{\rho}_1, \dots, \widetilde{\rho}_{n_{\mathrm{a}}}, \rho_{\mathrm{e}}\boldsymbol{v}_{\mathrm{e}}, \mathcal{E}_{\mathrm{e}} + \tfrac{1}{2}\rho_{\mathrm{e}}|\boldsymbol{v}_{\mathrm{e}}|^2\Big)^t,$$

and its components are associated with atom mass densities $\widetilde{\rho}_l = \langle \widetilde{\mathfrak{a}}_l, \varrho_{\mathrm{e}} \rangle$, $l \in \mathfrak{A}$, momentum $\rho_{\mathrm{e}}\boldsymbol{v}_{\mathrm{e}}$, and total energy $\mathcal{E}_{\mathrm{e}} + \frac{1}{2}\rho_{\mathrm{e}}|\boldsymbol{v}_{\mathrm{e}}|^2$. From Lemma 2.7, for given internal energy $\mathcal{E}_{\mathrm{e}}$ and atom densities $\widetilde{\varrho}$, there exists a unique equilibrium state ϱ_{e} and equilibrium temperature T_{e}. Equivalently, from Proposition 2.5, the equilibrium state $\varrho_{\mathrm{e}}(\widetilde{\varrho}, T_{\mathrm{e}}) = \big(\rho_{\mathrm{e}1}(\widetilde{\varrho}, T), \dots, \rho_{\mathrm{e}n}(\widetilde{\varrho}, T_{\mathrm{e}})\big)^t$ is the unique equilibrium species densities for given atom densities $\widetilde{\varrho} = (\widetilde{\rho}_1, \dots, \varrho_{n_{\mathrm{a}}})^t$ and temperature T_{e}. The internal energy at chemical equilibrium per unit volume may be written $\mathcal{E}_{\mathrm{e}}(\widetilde{\varrho}, T_{\mathrm{e}}) = \mathcal{E}(\varrho_{\mathrm{e}}(\widetilde{\varrho}, T_{\mathrm{e}}), T_{\mathrm{e}})$ and similarly, the entropy at equilibrium $\mathcal{S}_{\mathrm{e}}$ per unit volume is given by $\mathcal{S}_{\mathrm{e}}(\widetilde{\varrho}, T_{\mathrm{e}}) = \mathcal{S}(\varrho_{\mathrm{e}}(\widetilde{\varrho}, T_{\mathrm{e}}), T_{\mathrm{e}})$ and thermodynamics at chemical equilibrium has been investigated in Lemma 2.6 and Lemma 2.7 of Section 2.6.

The system at equilibrium may be rewritten in quasilinear form

$$\partial_t \mathsf{u}_{\mathrm{e}} + \sum_{i\in\mathcal{D}} \mathsf{A}_i^{\mathrm{e}}(\mathsf{u}_{\mathrm{e}})\partial_i \mathsf{u}_{\mathrm{e}} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\mathsf{B}_{ij}^{\mathrm{e}}(\mathsf{u}_{\mathrm{e}})\partial_j \mathsf{u}_{\mathrm{e}}\big) = 0, \tag{4.5}$$

where $\mathsf{A}_i^{\mathrm{e}}(\mathsf{u}_{\mathrm{e}}) = \Pi_{\mathsf{e}}^t\mathsf{A}_i\big(\mathsf{u}_{\mathrm{eq}}(\mathsf{u}_{\mathrm{e}})\big)\,\partial_{\mathsf{u}_{\mathrm{e}}}\mathsf{u}_{\mathrm{eq}}$ and $\mathsf{B}_{ij}^{\mathrm{e}}(\mathsf{u}_{\mathrm{e}}) = \Pi_{\mathsf{e}}^t\mathsf{B}_{ij}\big(\mathsf{u}_{\mathrm{eq}}(\mathsf{u}_{\mathrm{e}})\big)\,\partial_{\mathsf{u}_{\mathrm{e}}}\mathsf{u}_{\mathrm{eq}}$ for $i, j \in \mathcal{D}$. The formulation is conservative with $\mathsf{A}_i^{\mathrm{e}} = \partial_{\mathsf{u}_{\mathrm{e}}}\mathsf{F}_i^{\mathrm{e}}$, $i \in \mathcal{D}$, where $\mathsf{F}_i^{\mathrm{e}}(\mathsf{u}_{\mathrm{e}}) = (\Pi_{\mathsf{e}})^t\mathsf{F}_i\big(\mathsf{u}_{\mathrm{eq}}(\mathsf{u}_{\mathrm{e}})\big)$, $i \in \mathcal{D}$, denote the convective fluxes at equilibrium. Letting $\mathsf{n}_{\mathrm{e}} = n_{\mathrm{a}} + d + 1$, then $\mathsf{u}_{\mathrm{e}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{e}}}$ and the natural variable $\mathsf{z}_{\mathrm{e}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{e}}}$ is given by $\mathsf{z}_{\rm e} = \big(\widetilde{\rho}_1, \ldots, \widetilde{\rho}_{n_{\rm a}}, \boldsymbol{v}_{\rm e}, T_{\rm e}\big)^t$. The map $\mathsf{z}_{\rm e} \mapsto \mathsf{u}_{\rm e}$ is a $C^\varkappa$ diffeomorphism from the open set $\mathcal{O}_{\mathsf{z}_{\rm e}} = (0,\infty)^{n_{\rm a}} \times \mathbb{R}^d \times (0,\infty)$ onto an open set $\mathcal{O}_{\mathsf{u}_{\rm e}}$ keeping in mind that the specific heat at chemical equilibrium $c_{\rm ev} = \partial_{T_{\rm e}} \mathcal{E}_{\rm e}(\widetilde{\varrho}, T_{\rm e})/\rho_{\rm e}$ is positive from Lemma 2.6. For multicomponent flows, the open set $\mathcal{O}_{\mathsf{u}_{\rm e}}$ can be fully characterized and shown to be convex under stronger assumptions associated with stable versions of the atomic elements and the associated heat of reactions at zero temperature [18]. The natural initial condition for $\mathsf{u}_{\rm e}$ also reads $\mathsf{u}_{\rm e0} = \Pi_{\rm e}^t \mathsf{u}_0$ and when u_0 is an equilibrium state then $\mathsf{u}_0 = \mathsf{u}_{\rm eq}(\mathsf{u}_{\rm e0})$.

The governing equations at chemical equilibrium (4.5) are notably used in practical applications like astronautics, chemical engineering and combustion [1, 42, 31, 52, 24] and may also be obtained from a kinetic framework [12]. These equations lead in particular to an important reduction of the number of dependent variables from $n + d + 1$ down to $n_{\rm a} + d + 1$ and also suppress the stiffness associated with chemical sources. These models are valid when the chemical characteristic times are shorter than the fluid mechanic characteristic times.

The natural entropic symmetrized form is now evaluated as well as a normal form for the system of partial differential equations (4.5) modeling fluids at chemical equilibrium [18, 28]. The symmetric form may also be related to that out of chemical equilibrium and in the following theorem $\mathsf{v}_{\rm eq}$ is the entropic symmetric variable associated with $\mathsf{u}_{\rm eq}$.

Theorem 4.3. *Assume that* (H_1)-(H_5) *hold. Then the function* $\sigma_{\rm e} = -\mathcal{S}_{\rm e}/R$ *is a mathematical entropy for the system* (4.5) *and the corresponding entropic variable is*

$$\mathsf{v}_{\rm e} = \frac{1}{RT_{\rm e}}\Big(\gamma_1 - \tfrac{1}{2}|\boldsymbol{v}_{\rm e}|^2 \ldots, \gamma_{n_{\rm a}} - \tfrac{1}{2}|\boldsymbol{v}_{\rm e}|^2, \boldsymbol{v}_{\rm e}, -1\Big)^t, \tag{4.6}$$

where γ_l, $l \in \mathfrak{A}$, *are uniquely defined by* $\big(g_{1\rm e}, \ldots, g_{n\rm e}\big)^t = \sum_{l\in\mathfrak{A}} \gamma_l\, \widetilde{\mathfrak{a}}_l$ *and furthermore* $\mathsf{v}_{\rm eq} = \Pi_{\rm e}\mathsf{v}_{\rm e}$ *and* $\mathsf{v}_{\rm e} = \mathcal{J}_{\rm e}\Pi_{\rm e}^t\mathsf{v}_{\rm eq}$. *The map* $\mathsf{u}_{\rm e} \mapsto \mathsf{v}_{\rm e}$ *is a* $C^{\varkappa-1}$ *diffeomorphism from* $\mathcal{O}_{\mathsf{u}_{\rm e}}$ *onto the open set* $\mathcal{O}_{\mathsf{v}_{\rm e}} = \{\mathsf{v} \in \mathbb{R}^{\mathsf{n}_{\rm e}};\ \mathsf{v}_{n_{\rm a}+d+1} < 0\}$. *The system written in terms of the entropic variable* $\mathsf{v}_{\rm e}$ *is of the symmetric form*

$$\widetilde{\mathsf{A}}_0^{\rm e}(\mathsf{v}_{\rm e})\partial_t\mathsf{v}_{\rm e} + \sum_{i\in\mathcal{D}} \widetilde{\mathsf{A}}_i^{\rm e}(\mathsf{v}_{\rm e})\partial_i\mathsf{v}_{\rm e} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\widetilde{\mathsf{B}}_{ij}^{\rm e}(\mathsf{v}_{\rm e})\partial_j\mathsf{v}_{\rm e}\big) = 0, \tag{4.7}$$

where $\widetilde{\mathsf{A}}_0^{\rm e} = \partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e} = \Pi_{\mathsf{e}}^t\widetilde{\mathsf{A}}_0\Pi_{\mathsf{e}}$, $\widetilde{\mathsf{A}}_i^{\rm e} = \mathsf{A}_i^{\rm e}\partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e} = \Pi_{\mathsf{e}}^t\widetilde{\mathsf{A}}_i\Pi_{\mathsf{e}}$, $\widetilde{\mathsf{B}}_{ij}^{\rm e} = \mathsf{B}_{ij}^{\rm e}\partial_{\mathsf{v}_{\rm e}}\mathsf{u}_{\rm e} = \Pi_{\mathsf{e}}^t\widetilde{\mathsf{B}}_{ij}\Pi_{\mathsf{e}}$, *have at least regularity* $\varkappa - 2$.

The nullspace invariance property for the symmetrized system (4.7) modeling fluids at chemical equilibrium has been established with $N(\widetilde{\mathsf{B}}^{\rm e}) = \mathbb{R}(\mathbb{I}_{\rm e}, \mathbf{0}, 0,)^t$ where $\mathbb{I}_{\rm e} \in \mathbb{R}^{n_{\rm a}}$ and $\mathbb{I}_{\rm e} = (1, \ldots, 1)^t$ [18]. A normal variable $\mathsf{w}_{\rm e}$ similar to (3.20) is selected for convenience

$$\mathsf{w}_{\rm e} = \Big(\rho_{\rm e}, \frac{\gamma_2 - \gamma_1}{RT_{\rm e}}, \ldots, \frac{\gamma_{n_{\rm a}} - \gamma_1}{RT_{\rm e}}, \boldsymbol{v}_{\rm e}, T_{\rm e}\Big)^t, \tag{4.8}$$

and the commutative diagram of changes of variables at equilibrium is presented in Figure 1 where double arrows denote diffeomorphisms and where linear maps are mentioned.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathsf{u}_{\mathrm{eq}} \arrow[r, shift left] \arrow[d, shift right, "\Pi_{\mathrm{e}}^t"'] & \mathsf{v}_{\mathrm{eq}} \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift right, "\mathcal{J}_{\mathrm{e}}\Pi_{\mathrm{e}}^t"'] & \mathsf{w}_{\mathrm{eq}} \arrow[l, shift left] \arrow[d, shift right, "\overline{\mathcal{J}}_{\mathrm{e}}\overline{\Pi}_{\mathrm{e}}^t"'] \\
\mathsf{u}_{\mathrm{e}} \arrow[u, shift right] \arrow[r, shift left] & \mathsf{v}_{\mathrm{e}} \arrow[u, shift right, "\Pi_{\mathrm{e}}"'] \arrow[l, shift left] \arrow[r, shift left] & \mathsf{w}_{\mathrm{e}} \arrow[u, shift right, "\overline{\Pi}_{\mathrm{e}}"'] \arrow[l, shift left]
\end{tikzcd}$$

Figure 1: Schematic of the changes of variables at equilibrium with linear maps indicated.

More specifically, for any change of variable $\mathsf{a} \mapsto \mathsf{b}$, denoting by $\phi_{\mathsf{a}\mapsto\mathsf{b}}$ the diffeomorphism with $\mathsf{b} = \phi_{\mathsf{a}\mapsto\mathsf{b}}(\mathsf{a})$ from the corresponding open sets $\mathcal{O}_{\mathsf{a}} \mapsto \mathcal{O}_{\mathsf{b}}$, then $\mathsf{v}_{\mathrm{e}} = \phi_{\mathsf{u}_{\mathrm{e}}\mapsto\mathsf{v}_{\mathrm{e}}}(\mathsf{u}_{\mathrm{e}})$, $\mathsf{w}_{\mathrm{e}} = \phi_{\mathsf{v}_{\mathrm{e}}\mapsto\mathsf{w}_{\mathrm{e}}}(\mathsf{v}_{\mathrm{e}})$, $\mathsf{u}_{\mathrm{eq}} = \phi_{\mathsf{u}_{\mathrm{e}}\mapsto\mathsf{u}_{\mathrm{eq}}}(\mathsf{u}_{\mathrm{e}})$, $\mathsf{v}_{\mathrm{eq}} = \phi_{\mathsf{u}\mapsto\mathsf{v}}(\mathsf{u}_{\mathrm{eq}})$, and $\mathsf{w}_{\mathrm{eq}} = \phi_{\mathsf{v}\mapsto\mathsf{w}}(\mathsf{v}_{\mathrm{eq}})$. It is then remarkable that many maps are *linear* with $\mathsf{u}_{\mathrm{e}} = \Pi_{\mathrm{e}}^t\mathsf{u}_{\mathrm{eq}}$, $\mathsf{v}_{\mathrm{e}} = \mathcal{J}_{\mathrm{e}}\Pi_{\mathrm{e}}^t\mathsf{v}_{\mathrm{eq}}$, $\mathsf{w}_{\mathrm{e}} = \overline{\mathcal{J}}_{\mathrm{e}}\overline{\Pi}_{\mathrm{e}}^t\mathsf{w}_{\mathrm{eq}}$, $\mathsf{v}_{\mathrm{eq}} = \Pi_{\mathrm{e}}\mathsf{v}_{\mathrm{e}}$, and $\mathsf{w}_{\mathrm{eq}} = \overline{\Pi}_{\mathrm{e}}\mathsf{w}_{\mathrm{e}}$. The corresponding equations in normal form are investigated in the following theorem [18].

Theorem 4.4. *Assume that* (H_1)-(H_5) *hold. Then the map* $\mathsf{v}_{\mathrm{e}} \mapsto \mathsf{w}_{\mathrm{e}}$ *is a* $C^{\varkappa-1}$ *diffeomorphism from* $\mathcal{O}_{\mathsf{v}_{\mathrm{e}}}$ *onto the open set* $\mathcal{O}_{\mathsf{w}_{\mathrm{e}}} = (0,\infty)\times\mathbb{R}^{n_{\mathrm{a}}-1}\times\mathbb{R}^d\times(0,\infty)$. *The system in the* w_{e} *variable is of the normal form*

$$\overline{\mathsf{A}}_0^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}})\partial_t\mathsf{w}_{\mathrm{e}} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}})\partial_i\mathsf{w}_{\mathrm{e}} - \sum_{i,j\in\mathcal{D}}\partial_i\big(\overline{\mathsf{B}}_{ij}^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}})\partial_j\mathsf{w}_{\mathrm{e}}\big) = \overline{\mathsf{Q}}_{\mathrm{e}}(\mathsf{w}_{\mathrm{e}},\partial_x\mathsf{w}_{\mathrm{e}}), \qquad (4.9)$$

where $\overline{\mathsf{A}}_0^{\mathrm{e}} = (\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}})^t\,\widetilde{\mathsf{A}}_0^{\mathrm{e}}\,\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}}$, $\overline{\mathsf{A}}_i^{\mathrm{e}}(\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}})^t\,\widetilde{\mathsf{A}}_i^{\mathrm{e}}\,\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{e}} = (\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}})^t\,\widetilde{\mathsf{B}}_{ij}^{\mathrm{e}}\,\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}}$, *have at least regularity* $\varkappa-2$ *and the term* $\overline{\mathsf{Q}}_{\mathrm{e}}$ *is quadratic in the gradients* $\overline{\mathsf{Q}}_{\mathrm{e}} = -\sum_{i,j\in\mathcal{D}}\partial_i(\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}})^t(\partial_{\mathsf{v}_{\mathrm{e}}}\mathsf{w}_{\mathrm{e}})^t\,\overline{\mathsf{B}}_{ij}^{\mathrm{e}}\ \partial_j\mathsf{w}_{\mathrm{e}}$. *The quadratic residual may be written* $\overline{\mathsf{Q}}_{\mathrm{e}} = \sum_{i,j\in\mathcal{D}}\overline{\mathrm{M}}_{ij}^{\mathrm{e}}\partial_i\mathsf{w}_{\mathrm{e}}\partial_j\mathsf{w}_{\mathrm{e}}$ *with coefficients* $\overline{\mathrm{M}}_{ij}^{\mathrm{e}}$ *that have at least regularity* $\varkappa-3$ *and only involves the parabolic components* $\overline{\mathsf{Q}}_{\mathrm{e}} = \big(0,(\overline{\mathsf{Q}}_{\mathrm{e}})_{\mathrm{II}}\big)^t$. *The system coefficients are also given by* $\overline{\mathsf{A}}_0^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}}) = \overline{\Pi}_{\mathsf{e}}^t\overline{\mathsf{A}}_0(\overline{\Pi}_{\mathsf{e}}\mathsf{w}_{\mathrm{e}})\overline{\Pi}_{\mathsf{e}}$, $\overline{\mathsf{A}}_i^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}}) = \overline{\Pi}_{\mathsf{e}}^t\overline{\mathsf{A}}_i(\overline{\Pi}_{\mathsf{e}}\mathsf{w}_{\mathrm{e}})\overline{\Pi}_{\mathsf{e}}$, $i\in\mathcal{D}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}}) = \overline{\Pi}_{\mathsf{e}}^t\overline{\mathsf{B}}_{ij}(\overline{\Pi}_{\mathsf{e}}\mathsf{w}_{\mathrm{e}})\overline{\Pi}_{\mathsf{e}}$, $i,j\in\mathcal{D}$, *and* $\overline{\mathsf{Q}}_{\mathrm{e}}(\mathsf{w}_{\mathrm{e}},\partial_x\mathsf{w}_{\mathrm{e}}) = \Pi_{\mathsf{e}}^t\overline{\mathsf{Q}}(\mathsf{w}_{\mathrm{eq}},\partial_x\mathsf{w}_{\mathrm{eq}})$.

Proof. The equations at equilibrium in normal form are obtained through the usual change of variable $\mathsf{v}_{\mathrm{e}} = \mathsf{v}_{\mathrm{e}}(\mathsf{w}_{\mathrm{e}})$ in (4.7) after multiplication on the left by $(\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}})^t$. The relation $\overline{\mathsf{A}}_0^{\mathrm{e}} = \overline{\Pi}_{\mathsf{e}}^t\overline{\mathsf{A}}_0(\overline{\Pi}_{\mathsf{e}}\mathsf{w}_{\mathrm{e}})\overline{\Pi}_{\mathsf{e}}$ is next derived by using the commutative diagram of Figure 1. From this diagram, it is indeed obtained that $\mathsf{v}_{\mathrm{e}}(\mathsf{w}_{\mathrm{e}}) = \mathcal{J}_{\mathrm{e}}\Pi_{\mathrm{e}}^t\mathsf{v}(\overline{\Pi}_{\mathsf{e}}\mathsf{w}_{\mathrm{e}})$ and by differentiation one gets that $\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}} = \mathcal{J}_{\mathrm{e}}\Pi_{\mathrm{e}}^t\partial_{\mathsf{w}}\mathsf{v}\overline{\Pi}_{\mathsf{e}}$. Since $\mathsf{v}(\mathsf{w}_{\mathrm{e}})$ stays on the manifold $\mathfrak{E}$ we also have $\partial_{\mathsf{w}}\mathsf{v}\overline{\Pi}_{\mathrm{e}} = \Pi_{\mathrm{e}}\mathcal{J}_{\mathrm{e}}\Pi_{\mathrm{e}}^t\partial_{\mathsf{w}}\mathsf{v}\overline{\Pi}_{\mathrm{e}}$. Further using the expression of $\widetilde{\mathsf{A}}_0^{\mathrm{e}}$, a direct evaluation of $\overline{\mathsf{A}}_0^{\mathrm{e}} = (\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}})^t\widetilde{\mathsf{A}}_0^{\mathrm{e}}\partial_{\mathsf{w}_{\mathrm{e}}}\mathsf{v}_{\mathrm{e}}$ then yields that $\overline{\mathsf{A}}_0^{\mathrm{e}} = \overline{\Pi}_{\mathsf{e}}^t\overline{\mathsf{A}}_0(\overline{\Pi}_{\mathsf{e}}\mathsf{w}_{\mathrm{e}})\overline{\Pi}_{\mathsf{e}}$ and the relations expressing $\overline{\mathsf{A}}_i^{\mathrm{e}}$, $\overline{\mathsf{B}}_{ij}^{\mathrm{e}}$, and $\overline{\mathsf{Q}}_{\mathrm{e}}$ are obtained in a similar way. □

From the general expression of dissipation matrices at chemical equilibrium $\widetilde{\mathsf{B}}^{\mathrm{e}}_{ij}$ and $\overline{\mathsf{B}}^{\mathrm{e}}_{ij}$, the definition of atom vectors, and the dissipation matrices out of equilibrium associated with diffusive processes (3.14) and (3.25), we deduce that cross effects between the atom concentrations and energy also arise at chemical equilibrium. It is now established that the system in normal form at chemical equilibrium is strictly dissipative and that it is possible to find a compensating matrix in the form $K^{\mathrm{e}}(\boldsymbol{\xi}) = \sum_{j\in\mathcal{D}} K^{\mathrm{e}}_j \xi_j$ with $\boldsymbol{\xi} \in \Sigma^{d-1}$.

Proposition 4.5. *Let $\mathsf{w}_{\mathrm{e}} \in \mathcal{O}_{\mathsf{w}_{\mathrm{e}}}$ be fixed and $K^{\mathrm{e}}(\boldsymbol{\xi})$ be the matrix defined for $\boldsymbol{\xi} \in \Sigma^{d-1}$ by*

$$K^{\mathrm{e}}(\boldsymbol{\xi}) = \sum_{j\in\mathcal{D}} \xi_j K^{\mathrm{e}}_j = \delta \begin{bmatrix} 0 & \boldsymbol{\xi}^t & 0 \\ -\boldsymbol{\xi} & 0_{d,d} & 0_{d,1} \\ 0 & 0_{1,d} & 0 \end{bmatrix} \left(\overline{\mathsf{A}}^{\mathrm{e}}_0(\mathsf{w}_{\mathrm{e}})\right)^{-1}.$$

Then for sufficiently small positive δ, the map $\boldsymbol{\xi} \mapsto K^{\mathrm{e}}(\boldsymbol{\xi})$ is a compensating function for the system of partial differential equations in normal form at equilibrium, that is, the product $K^{\mathrm{e}}(\boldsymbol{\xi})\overline{\mathsf{A}}^{\mathrm{e}}_0(\mathsf{w}_{\mathrm{e}})$ is skew-symmetric, $K^{\mathrm{e}}(-\boldsymbol{\xi}) = -K^{\mathrm{e}}(\boldsymbol{\xi})$, and the matrix $K^{\mathrm{e}}(\boldsymbol{\xi})\overline{\mathsf{A}}^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}},\boldsymbol{\xi})+\overline{\mathsf{B}}^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}},\boldsymbol{\xi})$ is positive definite for $\boldsymbol{\xi} \in \Sigma^{d-1}$ where $\overline{\mathsf{B}}^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}},\boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}^{\mathrm{e}}_{ij}(\mathsf{w}_{\mathrm{e}})\xi_i\xi_j$.

Proof. The proof is similar to that of the nonequilibrium case. □

Denoting by w a solution of the out of equilibrium system (3.21) and by w_{e} a solution of the limiting equilibrium system (4.9), one of the goal of this paper is to establish that w is close to $\overline{\Pi}_{\mathsf{e}}\mathsf{w}_{\mathrm{e}}$ when ϵ is small on any fixed time interval of arbitrary size. Equivalently, according to the diagram of Figure 1, one may establish that the equilibrium projection $\overline{\mathcal{J}}_{\mathsf{e}}\overline{\Pi}^t_{\mathsf{e}}\mathsf{w}$ of the normal variable w out of chemical equilibrium is close to the normal variable w_{e} at chemical equilibrium. Such an asymptotic analysis first requires to establish a global existence theorem out of equilibrium uniformly with respect to the relaxation parameter ϵ and this is the object of the next section.

5. Hyperbolic-parabolic systems with stiff source terms

Existence theorems for symmetric hyperbolic-parabolic systems of partial differential equations are of fundamental importance in mathematical physics [30, 16, 48, 34, 35, 40, 36, 9, 21, 22, 23, 18, 10, 58, 13, 56, 37, 20, 4, 19, 29, 44, 45, 61]. Global existence theorems for hyperbolic-parabolic systems of partial differential equations in normal form with stiff sources are investigated in this section.

5.1. Structural assumptions

An abstract hyperbolic-parabolic system (3.1) with a mathematical entropy as in Definition 3.1, and such that the symmetrized form (3.2) satisfies the nullspace invariance property (N), is considered, as in Section 3. The system is written using a normal variable $\mathsf{w} \in \mathbb{R}^{\mathsf{n}}$ as in Theorem 3.5 and assuming a quasilinear stiff source term

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t \mathsf{w} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w})\partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_j \mathsf{w}\big) + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\mathsf{w} = \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}(\mathsf{w})\partial_i \mathsf{w}\partial_j \mathsf{w}, \tag{5.1}$$

where $\mathsf{w} = (\mathsf{w}_{\mathrm{I}}, \mathsf{w}_{\mathrm{II}})^t \in \mathcal{O}_{\mathsf{w}} \subset \mathbb{R}^{\mathsf{n}}$, $\mathsf{w}_{\mathrm{I}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{I}}}$ denotes the hyperbolic components, $\mathsf{w}_{\mathrm{II}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{II}}}$ the parabolic components, and $\epsilon \in (0, 1]$ is the relaxation parameter. The matrix $\overline{\mathsf{A}}_0(\mathsf{w}) \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$ is block diagonal and symmetric positive definite, the matrices $\overline{\mathsf{A}}_i(\mathsf{w}) \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$, $i \in \mathcal{D}$, are symmetric, the matrices $\overline{\mathsf{B}}_{ij}(\mathsf{w}) \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$, have non zero component only in the $\overline{\mathsf{B}}_{ij}^{\mathrm{II,II}} \in \mathbb{R}^{\mathsf{n}_{\mathrm{II}},\mathsf{n}_{\mathrm{II}}}$ lower block, $\overline{\mathsf{B}}_{ij}^t(\mathsf{w}) = \overline{\mathsf{B}}_{ji}(\mathsf{w})$ for $i, j \in \mathcal{D}$, are such that $\overline{\mathsf{B}}^{\mathrm{II,II}}(\mathsf{w}, \boldsymbol{\xi}) = \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\mathrm{II,II}}(\mathsf{w})\xi_i\xi_j$ is positive definite for $\mathsf{w} \in \mathcal{O}_{\mathsf{w}}$ and $\boldsymbol{\xi} \in \Sigma^{d-1}$. The matrix $\overline{\mathsf{L}}(\mathsf{w}) \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$ is positive semi-definite with a fixed nullspace $N\big(\overline{\mathsf{L}}(\mathsf{w})\big) = \overline{\mathcal{E}} \subset \mathbb{R}^{\mathsf{n}}$. The quadratic residual $\overline{\mathsf{Q}} = \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}(\mathsf{w})\partial_i \mathsf{w}\partial_j \mathsf{w} \in \mathbb{R}^{\mathsf{n}}$ may also be written $\overline{\mathsf{Q}} = -\sum_{i,j\in\mathcal{D}} \partial_i(\partial_{\mathsf{w}}\mathsf{v})^t\,(\partial_{\mathsf{v}}\mathsf{w})^t\,\overline{\mathsf{B}}_{ij}\partial_j \mathsf{w}$ and the third order tensors $\overline{\mathrm{M}}_{ij} \in \mathbb{R}^{\mathsf{n},\mathsf{n},\mathsf{n}}$ only involve parabolic components $\overline{\mathrm{M}}_{ij}(\mathsf{w})\partial_i \mathsf{w}\partial_j \mathsf{w} = \big(0, \overline{\mathrm{M}}_{ij}^{\mathrm{II,II,II}}(\mathsf{w})\partial_i \mathsf{w}_{\mathrm{II}}\partial_j \mathsf{w}_{\mathrm{II}}\big)^t$. According to Theorem 3.5 and Definition 3.4 the coefficients of (5.1) have at least regularity $\varkappa - 2$ and the coefficients $\overline{\mathrm{M}}_{ij}$, $i, j \in \mathcal{D}$, of $\overline{\mathsf{Q}}$ have at least regularity $\varkappa - 3$. The regularity class $\varkappa$ is assumed to be as large as required by the theorems established in Section 5 and more specifically such that $\varkappa - 3 \geq l + 1 \geq l_0 + 2$ where $l_0 = [d/2] + 1$.

Denoting by u and v the conservative and entropic variables associated with w, using the diffeomorphisms $\mathsf{w} \mapsto \mathsf{u}$ and $\mathsf{w} \mapsto \mathsf{v}$, then u satisfies (3.1) and v satisfies (3.2) that may be used for convenience along with (5.1). For such systems, a governing equation (3.9) for the fast variable $\pi\mathsf{w} \in \mathbb{R}^{\mathsf{n}}$ also holds as established in Proposition 3.9 where π denotes the orthogonal projector on the fast manifold $\overline{\mathcal{E}}^{\perp}$. Only the situation of well prepared initial data is considered in this work, that is, the initial condition w_0 is assumed to be close to the equilibrium manifold $\overline{\mathcal{E}}$ in such a way that $\pi\mathsf{w}_0$ is small.

Let $\mathsf{u}^\star \in \mathbb{R}^{\mathsf{n}}$, $\mathsf{v}^\star \in \mathbb{R}^{\mathsf{n}}$ and $\mathsf{w}^\star \in \mathbb{R}^{\mathsf{n}}$ denote a constant equilibrium state in the u, v and w variables respectively, so that $\mathsf{v}^\star \in \mathcal{O}_{\mathsf{v}} \cap \mathcal{E}$, $\mathsf{w}^\star \in \mathcal{O}_{\mathsf{w}} \cap \overline{\mathcal{E}}$ and $\pi\,\mathsf{w}^\star = 0$. The system of partial differential equations in normal form is assumed to be strictly parabolic dissipative at $\mathsf{w}^\star$ with a compensating matrix $K \in \mathbb{R}^{\mathsf{n},\mathsf{n}}$, compatible with the fast manifold, that is such that $K\pi = 0$. For the purpose of simplicity, the compensating matrix is assumed to be in the form $K = \sum_{j\in\mathcal{D}} K_j\xi_j$ where K_j is a constant matrix, as established in Proposition 3.13 in the situation of nonequilibrium fluids. The norm in the Sobolev space $H^l = H^l(\mathbb{R}^d)$ is denoted by $|\bullet|_l$ and otherwise by $|\bullet|_A$ in the functional space A. Similarly, $|\bullet|$ denotes the Euclidean norm in $\mathbb{R}$ or $\mathbb{R}^{\mathsf{n}}$, the Frobenius norm in

$\mathbb{R}^{\mathsf{n},\mathsf{n}}$, and the Euclidean distance between any $\mathfrak{w} \in \mathcal{O}_{\mathsf{w}}$ and the boundary $\partial\mathcal{O}_{\mathsf{w}}$ is denoted by $\mathrm{dist}(\mathfrak{w}, \partial\mathcal{O}_{\mathsf{w}})$.

If $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ is a multiindex, ∂^α denotes the differential operator $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $|\alpha|$ the order $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The square of k^{th} derivatives of scalar functions ϕ, like T, ρ, or v_i, $1 \le i \le d$, is defined by

$$|\partial^k \phi|^2 = \sum_{|\alpha|=k} \frac{|\alpha|!}{\alpha!} (\partial^\alpha \phi)^2 = \sum_{1 \le i_1, \ldots, i_k \le d} (\partial_{i_1} \cdots \partial_{i_k} \phi)^2, \tag{5.2}$$

where $|\alpha|!/\alpha!$ are the multinomial coefficients and similarly for a vector function like v the norm $|\partial^k v|^2$ stands for $|\partial^k v|^2 = \sum_{1 \le i \le d} |\partial^k v_i|^2$. Finally, for any map $\phi : [0, \bar{\tau}] \times \mathbb{R}^d \mapsto \mathbb{R}^{\mathsf{n}}$, where $\bar{\tau} > 0$ is positive, $\phi(\tau)$ denotes the partial map $x \mapsto \phi(\tau, x)$ for $\tau \in [0, \bar{\tau}]$.

5.2. Local existence

The local existence theorem previously established for symmetrized systems with small second order terms and stiff sources [29] is extended in this paper to the situation where the matrix $\overline{\mathsf{A}}_0(\mathsf{w})$ and the projector onto the fast manifold π *do not necessarily* commute. In other words, the slow manifold $\overline{\mathcal{E}}$ or the fast manifold $\overline{\mathcal{E}}^\perp$ are not assumed to be left invariant by $\overline{\mathsf{A}}_0$. The essential difference with the 'commutative case' is the derivation of new a priori estimates for linearized equations. These new estimates, established in Appendix A, are considerably more intricate to establish than in the 'commutative case' where $\pi\overline{\mathsf{A}}_0 = \overline{\mathsf{A}}_0\pi$ leads to the commutation of π with $\overline{\mathsf{A}}_0[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\overline{\mathsf{L}}]$ and thus to the relation $\overline{\mathsf{A}}_0[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\overline{\mathsf{L}}] = \pi\overline{\mathsf{A}}_0[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\overline{\mathsf{L}}]$ which drastically simplifies the analysis. The rest of the proof is essentially similar and is only sketched [29].

Theorem 5.1. *Let $d \ge 1$, $l \ge l_0 + 1$, $l_0 = [d/2] + 1$, be integers and let $b > 0$. Let $\mathcal{O}_0$ be such that $\overline{\mathcal{O}_0} \subset \mathcal{O}_{\mathsf{w}}$, a_1 such that $0 < a_1 < \mathrm{dist}(\overline{\mathcal{O}_0}, \partial\mathcal{O}_{\mathsf{w}})$, and $\mathcal{O}_1 = \{\, \mathsf{w} \in \mathcal{O}_{\mathsf{w}} ; \mathrm{dist}(\mathsf{w}, \overline{\mathcal{O}_0}) < a_1 \,\}$. There exists $\bar{\tau} > 0$ depending on $\mathcal{O}_1$ and b, and independent $\epsilon \in (0, 1]$, such that for any w_0 with $\mathsf{w}_0 \in \overline{\mathcal{O}_0}$, $\mathsf{w}_0 - \mathsf{w}^\star \in H^l$, and*

$$|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon} |\pi \mathsf{w}_0|_{l-1}^2 < b^2, \tag{5.3}$$

there exists a unique local solution w *to the system*

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t \mathsf{w} + \sum_{i \in \mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w})\partial_i \mathsf{w} - \sum_{i,j \in \mathcal{D}} \partial_i \big(\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_j \mathsf{w}\big) + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\,\mathsf{w} = \sum_{i,j \in \mathcal{D}} \overline{\mathrm{M}}_{ij}(\mathsf{w})\, \partial_i \mathsf{w}\, \partial_j \mathsf{w}, \tag{5.4}$$

with initial condition $\mathsf{w}(0, x) = \mathsf{w}_0(x)$, such that $\mathsf{w}(t, x) \in \mathcal{O}_1$ for $(t, x) \in [0, \bar{\tau}] \times \mathbb{R}^d$, and

$$\mathsf{w}_{\mathrm{I}} - \mathsf{w}_{\mathrm{I}}^\star \in C^0\big([0, \bar{\tau}], H^l\big) \cap C^1\big([0, \bar{\tau}], H^{l-1}\big),$$
$$\mathsf{w}_{\mathrm{II}} - \mathsf{w}_{\mathrm{II}}^\star \in C^0\big([0, \bar{\tau}], H^l\big) \cap C^1\big([0, \bar{\tau}], H^{l-2}\big) \cap L^2\big((0, \bar{\tau}), H^{l+1}\big).$$

Moreover, there exists $\mathsf{c}_{\rm loc} > 0$ *only depending on* $\mathcal{O}_1$ *and* b *such that*

$$\sup_{0\leq\tau\leq\bar\tau}\Big(|\mathsf{w}(\tau)-\mathsf{w}^\star|_l^2+\frac{1}{\epsilon}|\pi\mathsf{w}(\tau)|_{l-1}^2\Big)+\int_0^{\bar\tau}|\mathsf{w}_{\rm II}(\tau)-\mathsf{w}^\star_{\rm II}|_{l+1}^2\,d\tau+\frac{1}{\epsilon}\int_0^{\bar\tau}|\pi\mathsf{w}(\tau)|_l^2\,d\tau$$
$$+\frac{1}{\epsilon^2}\int_0^{\bar\tau}|\pi\mathsf{w}(\tau)|_{l-1}^2\,d\tau+\int_0^{\bar\tau}|\partial_t\mathsf{w}(\tau)|_{l-1}^2\,d\tau\leq \mathsf{c}_{\rm loc}^2\Big(|\mathsf{w}_0-\mathsf{w}^\star|_l^2+\frac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2\Big). \tag{5.5}$$

Proof. Solutions to the nonlinear system (5.4) are fixed points $\widetilde{\mathsf{w}}=\mathsf{w}$ of the linearized equations [34]

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}}\;+\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\widetilde{\mathsf{w}}-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}}+\frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\widetilde{\mathsf{w}}=\mathsf{g}(\mathsf{w},\partial_x\mathsf{w}), \tag{5.6}$$

with $\mathsf{g}(\mathsf{w},\partial_x\mathsf{w})=\sum_{i,j\in\mathcal{D}}\partial_i\big(\overline{\mathsf{B}}_{ij}(\mathsf{w})\big)\,\partial_j\mathsf{w}-\sum_{i,j\in\mathcal{D}}\partial_i(\partial_{\mathsf{w}}\mathsf{v})^t\,(\partial_{\mathsf{v}}\mathsf{w})^t\overline{\mathsf{B}}_{ij}\,\partial_j\mathsf{w}$. Fixed points are investigated in the space $\mathsf{w}\in\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1,M,M_1\big)$ that is defined by $\mathsf{w}_{\rm I}-\mathsf{w}^\star_{\rm I}\in C^0\big([0,\bar\tau],H^l\big)$, $\partial_t\mathsf{w}_{\rm I}\in C^0\big([0,\bar\tau],H^{l-1}\big)$, $\mathsf{w}_{\rm II}-\mathsf{w}^\star_{\rm II}\in C^0\big([0,\bar\tau],H^l\big)\cap L^2\big((0,\bar\tau),H^{l+1}\big)$, $\partial_t\mathsf{w}_{\rm II}\in C^0\big([0,\bar\tau],H^{l-2}\big)\cap L^2\big((0,\bar\tau),H^{l-1}\big)$, $\mathsf{w}(t,x)\in\mathcal{O}_1$, and

$$\sup_{0\leq\tau\leq\bar\tau}|\mathsf{w}(\tau)-\mathsf{w}^\star|_l^2+\int_0^{\bar\tau}|\mathsf{w}_{\rm II}(\tau)-\mathsf{w}^\star_{\rm II}|_{l+1}^2\,d\tau+\frac{1}{\epsilon}\int_0^{\bar\tau}|\pi\mathsf{w}(\tau)|_l^2\,d\tau\leq M^2,$$

$$\frac{1}{\epsilon}\sup_{0\leq\tau\leq\bar\tau}|\pi\mathsf{w}(\tau)|_{l-1}^2+\frac{1}{\epsilon^2}\int_0^{\bar\tau}|\pi\mathsf{w}(\tau)|_{l-1}^2\,d\tau+\int_0^{\bar\tau}|\partial_t\mathsf{w}(\tau)|_{l-1}^2\,d\tau\leq M_1^2.$$

For w in $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1,M,M_1\big)$, the solution $\widetilde{\mathsf{w}}$ of the linearized equations (5.6) is estimated with Theorem Appendix A.1 of Appendix A involving the constants $\mathsf{c}_1(\mathcal{O}_1)$ and $\mathsf{c}_2(\mathcal{O}_1,M)$. Letting $M_b=2\mathsf{c}_1(\mathcal{O}_1)b$, $M_{1b}=\mathsf{c}_2(\mathcal{O}_1,M_b)2\mathsf{c}_1(\mathcal{O}_1)b$, assuming that $\bar\tau\leq1$ is small enough such that $\exp\big(\mathsf{c}_2(\mathcal{O}_1,M_b)(\bar\tau+M_{1b}\sqrt{\bar\tau})\big)\leq 2$, $\mathsf{c}_2^2(\mathcal{O}_1,M_b)\bar\tau\big(2\mathsf{c}_1(\mathcal{O}_1)\big)^2\leq1$, and $\mathsf{c}_0M_{1b}\sqrt{\bar\tau}<a_1$ where c_0 is such that $\|\phi\|_{L^\infty}\leq\mathsf{c}_0|\phi|_{l-1}$, then $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1,M_b,M_{1b}\big)$ is a stable subspace. For any $\mathsf{w}\in\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1,M_b,M_{1b}\big)$, any w_0 such that $\mathsf{w}_0-\mathsf{w}^\star\in H^l$, $\mathsf{w}_0\in\overline{\mathcal{O}}_0$, and $|\mathsf{w}_0-\mathsf{w}^\star|_l^2+|\pi\mathsf{w}_0|_{l-1}^2/\epsilon<b^2$, and any $\epsilon\in(0,1]$, the solution $\widetilde{\mathsf{w}}$ to the linearized equations (5.6) with initial condition w_0 stays in the same space $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1,M_b,M_{1b}\big)$.

The sequence of successive approximations $\{\mathsf{w}^k\}_{k\geq0}$ starting at $\mathsf{w}^0=\mathsf{w}^\star$ is defined with $\mathsf{w}^{k+1}=\widetilde{\mathsf{w}}^k$, i.e., w^{k+1} is obtained as the solution $\widetilde{\mathsf{w}}=\mathsf{w}^{k+1}$ of linearized equations with $\mathsf{w}=\mathsf{w}^k$ and with the same initial condition w_0. Let $\delta^k\mathsf{w}$ denotes the difference $\delta^k\mathsf{w}=\mathsf{w}^{k+1}-\mathsf{w}^k$ for $k\geq0$. For a suitable τ_ϵ small enough, the sequence of approximations $\{\mathsf{w}^k\}_{k\geq0}$ is successively estimated over each interval $[j\tau_\epsilon,(j+1)\tau_\epsilon]\subset[0,\bar\tau]$ by induction on j. Uniqueness of the solution is also established over $[0,\tau_\epsilon]$ and gradually over each $[j\tau_\epsilon,(j+1)\tau_\epsilon]$ included in $[0,\bar\tau]$.

Consider w and $\widehat{\mathsf{w}}$ in $\mathsf{X}^l_{\bar\tau}\big(\mathcal{O}_1,M_b,M_{1b}\big)$, and define $\delta\mathsf{w}=\mathsf{w}-\widehat{\mathsf{w}}$ and $\delta\widetilde{\mathsf{w}}=\widetilde{\mathsf{w}}-\widetilde{\widehat{\mathsf{w}}}$ where $\widetilde{\mathsf{w}}$ and $\widetilde{\widehat{\mathsf{w}}}$ are the solutions of the corresponding linearized equations with initial condition w_0. Forming the difference between the linearized equations, one may obtain that

$$\overline{\mathsf{A}}_0(\widehat{\mathsf{w}})\partial_t\delta\widetilde{\mathsf{w}} \;+\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\widehat{\mathsf{w}})\partial_i\delta\widetilde{\mathsf{w}}-\sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}(\widehat{\mathsf{w}})\partial_i\partial_j\delta\widetilde{\mathsf{w}}+\frac{1}{\epsilon}\overline{\mathsf{L}}(\widehat{\mathsf{w}})\delta\widetilde{\mathsf{w}}=\delta\mathsf{f}+\delta\mathsf{g},$$

where $\delta\mathsf{f}$ and $\delta\mathsf{g}$ are such that $\delta\mathsf{g}_{\mathrm{I}}=0$, $|\delta\mathsf{f}_{\mathrm{I}}|_{l-1}^2+|\delta\mathsf{f}_{\mathrm{II}}|_{l-1}^2\leq(\mathsf{c}_2/\epsilon)|\delta\mathsf{w}|_{l-1}^2$ and $|\delta\mathsf{g}_{\mathrm{II}}|_{l-2}^2\leq\mathsf{c}_2|\delta\mathsf{w}|_{l-1}^2$ where the $1/\epsilon$ factor arises from stiff sources. Defining $\mathfrak{N}_{l-1}^2(a,a',\delta\widetilde{\mathsf{w}})$ for any $[a,a']\subset[0,\bar{\tau}]$ by

$$\mathfrak{N}_{l-1}^2(a,a',\delta\widetilde{\mathsf{w}})=\sup_{a\leq\tau\leq a'}\Big(|\delta\widetilde{\mathsf{w}}(\tau)|_{l-1}^2+\frac{1}{\epsilon}\big|\pi\delta\widetilde{\mathsf{w}}(\tau)\big|_{l-2}^2\Big)+\int_a^{a'}|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}(\tau)|_l^2\,d\tau$$
$$+\frac{1}{\epsilon}\int_a^{a'}|\pi\delta\widetilde{\mathsf{w}}(\tau)|_{l-1}^2\,d\tau+\frac{1}{\epsilon^2}\int_a^{a'}\big|\pi\delta\widetilde{\mathsf{w}}(\tau)\big|_{l-2}^2\,d\tau+\int_a^{a'}\big|\partial_t\delta\widetilde{\mathsf{w}}(\tau)\big|_{l-2}^2\,d\tau,$$

it is obtained, using the difference equations with $\mathsf{w}=\mathsf{w}^{k+1}$ and $\widehat{\mathsf{w}}=\mathsf{w}^k$, and the linearized estimates over $[j\tau_\epsilon,(j+1)\tau_\epsilon]$, that

$$\mathfrak{N}_{l-1}^2\big(j\tau_\epsilon,(j+1)\tau_\epsilon,\delta^{k+1}\mathsf{w}\big)\leq\mathsf{c}_2\Big(|\delta^{k+1}\mathsf{w}(j\tau_\epsilon)|_{l-1}^2+\frac{1}{\epsilon}\big|\pi\delta^{k+1}(j\tau_\epsilon)\big|_{l-2}^2\Big)$$
$$+\frac{\tau_\epsilon\mathsf{c}_2'}{\epsilon}\mathfrak{N}_{l-1}^2\big(j\tau_\epsilon,(j+1)\tau_\epsilon,\delta^k\mathsf{w}\big),$$

where c_2' is independent of j and k keeping in mind that all iterates are in $\mathsf{X}_{\bar{\tau}}^l\big(\mathcal{O}_1,M_b,M_{1b}\big)$. Assuming that τ_ϵ is small enough such that $\mathsf{c}_2'\tau_\epsilon/\epsilon<1/4$ while $\bar{\tau}/\tau_\epsilon$ is an integer denoted by $N_\epsilon+1$, using

$$|\delta^{k+1}\mathsf{w}(j\tau_\epsilon)|_{l-1}^2+\frac{1}{\epsilon}\big|\pi\delta^{k+1}\mathsf{w}(j\tau_\epsilon)\big|_{l-2}^2\leq\mathfrak{N}_{l-1}^2\big((j-1)\tau_\epsilon,j\tau_\epsilon,\delta^{k+1}\mathsf{w}\big),$$

letting $\beta_k^j=\mathfrak{N}_{l-1}^2(j\tau_\epsilon,(j+1)\tau_\epsilon,\delta^k\mathsf{w})$, for $0\leq k$ and $0\leq j\leq N_\epsilon$, and $\beta_k^{-1}=0$ for $0\leq k$, yields $\beta_{k+1}^j\leq\mathsf{c}_2\beta_{k+1}^{j-1}+\frac14\beta_k^j$. Multiplying by 2^{k+1} and letting $\gamma_k^j=2^k\beta_k^j$ for $0\leq k$ and $0\leq j\leq N_\epsilon$, we get

$$\gamma_{k+1}^j\leq\mathsf{c}_2\gamma_{k+1}^{j-1}+\tfrac12\gamma_k^j,\qquad 0\leq k,\quad 0\leq j\leq N_\epsilon,$$

while for the first interval $\gamma_k^{-1}=0$ for $k\geq0$. It is then easily obtained that $\gamma_k^i\leq\Gamma^i$ where the majorizing bounds Γ^i are defined by $\Gamma^0=\mathfrak{N}_{l-1}^2(0,\tau_\epsilon,\delta^0\mathsf{w})$ and $\Gamma^i=2\mathsf{c}_2\Gamma^{i-1}+\mathfrak{N}_{l-1}^2(i\tau_\epsilon,(i+1)\tau_\epsilon,\delta^0\mathsf{w})$ for $1\leq i\leq N_\epsilon$. Moreover, the differential inequalities implies uniqueness over each interval by induction on j. Letting for short $\mathsf{c}_\epsilon=\sum_{0\leq j\leq N_\epsilon}\Gamma_j$, it has been established that

$$\mathfrak{N}_{l-1}^2(0,\bar{\tau},\delta^k\mathsf{w})\leq\frac{\mathsf{c}_\epsilon}{2^k},\qquad 0\leq k,$$

where c_ϵ depends on ϵ, $\mathcal{O}_1$, b, and the data but is independent of k. The sequence of successive approximation $\{\mathsf{w}^k\}_{k\geq0}$ is thus convergent over $[0,\bar{\tau}]$ for the norm $\mathfrak{N}_{l-1}(0,\bar{\tau},\mathsf{w}^k-\overline{\mathsf{w}})$ towards a fixed point $\overline{\mathsf{w}}$. Since the sequence $\{\mathsf{w}^k\}_{k\geq 0}$ is bounded in the space $\mathsf{X}^l_{\bar{\tau}}\big(\mathcal{O}_1,M_b,M_{1b}\big)$, it follows from standard functional analysis arguments using weakly convergent subsequences that $\overline{\mathsf{w}}$ is the unique solution of the system of partial differential equations with the required regularity. The estimates (5.5) are next established by using that the solution is a fixed point $\widetilde{\mathsf{w}}=\mathsf{w}$ [29]. □

The estimates obtained in Theorem 5.1 will be sufficient to investigate the convergence of solutions as $\epsilon\to 0$, that is, to investigate a one term Chapman-Enskog expansion. Only two term Chapman-Enskog expansions would require stronger time derivative estimates, that is, estimates of $\partial_t\pi\mathsf{w}/\epsilon$ assuming that $\partial_t\mathsf{w}_0$ is close to the equilibrium manifold [29].

5.3. New a priori estimates

A priori estimates of solutions over time intervals of arbitrary length are now investigated when w remains close to a constant equilibrium state $\mathsf{w}^\star$. It is assumed that for some time interval $\bar{\tau}$

$$\mathsf{w}_{\mathrm{I}}-\mathsf{w}_{\mathrm{I}}^\star\in C^0\big([0,\bar{\tau}],H^l\big)\cap C^1\big([0,\bar{\tau}],H^{l-1}\big),\qquad \partial_x\mathsf{w}_{\mathrm{I}}\in L^2\big((0,\bar{\tau}),H^{l-1}\big),\tag{5.7}$$

$$\mathsf{w}_{\mathrm{II}}-\mathsf{w}_{\mathrm{II}}^\star\in C^0\big([0,\bar{\tau}],H^l\big)\cap C^1\big([0,\bar{\tau}],H^{l-2}\big),\qquad \partial_x\mathsf{w}_{\mathrm{II}}\in L^2\big((0,\bar{\tau}),H^{l}\big).\tag{5.8}$$

Let $N_l(t)$ be defined by for $0\leq t\leq\bar{t}$ by

$$\begin{aligned}N_l^2(t)=&\sup_{0\leq\tau\leq t}\Big(|\mathsf{w}(\tau)-\mathsf{w}^\star|_l^2+\frac{1}{\epsilon}|\pi\mathsf{w}(\tau)|_{l-1}^2\Big)\\&+\int_0^t\big(|\partial_x\mathsf{w}_{\mathrm{I}}(\tau)|_{l-1}^2+|\partial_x\mathsf{w}_{\mathrm{II}}(\tau)|_l^2\big)\,d\tau\\&+\frac{1}{\epsilon}\int_0^t|\pi\mathsf{w}(\tau)|_l^2\,d\tau+\frac{1}{\epsilon^2}\int_0^t|\pi\mathsf{w}(\tau)|_{l-1}^2\,d\tau+\int_0^t|\partial_t\mathsf{w}(\tau)|_{l-1}^2\,d\tau.\end{aligned}\tag{5.9}$$

This norm notably differs from that used for non stiff sources by the presence of the terms involving $\pi\mathsf{w}/\sqrt{\epsilon}$, $\pi\mathsf{w}/\epsilon$, and $\partial_t\mathsf{w}$, and differ also from $\mathfrak{N}_l^2$ used over bounded time intervals.

Lemma 5.2. *Consider the modified entropy $\overline{\sigma}$ defined over $\mathcal{O}_{\mathsf{u}}$ by*

$$\overline{\sigma}(\mathsf{u})=\overline{\sigma}(\mathsf{u})-\overline{\sigma}(\mathsf{u}^\star)-\partial_{\mathsf{u}}\sigma(\mathsf{u}^\star)\,(\mathsf{u}-\mathsf{u}^\star),\qquad \mathsf{u}\in\mathcal{O}_{\mathsf{u}},\tag{5.10}$$

and the diffeomorphism $\mathsf{w}\mapsto\mathsf{u}$ from $\mathcal{O}_{\mathsf{w}}$ onto $\mathcal{O}_{\mathsf{u}}$. There exists a neighborhood $\mathcal{O}_{\overline{\sigma}}=\{\,\mathsf{w}\in\mathcal{O}_{\mathsf{w}},\ |\mathsf{w}-\mathsf{w}^\star|<r\,\}$ of $\mathsf{w}^\star$ and $0<\delta_{\overline{\sigma}}<1$ such that $\overline{\mathcal{O}_{\overline{\sigma}}}\subset\mathcal{O}_{\mathsf{w}}$ and

$$\delta_{\overline{\sigma}}|\mathsf{w}-\mathsf{w}^\star|^2\leq\overline{\sigma}\big(\mathsf{u}(\mathsf{w})\big)\leq(1/\delta_{\overline{\sigma}})|\mathsf{w}-\mathsf{w}^\star|^2,\qquad \mathsf{w}\in\mathcal{O}_{\overline{\sigma}},\tag{5.11}$$

$$\delta_{\overline{\sigma}}|\pi\mathsf{w}|^2\leq-\partial_{\mathsf{u}}\sigma\big(\mathsf{u}(\mathsf{w})\big)\,\Omega\big(\mathsf{u}(\mathsf{w})\big),\qquad \mathsf{w}\in\mathcal{O}_{\overline{\sigma}},\tag{5.12}$$

Proof. The inequality (5.11) is a consequence of the the strict convexity of σ over $\mathcal{O}_{\mathsf{u}}$ and the smoothness of $\mathsf{u} \mapsto \mathsf{w}$. On the other hand, since $\overline{\mathsf{L}}$ is positive definite over $\overline{\mathcal{E}}^{\perp}$, there exists δ such that $\delta|\pi\mathsf{w}| \leq |\overline{\mathsf{L}}\pi\mathsf{w}| = |\overline{\Omega}| \leq c|\Omega|$ keeping in mind that $\overline{\Omega} = (\partial_{\mathsf{w}}\mathsf{v})^t\Omega$. We may then use the stability inequality $\delta|\Omega|^2 \leq -\partial_{\mathsf{u}}\sigma\,\Omega$ established from (S_5)-(S_7) in Proposition 3.2 of reference [37] and also introduced in [22].

This stability inequality may be obtained as follows using for convenience the variable v obtained with the diffeomorphism $\mathsf{w} \mapsto \mathsf{v}$ and the corresponding source term $\widetilde{\Omega}$. Denoting by $\tilde{\pi}$ the orthogonal projector onto $\mathcal{E}^{\perp}$ we first note that for any v in the neighborhood of $\mathsf{v}^\star$ we have $\mathsf{v}^\star + (\mathbb{I}-\tilde{\pi})(\mathsf{v}-\mathsf{v}^\star) \in \mathcal{E}$ keeping in mind that $\mathsf{v}^\star \in \mathcal{E}$ from (S_5). Therefore $\widetilde{\Omega}\big(\mathsf{v}^\star + (\mathbb{I}-\tilde{\pi})(\mathsf{v}-\mathsf{v}^\star)\big) = 0$ from (S_5) so that $\widetilde{\Omega}(\mathsf{v}) = \widetilde{\Omega}(\mathsf{v}) - \widetilde{\Omega}\big(\mathsf{v}^\star + (\mathbb{I}-\tilde{\pi})(\mathsf{v}-\mathsf{v}^\star)\big)$ and

$$\widetilde{\Omega}(\mathsf{v}) = \int_0^1 \partial_{\mathsf{v}}\widetilde{\Omega}\big(\mathsf{v}^\star + (\mathbb{I}-\tilde{\pi})(\mathsf{v}-\mathsf{v}^\star) + \theta\tilde{\pi}(\mathsf{v}-\mathsf{v}^\star)\big)\,\mathrm{d}\theta\;\tilde{\pi}(\mathsf{v}-\mathsf{v}^\star).$$

This relation first implies that $\delta|\widetilde{\Omega}| \leq |\tilde{\pi}(\mathsf{v}-\mathsf{v}^\star)|$ in a neighborhood of $\mathsf{v}^\star$. In addition, further using $\mathsf{v}^\star \in \mathcal{E}$ and $\widetilde{\Omega} \in \mathcal{E}^{\perp}$, we may rewrite $-\partial_{\mathsf{u}}\sigma\,\Omega = -\big\langle \mathsf{v}-\mathsf{v}^\star, \widetilde{\Omega}(\mathsf{v})\big\rangle$ in the form

$$-\partial_{\mathsf{u}}\sigma\,\Omega = -\Big\langle \tilde{\pi}(\mathsf{v}-\mathsf{v}^\star), \int_0^1 \partial_{\mathsf{v}}\widetilde{\Omega}\big(\mathsf{v}^\star + (\mathbb{I}-\tilde{\pi})(\mathsf{v}-\mathsf{v}^\star) + \theta\tilde{\pi}(\mathsf{v}-\mathsf{v}^\star)\big)\,\mathrm{d}\theta\;\tilde{\pi}(\mathsf{v}-\mathsf{v}^\star)\Big\rangle.$$

On the other hand, the matrix $\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v}^\star)$ is symmetric with nullspace $\mathcal{E}$ from (S_6) and it is easily deduced using (S_7) with vectors in the form $\mathsf{v} = \mathsf{v}^\star + av$ that $a^2\langle v, \partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v}^\star)v\rangle + \mathcal{O}(a^3) \leq 0$ so that letting $a \to 0$ we obtain that $\partial_{\mathsf{v}}\widetilde{\Omega}(\mathsf{v}^\star)$ is negative semi-definite and thus negative definite over $\mathcal{E}^{\perp}$. Combining this property with the above expression of $-\partial_{\mathsf{u}}\sigma\,\Omega$ yields that for v in the neighborhood of $\mathsf{v}^\star$ we have $\delta\big|\tilde{\pi}(\mathsf{v}-\mathsf{v}^\star)\big|^2 \leq -\partial_{\mathsf{u}}\sigma\,\Omega$ and this completes the proof of the stability inequality. □

Lemma 5.3. *Let* w *with* $\mathsf{w}-\mathsf{w}^\star \in C^0\big([0,\bar{\tau}], H^l\big)$ *such that* (5.7)(5.8) *hold. There exists* $b_{\overline{\sigma}}$ *such that*

$$N_l(t) \leq b_{\overline{\sigma}} \implies \mathsf{w}(\tau,x) \in \mathcal{O}_{\overline{\sigma}}, \quad 0 \leq \tau \leq t, \quad x \in \mathbb{R}^d. \tag{5.13}$$

Proof. This property (5.13) is a consequence of the assumption $l \geq l_0$. □

Lemma 5.4. *Assume that* w *is a solution of* (5.1) *over* $[0,\bar{\tau}]\times\mathbb{R}^d$ *with regularity* (5.7)(5.8)*. Using the diffeomorphism* $\mathsf{w} \mapsto \mathsf{u}$ *from* $\mathcal{O}_{\mathsf{w}}$ *onto* $\mathcal{O}_{\mathsf{u}}$*, let us consider the corresponding modified entropy* $\overline{\sigma}\big(\mathsf{u}(\mathsf{w})\big)$*. Then* $\overline{\sigma}\big(\mathsf{u}(\mathsf{w})\big)$ *satisfy the following partial differential equation*

$$\begin{aligned}\partial_t\overline{\sigma} + \sum_{i\in\mathcal{D}} \partial_i\big(\mathsf{q}_i - \mathsf{q}_i^\star - (\partial_{\mathsf{u}}\sigma)^\star(\mathsf{F}_i - \mathsf{F}_i^\star)\big) - \sum_{i,j\in\mathcal{D}} \partial_i\big((\partial_{\mathsf{u}}\sigma - \partial_{\mathsf{u}}\sigma^\star)\mathsf{B}_{ij}\partial_j\mathsf{u}\big)&\\ + \sum_{i,j\in\mathcal{D}} \langle \overline{\mathsf{B}}_{ij}^{\mathrm{II,II}}\partial_j\mathsf{w}_{\mathrm{II}}, \partial_i\mathsf{w}_{\mathrm{II}}\rangle - \frac{1}{\epsilon}\partial_{\mathsf{u}}\sigma\,\Omega = 0,&\end{aligned} \tag{5.14}$$

where $\mathsf{q}_i = \mathsf{q}_i\big(\mathsf{u}(\mathsf{w}(t,x)\big)$ *and* $(\partial_{\mathsf{u}}\sigma)^\star = \partial_{\mathsf{u}}\sigma(\mathsf{u}^\star)$.

Proof. Since w is a solution of the system in normal form, the corresponding conservative variable $\mathsf{u} = \mathsf{u}(\mathsf{w})$ obtained from the diffeomorphism $\mathsf{w} \mapsto \mathsf{u}$ is a solution of the system of equations in conservative form (3.1) and similarly $\mathsf{v} = \mathsf{v}(\mathsf{w})$ obtained from $\mathsf{w} \mapsto \mathsf{v}$ is a solution of the system in entropic symmetrized form (3.2).

In order to derive (5.14) we then multiply the u conservation equation by $\partial_{\mathsf{u}}\sigma - \partial_{\mathsf{u}}\sigma^\star = \mathsf{v}^t - \mathsf{v}^{\star t}$ and evaluate each term of the resulting equation. For the time derivative term, we note that

$$(\partial_{\mathsf{u}}\sigma - \partial_{\mathsf{u}}\sigma^\star)\partial_t\mathsf{u} = \partial_t\big(\sigma - \sigma^\star - \partial_{\mathsf{u}}\sigma^\star(\mathsf{u} - \mathsf{u}^\star)\big),$$

whereas for convective terms, using the property of entropy fluxes $\partial_{\mathsf{u}}\sigma\, \mathsf{A}_i = \partial_{\mathsf{u}}\mathsf{q}_i$, $i \in \mathcal{D}$, we obtain that

$$\sum_{i\in\mathcal{D}}(\partial_{\mathsf{u}}\sigma - \partial_{\mathsf{u}}\sigma^\star)\mathsf{A}_i\partial_i\mathsf{u} = \sum_{i\in\mathcal{D}}\partial_i\big(\mathsf{q}_i - \mathsf{q}_i^\star - \partial_{\mathsf{u}}\sigma^\star(\mathsf{F}_i - \mathsf{F}_i^\star)\big).$$

The dissipative terms are next integrated by part

$$\sum_{i,j\in\mathcal{D}}(\partial_{\mathsf{u}}\sigma - \partial_{\mathsf{u}}\sigma^\star)\partial_i(\mathsf{B}_{ij}\partial_j\mathsf{u}) =$$
$$\sum_{i,j\in\mathcal{D}}\partial_i\big((\partial_{\mathsf{u}}\sigma - \partial_{\mathsf{u}}\sigma^\star)\mathsf{B}_{ij}\partial_j\mathsf{u}\big) - \sum_{i,j\in\mathcal{D}}\partial_i(\partial_{\mathsf{u}}\sigma - \partial_{\mathsf{u}}\sigma^\star)\mathsf{B}_{ij}\partial_j\mathsf{u},$$

and the second term may be rewritten

$$\partial_i(\partial_{\mathsf{u}}\sigma - \partial_{\mathsf{u}}\sigma^\star)\mathsf{B}_{ij}\partial_j\mathsf{u} = \langle\partial_i\mathsf{v}, \mathsf{B}_{ij}\partial_j\mathsf{u}\rangle = \langle\partial_i\mathsf{v}, \widetilde{\mathsf{B}}_{ij}\partial_j\mathsf{v}\rangle,$$

since $\mathsf{B}_{ij}\partial_j\mathsf{u} = \widetilde{\mathsf{B}}_{ij}\partial_j\mathsf{v}$ from $\widetilde{\mathsf{B}}_{ij} = \mathsf{B}_{ij}\partial_{\mathsf{v}}\mathsf{u}$. Moreover, using $\partial_i\mathsf{v} = \partial_{\mathsf{w}}\mathsf{v}\,\partial_i\mathsf{w}$ and $\overline{\mathsf{B}}_{ij} = (\partial_{\mathsf{w}}\mathsf{v})^t\widetilde{\mathsf{B}}_{ij}\partial_{\mathsf{w}}\mathsf{v}$ as well as the block decomposition of $\overline{\mathsf{B}}_{ij}$, we obtain that

$$\langle\widetilde{\mathsf{B}}_{ij}\partial_j\mathsf{v}, \partial_i\mathsf{v}\rangle = \langle\overline{\mathsf{B}}_{ij}\partial_j\mathsf{w}, \partial_i\mathsf{w}\rangle = \langle\overline{\mathsf{B}}_{ij}^{\mathrm{II,II}}\partial_j\mathsf{w}_{\mathrm{II}}, \partial_i\mathsf{w}_{\mathrm{II}}\rangle.$$

Finally, using the equilibrium condition $(\partial_{\mathsf{u}}\sigma^\star)^t \in \mathcal{E}$ and $\partial_{\mathsf{u}}\sigma^\star\Omega = 0$ completes the proof of (5.14). □

The aim of this section is to establish estimates with constants independent of the time interval $[0, \bar{\tau}]$.

Theorem 5.5. *Assume that* w *is a solution of* (5.1) *over* $[0,\bar{\tau}] \times \mathbb{R}^d$ *with* (5.7)(5.8). *There exists contants* $b_{\mathrm{N}} \leq b_{\overline{\sigma}}$ *and* $\mathsf{c}_{\mathrm{N}} \geq 1$, *independent of* $\epsilon \in (0,1]$ *and independent of* $\bar{\tau}$, *such that*

$$N_l(\bar{\tau}) \leq b_{\mathrm{N}} \quad\Longrightarrow\quad N_l^2(\bar{\tau}) \leq \mathsf{c}_{\mathrm{N}}^2\big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2\big). \tag{5.15}$$

Proof. It is sufficient to establish that there exists constants $b' \leq b_{\overline{\sigma}}$, c', and c'' such that $N_l(t) \leq b'$ implies $N_l^2(t) \leq \mathsf{c}'\big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi \mathsf{w}_0|_{l-1}^2\big) + \mathsf{c}'' N_l^3(t)$ since it directly yields (5.15). It is also sufficient to consider smooth solutions of (5.1) since one may use mollifiers and convolution operators [34]. Since it is assumed that $b' \leq b_{\overline{\sigma}}$, one may further use the inequalities (5.11) and (5.12).

Step 0. In the following $\delta_1 = \delta(\mathcal{O}_{\overline{\sigma}}) \leq 1$ denotes a generic small constant only depending on $\mathcal{O}_{\overline{\sigma}}$, $\mathsf{c}_1 = \mathsf{c}_1(\mathcal{O}_{\overline{\sigma}}) \geq 1$ a generic large constant only depending on $\mathcal{O}_{\overline{\sigma}}$, and $\mathsf{c}_2 = \mathsf{c}_2(\mathcal{O}_{\overline{\sigma}}, b_{\overline{\sigma}}) \geq 1$ a generic large constant depending on $\mathcal{O}_{\overline{\sigma}}$ and $b_{\overline{\sigma}}$. The various occurrences of these constants may be distinguished and the minimum of all δ_1 and the maxima of all c_1 and c_2 may be taken at the end of the proof so that only single constants ultimately remain. In order to alleviate notation in the proof $\delta\mathsf{w}$ denotes for short $\delta\mathsf{w} = \mathsf{w} - \mathsf{w}^\star$. The classical nonlinear estimate $|f(\phi) - f(0)|_k \leq \mathsf{c}_0 \|f\|_{\mathcal{C}^k(\mathcal{O}_\phi)}(1 + \|\phi\|_{L^\infty})^{k-1}\,|\phi|_k$ where $k \geq 1$, $\phi \in H^k \cap L^\infty$, $\mathcal{O}_\phi$ is an open ball that contains the range of ϕ, f is a C^k function over $\mathcal{O}_\phi$, and c_0 denotes a generic constant independent of $\mathcal{O}_{\overline{\sigma}}$ and $b_{\overline{\sigma}}$ will be used repeatedly. We will also use the estimates $|uv|_k^2 \leq \mathsf{c}_0 |u|_{\overline{l}}^2 |v|_k^2$, where $0 \leq k \leq \overline{l}$ and $\|\phi\|_{L^\infty} \leq \mathsf{c}_0 |\phi|_{\overline{l}}$ valid for any $\overline{l} \geq l_0 = [d/2] + 1$. The commutator estimate $\sum_{0 \leq |\alpha| \leq l} \big|[\partial^\alpha, u]v\big|_0 \leq \mathsf{c}_0 |\partial_x u|_{l-1} |v|_{l-1}$ is also valid for any $l \geq l_0 + 1$ where $[\partial^\alpha, u]v = \partial^\alpha(uv) - u\partial^\alpha v$ denotes the commutator between ∂^α and u.

In the proof, we will successively derive a 0th order entropic type estimate, a lth order entropic type estimate, a$(l-1)$th order estimate for the time integral of hyperbolic terms, a $(l-1)$th estimate for the fast variable, a $(l-1)$th order estimates for the time derivative and combine them all in order to establish Theorem 5.5. The assumptions required for these new estimates are that of Section 5.1 and we will use in particular the normal form, the smoothness of the coefficients, the regularity properties (5.7)(5.8), the modified entropy and the related entropic estimate, the parabolic strict dissipativity, the compensating matrix compatible with the fast manifold, and the fast variable governing equation.

Step 1. The zeroth order estimate. The entropic type governing equation (5.14) is rewritten by using $\overline{\mathsf{B}}_{ij} = \overline{\mathsf{B}}_{ij}^\star + (\overline{\mathsf{B}}_{ij} - \overline{\mathsf{B}}_{ij}^\star)$, where $\overline{\mathsf{B}}_{ij}^\star = \overline{\mathsf{B}}_{ij}(\mathsf{w}^\star)$, and integrating over $x \in \mathbb{R}^d$ and $\tau \in [0,t]$ yields

$$\delta_1 \int_{\mathbb{R}^d} |\delta\mathsf{w}|^2\, \mathrm{d}x + \sum_{i,j \in \mathcal{D}} \int_0^t \int_{\mathbb{R}^d} \langle \overline{\mathsf{B}}_{ij}^{\mathrm{II,II}\,\star} \partial_j \mathsf{w}_{\mathrm{II}}, \partial_i \mathsf{w}_{\mathrm{II}} \rangle\, \mathrm{d}x\mathrm{d}\tau + \frac{\delta_1}{\epsilon} \int_0^t \int_{\mathbb{R}^d} |\pi \mathsf{w}|^2\, \mathrm{d}x\mathrm{d}\tau$$

$$\leq \mathsf{c}_1 \int_{\mathbb{R}^d} |\delta\mathsf{w}_0|^2\, \mathrm{d}x + \sum_{i,j \in \mathcal{D}} \int_0^t \int_{\mathbb{R}^d} \big| \langle (\overline{\mathsf{B}}_{ij}^{\mathrm{II,II}} - \overline{\mathsf{B}}_{ij}^{\mathrm{II,II}\,\star}) \partial_j \mathsf{w}_{\mathrm{II}}, \partial_i \mathsf{w}_{\mathrm{II}} \rangle \big|\, \mathrm{d}x\mathrm{d}\tau. \quad (5.16)$$

Noting that the last term in the right hand side is majorized by

$$\sum_{i,j \in \mathcal{D}} \int_0^t |\overline{\mathsf{B}}_{ij}^{\mathrm{II,II}} - \overline{\mathsf{B}}_{ij}^{\mathrm{II,II}\,\star}|_{L^\infty} |\mathsf{w}_{\mathrm{II}}|_0^2\, \mathrm{d}\tau \leq \mathsf{c}_1\, N_l(t) \int_0^t |\mathsf{w}_{\mathrm{II}}|_0^2\, \mathrm{d}\tau,$$

since $|\overline{\mathsf{B}}_{ij}^{\text{II,II}} - \overline{\mathsf{B}}_{ij}^{\text{II,II}\star}|_{L^\infty} \leq \mathsf{c}_1|\mathsf{w}-\mathsf{w}^\star|_{L^\infty} \leq \mathsf{c}_1|\mathsf{w}-\mathsf{w}^\star|_{l-1} \leq \mathsf{c}_1 N_l(t)$, the zeroth order estimate is obtained by further using Parseval identity and parabolicity

$$\delta_1|\delta\mathsf{w}|_0^2 + \delta_1\int_0^t |\partial_x \mathsf{w}_{\text{II}}|_0^2 \,\mathrm{d}\tau + \frac{\delta_1}{\epsilon}\int_0^t |\pi\mathsf{w}|_0^2 \,\mathrm{d}\tau \leq \mathsf{c}_1|\delta\mathsf{w}_0|_0^2 + \mathsf{c}_1 N_l^3(t). \tag{5.17}$$

Step 2. The lth order estimate. Let ∂^α denotes the αth derivative spatial operator. Differentiating the partial differential equation (5.1) yields

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\partial^\alpha\mathsf{w} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\partial^\alpha\mathsf{w} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_i\partial_j\partial^\alpha\mathsf{w} + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\partial^\alpha\mathsf{w} = \mathsf{h}^\alpha, \tag{5.18}$$

where the residual h^α reads

$$\begin{aligned}\mathsf{h}^\alpha = \overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\overline{\mathsf{Q}}'\big) - \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{A}}_i\big]\partial_i\mathsf{w} \\ - \frac{1}{\epsilon}\overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\big]\pi\mathsf{w} + \sum_{i,j\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}_{ij}\big]\partial_i\partial_j\mathsf{w},\end{aligned}$$

and $\overline{\mathsf{Q}}' = \overline{\mathsf{Q}} + \sum_{i,j\in\mathcal{D}}\partial_{\mathsf{w}}\overline{\mathsf{B}}_{ij}\partial_i\mathsf{w}\partial_j\mathsf{w}$ is quadratic in the gradients. Multiplying equation (5.18) by $\partial^\alpha\mathsf{w}$ and $|\alpha|!/\alpha!$, writing $\overline{\mathsf{B}}_{ij} = \overline{\mathsf{B}}_{ij}^\star + (\overline{\mathsf{B}}_{ij} - \overline{\mathsf{B}}_{ij}^\star)$, integrating over $x \in \mathbb{R}^d$, summing over $1 \leq |\alpha| \leq l$, integrating over $\tau \in [0,t]$, and adding the zeroth order estimate (5.17), yields that

$$\begin{aligned}&\delta_1|\delta\mathsf{w}|_l^2 + \delta_1\int_0^t |\partial_x\mathsf{w}_{\text{II}}|_l^2\,\mathrm{d}\tau + \frac{\delta_1}{\epsilon}\int_0^t |\pi\mathsf{w}|_l^2\,\mathrm{d}\tau \leq \mathsf{c}_1|\delta\mathsf{w}_0|_l^2 + \mathsf{c}_1 N_l(t)^3 \\ &+ \sum_{1\leq|\alpha|\leq l}\frac{|\alpha|!}{\alpha!}\int_0^t\int_{\mathbb{R}^d}|\langle\partial_t\overline{\mathsf{A}}_0\partial^\alpha\mathsf{w}, \partial^\alpha\mathsf{w}\rangle|\,\mathrm{d}x\mathrm{d}\tau \\ &+ \sum_{\substack{i\in\mathcal{D}\\ 1\leq|\alpha|\leq l}}\frac{|\alpha|!}{\alpha!}\int_0^t\int_{\mathbb{R}^d}|\langle\partial_i\overline{\mathsf{A}}_i\partial^\alpha\mathsf{w}, \partial^\alpha\mathsf{w}\rangle|\,\mathrm{d}x\mathrm{d}\tau \\ &+ \sum_{\substack{i,j\in\mathcal{D}\\ 1\leq|\alpha|\leq l}}\frac{|\alpha|!}{\alpha!}\int_0^t\int_{\mathbb{R}^d}\big|\big\langle\partial_j\big((\overline{\mathsf{B}}_{ij} - \overline{\mathsf{B}}_{ij}^\star)\partial^\alpha\mathsf{w}\big), \partial_i\partial^\alpha\mathsf{w}\big\rangle\big|\,\mathrm{d}x\mathrm{d}\tau \\ &+ \sum_{1\leq|\alpha|\leq l}\frac{|\alpha|!}{\alpha!}\int_0^t\int_{\mathbb{R}^d}|\langle\mathsf{h}^\alpha, \partial^\alpha\mathsf{w}\rangle|\,\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

The various terms in the right hand side are then estimated using $|\partial_t\mathsf{w}|_{L^\infty} \leq \mathsf{c}_0|\partial_t\mathsf{w}|_{l-1}$, $|\partial_x\mathsf{w}|_{L^\infty} \leq \mathsf{c}_0|\partial_x\mathsf{w}|_{l-1}$, keeping in mind that $l \geq l_0+1$, as well as $|\partial_{\mathsf{w}}\overline{\mathsf{A}}_0|_{L^\infty} \leq \mathsf{c}_1$, $|\partial_{\mathsf{w}}\overline{\mathsf{A}}_i|_{L^\infty} \leq \mathsf{c}_1$, and $|\partial_{\mathsf{w}}\overline{\mathsf{B}}_{ij}|_{L^\infty} \leq \mathsf{c}_1$, for $i,j \in \mathcal{D}$, since these maxima over $\mathcal{O}_{\overline{\sigma}}$ are independent of $b_{\overline{\sigma}}$. The right hand side terms estimates are then obtained in the form

$$\int_0^t\int_{\mathbb{R}^d}|\langle\partial_t\overline{\mathsf{A}}_0\partial^\alpha\delta\mathsf{w}, \partial^\alpha\delta\mathsf{w}\rangle|\,\mathrm{d}x\mathrm{d}\tau \leq \mathsf{c}_1\int_0^t |\partial_t\mathsf{w}|_{L^\infty}\,|\partial_x\mathsf{w}|_{l-1}^2\,\mathrm{d}\tau \leq \mathsf{c}_1 N_l^3(t).$$

$$\int_0^t\!\!\int_{\mathbb{R}^d} |\langle \partial_i \overline{\mathsf{A}}_i \partial^\alpha \delta \mathsf{w}, \partial^\alpha \delta \mathsf{w}\rangle| \,\mathrm{d}x\mathrm{d}\tau \le \mathsf{c}_1 \int_0^t |\partial_x \mathsf{w}|_{L^\infty} |\partial_x \mathsf{w}|_{l-1}^2 \,\mathrm{d}\tau \le \mathsf{c}_1 N_l^3(t),$$

$$\int_0^t\!\!\int_{\mathbb{R}^d} |\langle (\overline{\mathsf{B}}_{ij} - \overline{\mathsf{B}}^\star_{ij}) \partial_j \partial^\alpha \mathsf{w}, \partial_i \partial^\alpha \mathsf{w}\rangle| \,\mathrm{d}x\mathrm{d}\tau$$
$$\le \mathsf{c}_1 \int_0^t |\mathsf{w} - \mathsf{w}^\star|_{L^\infty} |\partial_x \mathsf{w}_{\mathrm{II}}|_l^2 \,\mathrm{d}\tau \le \mathsf{c}_1 N_l(t)^3,$$

$$\int_0^t\!\!\int_{\mathbb{R}^d} |\langle \partial_j (\overline{\mathsf{B}}_{ij} - \overline{\mathsf{B}}^\star_{ij})\, \partial^\alpha \mathsf{w}, \partial_i \partial^\alpha \mathsf{w}\rangle| \,\mathrm{d}x\mathrm{d}\tau$$
$$\le \mathsf{c}_1 \int_0^t |\partial_x \mathsf{w}|_{L^\infty} |\partial_x \mathsf{w}_{\mathrm{II}}|_{l-1} |\partial_x \mathsf{w}_{\mathrm{II}}|_l \,\mathrm{d}\tau \le \mathsf{c}_1 N_l(t)^3.$$

The first of the residual terms involving h^α is majorized by using that $\overline{\mathrm{Q}}'$ is quadratic in the gradients of w_{II}

$$\int_0^t\!\!\int_{\mathbb{R}^d} \big|\big\langle \overline{\mathsf{A}}_0 \partial^\alpha \big(\overline{\mathsf{A}}_0^{-1} \overline{\mathrm{Q}}' \big), \partial^\alpha \mathsf{w} \big\rangle\big| \,\mathrm{d}x\mathrm{d}\tau \le \mathsf{c}_2 \int_0^t |\partial_x \mathsf{w}_{\mathrm{II}}|_l^2 \, |\partial_x \mathsf{w}_{\mathrm{II}}|_{l-1} \,\mathrm{d}\tau \le \mathsf{c}_2 N_l(t)^3.$$

More specifically, $\overline{\mathrm{Q}}' = (0, \overline{\mathrm{Q}}'_{\mathrm{II}})^t$ and $\overline{\mathrm{Q}}'_{\mathrm{II}}$ is a sum of terms $\overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II},\mathrm{II}}(\mathsf{w}) \partial_i \mathsf{w}_{\mathrm{II}} \partial_j \mathsf{w}_{\mathrm{II}}$ and each term is majorized in the form

$$\int_0^t\!\!\int_{\mathbb{R}^d} \big|\big\langle \overline{\mathsf{A}}_0 \partial^\alpha \big(\overline{\mathsf{A}}_0^{-1} \overline{\mathrm{M}}_{ij} \partial_i \mathsf{w} \partial_j \mathsf{w} \big), \partial^\alpha \mathsf{w} \big\rangle\big| \,\mathrm{d}x\mathrm{d}\tau$$
$$\le |\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}}|_{L^\infty} \int_0^t |\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}-1} \overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II},\mathrm{II}} \partial_i \mathsf{w}_{\mathrm{II}} \partial_j \mathsf{w}_{\mathrm{II}}|_l |\partial_x \mathsf{w}_{\mathrm{II}}|_{l-1} \,\mathrm{d}\tau,$$

with $|\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}-1} \overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II},\mathrm{II}} \partial_i \mathsf{w}_{\mathrm{II}} \partial_j \mathsf{w}_{\mathrm{II}}|_l \le \mathsf{c}_2 |\partial_x \mathsf{w}_{\mathrm{II}}|_l^2$ as established by decomposing the product $\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}-1} \overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II},\mathrm{II}} \partial_i \mathsf{w}_{\mathrm{II}} \partial_j \mathsf{w}_{\mathrm{II}}$ as the sum

$$\overline{\mathsf{A}}_0^{\star\mathrm{II},\mathrm{II}-1} \overline{\mathrm{M}}_{ij}^{\star\ \mathrm{II},\mathrm{II},\mathrm{II}} \partial_i \mathsf{w}_{\mathrm{II}} \partial_j \mathsf{w}_{\mathrm{II}} + (\overline{\mathsf{A}}_0^{\mathrm{II},\mathrm{II}-1} \overline{\mathrm{M}}_{ij}^{\mathrm{II},\mathrm{II},\mathrm{II}} - \overline{\mathsf{A}}_0^{\star\mathrm{II},\mathrm{II}-1} \overline{\mathrm{M}}_{ij}^{\star\ \mathrm{II},\mathrm{II},\mathrm{II}}) \partial_i \mathsf{w}_{\mathrm{II}} \partial_j \mathsf{w}_{\mathrm{II}},$$

using $|uv|_l \le \mathsf{c}_0 |u|_l |v|_l$ for any $u, v \in H^l(\mathbb{R}^d)$, the classical nonlinear estimates, as well as $N_l(t) \le b_{\overline{\sigma}}$. The other residual terms involving h^α are majorized by using the commutator estimates

$$\int_0^t\!\!\int_{\mathbb{R}^d} \big|\big\langle \overline{\mathsf{A}}_0 \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1} \,\overline{\mathsf{A}}_i\big] \partial_i \mathsf{w}, \partial^\alpha \mathsf{w} \big\rangle\big| \,\mathrm{d}x\mathrm{d}\tau \le \mathsf{c}_2 \int_0^t |\partial_x \mathsf{w}|_{l-1}^3 \mathrm{d}\tau \le \mathsf{c}_2 N_l(t)^3,$$

$$\frac{1}{\epsilon} \int_0^t\!\!\int_{\mathbb{R}^d} \big|\big\langle \overline{\mathsf{A}}_0 \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1} \,\overline{\mathsf{L}}\big] \pi \mathsf{w}, \partial^\alpha \mathsf{w} \big\rangle\big| \,\mathrm{d}x\mathrm{d}\tau \le \frac{\mathsf{c}_2}{\epsilon} \int_0^t |\pi \mathsf{w}|_{l-1} |\partial_x \mathsf{w}|_{l-1}^2 \mathrm{d}\tau \le \mathsf{c}_2 N_l(t)^3,$$

$$\int_0^t\!\!\int_{\mathbb{R}^d} \big|\big\langle \overline{\mathsf{A}}_0 \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1} \,\overline{\mathsf{B}}_{ij}\big] \partial_i \partial_j \mathsf{w}, \partial^\alpha \mathsf{w} \big\rangle\big| \,\mathrm{d}x\mathrm{d}\tau$$
$$\le \mathsf{c}_2 \int_0^t |\partial_x \mathsf{w}_{\mathrm{II}}|_l |\partial_x \mathsf{w}_{\mathrm{II}}|_{l-1} |\partial_x \mathsf{w}|_{l-1} \mathrm{d}\tau \le \mathsf{c}_2 N_l(t)^3.$$

A *key point* for such estimates is the inclusion of the terms $\int_0^t |\partial_t \mathsf{w}|_{l-1}^2 \,\mathrm{d}\tau$ and $\int_0^t |\pi \mathsf{w}|_{l-1}^2 \,\mathrm{d}\tau/\epsilon^2$ in the norm $N_l^2(t)$. We have also used that the Sobolev norms of products in the form $|\partial_x(\overline{\mathsf{A}}_0^{-1}\overline{\mathsf{A}}_i)|_{l-1}$ is majorized by $|\partial_x(\overline{\mathsf{A}}_0^{-1}\overline{\mathsf{A}}_i)|_{l-1} \leq \mathsf{c}_2|\partial_x \mathsf{w}|_{l-1}$ as established by decomposing $\overline{\mathsf{A}}_0^{-1}\overline{\mathsf{A}}_i$ as the sum

$$\overline{\mathsf{A}}_0^{-1}\overline{\mathsf{A}}_i = \overline{\mathsf{A}}_0^{\star -1}\overline{\mathsf{A}}_i^{\star} + (\overline{\mathsf{A}}_0^{-1} - \overline{\mathsf{A}}_0^{\star -1})\overline{\mathsf{A}}_i^{\star} + \overline{\mathsf{A}}_0^{\star -1}(\overline{\mathsf{A}}_i - \overline{\mathsf{A}}_i^{\star}) + (\overline{\mathsf{A}}_0^{-1} - \overline{\mathsf{A}}_0^{\star -1})(\overline{\mathsf{A}}_i - \overline{\mathsf{A}}_i^{\star}),$$

using $|uv|_{l-1} \leq \mathsf{c}_0|u|_{l-1}|v|_{l-1}$ for any $u, v \in H^{l-1}(\mathbb{R}^d)$, the classical nonlinear estimates, as well as $N_l(t) \leq b_{\overline{\sigma}}$. Collecting all contributions it is established that

$$\delta_1|\delta \mathsf{w}|_l^2 + \delta_1 \int_0^t |\partial_x \mathsf{w}_{\mathrm{II}}|_l^2 \,\mathrm{d}\tau + \frac{\delta_1}{\epsilon}\int_0^t |\pi \mathsf{w}|_l^2 \,\mathrm{d}\tau \leq \mathsf{c}_1|\delta \mathsf{w}_0|_l^2 + \mathsf{c}_2 N_l(t)^3. \tag{5.19}$$

Step 3. Estimate of the time integral of hyperbolic terms [34]. Linearizing the system of governing equations in normal form yields

$$\overline{\mathsf{A}}_0^{\star}\partial_t \mathsf{w} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i^{\star}\partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \overline{\mathsf{B}}_{ij}^{\star}\partial_i\partial_j \mathsf{w} + \frac{1}{\epsilon}\overline{\mathsf{L}}^{\star}\mathsf{w} = \mathsf{h}^{\mathrm{lin}}, \tag{5.20}$$

with

$$\begin{aligned}\mathsf{h}^{\mathrm{lin}} = &-\sum_{i\in\mathcal{D}}\big(\overline{\mathsf{A}}_0^{\star}\overline{\mathsf{A}}_0^{-1}\overline{\mathsf{A}}_i - \overline{\mathsf{A}}_i^{\star}\big)\partial_i \mathsf{w} + \sum_{i,j\in\mathcal{D}}\big(\overline{\mathsf{A}}_0^{\star}\overline{\mathsf{A}}_0^{-1}\overline{\mathsf{B}}_{ij} - \overline{\mathsf{B}}_{ij}^{\star}\big)\partial_i\partial_j \mathsf{w} \\ &- \frac{1}{\epsilon}\big(\overline{\mathsf{A}}_0^{\star}\overline{\mathsf{A}}_0^{-1}\overline{\mathsf{L}} - \overline{\mathsf{L}}^{\star}\big)\pi \mathsf{w} + \sum_{i,j\in\mathcal{D}} \overline{\mathsf{A}}_0^{\star}\overline{\mathsf{A}}_0^{-1}\partial_i\overline{\mathsf{B}}_{ij}\partial_j \mathsf{w} + \overline{\mathsf{A}}_0^{\star}\overline{\mathsf{A}}_0^{-1}\overline{\mathsf{Q}}.\end{aligned}$$

Applying the differentiation operator ∂^α, multiplying on the left by K_j and $|\alpha|!/\alpha!$, using $K_j\overline{\mathsf{L}} = K_j\pi\overline{\mathsf{L}} = 0$, taking the scalar product with $\partial_j\partial^\alpha \mathsf{w}$, summing over $0 \leq |\alpha| \leq l-1$ and $j \in \mathcal{D}$, and integrating over $x \in \mathbb{R}^d$ and $\tau \in [0, t]$ next yields

$$\begin{aligned}&\sum_{\substack{j\in\mathcal{D}\\0\leq|\alpha|\leq l-1}} \int_0^t\int_{\mathbb{R}^d} \frac{|\alpha|!}{\alpha!}\langle K_j\overline{\mathsf{A}}_0^{\star}\partial_t\partial^\alpha \mathsf{w}, \partial_j\partial^\alpha \mathsf{w}\rangle \,\mathrm{d}x\mathrm{d}\tau \\ &+ \sum_{\substack{i,j\in\mathcal{D}\\0\leq|\alpha|\leq l-1}} \int_0^t\int_{\mathbb{R}^d} \frac{|\alpha|!}{\alpha!}\langle K_j\overline{\mathsf{A}}_i^{\star}\partial_i\partial^\alpha \mathsf{w}, \partial_j\partial^\alpha \mathsf{w}\rangle \,\mathrm{d}x\mathrm{d}\tau \\ &- \sum_{\substack{i,j,j'\in\mathcal{D}\\0\leq|\alpha|\leq l-1}} \int_0^t\int_{\mathbb{R}^d} \langle K_j\overline{\mathsf{B}}_{ij'}^{\star}\partial_i\partial_{j'}\partial^\alpha \mathsf{w}, \partial_j\partial^\alpha \mathsf{w}\rangle \,\mathrm{d}x\mathrm{d}\tau \\ &= \sum_{\substack{j\in\mathcal{D}\\0\leq|\alpha|\leq l-1}} \int_0^t\int_{\mathbb{R}^d} \frac{|\alpha|!}{\alpha!}\langle K_j\partial^\alpha \mathsf{h}^{\mathrm{lin}}, \partial_j\partial^\alpha \mathsf{w}\rangle \,\mathrm{d}x\mathrm{d}\tau.\end{aligned} \tag{5.21}$$

The terms $\int_0^t\!\int_{\mathbb{R}^d} \langle K_j \overline{\mathsf{A}}_0^\star \partial_t \partial^\alpha \mathsf{w}, \partial_j \partial^\alpha \mathsf{w}\rangle \, \mathrm{d}x\mathrm{d}\tau$ in (5.21) are rewritten with the identity

$$2\langle K_j \overline{\mathsf{A}}_0^\star \partial_t \partial^\alpha \mathsf{w}, \partial_j \partial^\alpha \mathsf{w}\rangle = \partial_t \langle K_j \overline{\mathsf{A}}_0^\star \partial^\alpha \delta\mathsf{w}, \partial_j \partial^\alpha \mathsf{w}\rangle + \partial_j \langle K_j \overline{\mathsf{A}}_0^\star \partial_t \partial^\alpha \mathsf{w}, \partial^\alpha \delta\mathsf{w}\rangle,$$

where $\delta\mathsf{w} = \mathsf{w} - \mathsf{w}^\star$, by using that $K_j \overline{\mathsf{A}}_0^\star$ are skew symmetric for $j \in \mathcal{D}$, so that

$$2\int_0^t\!\int_{\mathbb{R}^d} \langle K_j \overline{\mathsf{A}}_0^\star \partial_t \partial^\alpha \mathsf{w}, \partial_j \partial^\alpha \mathsf{w}\rangle \, \mathrm{d}x\mathrm{d}\tau = \int_{\mathbb{R}^d} \langle K_j \overline{\mathsf{A}}_0^\star \partial^\alpha \delta\mathsf{w}, \partial_j \partial^\alpha \mathsf{w}\rangle \, \mathrm{d}x \\ - \int_{\mathbb{R}^d} \langle K_j \overline{\mathsf{A}}_0^\star \partial^\alpha \delta\mathsf{w}_0, \partial_j \partial^\alpha \mathsf{w}_0\rangle \, \mathrm{d}x,$$

and the right hand side is directly majorized by $\mathsf{c}_1|\mathsf{w}-\mathsf{w}^\star|_l^2 + \mathsf{c}_1|\mathsf{w}_0-\mathsf{w}^\star|_l^2$. The term

$$\sum_{i,j\in\mathcal{D}} \sum_{0\le|\alpha|\le l-1} \int_0^t\!\int_{\mathbb{R}^d} \frac{|\alpha|!}{\alpha!} \langle K_j \overline{\mathsf{A}}_i^\star \partial_i \partial^\alpha \mathsf{w}, \partial_j \partial^\alpha \mathsf{w}\rangle \, \mathrm{d}x\mathrm{d}\tau,$$

associated with first order derivatives is rewritten using Parseval identity and, using the strict dissipativity, is further minorized as

$$\delta_1 \int_0^t |\partial_x \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau - \mathsf{c}_1 \int_0^t |\partial_x \mathsf{w}_{\mathrm{II}}|_{l-1}^2 \, \mathrm{d}\tau \\ \le \sum_{\substack{i,j\in\mathcal{D} \\ 0\le|\alpha|\le l-1}} \int_0^t\!\int_{\mathbb{R}^d} \frac{|\alpha|!}{\alpha!} \langle K_j \overline{\mathsf{A}}_i^\star \partial_i \partial^\alpha \mathsf{w}, \partial_j \partial^\alpha \mathsf{w}\rangle \, \mathrm{d}x\mathrm{d}\tau.$$

The terms associated with second order derivatives are majorized with

$$\int_0^t\!\int_{\mathbb{R}^d} |\langle K_j \overline{\mathsf{B}}_{ij'}^\star \partial_i \partial_{j'} \partial^\alpha \mathsf{w}, \partial_j \partial^\alpha \mathsf{w}\rangle| \, \mathrm{d}x\mathrm{d}\tau \le \mathsf{c}_1 \int_0^t |\partial_x \mathsf{w}|_{l-1} |\partial_x \mathsf{w}_{\mathrm{II}}|_l \, \mathrm{d}\tau.$$

It now remains to examine the residuals associated with $\langle K_j \partial^\alpha \mathsf{h}^{\mathrm{lin}}, \partial_j \partial^\alpha \mathsf{w}\rangle$. The first contribution associated with convective terms is majorized using

$$\int_0^t\!\int_{\mathbb{R}^d} |\langle K_j \partial^\alpha \big((\overline{\mathsf{A}}_0^\star \overline{\mathsf{A}}_0^{-1} \overline{\mathsf{A}}_i - \overline{\mathsf{A}}_i^\star) \partial_i \mathsf{w}\big), \partial_j \partial^\alpha \mathsf{w}\rangle| \, \mathrm{d}x\mathrm{d}\tau \\ \le \mathsf{c}_2 \int_0^t |\mathsf{w}-\mathsf{w}^\star|_{l-1} |\partial_x \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau \le \mathsf{c}_2 N_l^3(t),$$

with similar estimates for the second derivatives contributions. The terms associated with sources are estimated in the form

$$\frac{1}{\epsilon} \int_0^t\!\int_{\mathbb{R}^d} |\langle K_j \partial^\alpha \big((\overline{\mathsf{A}}_0^\star \overline{\mathsf{A}}_0^{-1} \overline{\mathsf{L}} - \overline{\mathsf{L}}^\star) \pi \mathsf{w}\big), \partial_j \partial^\alpha \mathsf{w}\rangle| \, \mathrm{d}x\mathrm{d}\tau \\ \le \frac{\mathsf{c}_2}{\epsilon} \int_0^t |\mathsf{w}-\mathsf{w}^\star|_{l-1} |\pi\mathsf{w}|_{l-1} |\partial_x \mathsf{w}|_{l-1} \, \mathrm{d}\tau \le \mathsf{c}_2 N_l^3(t).$$

Moreover, the contributions form $\langle K_j \partial^\alpha \mathsf{h}^{\mathrm{lin}}, \partial_j \partial^\alpha \mathsf{w} \rangle$ associated the terms of $\mathsf{h}^{\mathrm{lin}}$ quadratic in the gradients are cubic with respect to first order derivatives and easily majorized by $\mathsf{c}_2 N_l^3(t)$. Collecting previous estimates it has been established that

$$\delta_1 \int_0^t |\partial_x \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau \leq \mathsf{c}_1 \int_0^t |\partial_x \mathsf{w}_{\mathrm{II}}|_{l-1}^2 \, \mathrm{d}\tau + \mathsf{c}_1 |\mathsf{w} - \mathsf{w}^\star|_l^2 + \mathsf{c}_1 |\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \mathsf{c}_2 N_l^3(t). \tag{5.22}$$

Combining this estimate of the time integral $\int_0^t |\partial_x \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau$ with (5.19) finally yields

$$\delta_1 |\delta \mathsf{w}|_l^2 + \delta_1 \int_0^t \big(|\partial_x \mathsf{w}_{\mathrm{I}}|_{l-1}^2 + |\partial_x \mathsf{w}_{\mathrm{II}}|_l^2 \big) \, \mathrm{d}\tau + \frac{\delta_1}{\epsilon} \int_0^t |\pi \mathsf{w}|_l^2 \, \mathrm{d}\tau \leq \mathsf{c}_1 |\delta \mathsf{w}_0|_l^2 + \mathsf{c}_2 N_l(t)^3. \tag{5.23}$$

Step 4. Estimates of the fast variable $\pi\mathsf{w}$. A linearized equation for the fast variable $\pi\mathsf{w}$ is first derived by applying $\overline{\Pi}_{\mathsf{r}}^t$ and $\overline{\Pi}_{\mathsf{e}}^t$ to (5.20) and proceeding as in Section 3.4. Equivalently, one may linearize the governing equation (3.9) of the fast variable. This linearized equation is in the form

$$\overline{\mathsf{A}}_0^{\pi\star} \partial_t \pi \mathsf{w} + \sum_{i \in \mathcal{D}} \overline{\mathsf{A}}_i^{\pi\star} \partial_i \mathsf{w} - \sum_{i,j \in \mathcal{D}} \overline{\mathsf{B}}_{ij}^{\pi\star} \partial_i \partial_j \mathsf{w} + \frac{1}{\epsilon} \overline{\mathsf{L}}^\star \pi \mathsf{w} = \mathsf{h}^\pi,$$

where $\overline{\mathsf{A}}_0^{\pi\star} = \pi \pi_{\mathsf{A}}^\star \overline{\mathsf{A}}_0^\star \pi + (\mathbb{I} - \pi) \overline{\mathsf{A}}_0^\star (\mathbb{I} - \pi)$, $\overline{\mathsf{A}}_i^{\pi\star} = \pi \pi_{\mathsf{A}}^\star \overline{\mathsf{A}}_i^\pi$, $\overline{\mathsf{B}}_{ij}^{\pi\star} = \pi \pi_{\mathsf{A}}^\star \overline{\mathsf{B}}_{ij}^\star$, $\mathsf{h}^\pi = \pi \pi_{\mathsf{A}}^\star \mathsf{h}^{\mathrm{lin}}$, and $\pi_{\mathsf{A}}^\star = \mathbb{I} - \overline{\mathsf{A}}_0^\star \overline{\Pi}_{\mathsf{e}} (\overline{\Pi}_{\mathsf{e}}^t \overline{\mathsf{A}}_0^\star \overline{\Pi}_{\mathsf{e}})^{-1} \overline{\Pi}_{\mathsf{e}}^t$. Applying the derivative operator ∂^α to this equation then yields

$$\overline{\mathsf{A}}_0^{\pi\star} \partial_t \partial^\alpha \pi \mathsf{w} + \frac{1}{\epsilon} \overline{\mathsf{L}}^\star \partial^\alpha \pi \mathsf{w} = \mathsf{h}^{\pi\alpha}, \tag{5.24}$$

where $\mathsf{h}^{\pi\alpha} = \partial^\alpha \mathsf{h}^\pi - \sum_{i \in \mathcal{D}} \overline{\mathsf{A}}_i^{\pi\star} \partial_i \partial^\alpha \mathsf{w} + \sum_{i,j \in \mathcal{D}} \overline{\mathsf{B}}_{ij}^{\pi\star} \partial_i \partial_j \partial^\alpha \mathsf{w}$. Multiplying equation (5.24) by $\partial^\alpha(\pi\mathsf{w})/\epsilon$ and $|\alpha|!/\alpha!$, integrating over $x \in \mathbb{R}^d$, summing over $0 \leq |\alpha| \leq l-1$, and integrating over $\tau \in [0,t]$, it is obtained that

$$\begin{aligned} \frac{\delta_1}{\epsilon} |\pi \mathsf{w}|_{l-1}^2 + \frac{\delta_1}{\epsilon^2} \int_0^t |\pi \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau \leq & \frac{\mathsf{c}_1}{\epsilon} |\pi \mathsf{w}_0|_{l-1}^2 + \mathsf{c}_1 \int_0^t |\partial_x \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau \\ & + \mathsf{c}_1 \int_0^t |\partial_x \mathsf{w}_{\mathrm{II}}|_l^2 \, \mathrm{d}\tau + \mathsf{c}_1 \int_0^t |\mathsf{h}^\pi|_{l-1}^2 \, \mathrm{d}\tau. \end{aligned}$$

All contributions arising from $\mathsf{h}^\pi = \pi \pi_{\mathsf{A}}^\star \mathsf{h}^{\mathrm{lin}}$ may then be investigated as in previous steps and since h^π is quadratic in the solution, it is easily obtained that $\int_0^t |\mathsf{h}^\pi|_{l-1}^2 \, \mathrm{d}\tau \leq \mathsf{c}_2 N_l(t)^3$ in such a way that using the lth order estimate yields

$$\frac{\delta_1}{\epsilon} |\pi \mathsf{w}|_{l-1}^2 + \frac{\delta_1}{\epsilon^2} \int_0^t |\pi \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau \leq \mathsf{c}_1 \Big(|\mathsf{w}_0|_l^2 + \frac{1}{\epsilon} |\pi \mathsf{w}_0|_{l-1}^2 \Big) + \mathsf{c}_2 N_l(t)^3. \tag{5.25}$$

Step 5. Estimates of the time derivative. Using the above estimate of the fast variable (5.25) combined to the governing equations directly yields estimates for the time derivatives. Equivalently, one may multiply the derived equation (5.20) by $\partial_t\partial^\alpha \mathsf{w}$ and $|\alpha|!/\alpha!$, integrate over $x \in \mathbb{R}^d$, sum over $0 \le |\alpha| \le l-1$, and integrate over $\tau \in [0,t]$, to get that

$$\begin{aligned}\delta_1 \int_0^t |\partial_t \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau + \frac{\delta_1}{\epsilon}|\pi \mathsf{w}|_{l-1}^2 \le & \frac{\mathsf{c}_1}{\epsilon}|\pi \mathsf{w}_0|_{l-1}^2 + \mathsf{c}_1 \int_0^t |\partial_x \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau \\ & + \mathsf{c}_1 \int_0^t |\partial_x \mathsf{w}_{\mathrm{II}}|_l^2 \, \mathrm{d}\tau + \mathsf{c}_1 \int_0^t |\mathsf{h}^{\mathrm{lin}}|_{l-1}^2 \, \mathrm{d}\tau,\end{aligned}$$

and next that

$$\delta_1 \int_0^t |\partial_t \mathsf{w}|_{l-1}^2 \, \mathrm{d}\tau + \frac{\delta_1}{\epsilon}|\pi \mathsf{w}|_{l-1}^2 \le \mathsf{c}_1 \Big(|\mathsf{w}_0|_l^2 + \frac{1}{\epsilon}|\pi \mathsf{w}_0|_{l-1}^2 \Big) + \mathsf{c}_2 N_l(t)^3. \tag{5.26}$$

Combining the estimates (5.23) with (5.25) and (5.26) finally completes the proof. □

5.4. Global solutions

Global existence of solution uniformly with respect to ϵ may now be derived by combining local existence from Theorem 5.1 and the estimates of Theorem 5.5 [34].

Theorem 5.6. *Let $d \ge 1$ and $l \ge [d/2]+2$ be integers. There exists $\bar{b} > 0$ small enough such that if w_0 satisfies $\mathsf{w}_0 - \mathsf{w}^\star \in H^l(\mathbb{R}^d)$ and*

$$|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi \mathsf{w}_0|_{l-1}^2 < \bar{b}^2,$$

there exists a unique global solution to the Cauchy problem

$$\overline{\mathsf{A}}_0 \partial_t \mathsf{w} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i \partial_i \mathsf{w} - \sum_{i,j\in\mathcal{D}} \partial_i(\overline{\mathsf{B}}_{ij}\partial_j \mathsf{w}) + \frac{1}{\epsilon}\overline{\mathsf{L}}\mathsf{w} = \sum_{i,j\in\mathcal{D}} \overline{\mathrm{M}}_{ij}(\mathsf{w})\,\partial_i \mathsf{w}\,\partial_j \mathsf{w},$$

with initial condition $\mathsf{w}(0,x) = \mathsf{w}_0(x)$ and

$$\begin{aligned} &\mathsf{w}_{\mathrm{I}} - \mathsf{w}_{\mathrm{I}}^\star \in C^0\big([0,\infty), H^l\big) \cap C^1\big([0,\infty), H^{l-1}\big), && \partial_x \mathsf{w}_{\mathrm{I}} \in L^2\big((0,\infty), H^{l-1}\big),\\ &\mathsf{w}_{\mathrm{II}} - \mathsf{w}_{\mathrm{II}}^\star \in C^0\big([0,\infty), H^l\big) \cap C^1\big([0,\infty), H^{l-2}\big), && \partial_x \mathsf{w}_{\mathrm{II}} \in L^2\big((0,\infty), H^{l}\big).\end{aligned}$$

Furthermore, there exists a constant $\bar{c}$ independent of ϵ such that w satisfies the estimate

$$\begin{aligned}|\mathsf{w}(t) - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi \mathsf{w}(t)|_{l-1}^2 + \int_0^t |\partial_x \mathsf{w}_{\mathrm{I}}|_{l-1}^2 \, d\tau + \int_0^t |\partial_x \mathsf{w}_{\mathrm{II}}|_l^2 \, d\tau + \frac{1}{\epsilon}\int_0^t |\pi \mathsf{w}(\tau)|_l^2 \, d\tau \\ + \frac{1}{\epsilon^2}\int_0^t |\pi \mathsf{w}(\tau)|_{l-1}^2 \, d\tau + \int_0^t |\partial_t \mathsf{w}(\tau)|_{l-1}^2 \, d\tau \le \bar{c}^2 \Big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi \mathsf{w}_0|_{l-1}^2 \Big),\end{aligned} \tag{5.27}$$

and $\sup_{x\in\mathbb{R}^d} |\mathsf{w}(t,x) - \mathsf{w}^\star|$ goes to zero as $t \to \infty$.

Proof. Apply the local existence Theorem 5.1 with $\mathcal{O}_0 = \mathcal{O}_{\overline{\sigma}}$ where $0 < a_1 < \mathrm{dist}(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathsf{w}})$, and $b = b_{\overline{\sigma}}$. There exists a positive time $\bar{\tau} > 0$ and a local solution defined over $[0, \bar{\tau}]$ whenever $|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2 < b_{\overline{\sigma}}^2$. From Theorem 5.1 one gets the estimates

$$N_l(\bar{\tau}) \leq \mathsf{c}_{\mathrm{loc}}\big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \tfrac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2\big)^{1/2},$$

where $\mathsf{c}_{\mathrm{loc}} \geq 1$ depends on $\mathcal{O}_1$ and $b = b_{\overline{\sigma}}$. Let then

$$\bar{b} = \inf\Big(\frac{b_{\mathrm{N}}}{\mathsf{c}_{\mathrm{loc}}}, \frac{b_{\mathrm{N}}}{\mathsf{c}_{\mathrm{N}}\sqrt{1 + \mathsf{c}_{\mathrm{loc}}^2}}\Big),$$

where $b_{\mathrm{N}} \leq b_{\overline{\sigma}}$ and c_{N} are given by Theorem 5.5 and assume that $\big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2\big)^{1/2} < \bar{b}$. One first has a solution defined on $[0, \bar{\tau}]$ with

$$N_l(\bar{\tau}) \leq \mathsf{c}_{\mathrm{loc}}\big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \tfrac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2\big)^{1/2} < \mathsf{c}_{\mathrm{loc}}\bar{b} \leq b_{\mathrm{N}} \leq b_{\overline{\sigma}}.$$

Since $N_l(\bar{\tau}) \leq b_{\mathrm{N}}$ one also has $N_l(\bar{\tau}) \leq \mathsf{c}_{\mathrm{N}}\big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2\big)^{1/2} < \mathsf{c}_{\mathrm{N}}\bar{b}$. One can now start again from $\mathsf{w}(\bar{\tau})$ at $\bar{\tau}$ since $\big(|\mathsf{w}(\bar{\tau}) - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}(\bar{\tau})|_{l-1}^2\big)^{1/2} \leq N_l(\bar{\tau}) < b_{\overline{\sigma}}$ and this yields a solution defined on $[\bar{\tau}, 2\bar{\tau}]$ with $N_l'(\bar{\tau}, 2\bar{\tau}) \leq \mathsf{c}_{\mathrm{loc}}N_l(\bar{\tau})$ where $N_l'(\bar{\tau}, 2\bar{\tau})$ is naturally defined as

$$\begin{aligned}
N_l'^2(\bar{\tau}, 2\bar{\tau}) = &\sup_{\bar{\tau} \leq \tau \leq 2\bar{\tau}} \Big(|\mathsf{w}(\tau) - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}(\tau)|_{l-1}^2\Big) \\
&+ \int_{\bar{\tau}}^{2\bar{\tau}} \big(|\partial_x\mathsf{w}_{\mathrm{I}}(\tau)|_{l-1}^2 + |\partial_x\mathsf{w}_{\mathrm{II}}(\tau)|_l^2\big)\,d\tau \\
&+ \frac{1}{\epsilon}\int_{\bar{\tau}}^{2\bar{\tau}} |\pi\mathsf{w}(\tau)|_l^2\,d\tau + \frac{1}{\epsilon^2}\int_{\bar{\tau}}^{2\bar{\tau}} |\pi\mathsf{w}(\tau)|_{l-1}^2\,d\tau + \int_{\bar{\tau}}^{2\bar{\tau}} |\partial_t\mathsf{w}(\tau)|_{l-1}^2\,d\tau.
\end{aligned}$$

As a consequence, one obtains

$$N_l(2\bar{\tau}) \leq (1 + \mathsf{c}_{\mathrm{loc}}^2)^{1/2}N_l(\bar{\tau}) < (1 + \mathsf{c}_{\mathrm{loc}}^2)^{1/2}\mathsf{c}_{\mathrm{N}}\bar{b} \leq b_{\mathrm{N}} \leq b_{\overline{\sigma}},$$

so that from of Theorem 5.5 with $\tau = 2\bar{\tau}$ it is obtained $N_l(2\bar{\tau}) \leq \mathsf{c}_{\mathrm{N}}\bar{b} \leq b_{\mathrm{N}} < b_{\overline{\sigma}}$. This shows that $\mathsf{w}(2\bar{\tau}) \in \mathcal{O}_{\overline{\sigma}}$ and one can start again from $\mathsf{w}(2\bar{\tau})$ at $2\bar{\tau}$ and an easy induction shows that the solution is defined for all time intervals $\big[j\bar{\tau}, (j+1)\bar{\tau}\big]$ for any $j \geq 0$ and that $N_l(t) \leq \bar{c}\big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2\big)^{1/2}$ where $\bar{c} = \mathsf{c}_{\mathrm{N}}\sqrt{1 + \mathsf{c}_{\mathrm{loc}}^2}$ for any $t \geq 0$ and this yields (5.27).

Letting $\Phi(t) = |\partial_x\mathsf{w}(t)|_{l-2}^2$ and it is next established from (5.27) that

$$\int_0^\infty |\Phi(t)|\,dt + \int_0^\infty |\partial_t\Phi(t)|\,dt \leq \mathsf{c}\Big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \tfrac{1}{\epsilon}|\pi\mathsf{w}_0|_{l-1}^2\Big),$$

so that $\lim_{t\to\infty} |\partial_x\mathsf{w}(t)|_{l-2} = 0$. Using the interpolation inequality

$$\sup_{x\in\mathbb{R}^d} |\mathsf{w}(t,x) - \mathsf{w}^\star| \leq \mathsf{c}_0\ |\partial_x^{l-1}\mathsf{w}|_0^a\ |\mathsf{w}|_0^{1-a},$$

where $a = d/2(l-1) \in (0,1)$ then completes the proof. $\square$

Applying these results to the system of equations modeling reactive fluids, global existence theorems are obtained uniformly with respect to the chemistry relaxation parameter ϵ for well prepared data. The asymptotic behavior as $\epsilon \to 0$ of such solutions in investigated in the next section.

6. Convergence analysis

The difference $\mathsf{w}-\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathrm{e}}^t\mathsf{u})$ between the normal variable w out of equilibrium and the equilibrium state $\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathrm{e}}^t\mathsf{u})$ associated with the slow variable projection $\mathscr{u}_{\mathrm{e}} = \Pi_{\mathrm{e}}^t\mathsf{u}$ is first estimated in $\mathbb{R}^{\mathsf{n}}$ in an algebraic way. We then consider a solution of the equations in normal form, and following the diagram of Figure 1, it is next estimated to what extend the corresponding normal equilibrium variable $\mathscr{w}_{\mathrm{e}} = \overline{\mathcal{J}}_{\mathrm{e}}\overline{\Pi}_{\mathrm{e}}^t\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathrm{e}}^t\mathsf{u})$ satisfies the equations of the normal variable w_{e} at chemical equilibrium (4.5). Using a stability result at equilibrium the difference $\mathscr{w}_{\mathrm{e}} - \mathsf{w}_{\mathrm{e}}$ is then estimated and next $\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathrm{e}}^t\mathsf{u}) - \overline{\Pi}_{\mathrm{e}}\mathsf{w}_{\mathrm{e}}$ and $\mathsf{w} - \overline{\Pi}_{\mathrm{e}}\mathsf{w}_{\mathrm{e}}$. This yields a convergence theorem towards the chemical equilibrium fluid in the fast chemistry limit $\epsilon \to 0$ and a first rigorous justification of the chemical equilibrium limit system of partial differential equations (4.5).

The first idea is that since $\pi\mathsf{w}$ goes to zero, w should be close to an equilibrium state associated with the corresponding slow variables $\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathrm{e}}^t\mathsf{u})$. A second idea is that the equilibrium states $\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathrm{e}}^t\mathsf{u})$ is such that the corresponding residual for the equations at chemical equilibrium is small and stability at equilibrium completes the proof.

6.1. First estimates

In this section, the differences $\mathsf{u} - \mathsf{u}_{\mathrm{eq}}(\Pi_{\mathrm{e}}^t\mathsf{u})$ and $\mathsf{w} - \mathsf{w}_{\mathrm{eq}}(\Pi_{\mathrm{e}}^t\mathsf{u})$ are estimated is terms of the fast variable out of equilibrium $\pi\mathsf{w}$.

Lemma 6.1. *Let $\mathsf{u} \in \mathbb{R}^{\mathsf{n}}$ be in a neighborhood of $\mathsf{u}^\star$ and denote by $\mathsf{w} \in \mathbb{R}^{\mathsf{n}}$ the corresponding normal variable. Denote by $\mathsf{u}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t\mathsf{u})$ the unique equilibrium chemistry point u_{eq} with projection $\Pi_{\mathsf{e}}^t\mathsf{u}$ in the neighborhood of $\Pi_{\mathsf{e}}^t\mathsf{u}^\star$. There exists a smooth linear operator $\mathcal{L}(\mathsf{u})$ defined in a neighborhood of $\mathsf{u}^\star$ such that*

$$\mathsf{u} - \mathsf{u}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t\mathsf{u}) = \mathcal{L}(\mathsf{u})\pi\mathsf{w} = \epsilon\mathcal{L}(\mathsf{u})\frac{\pi\mathsf{w}}{\epsilon}, \tag{6.1}$$

and $\mathcal{L}$ is in the form

$$\mathcal{L} = \Pi_{\mathsf{r}}\Big\{\int_0^1 \Pi_{\mathsf{r}}^t\,\partial_{\mathsf{u}}\mathsf{v}\big(\mathsf{u}_{\mathrm{eq}} + \tau(\mathsf{u} - \mathsf{u}_{\mathrm{eq}})\big)\Pi_{\mathsf{r}}\,d\tau\Big\}^{-1}\overline{\Pi}_{\mathsf{r}}^t.$$

Proof. From Proposition 2.5 and Lemma 2.6, or equivalently from Proposition 4.1, for any $\mathscr{u}_{\mathrm{e}}$ in the neighborhood of $\mathsf{u}_{\mathrm{e}}^\star = \Pi_{\mathrm{e}}\mathsf{u}^\star$ there exists a unique equilibrium point $\mathsf{u}_{\mathrm{eq}}(\mathscr{u}_{\mathrm{e}})$ with projection $\mathscr{u}_{\mathrm{e}} = \Pi_{\mathrm{e}}^t\mathsf{u}_{\mathrm{eq}}$. We may thus introduce

$\mathsf{u}_{\rm eq}(\Pi_{\rm e}^t \mathsf{u})$ where u is in a neighborhood of $\mathsf{u}^\star$ and the corresponding entropic variable, obtained from the diffeomorphism $\mathsf{u} \mapsto \mathsf{v}$, is denoted by $\mathsf{v}_{\rm eq}$. The difference $\mathsf{v} - \mathsf{v}_{\rm eq}$ may then be written in the form $\mathsf{v} - \mathsf{v}_{\rm eq} = \int_0^1 \partial_{\mathsf{u}}\mathsf{v}\big(\mathsf{u}_{\rm eq} + \tau(\mathsf{u} - \mathsf{u}_{\rm eq})\big)\, {\rm d}\tau\, (\mathsf{u} - \mathsf{u}_{\rm eq})$ so that

$$\Pi_{\mathsf{r}}^t \mathsf{v} = \int_0^1 \Pi_{\mathsf{r}}^t \partial_{\mathsf{u}}\mathsf{v}\big(\mathsf{u}_{\rm eq} + \tau(\mathsf{u} - \mathsf{u}_{\rm eq})\big)\Pi_{\mathsf{r}}\, {\rm d}\tau\ \mathcal{J}_{\mathsf{r}}\Pi_{\mathsf{r}}^t\, (\mathsf{u} - \mathsf{u}_{\rm eq}),$$

since $\Pi_{\mathsf{r}}^t \mathsf{v}_{\rm eq}(\Pi_{\mathsf{e}}^t \mathsf{u}) = 0$, $\tilde{\pi} = \Pi_{\mathsf{r}}\mathcal{J}_{\mathsf{r}}\Pi_{\mathsf{r}}^t$ and $\mathsf{u} - \mathsf{u}_{\rm eq} = \tilde{\pi}(\mathsf{u} - \mathsf{u}_{\rm eq})$ when $\mathsf{u}_{\rm eq} = \mathsf{u}_{\rm eq}(\Pi_{\mathsf{e}}^t \mathsf{u})$. The linear operator $\int_0^1 \Pi_{\mathsf{r}}^t \partial_{\mathsf{u}}\mathsf{v}\big(\mathsf{u}_{\rm eq} + \tau(\mathsf{u} - \mathsf{u}_{\rm eq})\big)\Pi_{\mathsf{r}}\, {\rm d}\tau$ is invertible as an average of positive definite matrices and this yields

$$\mathcal{J}_{\mathsf{r}}\Pi_{\mathsf{r}}^t\, (\mathsf{u} - \mathsf{u}_{\rm eq}) = \Big\{ \int_0^1 \Pi_{\mathsf{r}}^t \partial_{\mathsf{u}}\mathsf{v}\big(\mathsf{u}_{\rm eq} + \tau(\mathsf{u} - \mathsf{u}_{\rm eq})\big)\Pi_{\mathsf{r}}\, {\rm d}\tau \Big\}^{-1} \Pi_{\mathsf{r}}^t \mathsf{v}.$$

Using then $\tilde{\pi} = \Pi_{\mathsf{r}}\mathcal{J}_{\mathsf{r}}\Pi_{\mathsf{r}}^t$ and $\mathsf{u} - \mathsf{u}_{\rm eq} = \tilde{\pi}(\mathsf{u} - \mathsf{u}_{\rm eq})$ one gets

$$\mathsf{u} - \mathsf{u}_{\rm eq}(\Pi_{\mathsf{e}}^t \mathsf{u}) = \Pi_{\mathsf{r}} \Big\{ \int_0^1 \Pi_{\mathsf{r}}^t \partial_{\mathsf{u}}\mathsf{v}\big(\mathsf{u}_{\rm eq} + \tau(\mathsf{u} - \mathsf{u}_{\rm eq})\big)\Pi_{\mathsf{r}}\, {\rm d}\tau \Big\}^{-1} \Pi_{\mathsf{r}}^t \mathsf{v}.$$

The linear operator Π_{r} may be written $\Pi_{\mathsf{r}} = [\mathsf{a}_{\mathsf{n}_{\rm e}+1}, \ldots, \mathsf{a}_{\mathsf{n}}]$ where the basis vectors of the fast manifold $\mathcal{E}^\perp$ are associated with the formation reaction vectors $\mathsf{a}_{k+d+1} = (M\mathfrak{v}_k, \mathbf{0}, 0)^t$, for $k \in \mathfrak{S}\backslash\mathfrak{A} = \{\, n_{\rm a} + 1, \ldots, n \,\}$. The mass weighted formation vectors may be written

$$M\mathfrak{v}_k = (-m_1\mathfrak{a}_{k1}, \ldots, -m_{n_{\rm a}}\mathfrak{a}_{kn_{\rm a}}, 0, \ldots, 0, m_k, 0, \ldots, 0)^t,$$

for $k \in \mathfrak{S}\backslash\mathfrak{A}$ in such a way that

$$(\Pi_{\mathsf{r}}^t \mathsf{v})_{k+d+1} = \langle \mathsf{a}_{k+d+1}, \mathsf{v} \rangle = \frac{1}{RT}\Big(m_k g_k - \sum_{l\in\mathfrak{A}} m_l \mathfrak{a}_{kl} g_l \Big).$$

On the other hand, the linear operator $\overline{\Pi}_{\mathsf{r}}$ may be written $\overline{\Pi}_{\mathsf{r}} = [\overline{\mathsf{a}}_{\mathsf{n}_{\rm e}+1}, \ldots, \overline{\mathsf{a}}_{\mathsf{n}}]$ where the basis vectors of the fast manifold $\overline{\mathcal{E}}^\perp$ are in the form $\overline{\mathsf{a}}_{k+d+1} = (M\mathfrak{v}_k - m_1\mathfrak{v}_{1k}\mathsf{e}_1, \mathbf{0}, 0)^t$, for $k \in \mathfrak{S}\backslash\mathfrak{A} = \{\, n_{\rm a} + 1, \ldots, n \,\}$ where e_i, $i \in \mathfrak{S}$, are the base vectors of $\mathbb{R}^n$. The modified mass weighted formation vectors may be written

$$M\mathfrak{v}_k - m_1\mathfrak{v}_{1k}\mathsf{e}_1 = (0, -m_2\mathfrak{a}_{k2}, \ldots, -\mathfrak{a}_{kn_{\rm a}}, 0, \ldots, 0, m_k, 0, \ldots, 0)^t,$$

for $k \in \mathfrak{S}\backslash\mathfrak{A} = \{\, n_{\rm a} + 1, \ldots, n \,\}$ in such a way that

$$\begin{aligned} (\overline{\Pi}_{\mathsf{r}}^t \mathsf{w})_{k+d+1} &= \langle \overline{\mathsf{a}}_{k+d+1}, \mathsf{w} \rangle \\ &= \frac{1}{RT}\Big(m_k(g_k - g_1) - \sum_{l\in\mathfrak{A}\backslash\{1\}} m_l \mathfrak{a}_{kl}(g_l - g_1) \Big) = (\Pi_{\mathsf{r}}^t \mathsf{v})_{k+d+1}, \end{aligned}$$

using the mass relation $m_k = m_1\mathfrak{a}_{k1} + \sum_{l\in\mathfrak{A}\backslash\{1\}} m_l\mathfrak{a}_{kl}$. This shows that $\Pi_{\mathsf{r}}^t \mathsf{v} = \overline{\Pi}_{\mathsf{r}}^t \mathsf{w}$ and using the above expression of $\mathsf{u} - \mathsf{u}_{\rm eq}(\Pi_{\mathsf{e}}^t \mathsf{u})$ completes the proof. $\square$

Lemma 6.2. *Let* $\mathsf{u} \in \mathbb{R}^{\mathsf{n}}$ *be in a neighborhood of* $\mathsf{u}^\star$ *and denote by* $\mathsf{w} \in \mathbb{R}^{\mathsf{n}}$ *the corresponding normal variable. Denote by* $\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t \mathsf{u})$ *the unique equilibrium chemistry point* w_{eq} *associated with the slow variable* $\Pi_{\mathsf{e}}^t \mathsf{u}$. *There exists a smooth linear operator* $\mathcal{L}'(\mathsf{u})$ *defined in the neighborhood of* $\mathsf{u}^\star$ *such that*

$$\mathsf{w} - \mathsf{w}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t \mathsf{u}) = \mathcal{L}'(\mathsf{u})\pi \mathsf{w} = \epsilon \mathcal{L}'(\mathsf{u}) \frac{\pi \mathsf{w}}{\epsilon}, \tag{6.2}$$

and $\mathcal{L}'$ *is a smooth function of* u *in the neighborhood of* $\mathsf{u}^\star$, *or equivalently a smooth function of* w *in the neighborhood of* $\mathsf{w}^\star$.

Proof. Since w is a smooth function of u, the lemma is a direct consequence of Lemma 6.1 with $\mathcal{L}' = Z\mathcal{L}$ and $Z = \int_0^1 \partial_{\mathsf{u}} \mathsf{w}\big(\mathsf{u}_{\mathrm{eq}} + \tau(\mathsf{u} - \mathsf{u}_{\mathrm{eq}})\big)\, \mathrm{d}\tau$. $\square$

6.2. Residual estimates

The equations governing fluids at chemical equilibrium have been investigated in Section 4.2 and the corresponding system in normal form (4.9) reads

$$\overline{\mathsf{A}}_0^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}})\partial_t \mathsf{w}_{\mathrm{e}} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}})\partial_i \mathsf{w}_{\mathrm{e}} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\overline{\mathsf{B}}_{ij}^{\mathrm{e}}(\mathsf{w}_{\mathrm{e}})\partial_j \mathsf{w}_{\mathrm{e}}\big) - \overline{\mathsf{Q}}_{\mathrm{e}}(\mathsf{w}_{\mathrm{e}}, \partial_x \mathsf{w}_{\mathrm{e}}) = 0, \tag{6.3}$$

where w_{e} denotes the normal variable at equilibrium (4.8). On the other hand, let w be a solution of the system in normal form (5.1) and denote by u the corresponding solution of the equations in conservative form (3.1). We may then consider the natural slow variable $\Pi_{\mathsf{e}}^t \mathsf{u}$ and the corresponding equilibrium state in normal form $\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t \mathsf{u})$ and conservative form $\mathsf{u}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t \mathsf{u})$. Then, according to the diagram of Figure 1, the proper projection of the normal variable $\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t \mathsf{u})$ that should be close to w_{e} is $\mathscr{w}_{\mathrm{e}} = \mathcal{J}_{\mathrm{e}} \overline{\Pi}_{\mathrm{e}}^t \mathsf{w}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t \mathsf{u})$. In order to establish that $\mathscr{w}_{\mathrm{e}}$ is close to w_{e}, that satisfies (6.3), we thus have to investigate the 'default to equilibrium residual' h of $\mathscr{w}_{\mathrm{e}}$ defined by

$$\overline{\mathsf{A}}_0^{\mathrm{e}}(\mathscr{w}_{\mathrm{e}})\partial_t \mathscr{w}_{\mathrm{e}} + \sum_{i\in\mathcal{D}} \overline{\mathsf{A}}_i^{\mathrm{e}}(\mathscr{w}_{\mathrm{e}})\partial_i \mathscr{w}_{\mathrm{e}} - \sum_{i,j\in\mathcal{D}} \partial_i\big(\overline{\mathsf{B}}_{ij}^{\mathrm{e}}(\mathscr{w}_{\mathrm{e}})\partial_j \mathscr{w}_{\mathrm{e}}\big) - \overline{\mathsf{Q}}_{\mathrm{e}}\big(\mathscr{w}_{\mathrm{e}}, \partial_x \mathscr{w}_{\mathrm{e}}\big) = \mathsf{h}. \tag{6.4}$$

From bounds on h one may then estimate the difference $\mathscr{w}_{\mathrm{e}} - \mathsf{w}_{\mathrm{e}}$ and thus $\mathsf{w}_{\mathrm{eq}}(\Pi_{\mathrm{e}}^t \mathsf{u}) - \overline{\Pi}_{\mathrm{e}} \mathsf{w}_{\mathrm{e}}$ and finally $\mathsf{w} - \overline{\Pi}_{\mathrm{e}} \mathsf{w}_{\mathrm{e}}$ using the results of Section 6.1. The residual h is now estimated in the function space $L^2\big((0,\infty), H^{l-3}\big)$ uniformly with respect to the parameter $\epsilon \in (0,1]$. The regularity class $\varkappa$ is still assumed to be such that $\varkappa - 3 \geq l + 1$ but in this section $l \geq l_0 + 3$ with $l_0 = [d/2] + 1$, as more regularity is required for estimating h. The main ideas for estimating this residual h is to evaluate the related residual of the conservative form at equilibrium (4.5), to subtract the equilibrium projection Π_{e}^t of the out of equilibrium equations in conservative form (3.1), and to use the estimates of Section 6.1 and Theorem 5.6.

Theorem 6.3. *Assume that $l \geq l_0 + 3$ and that w_0 is such that $|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \frac{1}{\epsilon}|\pi \mathsf{w}_0|_{l-1}^2$ is uniformly bounded independently of $\epsilon \in (0,1]$. Then the residual h may be written in the form*

$$\mathsf{h} = \epsilon(\overline{\mathsf{h}} + \overline{\mathsf{h}}') \tag{6.5}$$

where $\overline{\mathsf{h}} \in C^0\big([0,\infty), H^{l-2}\big)$ is bounded in $L^2\big((0,\infty), H^{l-2}\big)$ independently of $\epsilon \in (0,1]$ whereas $\overline{\mathsf{h}}' \in C^0\big([0,\infty), H^{l-3}\big)$ with $\overline{\mathsf{h}}'_{\mathrm{I}} = 0$ is bounded in $L^2\big((0,\infty), H^{l-3}\big)$ independently of $\epsilon \in (0,1]$.

Proof. Using the diffeomorphism $\mathsf{w} \mapsto \mathsf{u}$ from $\mathcal{O}_{\mathsf{w}}$ onto $\mathcal{O}_{\mathsf{u}}$ we associate to the solution w its conservative variable $\mathsf{u}(\mathsf{w})$. Then $\mathsf{u}(\mathsf{w})$ is the solution of the system in conservative form (3.1). On the other hand, by definition of the normal form the residual $\mathsf{h} = \mathsf{h}_{w_{\mathrm{e}}}$ defined by (6.4) associated with the normal form (6.3) or (4.9) is directly related to the residual $\mathsf{h}_{u_{\mathrm{e}}}$ associated with the conservative form (4.5) by the relation $\mathsf{h} = \mathsf{h}_{w_{\mathrm{e}}} = \big(\partial_{w_{\mathrm{e}}} v_{\mathrm{e}}\big)^t \mathsf{h}_{u_{\mathrm{e}}}$ where $u_{\mathrm{e}} = \Pi_{\mathrm{e}}^t \mathsf{u}$, and v_{e} and w_{e} are the symmetric and normal variables associated with u_{e}. The residual $\mathsf{h}_{u_{\mathrm{e}}}$ with respect to the conservative variable form is by definition

$$\mathsf{h}_{u_{\mathrm{e}}} = \partial_t \Pi_{\mathsf{e}}^t \mathsf{u} + \sum_{i \in \mathcal{D}} \mathsf{A}_i^{\mathrm{e}}(\Pi_{\mathsf{e}}^t \mathsf{u}) \partial_i \Pi_{\mathsf{e}}^t \mathsf{u} - \sum_{i,j \in \mathcal{D}} \partial_i \big(\mathsf{B}_{ij}^{\mathrm{e}}(\Pi_{\mathsf{e}}^t \mathsf{u}) \partial_j \Pi_{\mathsf{e}}^t \mathsf{u}\big),$$

where $\mathsf{A}_i^{\mathrm{e}}(\mathfrak{r}_{\mathrm{e}}) = \Pi_{\mathsf{e}}^t \mathsf{A}_i\big(\mathsf{u}_{\mathrm{eq}}(\mathfrak{r}_{\mathrm{e}})\big)\partial_{\mathfrak{r}_{\mathrm{e}}} \mathsf{u}_{\mathrm{eq}}(\mathfrak{r}_{\mathrm{e}})$ and $\mathsf{B}_{ij}^{\mathrm{e}}(\mathfrak{r}_{\mathrm{e}}) = \Pi_{\mathsf{e}}^t \mathsf{B}_{ij}\big(\mathsf{u}_{\mathrm{eq}}(\mathfrak{r}_{\mathrm{e}})\big)\partial_{\mathfrak{r}_{\mathrm{e}}} \mathsf{u}_{\mathrm{eq}}(\mathfrak{r}_{\mathrm{e}})$ for any $\mathfrak{r}_{\mathrm{e}}$ in the neighborhood of $\mathsf{u}_{\mathrm{e}}^\star$. Using these expressions of $\mathsf{A}_i^{\mathrm{e}}$ and $\mathsf{B}_{ij}^{\mathrm{e}}$ it is next obtained that

$$\mathsf{h}_{u_{\mathrm{e}}} = \partial_t \Pi_{\mathsf{e}}^t \mathsf{u} + \sum_{i \in \mathcal{D}} \Pi_{\mathsf{e}}^t \mathsf{A}_i(\mathsf{u}_{\mathrm{eq}}) \partial_i \mathsf{u}_{\mathrm{eq}} - \sum_{i,j \in \mathcal{D}} \partial_i \big(\Pi_{\mathsf{e}}^t \mathsf{B}_{ij}(\mathsf{u}_{\mathrm{eq}}) \partial_j \mathsf{u}_{\mathrm{eq}}\big),$$

where $\mathsf{u}_{\mathrm{eq}} = \mathsf{u}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t \mathsf{u})$ is the equilibrium point associated with the slow variable $\Pi_{\mathsf{e}}^t \mathsf{u}$. Since u is the solution of the out of equilibrium equations, one may subtract from the above expression of $\mathsf{h}_{u_{\mathrm{e}}}$ the projection on the slow manifold of the nonequilibrium equations which are zero since u is a solution of the full out of equilibrium system. This directly yields that

$$\begin{aligned}\mathsf{h}_{u_{\mathrm{e}}} = &\sum_{i \in \mathcal{D}} \Pi_{\mathsf{e}}^t \mathsf{A}_i(\mathsf{u}_{\mathrm{eq}}) \partial_i \mathsf{u}_{\mathrm{eq}} - \sum_{i \in \mathcal{D}} \Pi_{\mathsf{e}}^t \mathsf{A}_i(\mathsf{u}) \partial_i \mathsf{u} \\ &- \sum_{i,j \in \mathcal{D}} \partial_i \big(\Pi_{\mathsf{e}}^t \mathsf{B}_{ij}(\mathsf{u}_{\mathrm{eq}}) \partial_j \mathsf{u}_{\mathrm{eq}}\big) + \sum_{i,j \in \mathcal{D}} \partial_i \big(\Pi_{\mathsf{e}}^t \mathsf{B}_{ij}(\mathsf{u}) \partial_j \mathsf{u}\big).\end{aligned}$$

One may thus write $\mathsf{h}_{u_{\mathrm{e}}}$ in the form

$$\begin{aligned}\mathsf{h}_{u_{\mathrm{e}}} = &\sum_{i \in \mathcal{D}} \Pi_{\mathsf{e}}^t \big(\mathsf{A}_i(\mathsf{u}_{\mathrm{eq}}) - \mathsf{A}_i(\mathsf{u})\big) \partial_i \mathsf{u}_{\mathrm{eq}} + \sum_{i \in \mathcal{D}} \Pi_{\mathsf{e}}^t \mathsf{A}_i(\mathsf{u}) \partial_i (\mathsf{u}_{\mathrm{eq}} - \mathsf{u}) \\ &- \sum_{i,j \in \mathcal{D}} \partial_i \Big(\Pi_{\mathsf{e}}^t \big(\mathsf{B}_{ij}(\mathsf{u}_{\mathrm{eq}}) - \mathsf{B}_{ij}(\mathsf{u})\big) \partial_j \mathsf{u}_{\mathrm{eq}}\Big) - \sum_{i,j \in \mathcal{D}} \partial_i \big(\Pi_{\mathsf{e}}^t \mathsf{B}_{ij}(\mathsf{u}) \partial_j (\mathsf{u}_{\mathrm{eq}} - \mathsf{u})\big).\end{aligned}$$

Keeping in mind that $\mathsf{u}_{\mathrm{eq}} = \mathsf{u}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t \mathsf{u})$ is the equilibrium point associated with the slow variable $\Pi_{\mathsf{e}}^t \mathsf{u}$, from the expression of $\mathsf{u} - \mathsf{u}_{\mathrm{eq}}(\Pi_{\mathsf{e}}^t \mathsf{u}) = \epsilon \mathcal{L}(\mathsf{u}) \frac{\pi \mathsf{w}}{\epsilon}$ of Lemma 6.1 one deduces that $\mathsf{h}_{u_{\mathrm{e}}} = \epsilon(\widehat{\mathsf{h}} + \widehat{\mathsf{h}}')$ where

$$\widehat{\mathsf{h}} = -\sum_{i\in\mathcal{D}} \Pi_{\mathsf{e}}^t \big\langle \partial_{\mathsf{u}} \mathsf{A}_i(\mathsf{u}_{\mathrm{eq}}, \mathsf{u}) \big\rangle \mathcal{L}(\mathsf{u}) \frac{\pi \mathsf{w}}{\epsilon} \partial_i \mathsf{u}_{\mathrm{eq}} - \sum_{i\in\mathcal{D}} \Pi_{\mathsf{e}}^t \mathsf{A}_i(\mathsf{u}) \partial_i \big(\mathcal{L}(\mathsf{u}) \frac{\pi \mathsf{w}}{\epsilon} \big), \tag{6.6}$$

$$\widehat{\mathsf{h}}' = \sum_{i,j\in\mathcal{D}} \partial_i \Big(\Pi_{\mathsf{e}}^t \big\langle \partial_{\mathsf{u}} \mathsf{B}_{ij}(\mathsf{u}_{\mathrm{eq}}, \mathsf{u}) \big\rangle \mathcal{L}(\mathsf{u}) \frac{\pi \mathsf{w}}{\epsilon} \partial_j \mathsf{u}_{\mathrm{eq}} \Big) + \sum_{i,j\in\mathcal{D}} \partial_i \Big(\Pi_{\mathsf{e}}^t \mathsf{B}_{ij}(\mathsf{u}) \partial_j \big(\mathcal{L}(\mathsf{u}) \frac{\pi \mathsf{w}}{\epsilon} \big) \Big), \tag{6.7}$$

and the average operator is defined by $\big\langle \phi(\mathsf{u}_{\mathrm{eq}}, \mathsf{u}) \big\rangle = \int_0^1 \phi\big(\theta \mathsf{u}_{\mathrm{eq}} + (1-\theta)\mathsf{u}\big)\, \mathrm{d}\theta$. One may then use the L^∞ bounds of u as well as the uniform estimates of $\pi\mathsf{w}/\epsilon$ in $L^2\big((0,\infty), H^{l-1}\big)$ to conclude that $\widehat{\mathsf{h}}$ is uniformy bounded in $L^2\big((0,\infty), H^{l-2}\big)$ *independently* of $\epsilon \in (0,1]$ whereas $\widehat{\mathsf{h}}'$ is uniformy bounded in $L^2\big((0,\infty), H^{l-3}\big)$ *independently* of $\epsilon \in (0,1]$.

Letting next $\overline{\mathsf{h}} = \big(\partial_{w_{\mathrm{e}}} v_{\mathrm{e}}\big)^t \widehat{\mathsf{h}}$ and $\overline{\mathsf{h}}' = \big(\partial_{w_{\mathrm{e}}} v_{\mathrm{e}}\big)^t \widehat{\mathsf{h}}'$, we have $\mathsf{h} = \mathsf{h}_{w_{\mathrm{e}}} = \epsilon(\overline{\mathsf{h}} + \overline{\mathsf{h}}')$ as well as similar uniform estimates for $\overline{\mathsf{h}}$ and $\overline{\mathsf{h}}'$ as for $\widehat{\mathsf{h}}$ and $\widehat{\mathsf{h}}'$ and moreover $\overline{\mathsf{h}}'_{\mathrm{I}} = 0$ since $\widehat{\mathsf{h}}'_{\mathrm{I}} = 0$ and the left lower $(\mathrm{II}, \mathrm{I})$ block of $\big(\partial_{w_{\mathrm{e}}} v_{\mathrm{e}}\big)^t$ is zero since $(v_{\mathrm{e}})_{\mathrm{II}}$ only depends on $(w_{\mathrm{e}})_{\mathrm{II}}$ and the proof is complete. □

6.3. Stability at equilibrium and convergence

Using estimates of the out of equilibrium solution and stability at chemical equilibrium valid on any finite time interval $[0, \bar{\tau}]$ convergence of the out of equilibrium solution towards the chemical equilibrium solution is now established.

Theorem 6.4. *Let $d \geq 1$, $l \geq l_0 + 3$, $l_0 = [d/2] + 1$, be integers and let $\bar{b}$ from Theorem 5.6. For any $\mathsf{w}_0 \in \overline{\mathcal{O}}_0$ with*

$$\pi \mathsf{w}_0 = 0, \qquad |\mathsf{w}_0 - \mathsf{w}^\star|_l^2 < \bar{b}^2, \tag{6.8}$$

there exists a unique solution w of the out of equilibrium system such that the estimates (5.27) *hold. For $\bar{b}$ small enough there exists also a unique global solution w_{e} of the equilibrium system starting from $\overline{\mathcal{J}}_{\mathrm{e}} \overline{\Pi}_{\mathrm{e}}^t \mathsf{w}_0$. Then the out of equilibrium solution converges toward the chemical equilibrium solution pointwise*

$$\lim_{\epsilon \to 0} \mathsf{w}(t,x) = \overline{\Pi}_{\mathrm{e}} \mathsf{w}_{\mathrm{e}}(t,x), \tag{6.9}$$

and for any time $\bar{\tau}$ there exists a constant c depending on $\bar{\tau}$ with the error estimate

$$\sup_{\tau \in [0,\bar{\tau}]} |\mathsf{w} - \overline{\Pi}_{\mathrm{e}} \mathsf{w}_{\mathrm{e}}|_{l-2} \leq \mathsf{c}\, \epsilon.$$

Proof. Global existence for reactive fluids with fast chemistry is first obtained by combining the symmetrized form of Theorem 3.12 with the existence result of Theorem 5.6. On the other hand, at chemical equilibrium, when there are not anymore sources, one may directly use Kawashima's theory [34] in order to obtain a global existence theorem starting from $\overline{\mathcal{J}}_{\mathrm{e}}\overline{\Pi}_{\mathrm{e}}^{t}\mathsf{w}_0$ assuming that $\bar{b}$ is small enough.

Using estimates of the out of equilibrium solution, it is then deduced that the residual h defined by (6.4) is in the form $\mathsf{h} = \epsilon(\overline{\mathsf{h}} + \overline{\mathsf{h}}')$ where $\overline{\mathsf{h}}$ is uniformy bounded in $L^2\big((0,\infty), H^{l-2}\big)$ and $\overline{\mathsf{h}}'$ uniformy bounded in $L^2\big((0,\infty), H^{l-3}\big)$. One may then use the *local stability theorem* established in Appendix B for the chemical equilibrium equations in normal form with $\mathsf{f} = \epsilon\overline{\mathsf{h}}$, $\mathsf{g} = \epsilon\overline{\mathsf{h}}'$, and l replaced by $l-1$, to deduce that $\mathcal{w}_{\mathrm{e}} - \mathsf{w}_{\mathrm{e}}$ is small in $C\big([0,\bar{\tau}], H^{l-2}\big)$ for any interval since the stability theorem may be used iteratively. We thus obtain that on any time $\bar{\tau}$ there exists c depending on $\bar{\tau}$ with the error estimate $\sup_{\tau\in[0,\bar{\tau}]} |\mathcal{w}_{\mathrm{e}} - \mathsf{w}_{\mathrm{e}}|_{l-2} \leq \mathsf{c}\,\epsilon$ or equivalently $\sup_{\tau\in[0,\bar{\tau}]} |\mathsf{w} - \overline{\Pi}_{\mathrm{e}}\mathsf{w}_{\mathrm{e}}|_{l-2} \leq \mathsf{c}\,\epsilon$. □

7. Conclusion

Global existence theorem have been obtained for multicomponent reactive fluids uniformly with respect to chemical characteristic times. Convergence of the out of equilibrium solution towards the chemical equilibrium fluid solution has been established as well as an error estimate. The limiting system of partial differential equations governing fluids at chemical equilibrium has been rigorously justified in the fast chemistry limit with arbitrary complex chemistry and detailed transport derived from the kinetic theory. Various generalizations may further be investigated as for instance the situation of partial equilibrium chemistry [24]. Higher order Chapman-Enskog expansions may also be investigated [29]. Boundary value problems in domains with boundaries involving surface reactions and surface heat transfer are also of high scientific interest as well as initial layers for ill prepared initial data.

Appendix A. Linearized equations estimates

The linearized estimates used in the local existence theorem of Section 5.2 are established in this Appendix A. In the non stiff case, such estimates have been established notably by Kawashima [34] and the estimates in the stiff case differ by the inclusion of new terms associated with the fast variable involving $\pi\mathsf{w}/\sqrt{\epsilon}$ and $\pi\mathsf{w}/\epsilon$ as well as for the coupling time derivatives. In comparison with previous work [29] it is not assumed that the matrix $\overline{\mathsf{A}}_0$ commutes with the orthogonal projector on the fast manifold π and the derivation is considerably more intricate.

The linearized equations are in the form

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_i\partial_j\widetilde{\mathsf{w}} + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\widetilde{\mathsf{w}} = \mathsf{f} + \mathsf{g}, \tag{A.1}$$

and it is assumed that w is such that

$$\begin{cases} \mathsf{w}_{\mathrm{I}} - \mathsf{w}_{\mathrm{I}}^{\star} \in C^0\big([0,\bar{\tau}], H^l\big) \cap C^1\big([0,\bar{\tau}], H^{l-1}\big), \\ \mathsf{w}_{\mathrm{II}} - \mathsf{w}_{\mathrm{II}}^{\star} \in C^0\big([0,\bar{\tau}], H^l\big) \cap C^1\big([0,\bar{\tau}], H^{l-2}\big) \cap H^1\big((0,\bar{\tau}), H^{l-1}\big), \end{cases} \tag{A.2}$$

where $\bar{\tau} > 0$, $l \geq l_0 + 1$, $l_0 = [d/2] + 1$. The quantities M and M_1 are defined by

$$\sup_{0 \leq \tau \leq \bar{\tau}} |\mathsf{w}(\tau) - \mathsf{w}^{\star}|_l^2 = M^2, \qquad \int_0^{\bar{\tau}} |\partial_t \mathsf{w}(\tau)|_{l-1}^2 \, d\tau = M_1^2. \tag{A.3}$$

It is assumed that $\mathcal{O}_0 \subset \overline{\mathcal{O}}_0 \subset \mathcal{O}_{\mathsf{w}}$, $0 < a_1 < \mathrm{dist}(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathsf{w}})$, and

$$\mathcal{O}_1 = \{\, \mathsf{w} \in \mathcal{O}_{\mathsf{w}};\ \mathrm{dist}(\mathsf{w}, \overline{\mathcal{O}}_0) < a_1 \,\}, \tag{A.4}$$

$\mathsf{w}_0(x) = \mathsf{w}(0,x) \in \mathcal{O}_0$, and $\mathsf{w}(t,x) \in \mathcal{O}_1$, for $(t,x) \in [0,\bar{\tau}] \times \mathbb{R}^d$. Moreover, it is assumed that

$$\mathsf{f} \in C^0\big([0,\bar{\tau}], H^{l-1}\big) \cap L^1\big([0,\bar{\tau}], H^l\big), \tag{A.5}$$

$$\mathsf{g} \in C^0\big([0,\bar{\tau}], H^{l-1}\big), \qquad \mathsf{g}_{\mathrm{I}} = 0. \tag{A.6}$$

Theorem Appendix A.1. *Let $l \geq l_0 + 1$ with $l_0 = [d/2] + 1$ and assume that the solution $\widetilde{\mathsf{w}}$ of the linearized system* (A.1) *is such that*

$$\begin{aligned} &\widetilde{\mathsf{w}}_{\mathrm{I}} - \widetilde{\mathsf{w}}_{\mathrm{I}}^{\star} \in C^0\big([0,\bar{\tau}], H^l\big) \cap C^1\big([0,\bar{\tau}], H^{l-1}\big), \\ &\widetilde{\mathsf{w}}_{\mathrm{II}} - \widetilde{\mathsf{w}}_{\mathrm{II}}^{\star} \in C^0\big([0,\bar{\tau}], H^l\big) \cap C^1\big([0,\bar{\tau}], H^{l-2}\big) \cap L^2\big((0,\bar{\tau}), H^{l+1}\big), \end{aligned} \tag{A.7}$$

where $\widetilde{\mathsf{w}}^{\star} = (\widetilde{\mathsf{w}}_{\mathrm{I}}^{\star}, \widetilde{\mathsf{w}}_{\mathrm{II}}^{\star})^t$ is a constant state $\widetilde{\mathsf{w}}^{\star} \in \overline{\mathcal{E}}$ and denote by $\widetilde{\mathsf{w}}_0$ the initial state $\widetilde{\mathsf{w}}_0(x) = \widetilde{\mathsf{w}}(0,x)$. Then there exists constants $\mathsf{c}_1(\mathcal{O}_1) \geq 1$ and $\mathsf{c}_2(\mathcal{O}_1, M) \geq 1$, with $\mathsf{c}_2(\mathcal{O}_1, M)$ increasing with M, such that for any $t \in [0,\bar{\tau}]$

$$\begin{aligned} \sup_{0 \leq \tau \leq t} \Big\{ |\widetilde{\mathsf{w}}(\tau) - \widetilde{\mathsf{w}}^{\star}|_l^2 + \frac{1}{\epsilon} |\pi \widetilde{\mathsf{w}}(\tau)|_0^2 \Big\} + \int_0^t |\widetilde{\mathsf{w}}_{\mathrm{II}}(\tau) - \widetilde{\mathsf{w}}_{\mathrm{II}}^{\star}|_{l+1}^2 \, d\tau + \frac{1}{\epsilon} \int_0^t |\pi \widetilde{\mathsf{w}}(\tau)|_l^2 \, d\tau \\ \leq\ \mathsf{c}_1^2 \exp\big(\mathsf{c}_2(t + M_1\sqrt{t})\big) \Big(|\widetilde{\mathsf{w}}_0 - \widetilde{\mathsf{w}}^{\star}|_l^2 + \frac{1}{\epsilon} |\pi \widetilde{\mathsf{w}}_0|_{l-1}^2 \\ + \mathsf{c}_2 \Big\{ \int_0^t |\mathsf{f}|_l \, d\tau \Big\}^2 + \mathsf{c}_2 \int_0^t |\mathsf{f}|_{l-1}^2 \, d\tau + \mathsf{c}_2 \int_0^t |\mathsf{g}_{\mathrm{II}}|_{l-1}^2 \, d\tau \Big), \end{aligned} \tag{A.8}$$

$$\begin{aligned} \frac{1}{\epsilon} \sup_{0 \leq \tau \leq t} |\pi \widetilde{\mathsf{w}}(\tau)|_{l-1}^2 + \frac{1}{\epsilon^2} \int_0^t |\pi \widetilde{\mathsf{w}}(\tau)|_{l-1}^2 \, d\tau + \int_0^t |\partial_t \widetilde{\mathsf{w}}(\tau)|_{l-1}^2 \, d\tau \\ \leq \mathsf{c}_2 \exp\big(\mathsf{c}_2(t + M_1\sqrt{t})\big) \Big(|\widetilde{\mathsf{w}}_0 - \widetilde{\mathsf{w}}^{\star}|_l^2 + \frac{1}{\epsilon} |\pi \widetilde{\mathsf{w}}_0|_{l-1}^2 \\ + \Big\{ \int_0^t |\mathsf{f}|_l \, d\tau \Big\}^2 + \int_0^t |\mathsf{f}|_{l-1}^2 \, d\tau + \int_0^t |\mathsf{g}_{\mathrm{II}}|_{l-1}^2 \, d\tau \Big). \end{aligned} \tag{A.9}$$

Proof. In the following $\delta_1 = \delta(\mathcal{O}_1) \leq 1$ denotes a generic small constant only depending on $\mathcal{O}_1$, $\mathsf{c}_1 = \mathsf{c}_1(\mathcal{O}_1) \geq 1$ a generic large constant only depending on $\mathcal{O}_1$, and $\mathsf{c}_2 = \mathsf{c}_2(\mathcal{O}_1, M) \geq 1$ a generic large constant depending on $\mathcal{O}_1$ and M. The various occurrences of these constants may be distinguished and the minimum of all δ_1 and the maxima of all c_1 and c_2 may be taken at the end of the proof so that only single constants ultimately remain. The dependence on d, l, n of these estimating constants is left implicit. For $k \geq 0$ and $\phi \in H^k$ the norms $E_k^2(\phi)$ and $\widehat{E}_k^2(\phi)$ are defined by

$$E_k^2(\phi) = \sum_{0 \leq |\alpha| \leq k} \frac{|\alpha|!}{\alpha!} \int_{\mathbb{R}^d} \big\langle \overline{\mathsf{A}}_0(\mathsf{w}) \partial^\alpha \phi, \partial^\alpha \phi \big\rangle \, \mathrm{d}x, \tag{A.10}$$

$$\widehat{E}_k^2(\phi) = \sum_{0 \leq |\alpha| \leq k} \frac{|\alpha|!}{\alpha!} \int_{\mathbb{R}^d} \big\langle \overline{\mathsf{L}}(\mathsf{w}) \partial^\alpha \phi, \partial^\alpha \phi \big\rangle \, \mathrm{d}x. \tag{A.11}$$

In order to alleviate notation in the proof $\delta\widetilde{\mathsf{w}}$ denotes for short $\delta\widetilde{\mathsf{w}} = \widetilde{\mathsf{w}} - \widetilde{\mathsf{w}}^\star$. The estimates $|f(\phi) - f(0)|_k \leq \mathsf{c}_0 \|f\|_{\mathcal{C}^k(\mathcal{O}_\phi)} (1 + \|\phi\|_{L^\infty})^{k-1} \, |\phi|_k$ where $k \geq 1$, $\phi \in H^k(\mathbb{R}^d)$, $\mathcal{O}_\phi$ is an open ball that contains the range of ϕ, f is a C^k function over $\mathcal{O}_\phi$, $|uv|_k^2 \leq \mathsf{c}_0 |u|_{\bar{l}}^2 |v|_k^2$, for $0 \leq k \leq \bar{l}$, and $\|\phi\|_{L^\infty} \leq \mathsf{c}_0 |\phi|_{\bar{l}}$ for any $\bar{l} \geq l_0 = [d/2] + 1$ are used in the following where c_0 denotes a generic constant independent of $\mathcal{O}_1$ and M. The commutator inequality $\sum_{0 \leq |\alpha| \leq l} \big|[\partial^\alpha, u]v\big|_0 \leq \mathsf{c}_0 |\partial_x u|_{l-1} |v|_{l-1}$ also holds for any $l \geq l_0 + 1$ where $[\partial^\alpha, u]v = \partial^\alpha(uv) - u\partial^\alpha v$ denotes the commutator between ∂^α and u. The Garding inequality also reads [53]

$$\delta_1 |\phi_{\mathrm{II}}|_1^2 \leq \sum_{i,j \in \mathcal{D}} \int_{\mathbb{R}^d} \langle \overline{\mathsf{B}}_{ij}^{\mathrm{II},\mathrm{II}}(\mathsf{w}) \partial_i \phi_{\mathrm{II}}, \partial_j \phi_{\mathrm{II}} \rangle \, \mathrm{d}x + \mathsf{c}_2 |\phi_{\mathrm{II}}|_0^2,$$

for ϕ_{II} vector valued function $\phi_{\mathrm{II}} : \mathbb{R}^d \mapsto \mathbb{R}^{\mathsf{n}_{\mathrm{II}}}$ in the space H^1.

The projected fast normal variable $\pi\widetilde{\mathsf{w}}$ also satisfies a system of partial differential equations obtained in a way similar to (3.9) by applying the proper projectors that reads

$$\overline{\mathsf{A}}_0^\pi(\mathsf{w}) \partial_t \pi\widetilde{\mathsf{w}} \ + \sum_{i \in \mathcal{D}} \overline{\mathsf{A}}_i^\pi(\mathsf{w}) \partial_i \widetilde{\mathsf{w}} - \sum_{i,j \in \mathcal{D}} \overline{\mathsf{B}}_{ij}^\pi(\mathsf{w}) \partial_i \partial_j \widetilde{\mathsf{w}} + \frac{1}{\epsilon} \overline{\mathsf{L}}(\mathsf{w}) \pi\widetilde{\mathsf{w}} = \mathsf{f}^\pi + \mathsf{g}^\pi, \tag{A.12}$$

keeping the notation of Proposition 3.9 for $\overline{\mathsf{A}}_0^\pi$, $\overline{\mathsf{A}}_i^\pi$, $\overline{\mathsf{B}}_{ij}^\pi$, whereas $\mathsf{f}^\pi = \pi\pi_{\mathsf{A}}\mathsf{f}$ and $\mathsf{g}^\pi = \pi\pi_{\mathsf{A}}\mathsf{g}$. The corresponding norm is denoted by

$$\overline{E}_k^2(\phi) = \sum_{0 \leq |\alpha| \leq k} \frac{|\alpha|!}{\alpha!} \int_{\mathbb{R}^d} \big\langle \overline{\mathsf{A}}_0^\pi(\mathsf{w}) \partial^\alpha \phi, \partial^\alpha \phi \big\rangle \, \mathrm{d}x. \tag{A.13}$$

Step 1. Zeroth order inequalities. Multiplying (A.1) by $\delta\widetilde{\mathsf{w}} = \widetilde{\mathsf{w}} - \widetilde{\mathsf{w}}^\star$ integrating over $\mathbb{R}^d$, using the symmetry of $\overline{\mathsf{A}}_0$ and $\overline{\mathsf{A}}_i$ and Garding inequality, noting that

$|\partial_x \mathsf{w}|_{l-1} \leq M$ and that c_2 may depend on M, yields after some algebra that

$$\partial_t E_0^2(\delta\widetilde{\mathsf{w}}) + \delta_1 |\delta\widetilde{\mathsf{w}}_{\text{II}}|_1^2 + \frac{\delta_1}{\epsilon}|\pi\widetilde{\mathsf{w}}|_0^2 \leq \mathsf{c}_1|\mathsf{f}|_0|\delta\widetilde{\mathsf{w}}|_0 + \mathsf{c}_1|\mathsf{g}_{\text{II}}|_0^2 + \mathsf{c}_2(1+|\partial_t \mathsf{w}|_{l-1})E_0^2(\delta\widetilde{\mathsf{w}}). \tag{A.14}$$

Multiplying the projected governing equation (A.12) by $(1/\epsilon)\pi\widetilde{\mathsf{w}}$ and proceeding similarly yields

$$\frac{1}{\epsilon}\partial_t \overline{E}_0^2(\pi\widetilde{\mathsf{w}}) + \frac{\delta_1}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_0^2 \leq \mathsf{c}_1|\mathsf{f}|_0^2 + \mathsf{c}_1|\mathsf{g}_{\text{II}}|_0^2 + \frac{\mathsf{c}_1}{\epsilon}|\partial_t \mathsf{w}|_{l-1}\overline{E}_0^2(\pi\widetilde{\mathsf{w}}) + \mathsf{c}_1|\delta\widetilde{\mathsf{w}}|_1^2 + \mathsf{c}_1|\delta\widetilde{\mathsf{w}}_{\text{II}}|_2^2. \tag{A.15}$$

In addition, multiplying the governing equation by $\partial_t\widetilde{\mathsf{w}}$, integrating over $\mathbb{R}^d$, using the symmetry of $\overline{\mathsf{L}}$, one obtains that

$$\delta_1|\partial_t\widetilde{\mathsf{w}}|_0^2 + \frac{1}{\epsilon}\partial_t \widehat{E}_0^2(\pi\widetilde{\mathsf{w}}) \leq \mathsf{c}_1|\mathsf{f}|_0^2 + \mathsf{c}_1|\mathsf{g}_{\text{II}}|_0^2 + \frac{\mathsf{c}_1}{\epsilon}|\partial_t \mathsf{w}|_{l-1}\widehat{E}_0^2(\pi\widetilde{\mathsf{w}}) + \mathsf{c}_1|\delta\widetilde{\mathsf{w}}|_1^2 + \mathsf{c}_1|\delta\widetilde{\mathsf{w}}_{\text{II}}|_2^2. \tag{A.16}$$

Step 2. The lth order inequality. Applying the αth spatial derivative operator ∂^α to (A.1) yields

$$\overline{\mathsf{A}}_0(\mathsf{w})\partial_t\partial^\alpha\widetilde{\mathsf{w}} + \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_i(\mathsf{w})\partial_i\partial^\alpha\widetilde{\mathsf{w}} - \sum_{i,j\in\mathcal{D}}\overline{\mathsf{B}}_{ij}(\mathsf{w})\partial_i\partial_j\partial^\alpha\widetilde{\mathsf{w}} + \frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\partial^\alpha\widetilde{\mathsf{w}} = \mathsf{h}^\alpha, \tag{A.17}$$

where

$$\begin{aligned}\mathsf{h}^\alpha =& \overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{f}\big) + \overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big) - \sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{A}}_i\big]\partial_i\widetilde{\mathsf{w}}\\ &- \frac{1}{\epsilon}\overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\big]\pi\widetilde{\mathsf{w}} + \sum_{i,j\in\mathcal{D}}\overline{\mathsf{A}}_0\big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}_{ij}\big]\partial_i\partial_j\widetilde{\mathsf{w}}.\end{aligned}$$

Multiplying (A.17) by $\partial^\alpha\delta\widetilde{\mathsf{w}}$ and $|\alpha|!/\alpha!$, integrating over $\mathbb{R}^d$, summing over $0 \leq |\alpha| \leq l$, and proceeding as for the zeroth order estimate (A.14), it is obtained that

$$\begin{aligned}\partial_t E_l^2(\delta\widetilde{\mathsf{w}}) + \delta_1|\delta\widetilde{\mathsf{w}}_{\text{II}}|_{l+1}^2 + \frac{\delta_1}{\epsilon}|\pi\widetilde{\mathsf{w}}|_l^2 \leq& \mathsf{c}_2(1+|\partial_t\mathsf{w}|_{l-1})E_l^2(\delta\widetilde{\mathsf{w}})\\ &+ \sum_{0\leq|\alpha|\leq l}\frac{|\alpha|!}{\alpha!}\int_{\mathbb{R}^d}\langle\mathsf{h}^\alpha, \partial^\alpha\delta\widetilde{\mathsf{w}}\rangle\,\mathrm{d}x.\end{aligned}$$

Keeping in mind that the zeroth order terms with $\alpha = 0$ in the residuals $\int_{\mathbb{R}^d}\langle\mathsf{h}^\alpha, \partial^\alpha\delta\widetilde{\mathsf{w}}\rangle\,\mathrm{d}x$ have already been examined with (A.14) it is sufficient to analyze the terms such that $1 \leq |\alpha| \leq l$. The nonstiff terms are estimated using commutator estimates (and integration by parts for the terms $\overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big)$ when $|\alpha| = l$) and this yields [34]

$$\Big|\int_{\mathbb{R}^d}\big\langle\overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{f}\big), \partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,\mathrm{d}x\Big| \leq |\overline{\mathsf{A}}_0|_\infty\,|\overline{\mathsf{A}}_0^{-1}\mathsf{f}|_l\,|\delta\widetilde{\mathsf{w}}|_l \leq \mathsf{c}_2|\mathsf{f}|_l\,|\delta\widetilde{\mathsf{w}}|_l,$$

$$\Big|\int_{\mathbb{R}^d}\big\langle\overline{\mathsf{A}}_0\partial^\alpha\big(\overline{\mathsf{A}}_0^{-1}\mathsf{g}\big),\partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,\mathrm{d}x\Big|\leq \mathsf{c}_2|\mathsf{g}_{\mathrm{II}}|_{l-1}\,|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l+1},$$

$$\Big|\int_{\mathbb{R}^d}\big\langle\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{A}}_i\big]\partial_i\widetilde{\mathsf{w}},\partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,\mathrm{d}x\Big|\leq \mathsf{c}_2|\delta\widetilde{\mathsf{w}}|_l^2,$$

$$\Big|\int_{\mathbb{R}^d}\big\langle\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{B}}_{ij}\big]\partial_i\partial_j\widetilde{\mathsf{w}},\partial^\alpha\delta\widetilde{\mathsf{w}}\big\rangle\,\mathrm{d}x\Big|\leq \mathsf{c}_2\,|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l+1}\,|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_l.$$

On the other hand, for the stiff terms $\frac{1}{\epsilon}\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\big]\pi\widetilde{\mathsf{w}}$ it is obtained that

$$\frac{1}{\epsilon}\Big|\int_{\mathbb{R}^d}\big\langle\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\big]\pi\widetilde{\mathsf{w}},\partial^\alpha\widetilde{\mathsf{w}}\big\rangle\,\mathrm{d}x\Big|\leq\frac{\mathsf{c}_2}{\epsilon}|\pi\widetilde{\mathsf{w}}|_{l-1}|\delta\widetilde{\mathsf{w}}|_l,$$

and this is an important difference with the 'commutative case'. Indeed, when $\pi\overline{\mathsf{A}}_0=\overline{\mathsf{A}}_0\pi$ then $\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\big]=\pi\overline{\mathsf{A}}_0\big[\partial^\alpha,\overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\big]$ so that the right hand side is simplified into $(\mathsf{c}_2/\epsilon)|\pi\widetilde{\mathsf{w}}|_{l-1}|\pi\widetilde{\mathsf{w}}|_l$ that is much easier to handle using the term $|\pi\widetilde{\mathsf{w}}|_l^2/\epsilon$ of the lth order entropic estimate, interpolation inequalities, and the 0th order entropic estimate. In the noncommutative case, the upper bound thus involves the product $\mathsf{c}_2|\pi\widetilde{\mathsf{w}}|_{l-1}|\delta\widetilde{\mathsf{w}}|_l/\epsilon$ instead of the easier $\mathsf{c}_2|\pi\widetilde{\mathsf{w}}|_{l-1}|\pi\widetilde{\mathsf{w}}|_l/\epsilon$. Collecting all contributions it has been established that

$$\begin{aligned}\partial_t E_l^2(\delta\widetilde{\mathsf{w}})+\delta_1|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l+1}^2+&\frac{\delta_1}{\epsilon}|\pi\widetilde{\mathsf{w}}|_l^2\leq\mathsf{c}_2(1+|\partial_t\mathsf{w}|_{l-1})E_l^2(\delta\widetilde{\mathsf{w}})\\&+\mathsf{c}_2|\mathsf{f}|_l E_l(\delta\widetilde{\mathsf{w}})+\mathsf{c}_2|\mathsf{g}_{\mathrm{II}}|_{l-1}^2+\frac{\mathsf{c}_2}{\epsilon}|\pi\widetilde{\mathsf{w}}|_{l-1}E_l(\delta\widetilde{\mathsf{w}}).\end{aligned}\tag{A.18}$$

In order to handle the term $\frac{\mathsf{c}_2}{\epsilon}|\pi\widetilde{\mathsf{w}}|_{l-1}E_l(\delta\widetilde{\mathsf{w}})$ in the right hand side, a new inequality involving the projected normal variable $\pi\mathsf{w}$ is now required.

Step 3. The $(l-1)$th order projected inequality. Differentiating (A.12) with respect to the space variable yields

$$\overline{\mathsf{A}}_0^\pi(\mathsf{w})\partial_t\partial^\alpha\pi\widetilde{\mathsf{w}}\;+\frac{1}{\epsilon}\overline{\mathsf{L}}(\mathsf{w})\partial^\alpha\pi\widetilde{\mathsf{w}}=\mathsf{h}^{\pi\alpha},\tag{A.19}$$

where

$$\begin{aligned}\mathsf{h}^{\pi\alpha}=&\overline{\mathsf{A}}_0^\pi\partial^\alpha\big(\,(\overline{\mathsf{A}}_0^\pi)^{-1}\mathsf{f}^\pi\,\big)+\overline{\mathsf{A}}_0^\pi\partial^\alpha\big(\,(\overline{\mathsf{A}}_0^\pi)^{-1}\mathsf{g}^\pi\,\big)-\sum_{i\in\mathcal{D}}\overline{\mathsf{A}}_0^\pi\partial^\alpha\big((\overline{\mathsf{A}}_0^\pi)^{-1}\,\overline{\mathsf{A}}_i^\pi\partial_i\widetilde{\mathsf{w}}\big)\\&-\frac{1}{\epsilon}\overline{\mathsf{A}}_0^\pi\big[\partial^\alpha,(\overline{\mathsf{A}}_0^\pi)^{-1}\,\overline{\mathsf{L}}\big]\pi\widetilde{\mathsf{w}}+\sum_{i,j\in\mathcal{D}}\overline{\mathsf{A}}_0^\pi\partial^\alpha\big((\overline{\mathsf{A}}_0^\pi)^{-1}\,\overline{\mathsf{B}}_{ij}^\pi\partial_i\partial_j\widetilde{\mathsf{w}}\big).\end{aligned}\tag{A.20}$$

Multiplying (A.19) by $\partial^\alpha\pi\widetilde{\mathsf{w}}/\epsilon$ and $|\alpha|!/\alpha!$, integrating over $x\in\mathbb{R}^d$, summing over $0\leq|\alpha|\leq l-1$, and proceeding as above yields that

$$\frac{1}{\epsilon}\partial_t\overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}})+\frac{\delta_1}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_{l-1}^2\leq\frac{\mathsf{c}_2}{\epsilon}|\partial_t\mathsf{w}|_{l-1}\overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}})+\sum_{0\leq|\alpha|\leq l}\mathsf{c}_1\,|\mathsf{h}^{\pi\alpha}|_0^2.$$

Since the zeroth order residuals $|\mathsf{h}^{\pi 0}|_0^2$ for $|\alpha| = 0$ has already been examined with (A.15), it is sufficient to analyze the terms $|\mathsf{h}^{\pi\alpha}|_0^2$ for $1 \leq |\alpha| \leq l-1$. The nonstiff terms are estimated directly (using commutator estimates) and it is obtained that

$$\int_{\mathbb{R}^d} \big|\overline{\mathsf{A}}_0^\pi \partial^\alpha \big((\overline{\mathsf{A}}_0^\pi)^{-1} \mathsf{f}^\pi \big)\big|^2 \,\mathrm{d}x \leq \mathsf{c}_2 |\mathsf{f}|_{l-1}^2,$$

$$\int_{\mathbb{R}^d} \big|\overline{\mathsf{A}}_0^\pi \partial^\alpha \big((\overline{\mathsf{A}}_0^\pi)^{-1} \mathsf{g}^\pi \big)\big|^2 \,\mathrm{d}x \leq \mathsf{c}_2 |\mathsf{g}|_{l-1}^2,$$

$$\sum_{i\in\mathcal{D}} \int_{\mathbb{R}^d} \big|\overline{\mathsf{A}}_0^\pi \partial^\alpha \big((\overline{\mathsf{A}}_0^\pi)^{-1} \, \overline{\mathsf{A}}_i^\pi \partial_i \widetilde{\mathsf{w}} \big)\big|^2 \,\mathrm{d}x \leq \mathsf{c}_1 |\delta\widetilde{\mathsf{w}}|_l^2 + \mathsf{c}_2 |\delta\widetilde{\mathsf{w}}|_{l-1}^2,$$

$$\int_{\mathbb{R}^d} \Big| \sum_{i,j\in\mathcal{D}} \overline{\mathsf{A}}_0^\pi \partial^\alpha \big((\overline{\mathsf{A}}_0^\pi)^{-1} \, \overline{\mathsf{B}}_{ij}^\pi \partial_i \partial_j \widetilde{\mathsf{w}} \big)\Big|^2 \,\mathrm{d}x \leq \mathsf{c}_1 |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l+1}^2 + \mathsf{c}_2 |\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_l^2.$$

On the other hand, for the stiff term $\frac{1}{\epsilon}\overline{\mathsf{A}}_0 \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\,\big]\pi\widetilde{\mathsf{w}}$ it is obtained that

$$\frac{1}{\epsilon^2} \int_{\mathbb{R}^d} \big|\overline{\mathsf{A}}_0 \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1}\,\overline{\mathsf{L}}\,\big]\pi\widetilde{\mathsf{w}}\big|^2 \,\mathrm{d}x \leq \frac{\mathsf{c}_2}{\epsilon^2} |\pi\widetilde{\mathsf{w}}|_{l-2}^2.$$

Collecting all contributions yields

$$\begin{aligned} \frac{1}{\epsilon}\partial_t \overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}}) + \frac{\delta_1}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_{l-1}^2 \leq & \frac{\mathsf{c}_2}{\epsilon}|\partial_t \mathsf{w}|_{l-1} \overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}}) + \mathsf{c}_2|\mathsf{f}|_{l-1}^2 + \mathsf{c}_2|\mathsf{g}_{\mathrm{II}}|_{l-1}^2 \\ & + \mathsf{c}_2|\delta\widetilde{\mathsf{w}}|_l^2 + \mathsf{c}_1|\delta\widetilde{\mathsf{w}}_{\mathrm{II}}|_{l+1}^2 + \frac{\mathsf{c}_2}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_{l-2}^2. \end{aligned} \tag{A.21}$$

Step 4. The combined estimate. In order to handle the term $\frac{\mathsf{c}_2}{\epsilon}|\pi\widetilde{\mathsf{w}}|_{l-1}E_l(\delta\widetilde{\mathsf{w}})$ arising in the right hand side of (A.18) the following inequality is used where $\delta > 0$ is a positive number

$$\frac{\mathsf{c}_2}{\epsilon}|\pi\widetilde{\mathsf{w}}|_{l-1}E_l(\delta\widetilde{\mathsf{w}}) \leq \delta\frac{|\pi\widetilde{\mathsf{w}}|_{l-1}^2}{\epsilon^2} + \frac{\mathsf{c}_2^2}{\delta}E_l^2(\delta\widetilde{\mathsf{w}}). \tag{A.22}$$

From the interpolation inequality $|\phi|_{l-2} \leq \mathsf{c}_0|\phi|_{l-1}^{\frac{l-2}{l-1}}|\phi|_0^{\frac{1}{l-1}}$ and Holder's inequality one also obtains the following estimate where $\theta > 0$ and $\delta' > 0$ are positive numbers

$$\theta\mathsf{c}_2|\pi\widetilde{\mathsf{w}}|_{l-2}^2 \leq \delta'|\pi\widetilde{\mathsf{w}}|_{l-1}^2 + \frac{(\theta\mathsf{c}_2\mathsf{c}_0)^{l-1}}{\delta'^{\,l-2}}|\pi\widetilde{\mathsf{w}}|_0^2. \tag{A.23}$$

Forming the combination (A.18) + θ (A.21) + θ' (A.15) now yields that

$$\partial_t E_l^2(\delta\widetilde{\mathsf{w}}) + \delta_1|\delta\widetilde{\mathsf{w}}_{\text{II}}|_{l+1}^2 + \frac{\delta_1}{\epsilon}|\pi\widetilde{\mathsf{w}}|_l^2 + \frac{\theta}{\epsilon}\partial_t\overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}}) + \frac{\theta\delta_1}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_{l-1}^2$$
$$+ \frac{\theta'}{\epsilon}\partial_t\overline{E}_0^2(\pi\widetilde{\mathsf{w}}) + \frac{\theta'\delta_1}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_0^2 \leq \mathsf{c}_2(1+|\partial_t\mathsf{w}|_{l-1})E_l^2(\delta\widetilde{\mathsf{w}}) + \mathsf{c}_2|\mathsf{f}|_l E_l(\delta\widetilde{\mathsf{w}})$$
$$+ \mathsf{c}_2|\mathsf{g}_{\text{II}}|_{l-1}^2 + \frac{\mathsf{c}_2}{\epsilon}|\pi\widetilde{\mathsf{w}}|_{l-1}E_l(\delta\widetilde{\mathsf{w}}) + \theta\Big(\frac{\mathsf{c}_2}{\epsilon}|\partial_t\mathsf{w}|_{l-1}\overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}})$$
$$+ \mathsf{c}_2|\mathsf{f}|_{l-1}^2 + \mathsf{c}_2|\mathsf{g}_{\text{II}}|_{l-1}^2 + \mathsf{c}_2|\delta\widetilde{\mathsf{w}}|_l^2 + \mathsf{c}_1|\delta\widetilde{\mathsf{w}}_{\text{II}}|_{l+1}^2 + \frac{\mathsf{c}_2}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_{l-2}^2\Big)$$
$$+ \theta'\Big(\mathsf{c}_1|\mathsf{f}|_0^2 + \mathsf{c}_2|\mathsf{g}_{\text{II}}|_0^2 + \frac{\mathsf{c}_1}{\epsilon}|\partial_t\mathsf{w}|_{l-1}\overline{E}_0^2(\pi\widetilde{\mathsf{w}}) + \mathsf{c}_1|\delta\widetilde{\mathsf{w}}|_1^2 + \mathsf{c}_1|\delta\widetilde{\mathsf{w}}_{\text{II}}|_2^2\Big). \qquad \text{(A.24)}$$

Combining (A.24) with (A.22) and (A.23) next gives that

$$\partial_t\Big(E_l^2(\delta\widetilde{\mathsf{w}}) + \frac{\theta}{\epsilon}\overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}}) + \frac{\theta'}{\epsilon}\overline{E}_0^2(\pi\widetilde{\mathsf{w}})\Big) + (\delta_1 - \theta\mathsf{c}_1)|\delta\widetilde{\mathsf{w}}_{\text{II}}|_{l+1}^2$$
$$+ \frac{\delta_1}{\epsilon}|\pi\widetilde{\mathsf{w}}|_l^2 + \frac{\theta\delta_1 - \delta - \delta'}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_{l-1}^2 + \frac{\theta'\delta_1 - \frac{(\theta\mathsf{c}_2\mathsf{c}_0)^{l-1}}{\delta'^{\,l-2}}}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_{l-1}^2$$
$$\leq \mathsf{c}_2(1+|\partial_t\mathsf{w}|_{l-1})\Big(E_l^2(\delta\widetilde{\mathsf{w}}) + \frac{\theta}{\epsilon}\overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}}) + \frac{\theta'}{\epsilon}\overline{E}_0^2(\pi\widetilde{\mathsf{w}})\Big)$$
$$+ \mathsf{c}_2|\mathsf{f}|_l E_l(\delta\widetilde{\mathsf{w}}) + \mathsf{c}_2|\mathsf{f}|_{l-1}^2 + \mathsf{c}_2|\mathsf{g}_{\text{II}}|_{l-1}^2. \qquad \text{(A.25)}$$

In order to control the terms in the left hand side of (A.25) and it is required to insure that

$$\delta_1 > \theta\mathsf{c}_1, \qquad \theta\delta_1 > \delta + \delta', \qquad \theta'\delta_1 > \frac{(\theta\mathsf{c}_2\mathsf{c}_0)^{l-1}}{\delta'^{\,l-2}}. \qquad \text{(A.26)}$$

To this aim, one may chose

$$\theta = \frac{4\delta}{\delta_1}, \qquad \delta = \delta', \qquad \theta' = \frac{1}{\delta_1},$$

with δ small enough such that

$$\delta < \min\Big\{\frac{\delta_1^2}{4\mathsf{c}_1}, \Big(\frac{\delta_1}{4\mathsf{c}_0\mathsf{c}_2}\Big)^{l-1}, 1\Big\},$$

and this choice guarantee that (A.26) holds in such a way that

$$\partial_t\Big(E_l^2(\delta\widetilde{\mathsf{w}}) + \frac{\theta}{\epsilon}\overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}}) + \frac{\theta'}{\epsilon}\overline{E}_0^2(\pi\widetilde{\mathsf{w}})\Big) + \delta_1|\delta\widetilde{\mathsf{w}}_{\text{II}}|_{l+1}^2 + \frac{\delta_1}{\epsilon}|\pi\widetilde{\mathsf{w}}|_l^2 + \frac{\theta\delta_1}{\epsilon^2}|\pi\widetilde{\mathsf{w}}|_{l-1}^2$$
$$\leq \mathsf{c}_2(1+|\partial_t\mathsf{w}|_{l-1})\Big(E_l^2(\delta\widetilde{\mathsf{w}}) + \frac{\theta}{\epsilon}\overline{E}_{l-1}^2(\pi\widetilde{\mathsf{w}}) + \frac{\theta'}{\epsilon}\overline{E}_0^2(\pi\widetilde{\mathsf{w}})\Big)$$
$$+ \mathsf{c}_2|\mathsf{f}|_l E_l(\delta\widetilde{\mathsf{w}}) + \mathsf{c}_2|\mathsf{f}|_{l-1}^2 + \mathsf{c}_2|\mathsf{g}_{\text{II}}|_{l-1}^2. \qquad \text{(A.27)}$$

Using Gronwall lemma then yields the first estimates (A.8).

Step 5. The lth order derived estimate. Multiplying equation (A.17) by $\partial_t \partial^\alpha \widetilde{\mathsf{w}}$ and $|\alpha|!/\alpha!$, integrating over $\mathbb{R}^d$, summing over $0 \le |\alpha| \le l-1$, and proceeding as for the zeroth order derived estimate, it is obtained that

$$\begin{aligned}\delta_1 |\partial_t \widetilde{\mathsf{w}}|_{l-1}^2 + \frac{1}{\epsilon} \partial_t \widehat{E}_{l-1}^2(\pi \widetilde{\mathsf{w}}) \le & \mathsf{c}_1 |\delta \widetilde{\mathsf{w}}|_l^2 + \mathsf{c}_1 |\delta \widetilde{\mathsf{w}}_{\mathrm{II}}|_{l+1}^2 \\ & + \frac{\mathsf{c}_1}{\epsilon} |\partial_t \mathsf{w}|_{l-1} \widehat{E}_{l-1}^2(\pi \widetilde{\mathsf{w}}) + \sum_{0 \le |\alpha| \le l-1} \mathsf{c}_1 |\mathsf{h}^\alpha|_0^2.\end{aligned}$$

Keeping in mind that the zeroth order residuals $|\mathsf{h}^0|_0^2$ with $|\alpha| = 0$ have already been examined with (A.16), it is sufficient to analyze the terms $|\mathsf{h}^\alpha|_0^2$ when $1 \le |\alpha| \le l-1$.

The nonstiff terms in h^α are estimated as usual whereas the stiff terms are estimated with

$$\frac{1}{\epsilon^2} \int_{\mathbb{R}^d} \big| \overline{\mathsf{A}}_0 \big[\partial^\alpha, \overline{\mathsf{A}}_0^{-1} \,\overline{\mathsf{L}}\, \big] \pi \widetilde{\mathsf{w}} \big|^2 \, \mathrm{d}x \le \frac{\mathsf{c}_0}{\epsilon^2} |\overline{\mathsf{A}}_0|_\infty^2 \, \big| \partial_x (\overline{\mathsf{A}}_0^{-1} \,\overline{\mathsf{L}}) \big|_{l-2}^2 |\pi \widetilde{\mathsf{w}}|_{l-2}^2 \le \frac{\mathsf{c}_2}{\epsilon^2} |\pi \widetilde{\mathsf{w}}|_{l-2}^2,$$

and collecting all contributions it has been established that

$$\sum_{0 \le |\alpha| \le l-1} |\mathsf{h}^\alpha|_0^2 \le \mathsf{c}_2 \big(|\mathsf{f}|_{l-1}^2 + |\mathsf{g}_{\mathrm{II}}|_{l-1}^2 + |\delta \widetilde{\mathsf{w}}|_{l-1}^2 + |\delta \widetilde{\mathsf{w}}_{\mathrm{II}}|_l^2 \big) + \frac{\mathsf{c}_2}{\epsilon^2} |\pi \widetilde{\mathsf{w}}|_{l-2}^2.$$

This inequality is now combined to (A.27) multiplied by a large constant κ_2 so as to compensate the terms $\mathsf{c}_1 |\delta \widetilde{\mathsf{w}}_{\mathrm{II}}|_{l+1}^2$ and $\frac{\mathsf{c}_2}{\epsilon^2} |\pi \widetilde{\mathsf{w}}|_{l-2}^2$ of the right hand side. From the Gronwall inequality one gets after some algebra the second estimate (A.9).

Finally, the various occurencies of the constant c_2 in the proof all involve simple polynomials in M with positive coefficients, either arising as simple multiplication by M or through the estimates of nonlinear terms, so that the final constant c_2 is an increasing function of M and the proof is complete. □

Appendix B. Local stability at equilibrium

Theorem Appendix B.1. *Let $d \ge 1$ and $l \ge [d/2] + 2$ be integers and let $b > 0$ be given and consider the perturbed system of equations*

$$\overline{\mathsf{A}}_0(\mathsf{w}) \partial_t \mathsf{w} + \sum_{i \in \mathcal{D}} \overline{\mathsf{A}}_i(\mathsf{w}) \partial_i \mathsf{w} - \sum_{i,j \in \mathcal{D}} \partial_i \big(\overline{\mathsf{B}}_{ij}(\mathsf{w}) \partial_j \mathsf{w} \big) - \overline{\mathsf{Q}}(\mathsf{w}, \partial_x \mathsf{w}) = \mathsf{f} + \mathsf{g}, \quad \text{(B.1)}$$

where $\overline{\mathsf{Q}} = - \sum_{i,j \in \mathcal{D}} \partial_i (\partial_{\mathsf{w}} \mathsf{v})^t \, (\partial_{\mathsf{v}} \mathsf{w})^t \overline{\mathsf{B}}_{ij} \, \partial_j \mathsf{w}$ *and where for some positive* $\bar{\tau}_{\mathrm{m}} > 0$

$$\mathsf{f} \in C^0\big([0, \bar{\tau}_{\mathrm{m}}], H^{l-1}\big) \cap L^1\big([0, \bar{\tau}_{\mathrm{m}}], H^l\big), \quad \text{(B.2)}$$

$$\mathsf{g} \in C^0\big([0,\bar\tau_{\rm m}], H^{l-1}\big), \qquad \mathsf{g}_{\rm I} = 0. \tag{B.3}$$

Let $\mathcal{O}_0$ be given such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}_{\mathsf{w}}$, a_1 such that $0 < a_1 < \operatorname{dist}(\overline{\mathcal{O}}_0, \partial\mathcal{O}_{\mathsf{w}})$, and define

$$\mathcal{O}_1 = \{\, \mathsf{w} \in \mathcal{O}_{\mathsf{w}};\ \operatorname{dist}(\mathsf{w}, \overline{\mathcal{O}}_0) < a_1 \,\}.$$

There exists $\bar\tau$ with $0 < \bar\tau \le \bar\tau_{\rm m}$ and $\chi > 0$ depending on $\mathcal{O}_1$ and b, such that for any w_0 with $\mathsf{w}_0 \in \overline{\mathcal{O}}_0$ and any f and g satisfying (B.2)(B.3) *with*

$$|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 < b^2, \qquad \Big\{\int_0^t |\mathsf{f}|_l \,\mathrm{d}\tau\Big\}^2 + \int_0^t |\mathsf{f}|_{l-1}^2 \,\mathrm{d}\tau < \chi b^2, \qquad \int_0^t |\mathsf{g}|_{l-1}^2 \,\mathrm{d}\tau < \chi b^2, \tag{B.4}$$

there exists a unique local solution w to the perturbed system (B.1) *with initial condition $\mathsf{w}(0,x) = \mathsf{w}_0(x)$ such that $\mathsf{w}(t,x) \in \mathcal{O}_1$, for $t \in [0,\bar\tau]$ and $x \in \mathbb{R}^d$, and*

$$\begin{aligned}
&\mathsf{w}_{\rm I} - \mathsf{w}_{\rm I}^\star \in C^0\big([0,\bar\tau], H^l\big) \cap C^1\big([0,\bar\tau], H^{l-1}\big),\\
&\mathsf{w}_{\rm II} - \mathsf{w}_{\rm II}^\star \in C^0\big([0,\bar\tau], H^l\big) \cap C^1\big([0,\bar\tau], H^{l-2}\big) \cap L^2\big((0,\bar\tau), H^{l+1}\big).
\end{aligned}$$

In addition, there exists $C > 0$ only depending on $\mathcal{O}_1$ and b, such that

$$\begin{aligned}
\sup_{0\le\tau\le\bar\tau} |\mathsf{w}(\tau) - \mathsf{w}^\star|_l^2 + &\int_0^{\bar\tau} |\mathsf{w}_{\rm II}(\tau) - \mathsf{w}_{\rm II}^\star|_{l+1}^2 \, d\tau \\
&\le C\Big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \Big\{\int_0^{\bar\tau} |\mathsf{f}|_l \,\mathrm{d}\tau\Big\}^2 + \int_0^{\bar\tau} |\mathsf{g}|_{l-1}^2 \,\mathrm{d}\tau\Big),
\end{aligned} \tag{B.5}$$

$$\int_0^t |\partial_t \mathsf{w}(\tau)|_{l-1}^2 \, d\tau \le C\Big(|\mathsf{w}_0 - \mathsf{w}^\star|_l^2 + \Big\{\int_0^{\bar\tau} |\mathsf{f}|_l \,\mathrm{d}\tau\Big\}^2 + \int_0^{\bar\tau} |\mathsf{f}|_{l-1}^2 \,\mathrm{d}\tau + \int_0^{\bar\tau} |\mathsf{g}|_{l-1}^2 \,\mathrm{d}\tau\Big). \tag{B.6}$$

Moreover, if w and w' correspond to two different inital conditions and different perturbations, letting $\delta\mathsf{w} = \mathsf{w} - \mathsf{w}'$, $\delta\mathsf{f} = \mathsf{f} - \mathsf{f}'$, $\delta\mathsf{g} = \mathsf{g} - \mathsf{g}'$, then

$$\sup_{0\le\tau\le\bar\tau} |\delta\mathsf{w}(\tau)|_{l-1}^2 + \int_0^{\bar\tau} |\delta\mathsf{w}_{\rm II}(\tau)|_l^2 \, d\tau \le C\Big(|\delta\mathsf{w}_0|_{l-1}^2 + \Big\{\int_0^{\bar\tau} |\delta\mathsf{f}|_{l-1} \,\mathrm{d}\tau\Big\}^2 + \int_0^{\bar\tau} |\delta\mathsf{g}|_{l-2}^2 \,\mathrm{d}\tau\Big). \tag{B.7}$$

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