ASYMPTOTIC STABILITY OF EQUILIBRIUM STATES FOR MULTICOMPONENT REACTIVE FLOWS

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We consider the equations governing multicomponent reactive flows derived from the kinetic theory of dilute polyatomic reactive gas mixtures. Using an entropy function, we derive a symmetric conservative form of the system. In the framework of Kawashima's and Shizuta's theory, we recast the resulting system into a normal form, that is, in the form of a symmetric hyperbolic-parabolic composite system. We also characterize all normal forms for symmetric systems of conservation laws such that the nullspace associated with dissipation matrices is invariant. We then investigate an abstract second order quasilinear system with a source term, around a constant equilibrium state. Assuming the existence of a generalized entropy function, the invariance of the nullspace naturally associated with dissipation matrices, stability conditions for the source term, and a dissipative structure for the linearized equations, we establish global existence and asymptotic stability around the constant equilibrium state in all space dimensions and we obtain decay estimates. These results are then applied to multicomponent reactive flows using a normal form and the properties of Maxwellian chemical source terms.

1. Introduction

In this paper, we investigate the system of equations modeling multicomponent reactive flows. We derive various symmetric forms of the system and we establish global existence and asymptotic stability around constant equilibrium states for the Cauchy problem in all space dimensions.

We first present the governing equations for multicomponent gaseous flows derived from the kinetic theory of dilute polyatomic reactive gas mixtures [EG94]. We express the conservation equations, the transport fluxes and the thermodynamic properties. An important point in these equations is that the transport fluxes have their natural symmetry properties. We use in particular the symmetric diffusion coefficients introduced by Waldmann and Trübenbacher [WT62] [Gi91]. On the contrary, Hirschfelder, Curtiss and Bird [HCB64] have artificially destroyed this symmetry [Gi91] [EG94]. The detailed form of the chemistry source term derived from the kinetic theory is not needed in the first sections of the paper and is postponed to Section 6.

We then discuss symmetrizability for an abstract system of conservation laws. This property is closely related to the existence of an entropy function as shown by Kawashima and Shizuta in the case of hyperbolic–parabolic systems [KS88], extending previous work on hyperbolic systems [Go62] [FL71] [Mo80]. Starting from an entropy related conservative symmetric form, Kawashima and Shizuta have further investigated normal forms, that is, symmetric hyperbolic–parabolic composite forms. These authors have shown in particular that symmetric systems of conservation laws such that the nullspace naturally associated with dissipation matrices is invariant can be recast into a normal form. In the framework of their theory, we further characterize, in this paper, all normal forms for systems satisfying the invariance property.

As an application, we exhibit an entropy function for the system of equations modeling multicomponent flows. By using the corresponding entropic variable, we derive a symmetric conservative formulation of the system. Chalot, Hughes and Shakib [CHS90] have carried out similar calculations in the case of flows in thermochemical nonequilibrium. However, they have used a multicomponent diffusion matrix which is not symmetric [HCB54] and which prohibits complete symmetrization. These authors have thus advocated Onsager's phenomenological coefficients in order to achieve symmetrization. On the contrary, by using the symmetric form of the transport fluxes and of the diffusion coefficients [WT62] [Gi91] [EG94], we have obtained a naturally symmetric conservative formulation. The symmetrized multicomponent reactive flows governing equations are then shown to satisfy the invariance property and are recast into two different normal forms. The first normal form has simpler matrix coefficients and generalizes previous results from Kawashima and Shizuta [KS88]. The second normal form has dissipative terms in conservative form and leaves unchanged the structure of the source term. Both forms can be used for the asymptotic stability of constant equilibrium states investigated in the paper.

We then consider an abstract second order quasilinear system which admits an entropy function and satisfies the nullspace invariance property so that it can be recast into a normal form. Under stability conditions on the source term, and assuming conditions which guarantee the dissipative structure of the linearized normal system around the constant equilibrium state, we obtain global existence and asymptotic stability of the stationary state. As stability conditions, we assume that the chemical entropy production is nonnegative and that the source term lies in the range of its derivative at equilibrium. Our method of proof relies on Kawashima's theory [Ka84], on a priori estimates provided by the entropy conservation law and on stability properties of the source term. Decay estimates towards the constant stationary state are also obtained in all space dimensions. This work extends previous results of Kawashima [Ka84] in space dimension $d \geq 3$ for general source term. It also extend results of Kawashima [Ka84] concerning entropic systems with no source term in space dimension $d \geq 1$.

We then apply these results to the system modeling multicomponent reactive flows. We first investigate Maxwellian chemical source terms provided by the kinetic theory. We restate the existence of constant stationary states and investigate the structure of the corresponding linearized equations at equilibrium. We then obtain global existence, asymptotic stability of the constant equilibrium state together with decay estimates. To the authors' knowledge, these results on multicomponent reactive flows, obtained in a mathematical framework deduced from the kinetic theory of gas mixtures [Gi91] [EG94], are new.

The governing equation for multicomponent reactive flows are presented in Section 2. Symmetrizability for systems of conservation laws, entropy functions and normal forms are investigated in Section 3. These results are then applied to multicomponent reacting flows in Section 4. In Section 5 we investigate an abstract hyperbolic-parabolic composite system with a source term. Finally, Maxwellian chemical source terms and equilibrium states are discussed in Section 6 and asymptotic stability is obtained in Section 7.

2. Governing Equations

2.1. Conservation equations

The equations modeling multicomponent reactive flows express the conservation of species mass, momentum and energy. These equations can be written in the form [WT62] [EG94]

$$\partial_t U + \sum_{i \in C} \partial_i F_i + \sum_{i \in C} \partial_i \mathcal{F}_i = \Omega, \tag{2.1}$$

where ∂_t is the time derivative operator, U the conservative variable, ∂_i the space derivative operator in the i^{th} direction, $C = \{1, 2, 3\}$ the set of direction indices, F_i the advective flux in the i^{th} direction, $\mathcal{F}_i$ the dissipative flux in the i^{th} direction and Ω the source term. The variable U and the advective fluxes F_i, $i \in \mathcal{C}$, are given by

$$U = \Big(\rho_1, \dots, \rho_{n_S}, \rho v_1, \rho v_2, \rho v_3, \rho e^{\mathrm{tot}}\Big)^t, \tag{2.2}$$

and

$$F_i = \Big(\rho_1 v_i, \dots, \rho_{n_S} v_i, \rho v_1 v_i + \delta_{i1} p, \rho v_2 v_i + \delta_{i2} p, \rho v_3 v_i + \delta_{i3} p, \rho e^{\mathrm{tot}} v_i + p v_i\Big)^t, \tag{2.3}$$

where ρ_k is the density of the k^{th} species, n_S the number of species, $S = [1, n_S]$ the set of species indices, $\rho = \sum_{k \in S} \rho_k$ the total density, v_i the mass averaged flow velocity in the i^{th} direction, e^{tot} the total energy per unit mass of the mixture, and p the thermodynamical pressure. For convenience, the dissipative flux $\mathcal{F}_i$ is splitted between the mass and heat diffusion flux $\mathcal{F}_i^{D\lambda}$ and the viscous flux $\mathcal{F}_i^{\kappa\eta}$ so that

$$\mathcal{F}_i = \mathcal{F}_i^{D\lambda} + \mathcal{F}_i^{\kappa\eta}. \tag{2.4}$$

The fluxes $\mathcal{F}_i^{D\lambda}$ and $\mathcal{F}_i^{\kappa\eta}$, $i \in C$, and the source term Ω, are given by

$$\mathcal{F}_i^{\kappa\eta} = \Big(0, \ldots, 0, \Pi_{i1}, \Pi_{i2}, \Pi_{i3}, \sum_{j \in C} \Pi_{ij} v_j\Big)^t, \tag{2.5}$$

$$\mathcal{F}_i^{D\lambda} = \Big(\rho_1 \mathcal{V}_{1i}, \ldots, \rho_{n_S} \mathcal{V}_{n_S i}, 0, 0, 0, q_i\Big)^t, \tag{2.6}$$

$$\Omega = \Big(m_1\omega_1, \ldots, m_{n_S}\omega_{n_S}, \rho g_1, \rho g_2, \rho g_3, \rho g \cdot v\Big)^t, \tag{2.7}$$

where $\Pi = (\Pi_{ij})_{i,j \in C}$ is the viscous stress tensor, $\mathcal{V}_k = (\mathcal{V}_{k1}, \mathcal{V}_{k2}, \mathcal{V}_{k3})^t$ the diffusion velocity of the k^{th} species, $q = (q_1, q_2, q_3)^t$ the heat flux vector, m_k the molar mass of the k^{th} species, ω_k the molar production rate of the k^{th} species, $g = (g_1, g_2, g_3)^t$ the external force per unit mass acting on the species, $v = (v_1, v_2, v_3)^t$ the velocity vector and ^{t} the transposition symbol.

These equations have to be completed by the relations expressing the transport fluxes Π, $\mathcal{V}_k$, $k \in S$, and q, the thermodynamic properties p and e^{tot}, the chemical source terms ω_k, $k \in S$, and the specific force g.

2.2. Transport fluxes

The expressions for the transport fluxes rigorously derived from the kinetic theory of dilute polyatomic gas mixture can be written as [WT62] [EG94]

$$\mathcal{V}_k = -\sum_{l \in S} D_{kl}\, d_l - \theta_k \partial_x \log T, \qquad k \in S, \tag{2.8}$$

$$\Pi = -(\kappa - \tfrac{2}{3}\eta)(\partial_x \cdot v) I - \eta\big(\partial_x v + (\partial_x v)^t\big), \tag{2.9}$$

$$q = -\lambda' \partial_x T - p \sum_{k \in S} \theta_k d_k + \sum_{k \in S} \rho_k h_k \mathcal{V}_k, \tag{2.10}$$

where $D = (D_{kl})_{k,l \in S}$ is the diffusion matrix, d_k the diffusion driving force of the k^{th} species, $\theta = (\theta_1, \ldots, \theta_{n_S})^t$ the thermal diffusion vector, $\partial_x = (\partial_1, \partial_2, \partial_3)^t$ the usual differential operator, T the absolute temperature, κ the volume viscosity, η the shear viscosity, λ' the partial thermal conductivity and h_k the enthalpy per unit mass of the k^{th} species. The vectors d_k, $k \in S$, take into account the effects of various state variable gradients and are given by

$$d_k = \partial_x X_k + X_k \frac{\partial_x p}{p}, \tag{2.11}$$

where X_k denotes the mole fraction of the k^{th} species. Alternate expressions for the diffusion velocities and the heat flux vector are

$$\mathcal{V}_k = -\sum_{l \in S} D_{kl}(d_l + \chi_l \partial_x \log T), \qquad k \in S, \tag{2.12}$$

$$q = -\lambda \partial_x T + p \sum_{l \in S} \chi_l \mathcal{V}_l + \sum_{l \in S} \rho_l h_l \mathcal{V}_l, \tag{2.13}$$

where $\chi = (\chi_1, \ldots, \chi_{n_S})^t$ is the thermal diffusion ratio vector and λ the thermal conductivity. Both expressions (2.8) (2.12) for the diffusion velocities and (2.10) (2.13) for the heat flux vector will be used in the following.

These formulations of the pressure tensor (2.9), the diffusion velocities (2.8) (2.12), and the heat flux (2.10) (2.13) are due to Waldmann and Trübenbacher [WT62] [EG94]. In particular, the diffusion matrix D associated with these fluxes is symmetric as specified in Section 2.5 where the properties of the various transport coefficients are expressed.

2.3. Thermodynamic properties

From the kinetic theory, the state law expressing the pressure p is

$$p = \rho\, r\, T, \tag{2.14}$$

where

$$\rho\, r = R_{\mathrm{g}} \sum_{k \in S} \frac{\rho_k}{m_k}. \tag{2.15}$$

In these expressions, $\rho = \sum_{k \in S} \rho_k$ is the total mass density, r the specific gas constant of the mixture, and R_{g} the universal gas constant. The specific total energy e^{tot} and the specific internal energy e of the mixture are given by

$$e^{\text{tot}} = e + \tfrac{1}{2} v \cdot v, \tag{2.16}$$

where

$$\rho e = \sum_{k \in S} \rho_k e_k. \tag{2.17}$$

The quantity e_k is the internal energy per unit mass of the k^{th} species and can be written

$$e_k(T) = e_k^0 + \int_{T_0}^{T} c_{vk}(T')\, dT', \tag{2.18}$$

where e_k^0 is the energy of formation of the k^{th} species at the positive reference temperature T_0 and c_{vk} is the specific heat at constant volume of the k^{th} species. The mixture specific heat at constant volume c_v is also defined by

$$\rho c_v = \sum_{k \in S} \rho_k c_{vk}. \tag{2.19}$$

Similarly, the specific total enthalpy h^{tot} and specific enthalpy h are written

$$h^{\mathrm{tot}} = h + \tfrac{1}{2} v \cdot v, \qquad \rho h = \sum_{k \in S} \rho_k h_k, \tag{2.20}$$

where h_k, the enthalpy per unit mass of the k^{th} species, reads

$$h_k(T) = e_k(T) + r_k T, \tag{2.21}$$

and where $r_k = R_{\mathrm{g}}/m_k$ is the specific gas constant of the k^{th} species. It is also convenient to denote by e_k^{tot} the total energy per unit mass of the k^{th} species and by h_k^{tot} the total enthalpy per unit mass of the k^{th} species

$$e_k^{\mathrm{tot}} = e_k + \tfrac{1}{2} v \cdot v, \qquad h_k^{\mathrm{tot}} = h_k + \tfrac{1}{2} v \cdot v. \tag{2.22}$$

The kinetic theory also yields the specific (physical) entropy of the k^{th} species

$$s_k(\rho_k, T) = s_k^{00} + \int_{T_0}^{T} \frac{c_{vk}(T')}{T'}\, dT' - r_k \log\left(\frac{\rho_k}{\gamma_0 m_k}\right), \tag{2.23}$$

where s_k^{00} is the standard formation entropy at the positive reference temperature T_0 and positive reference pressure $\gamma_0 R_{\mathrm{g}} T_0$. Note that $\rho_k/m_k = \gamma_k$ is the molar concentration of the k^{th} species and γ_0 is the reference concentration. Finally, we will also need the expression of the chemical potential $\mu_k(\rho_k, T)$ of the k^{th} species

$$\mu_k = e_k + r_k T - s_k T. \tag{2.24}$$

2.4. Source terms

The detailed description of the chemical source terms ω_k, $k \in S$, is not needed in the first sections of the paper. It will only be needed for investigating global existence results and asymptotic stability of constant equilibrium states. Therefore, the detailed description of the terms ω_k, $k \in S$, is postponed to Section 6. In the following, we only require that the chemical source terms ω_k, $k \in S$, are functions of the natural variable $Y = (\rho_1, \ldots, \rho_{n_S}, v_1, v_2, v_3, T)^t$

$$\omega_k = \omega_k(Y), \tag{2.25}$$

with a similar assumption for the specific force term g

$$g = g(Y). \tag{2.26}$$

Remark 2.1. In this paper, for sake of simplicity, we only consider a species independent specific force, as gravity for instance. When the specific forces are species dependent, the overall force term reads $g = \sum_{k \in S} \rho_k \mathfrak{g}_k$, and the diffusion driving force terms become $d'_k = d_k - (\rho_k/p)\mathfrak{g}_k$, where $\mathfrak{g}_k$ denotes the specific force acting on the $k^{\rm th}$ species. In this situation, there are extra fluxes arising from the modified diffusion driving force terms, and symmetrization of these fluxes can only be achieved provided that special compatibility relations hold between the specific forces and the transport coefficients [Ma96].

2.5. Mathematical assumptions

We introduce here the mathematical assumptions concerning the transport coefficients, the thermodynamic properties and the simplified source terms. We assume that the natural variable $Y = (\rho_1, \dots, \rho_{n_S}, v_1, v_2, v_3, T)^t$ takes its values in the open convex set $\mathcal{O}_Y$

$$\mathcal{O}_Y = (0,\infty)^{n_S} \times \mathbb{R}^3 \times (T_0, \infty), \tag{2.27}$$

where T_0 is positive and we assume the following dependence and regularity properties.

- (H_1) The transport coefficients $(D_{kl})_{k,l\in S}$, $(\theta_1, \dots, \theta_{n_S})^t$, $(\chi_1, \dots, \chi_{n_S})^t$, κ, η, λ', and λ are C^∞ functions of $(\rho_1, \dots, \rho_{n_S}, T) \in (0,\infty)^{n_S} \times (T_0, \infty)$.

- (H_2) The specific heats c_{vk}, $k \in S$, are C^∞ functions of $T \in [T_0, \infty)$. Moreover there exists a positive constant a with $0 < a \le c_{vk}(\xi)$, for $\xi \ge T_0$ and $k \in S$.

- (H_3) The force term g is a C^∞ function of $Y \in \mathcal{O}_Y$.

- (H_4) The chemical production rate vector $\omega = (\omega_1, \dots, \omega_{n_S})^t$ is a C^∞ function of $Y \in \mathcal{O}_Y$.

- (H_5) The shear viscosity η, the thermal conductivity λ, and the partial thermal conductivity λ' are positive and the volume viscosity κ is nonnegative.

- (H_6) The matrix $D = (D_{kl})_{k,l\in S}$ is symmetric positive semi-definite and its nullspace is spanned by $\varrho = (\rho_1, \dots, \rho_{n_S})^t$. We have in particular the mass constraints
$$\sum_{k\in S} \rho_k D_{kl} = 0, \qquad l \in S.$$

- (H_7) The thermal diffusion vector $\theta = (\theta_1, \dots, \theta_{n_S})^t$ satisfies the mass constraint
$$\sum_{k\in S} \rho_k \theta_k = 0.$$

- (H_8) The thermal diffusion ratio vector $\chi = (\chi_1, \dots, \chi_{n_S})^t$ satisfies the relations
$$\sum_{l\in S} D_{kl}\chi_l = \theta_k, \quad k \in S, \qquad\qquad \sum_{k\in S} \chi_k = 0.$$

The partial thermal conductivity λ' is given by

$$\lambda = \lambda' - \frac{p}{T} \sum_{l \in S} \chi_l \theta_l .$$

(H_9) The chemical production rate vector $\omega = (\omega_1, \ldots, \omega_{n_S})^t$ satisfies the mass conservation relation

$$\sum_{k \in S} m_k \omega_k = 0.$$

We point out that all these assumptions are suggested by the semi-classical kinetic theory of dilute polyatomic reactive gas mixtures [Gi91] [EG94]. As previously mentioned, the diffusion coefficients considered here are symmetric and, therefore, are consistent with Onsager reciprocal relations. On the contrary, Hirschifelder, Curtiss and Bird have introduced an alternate definition [HCB54] and artificially destroyed the symmetry of the diffusion process [Gi91] [EG94]. The mass constraints of the diffusion matrix and the thermal diffusion vector also imply the mass conservation relation $\sum_{k \in S} \rho_k \mathcal{V}_k = 0$. In addition, the positivity properties of the transport coefficients are associated with the positivity of the entropy production quadratic form [Gi91] [EG94].

Further note that the gas species specific heats—and therefore the energies and enthalpies—obtained from the kinetic theory, could also be extended—from a mathematical point of view—up to zero temperature, but not the gas entropy which explodes like $\log T$. However, since the basic assumptions of the kinetic theory of dilute gas mixtures are not valid at low temperatures, where the gases are ultimately transformed into liquids and then into solids, we have chosen to restrict the temperature domain to $[T_0, \infty)$, where T_0 is positive, for modeling gas mixtures.

2.6. The quasilinear form

By expressing the natural variable Y in term of the conservative variable U, we now rewrite the system of conservation equations (2.1) in a quasilinear form. For this purpose, we first investigate the map $Y \to U$ and its range.

Proposition 2.2. The map $Y \longmapsto U$ is a C^∞ diffeomorphism from the open set $\mathcal{O}_Y = (0, \infty)^{n_S} \times \mathbb{R}^3 \times (T_0, \infty)$ onto an open set $\mathcal{O}_U$. The open set $\mathcal{O}_U$ is convex and given by

$$\mathcal{O}_U = \{\ z \in \mathbb{R}^{n_S+4} ; z_i > 0,\ 1 \leq i \leq n_S, \quad z_{n_S+4} - \phi(z_1, \ldots, z_{n_S+3}) > 0\ \}, \quad (2.28)$$

where

$$\phi(z_1, \ldots, z_{n_S+3}) = \tfrac{1}{2} \frac{z_{n_S+1}^2 + z_{n_S+2}^2 + z_{n_S+3}^2}{\sum_{i \in S} z_i} + \sum_{i \in S} z_i e_i^0 .$$

Proof. From Assumption (H_2) and the expression (2.2) we first deduce that the map $Y \rightarrow U$ is C^∞ over the domain $\mathcal{O}_Y$. The matrix $\partial_Y U$ is easily shown to be nonsingular over $\mathcal{O}_Y$ thanks to its triangular structure. Finally, it is straightforward to show that the map $Y \rightarrow U$ is one to one so that $Y \rightarrow U$ is a C^∞ diffeomorphism onto an open set $\mathcal{O}_U$. From (H_2), it is then easily established that $\mathcal{O}_U$ is given by (2.28). The convexity of $\mathcal{O}_U$ is then a direct consequence of the convexity of ϕ which is established by evaluating its second derivative. $\square$

From (2.8)–(2.11) and Proposition 2.2, the dissipation fluxes can then be expressed as functions of the conservative variable gradients

$$\mathcal{F}_i = -\sum_{j\in C} G_{ij}(U)\partial_j U \tag{2.29}$$

where $G_{ij}(U)$, $i,j \in C$, are the dissipation matrices. These matrices are square matrices of dimension $n_S + 4$, and, from (2.4)–(2.6), they admit the following decomposition

$$G_{ij} = G_{ij}^{\kappa\eta} + G_{ij}^{D\lambda}, \tag{2.30}$$

where

$$\mathcal{F}_i^{\kappa\eta} = -\sum_{j\in C} G_{ij}^{\kappa\eta}(U)\partial_j U, \qquad \mathcal{F}_i^{D\lambda} = -\sum_{j\in C} G_{ij}^{D\lambda}(U)\partial_j U.$$

We may further introduce the jacobian matrices A_i, $i \in C$, of the advection fluxes F_i, $i \in C$,

$$A_i = \partial_U F_i, \tag{2.31}$$

and finally rewrite the system into the quasilinear form

$$\partial_t U + \sum_{i\in C} A_i \partial_i U = \sum_{i,j\in C} \partial_i\big(G_{ij}\partial_j U\big) + \Omega(U), \tag{2.32}$$

where the matrix coefficients are defined on the open convex set $\mathcal{O}_U$. The detailed form of the coefficient matrices A_i, $i \in C$, and $G_{ij}(U)$, $i,j \in C$, will not be needed in the following, and, therefore, will not be given and we refer to [Ma96] for more details.

3. Symmetrization and Normal Forms

For hyperbolic systems of conservation laws, the existence of a conservative symmetric formulation has been shown to be equivalent to the existence of an entropy function [Go62] [FL71] [Mo80]. These results have been generalized to the case of second order quasilinear systems of equations by Kawashima and Shizuta [KS88], using Kawashima's definition of an entropy function [Ka84]. In this section, we first restate these results on conservative symmetrizability. Following Kawashima and Shizuta [KS88], we then investigate normal forms of the system, i.e., symmetric hyperbolic-parabolic composite forms. Kawashima and Shizuta [KS88] have shown that, when the nullspace naturally associated with dissipation matrices is a fixed subspace, a symmetric system of conservation equations can be put into a normal form. In the framework of their theory, we further characterize in this section all normal forms for such symmetric systems of conservation laws.

3.1. Entropy functions and symmetric conservative forms

We consider an abstract second order quasilinear system in the form

$$\partial_t U^* + \sum_{i\in C^*} \partial_i F_i^*(U^*) = \sum_{i,j\in C^*} \partial_i \big(G_{ij}^*(U^*)\partial_j U^*\big) + \Omega^*(U^*), \tag{3.1}$$

where $U^* \in \mathcal{O}_{U^*}$, $\mathcal{O}_{U^*}$ is an open convex set of $\mathbb{R}^n$, and $C^* = \{1, \ldots, d\}$ is the set of direction indices of $\mathbb{R}^d$. Note that the superscript * is used in order to distinguish between the abstract second order system (3.1) of size n in $\mathbb{R}^d$ and the particular multicomponent reactive flows system (2.32) of size $n_S + 4$ in $\mathbb{R}^3$. All quantities associated with the abstract system have the corresponding superscript *, so that for instance the unknown vector is U^*. We assume that the following properties hold for system (3.1).

(A_1) The flux vectors F_i^*, $i \in C^*$, the dissipation matrices G_{ij}^*, $i,j \in C^*$, and the source term Ω^*, are smooth functions of the variable $U^* \in \mathcal{O}_{U^*}$, where $\mathcal{O}_{U^*}$ is a convex open set of $\mathbb{R}^n$.

The following definition of a symmetric (conservative) form, for the system (3.1), is due to Kawashima and Shizuta [KS88].

Definition 3.1. Assume that $V^* \to U^*$ is a diffeomorphism from $\mathcal{O}_{V^*}$ onto $\mathcal{O}_{U^*}$, and consider the system in the V^* variable

$$\widetilde{A}_0^*\partial_t V^* + \sum_{i\in C^*} \widetilde{A}_i^* \partial_i V^* = \sum_{i,j\in C^*} \partial_i \Big(\widetilde{G}_{ij}^* \partial_j V^*\Big) + \widetilde{\Omega}^*, \tag{3.2}$$

where

$$\begin{cases} \widetilde{A}_0^* = \partial_{V^*} U^*, & \widetilde{A}_i^* = A_i^* \widetilde{A}_0^*, \\ \widetilde{G}_{ij}^* = G_{ij}^* \widetilde{A}_0^*, & \widetilde{\Omega}^* = \Omega^*. \end{cases} \tag{3.3}$$

The system (3.2) is said to be of the symmetric form if the matrix coefficients satisfy the following properties.

(S_1) The matrix $\widetilde{A}_0^*(V^*)$ is symmetric and positive definite for $V^* \in \mathcal{O}_{V^*}$,

(S_2) The matrices $\widetilde{A}_i^*(V^*)$, $i \in C^*$, are symmetric for $V^* \in \mathcal{O}_{V^*}$,

(S_3) We have $\widetilde{G}_{ij}^*(V^*)^t = \widetilde{G}_{ji}^*(V^*)$ for $i,j \in C^*$ and $V^* \in \mathcal{O}_{V^*}$,

- (S_4) The matrix $\widetilde{B}^*(V^*,w) = \sum_{i,j\in C^*} \widetilde{G}^*_{ij}(V^*)w_iw_j$ is symmetric and positive semi-definite, for $V^* \in \mathcal{O}_{V^*}$ and $w \in \mathcal{S}^{d-1}$, where $\mathcal{S}^{d-1}$ is the unit sphere in d dimensions.

Note that both first and second order derivative terms are in conservative form in (3.2). On the other hand, the following generalized definition of an entropy function has been given by Kawashima [Ka84] [KS88].

Definition 3.2. A real-valued smooth function $\mathcal{H}^*(U^*)$ defined on a convex set $\mathcal{O}_{U^*}$ is said to be an entropy function for system (3.1) if the following properties hold

- (E_1) The function $\mathcal{H}^*$ is a strictly convex function on $\mathcal{O}_{U^*}$ in the sense that the Hessian matrix is positive definite on $\mathcal{O}_{U^*}$,
- (E_2) There exists real-valued smooth functions $\sigma^*_i = \sigma^*_i(U^*)$ such that

$$\left(\partial_{U^*}\mathcal{H}^*\right) A^*_i = \partial_{U^*}\sigma^*_i, \quad i \in C^*, \qquad U^* \in \mathcal{O}_{U^*}, \tag{3.4}$$

- (E_3) We have the property

$$\left(\partial^2_{U^*}\mathcal{H}^*(U^*)\right)^{-1} (G^*_{ij})^t = G^*_{ji} \left(\partial^2_{U^*}\mathcal{H}^*(U^*)\right)^{-1}, \quad i,j \in C^*, \quad U^* \in \mathcal{O}_{U^*}. \tag{3.5}$$

- (E_4) The matrix $\widetilde{B}^*(V^*,w) = \sum_{i,j\in C^*} G^*_{ij}(U^*) \left(\partial^2_{U^*}\mathcal{H}^*(U^*)\right)^{-1} w_iw_j$ is symmetric positive semi-definite for $U^* \in \mathcal{O}_{U^*}$ and $w \in \mathcal{S}^{d-1}$.

Kawashima and Shizuta have established the equivalence between conservative symmetrizability and the existence of an entropy function for the system [KS88].

Theorem 3.3. The system (3.1) can be symmetrized on the open convex set $\mathcal{O}_{U^*}$ if and only if the system admits an entropy function $\mathcal{H}^*$ on $\mathcal{O}_{U^*}$. In this situation, the symmetrizing variable V^* can be expressed in terms of the gradient of the entropy function $\mathcal{H}^*$

$$V^* = \left(\partial_{U^*}\mathcal{H}^*\right)^t. \tag{3.6}$$

The mathematical entropy $\mathcal{H}^*$ is generally taken to be the opposite of the physical mixture entropy density per unit volume. The variable V^* is usually termed the entropic variable associated with the variable U^*.

Remark 3.4. Note that, for convenience, we have considered source terms in the previous definitions, which are minor modifications of [KS88]. Properties of entropy functions associated with source terms are discussed in Section 5.2.

3.2. Normal forms

We now assume that the abstract second order quasilinear system (3.1) is symmetrizable in the sense of Definition 3.1, that is, we assume

(A_2) The system (3.1) admits an entropy function $\mathcal{H}^*$ over the domain $\mathcal{O}_{U^*}$.

Introducing the symmetrizing variable $V^{*t} = \partial_{U^*}\mathcal{H}^*$, the corresponding system (3.2) then satisfies Properties (S_1)–(S_4). However, depending on the range of the dissipation matrices $\widetilde{G}^*_{ij}$, this system lies between the two limit cases of an hyperbolic system and a strongly parabolic one. In this section, we use a sufficient condition on the matrices $\widetilde{G}^*_{ij}$, $i,j \in C^*$, the Condition N introduced by Kawashima and Shizuta [KS88], under which the system can be recast in the form of a symmetric hyperbolic-parabolic composite system, defined as a normal form of the system. We then characterize all normal forms for symmetric systems of conservation laws satisfying Condition N.

Introducing a new variable W^*, associated with a diffeomorphism from $\mathcal{O}_{W^*}$ onto $\mathcal{O}_{V^*}$, and multiplying the conservative symmetric form on the left side by the transpose of the matrix $\partial_{W^*}V^*$ we then get a new system in the variable W^* and we have the following definition of a normal form [KS88].

Definition 3.5. Consider a system in symmetric form as in Definition 3.1 and a diffeomorphism $W^* \to V^*$ from $\mathcal{O}_{W^*}$ to $\mathcal{O}_{V^*}$. The system in the new variable W^*

$$\overline{A}^*_0 \partial_t W^* + \sum_{i\in C^*} \overline{A}^*_i \partial_i W^* = \sum_{i,j\in C^*} \partial_i\Big(\overline{G}^*_{ij}\partial_j W^*\Big) + \overline{H}^* + \overline{\Omega}^*, \tag{3.7}$$

where

$$\left\{\begin{aligned} &\overline{A}^*_0 = (\partial_{W^*}V^*)^t\ \widetilde{A}^*_0\ (\partial_{W^*}V^*), \qquad &&\overline{G}^*_{ij} = (\partial_{W^*}V^*)^t\ \widetilde{G}^*_{ij}\ (\partial_{W^*}V^*),\\ &\overline{A}^*_i = (\partial_{W^*}V^*)^t\ \widetilde{A}^*_i\ (\partial_{W^*}V^*), \qquad &&\overline{\Omega}^* = (\partial_{W^*}V^*)^t\widetilde{\Omega}^*,\\ &\overline{H}^* = -\sum_{i,j\in C^*} \partial_i(\partial_{W^*}V^*)^t\ \widetilde{G}^*_{ij}\ (\partial_{W^*}V^*)\partial_j W^*, && \end{aligned}\right. \tag{3.8}$$

satisfies

($\overline{S}_1$) The matrix $\overline{A}^*_0(W^*)$ is symmetric and positive definite for $W^* \in \mathcal{O}_{W^*}$,

($\overline{S}_2$) The matrices $\overline{A}^*_i(W^*)$, $i \in C^*$, are symmetric for $W^* \in \mathcal{O}_{W^*}$,

($\overline{S}_3$) We have $\overline{G}^*_{ij}(W^*)^t = \overline{G}^*_{ji}(W^*)$ for $i,j \in C^*$ and $W^* \in \mathcal{O}_{W^*}$,

($\overline{S}_4$) The matrix $\overline{B}^*(W^*,w) = \sum_{i,j\in C^*} \overline{G}^*_{ij}(W^*)w_iw_j$ is symmetric positive semi-definite, for $W^* \in \mathcal{O}_{W^*}$ and $w \in \mathcal{S}^{d-1}$.

This system is then said to be of the normal form if there exists a partition of $\{1,\dots,n\}$ into $\mathrm{I} = \{1,\dots,n_0\}$ and $\mathrm{II} = \{n_0+1,\dots,n\}$, such that the following properties hold.

(N_1) The matrices $\overline{A}_0^*$ and $\overline{G}_{ij}^*$ have the block structure

$$\overline{A}_0^* = \begin{pmatrix} \overline{A}_0^{*I,I} & 0 \\ 0 & \overline{A}_0^{*II,II} \end{pmatrix}, \qquad \overline{G}_{ij}^* = \begin{pmatrix} 0 & 0 \\ 0 & \overline{G}_{ij}^{*II,II} \end{pmatrix},$$

(N_2) The matrix $\overline{B}^{*II,II}(W^*, w) = \sum_{i,j\in C^*} \overline{G}_{ij}^{*II,II}(W^*)w_i w_j$ is positive definite, for $W^* \in \mathcal{O}_{W^*}$ and $w \in \mathcal{S}^{d-1}$,

(N_3) We have $\overline{H}^*(W^*, \partial_x W^*) = \left(\overline{H}_I^*(W^*, \partial_x W_{II}^*), \overline{H}_{II}^*(W^*, \partial_x W^*)\right)^t$,

where we have used the vector and matrix block structure induced by the partitioning of $\{1,\dots,n\}$ into $I = \{1,\dots,n_0\}$ and $II = \{n_0+1,\dots,n\}$, so that we have $W^* = (W_I^*, W_{II}^*)^t$ for instance.

Remark 3.6. Note that, for convenience, we have kept the dissipative terms $\overline{G}_{ij}^*$ in conservation form and we have considered a source term $\overline{\Omega}^*$ in Definition 3.5, which are minor modifications of [KS88].

A sufficient condition for system (3.2) to be recast into a normal form is that the nullspace naturally associated with dissipation matrices is a fixed subspace of $\mathbb{R}^n$. This is Condition N introduced by Kawashima and Shizuta which is now assumed to hold.

(A_3) The null space of the matrix

$$\widetilde{B}^*(V^*, w) = \sum_{i,j\in C^*} \widetilde{G}_{ij}^*(V^*)w_i w_j,$$

does not depend on $V^* \in \mathcal{O}_{V^*}$ and $w \in \mathcal{S}^{d-1}$ and we denote by n_0 its dimension $n_0 = \dim\big(N(\widetilde{B}^*)\big)$.

In order to characterize more easily normal forms for symmetric systems of conservation laws satisfying Condition N, that is, satisfying (A_3), we introduce the auxiliary variables $U^{*\prime}$ and $V^{*\prime}$, depending linearly on U^* and V^*, respectively. The dissipation matrices corresponding to these auxiliary variables have nonzero coefficients only in the lower right block of size $n - n_0$, where $n_0 = \dim\big(N(\widetilde{B}^*)\big)$. Normal forms are then equivalently—and more easily—obtained from the $V^{*\prime}$ symmetric equation.

Lemma 3.7. Consider a system of conservation laws (3.2) which is symmetric in the sense of Definition 3.1. Denote by $\mathcal{H}^*$ the associated entropy function and by $V^* = (\partial_{U^*}\mathcal{H}^*)^t$ the symmetrizing variable, and assume that Condition N is satisfied over $\mathcal{O}_{V^*}$.

Further consider *any* constant nonsingular matrix P of dimension n, such that its first n_0 columns span the nullspace $N(\widetilde{B}^*)$ associated with Condition N, where $n_0 = \dim\big(N(\widetilde{B}^*)\big)$. More specifically, assume that P is such that

$$\mathrm{span}\Big\{ (P_{1j},\dots,P_{nj})^t;\ 1 \le j \le n_0 \Big\} = N(\widetilde{B}^*).$$

The auxiliary variable

$$U^{*\prime} = P^t U^*,$$

satisfies the equation

$$\partial_t U^{*\prime} + \sum_{i \in C^*} A_i^{*\prime} \partial_i U^{*\prime} = \sum_{i,j \in C^*} \partial_i \left(G_{ij}^{*\prime} \partial_j U^{*\prime} \right) + \Omega^{*\prime}, \tag{3.9}$$

where $A_i^{*\prime} = P^t A_i^* (P^t)^{-1}$, $G_{ij}^{*\prime} = P^t G_{ij}^* (P^t)^{-1}$, and $\Omega^{*\prime} = P^t \Omega^*$. The corresponding entropy is then the functional $U^{*\prime} \to \mathcal{H}^*\left((P^t)^{-1} U^{*\prime}\right)$ and the associated entropic variable $V^{*\prime} = (\partial_{U^{*\prime}} \mathcal{H}^*)^t$ is given by

$$V^{*\prime} = P^{-1} V^*,$$

and satisfies the equation

$$\widetilde{A}_0^{*\prime} \partial_t V^{*\prime} + \sum_{i \in C^*} \widetilde{A}_i^{*\prime} \partial_i V^{*\prime} = \sum_{i,j \in C^*} \partial_i \left(\widetilde{G}_{ij}^{*\prime} \partial_j V^{*\prime} \right) + \widetilde{\Omega}^{*\prime}, \tag{3.10}$$

where $\widetilde{A}_0^{*\prime} = P^t \widetilde{A}_0^* P$, $\widetilde{A}_i^{*\prime} = P^t \widetilde{A}_i^* P$, $\widetilde{G}_{ij}^{*\prime} = P^t \widetilde{G}_{ij}^* P$, and $\widetilde{\Omega}^{*\prime} = P^t \widetilde{\Omega}^*$. In particular, $\widetilde{G}_{ij}^{*\prime}$ is in the form

$$\widetilde{G}_{ij}^{*\prime} = \begin{pmatrix} 0_{n_0 \times n_0} & 0_{n_0 \times (n-n_0)} \\ 0_{(n-n_0) \times n_0} & \widetilde{G}_{ij}^{*\prime II,II} \end{pmatrix}, \tag{3.11}$$

and $\widetilde{B}^{*\prime II,II}(V^{*\prime}, w) = \sum_{i,j \in S} \widetilde{G}_{ij}^{*\prime II,II} w_i w_j$ is positive definite over $\mathcal{O}_{V^{*\prime}} \times \mathcal{S}^{d-1}$.

Finally, the normal form (3.7) is equivalently obtained by multiplying the V^* equation (3.2) by $(\partial_{W^*} V^*)^t$ or the $V^{*\prime}$ equation (3.10) by $(\partial_{W^*} V^{*\prime})^t$.

Proof. Eq. (3.9) is easily established by multiplying Eq. (3.1) on the left by P^t. This also yields the relations $A_i^{*\prime} = P^t A_i^* (P^t)^{-1}$ and $G_{ij}^{*\prime} = P^t G_{ij}^* (P^t)^{-1}$, and $\Omega^{*\prime} = P^t \Omega^*$. It is also easily checked that the functional $U^{*\prime} \to \mathcal{H}^*\left((P^t)^{-1} U^{*\prime}\right)$ is the corresponding entropy. From $V^{*\prime} = (\partial_{U^{*\prime}} \mathcal{H}^*)^t$ and the chain rule, we then get that $V^{*\prime} = P^{-1} V^*$ and Eq. (3.10) is obtained as in (3.2) (3.3). Since $\widetilde{B}^{*\prime} = P^t \widetilde{B}^* P$ and since the first n_0 columns of P span $N(\widetilde{B}^*)$, we next deduce that $\widetilde{B}^{*\prime}$ is in the form

$$\widetilde{B}^{*\prime} = \begin{pmatrix} 0_{n_0 \times n_0} & 0_{n_0 \times (n-n_0)} \\ 0_{(n-n_0) \times n_0} & \widetilde{B}^{*\prime II,II} \end{pmatrix}, \tag{3.12}$$

so that all the matrices $\widetilde{G}_{ij}^{*\prime}$, $i, j \in C^*$, are also in the form (3.11). Moreover, the matrix $\widetilde{B}^{*\prime II,II}(V^{*\prime}, w)$ is positive definite since the $n - n_0$ last columns of P span a subspace complementary to $N(\widetilde{B}^*)$. ☐

Normal forms for symmetrizable systems of conservation laws satisfying Condition N are now completely characterized in the following theorem, in terms of the auxiliary variables $U^{*\prime}$ and $V^{*\prime}$.

Theorem 3.8. Keeping the assumptions and notation of Lemma 3.7, any normal form of the system (3.2) is given by a change of variable in the form

$$W^* = \big(\psi_{I}(U_{I}^{*\prime}), \varphi_{II}(V_{II}^{*\prime})\big)^t, \tag{3.13}$$

where ψ_{I} and φ_{II} are two diffeomorphisms of $\mathbb{R}^{n_0}$ and $\mathbb{R}^{n-n_0}$, respectively. The n_0 components of $U_{I}^{*\prime}$ can thus be termed hyperbolic components and the $n-n_0$ components of $V_{II}^{*\prime}$ parabolic components. Furthermore we have

$$\overline{H}^*(W^*, \partial_x W^*) = \big(0,\, \overline{H}_{II}^*(W^*, \partial_x W_{II}^*)\big)^t,$$

and when φ_{II} is a constant linear mapping, we also have $\overline{H}_{II}^* = 0$ and the dissipative terms are in conservative form.

Proof. The mapping $U^{*\prime} \longmapsto W^*$ is smooth from assumptions and a straightforward calculation yields that

$$\partial_{U^{*\prime}} W^* = \begin{pmatrix} \partial_{U_I^{*\prime}} \psi_{I} & 0_{(n-n_0)\times n} \\ \partial_{V_{II}^{*\prime}} \varphi_{II} \big(\widetilde{A}_0^{*\prime -1}\big)^{II,I} & \partial_{V_{II}^{*\prime}} \varphi_{II} \big(\widetilde{A}_0^{*\prime -1}\big)^{II,II} \end{pmatrix},$$

so that $\partial_{U^{*\prime}} W^*$ is invertible. On the other hand, the derivative of the mapping $U_{II}^{*\prime} \longmapsto V_{II}^{*\prime}(U_{I}^{*\prime}, U_{II}^{*\prime})$, for a fixed value of $U_{I}^{*\prime}$, is given by $\partial_{U_{II}^{*\prime}} V_{II}^{*\prime} = \big(\widetilde{A}_0^{*\prime -1}\big)^{II,II}$ and is positive definite. Since $\mathcal{O}_{U^{*\prime}}$ is an open convex set, we deduce that $U^{*\prime} \longmapsto W^*$ is one to one. As a consequence, the mapping $U^{*\prime} \longmapsto W^*$ is a diffeomorphism from $\mathcal{O}_{U^{*\prime}} = P^t \mathcal{O}_{U^*}$ onto an open set denoted by $\mathcal{O}_{W^*}$ and, similarly, the mapping $V^{*\prime} \longmapsto W^*$ is a diffeomorphism from $\mathcal{O}_{V^{*\prime}} = P^{-1}\mathcal{O}_{V^*}$ onto $\mathcal{O}_{W^*}$.

Evaluating and inverting the matrix $\partial_{V^{*\prime}} W^*$, we next obtain the following expression for $\partial_{W^*} V^{*\prime}$

$$\partial_{W^*} V^{*\prime} = \begin{pmatrix} \big(\widetilde{A}_0^{*\prime\, I,I}\big)^{-1} \big(\partial_{U_I^{*\prime}} \psi_{I}\big)^{-1} & -\big(\widetilde{A}_0^{*\prime\, I,I}\big)^{-1} \widetilde{A}_0^{*\prime\, I,II} \big(\partial_{V_{II}^{*\prime}} \varphi_{II}\big)^{-1} \\ 0_{(n-n_0)\times n} & \big(\partial_{V_{II}^{*\prime}} \varphi_{II}\big)^{-1} \end{pmatrix}, \tag{3.14}$$

and a direct calculation using (3.14) then shows that Properties (N_1)–(N_3) are satisfied, keeping in mind that normal forms are equivalently obtained from the $V^{*\prime}$ equation (3.10) or the V^* equation (3.2).

Consider a diffeomorphism $V^{*\prime} \longmapsto Z^* = \big(\varphi_{I}(V_{I}^{*\prime}, V_{II}^{*\prime}), \varphi_{II}(V_{I}^{*\prime}, V_{II}^{*\prime})\big)^t$ and assume that the system (3.7) in the variable Z^* is of the normal form. By using the definition of $\overline{B}^*$ and the auxiliary variable $V^{*\prime}$, it is easily established that

$$\widetilde{B}^{*\prime} = \big(\partial_{V^{*\prime}} Z^*\big)^t\, \overline{B}^*\, \big(\partial_{V^{*\prime}} Z^*\big).$$

Using the block structure of $\overline{B}^*$ and $\widetilde{B}^{*\prime}$ then yields that

$$\begin{pmatrix} \big(\partial_{V_I^{*\prime}} Z_{II}^*\big)^t \overline{B}^{*\, II,II} \big(\partial_{V_I^{*\prime}} Z_{II}^*\big) & \big(\partial_{V_I^{*\prime}} Z_{II}^*\big)^t \overline{B}^{*\, II,II} \big(\partial_{V_{II}^{*\prime}} Z_{II}^*\big) \\ \big(\partial_{V_{II}^{*\prime}} Z_{II}^*\big)^t \overline{B}^{*\, II,II} \big(\partial_{V_I^{*\prime}} Z_{II}^*\big) & \big(\partial_{V_{II}^{*\prime}} Z_{II}^*\big)^t \overline{B}^{*\, II,II} \big(\partial_{V_{II}^{*\prime}} Z_{II}^*\big) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{B}^{*\prime\, II,II} \end{pmatrix}, \tag{3.15}$$

which implies that $\partial_{V_{I}^{*\prime}} Z_{II}^{*} = 0$ since $\widetilde{B}^{*II,II}$ is positive definite, and thus that

$$Z_{II}^{*} = \varphi_{II}(V_{II}^{*\prime}). \tag{3.16}$$

On the other hand, by using the definition of $\overline{A}_0^{*}$, it is easily establish that

$$\overline{A}_0^{*}\big(\partial_{U^{*\prime}} Z^{*}\big) = \big(\partial_{Z^{*}} V^{*\prime}\big)^{t}.$$

Using the block structure of $\overline{A}_0^{*}$ and the relation $\partial_{Z_{I}^{*}} V_{II}^{\prime *} = 0$, derived from $\partial_{V_{I}^{\prime *}} Z_{II}^{*} = 0$, we then obtain that $\partial_{U_{II}^{*\prime}} Z_{I}^{*} = 0$ and thus that

$$Z_{I}^{*} = \psi_{I}(U_{I}^{*\prime}), \tag{3.17}$$

where we have defined $\varphi_{I}(V_{I}^{*\prime}, V_{II}^{*\prime}) = \psi_{I}(U_{I}^{*\prime}, U_{II}^{*\prime})$. This shows that any diffeomorphism associated with a normal form is of type (3.13).

Moreover, a direct calculation yields

$$\overline{H}^{*} = \Big(0, - \sum_{i,j \in C^{*}} \partial_i \big((\partial_{V_{II}^{*\prime}} \varphi_{II})^{-1}\big)^{t} \widetilde{G}_{ij}^{*\prime II,II} \big(\partial_{V_{II}^{*\prime}} \varphi_{II}\big)^{-1} \partial_j W_{II}^{*}\Big)^{t}. \tag{3.18}$$

This shows that $\overline{H}_{I}^{*} = 0$, and that $\overline{H}_{II}^{*} = 0$ when φ_{II} is constant linear mapping as was to be shown. $\square$

Remark 3.9. Theorem 3.8 shows in particular that the general form (3.13) is independent of the choice of P. It is also possible, however, to check it directly. Consider indeed another matrix Q, as in Lemma 3.7, and define $U^{*\prime\prime} = Q^{t} U^{*}$ and $V^{*\prime\prime} = Q^{-1} V^{*}$. Denoting by $\mathfrak{P}$ the matrix $\mathfrak{P} = P^{-1} Q$, we thus get $U^{*\prime\prime} = \mathfrak{P}^{t} U^{*\prime}$ and $V^{*\prime\prime} = \mathfrak{P}^{-1} V^{*\prime}$. Since the nullspace $N(\widetilde{B})$ is spanned by the n_0 first columns of P and Q, it is easily checked that $\mathfrak{P}^{II,I} = 0$. This implies that $\mathfrak{P}^{I,I}$ and $\mathfrak{P}^{II,II}$ are invertible so that $U_{I}^{*\prime\prime} = (\mathfrak{P}^{I,I})^{t} U_{I}^{*\prime}$ and $V_{II}^{*\prime\prime} = (\mathfrak{P}^{II,II})^{-1} V_{II}^{*\prime}$ and we recover that the general form (3.13) is independent of P.

4. Symmetrization for Multicomponent Flows

We now apply the general results of Section 3 to the system of equations governing multicomponent reactive flows (2.32). We first exhibit an entropy function and derive the corresponding conservative symmetric form. We then establish that Condition N is satisfied. As a result, we recast the symmetrized system into two normal forms. The first normal form has simpler matrix coefficients and generalizes the normal form of the Navier-Stokes equations previously obtained by Kawashima and Shizuta [KS88]. This form also perturbs the structure of the source term. The second normal form, given in the Appendix, is more natural but also more complex, and has dissipative terms in conservative form. Both forms can be used for the asymptotic stability of constant equilibrium states investigated in the next sections.

4.1. Entropy and symmetric conservative form

We first note that Property (A_1) is a direct consequence of Assumptions (H_1)–(H_4). We next define the mathematical entropy function $\mathcal{H}$ as the opposite of the physical mixture entropy density per unit volume

$$\mathcal{H} = -\sum_{k\in S} \rho_k s_k. \tag{4.1}$$

The corresponding entropic variable

$$V = \Big(\partial_U \mathcal{H}\Big)^t, \tag{4.2}$$

is then easily obtained and is in the form

$$V = \frac{1}{T}\Big(\mu_1 - \tfrac{1}{2}v\cdot v,\ \ldots,\ \mu_{n_S} - \tfrac{1}{2}v\cdot v,\ v_1,\ v_2,\ v_3,\ -1\Big), \tag{4.3}$$

where μ_k is the chemical potential of the k^{th} species.

Proposition 4.1. The change of variable $U \longmapsto V$ from the open convex set $\mathcal{O}_U$ onto the open set $\mathcal{O}_V = \mathbb{R}^{n_S+3} \times (-\infty, -1/T_0)$ is a C^∞ diffeomorphism.

Proof. From Proposition 2.2, $Y \to U$ is a C^∞ diffeomorphism from $\mathcal{O}_Y$ onto $\mathcal{O}_U$, so that we only have to show that $Y \to V$ is a C^∞ diffeomorphism from $\mathcal{O}_Y$ onto the open set $\mathcal{O}_V$. From Assumption (H_2) and the expressions (2.18) (2.23) (2.24), we first deduce that the map $Y \to V$ is C^∞ over the domain $\mathcal{O}_Y$. The matrix $\partial_Y V$ is then easily shown to be nonsingular over $\mathcal{O}_Y$ thanks to its triangular structure. Finally, it is straightforward to show that the map $Y \to V$ is one to one—since $-s_k$ is an increasing function of ρ_k at fixed T—and that its range is $\mathcal{O}_V$, and the proof is complete. $\square$

The conservative symmetric form is now investigated in the following theorem.

Theorem 4.2. The function $\mathcal{H}$ is an entropy for system (2.32), that is, $\mathcal{H}$ satisties Properties (E_1)–(E_4) of Definition 3.2. The system associated with the entropic variable $V \in \mathcal{O}_V$ can then be written

$$\widetilde{A}_0\partial_t V + \sum_{i\in C}\widetilde{A}_i\partial_i V = \sum_{i,j\in C}\partial_i\Big(\widetilde{G}_{ij}\partial_j V\Big) + \widetilde{\Omega}, \tag{4.4}$$

and satisfies Properties (S_1)–(S_4) of Definition 3.1. The matrix $\widetilde{A}_0$ is given by

$$\widetilde{A}_0 = \begin{pmatrix} \left(\dfrac{\rho_k}{r_k}\delta_{kl}\right)_{k,l\in S} & & Sym \\ \left(\dfrac{\rho_l}{r_l}v_i\right)_{i\in C,l\in S} & \left(\Sigma_\rho v_i v_j + \rho T\delta_{ij}\right)_{i,j\in C} & \\ \left(\dfrac{\rho_l}{r_l}e_l^{\mathrm{tot}}\right)_{l\in S} & \left(\Sigma_e v_j + \rho T v_j\right)_{j\in C} & \Upsilon_e \end{pmatrix} \tag{4.5}$$

where

$$\Sigma_\rho = \sum_{k\in S}\frac{\rho_k}{r_k}, \qquad \Sigma_e = \sum_{k\in S}\frac{\rho_k}{r_k}e_k^{\mathrm{tot}}, \qquad \Upsilon_e = \sum_{k\in S}\frac{\rho_k}{r_k}e_k^{\mathrm{tot}\,2} + \rho T\left(v\cdot v + c_v T\right).$$

Since this matrix is symmetric, we only give its block lower triangular part and write "*Sym*" in the upper triangular part. On the other hand, denoting by $\xi = (\xi_1, \xi_2, \xi_3)^t$ an arbitrary vector of $\mathbb{R}^3$, the matrices $\widetilde{A}_i$, $i \in C$, are given by

$$\sum_{i\in C}\widetilde{A}_i\xi_i =$$

$$\begin{pmatrix} \left(\delta_{kl}\dfrac{\rho_l}{r_l}v\cdot\xi\right)_{k,l\in S} & & Sym \\ \left(\rho_l T\xi_i + \dfrac{\rho_l}{r_l}v_i\, v\cdot\xi\right)_{i\in C,l\in S} & \left(\Sigma_\rho v_i v_j v\cdot\xi + \rho T(v_i\xi_j + v_j\xi_i + v\cdot\xi\delta_{ij})\right)_{i,j\in C} & \\ \left(h_l^{\mathrm{tot}}\dfrac{\rho_l}{r_l}\, v\cdot\xi\right)_{l\in S} & \left(\Sigma_h v_j v\cdot\xi + \rho T v_j v\cdot\xi + \rho T h^{\mathrm{tot}}\xi_j\right)_{j\in C} & \Upsilon_h v\cdot\xi \end{pmatrix} \tag{4.6}$$

where

$$\Sigma_h = \sum_{k\in S}\frac{\rho_k}{r_k}h_k^{\mathrm{tot}}, \qquad \Upsilon_h = \sum_{k\in S}\frac{\rho_k}{r_k}h_k^{\mathrm{tot}\,2} + \rho T\left(v\cdot v + (c_v + r)T\right).$$

Furthermore, concerning the dissipation matrices, we have the usual decomposition

$$\widetilde{G}_{ij} = \widetilde{G}_{ij}^{\kappa\eta} + \widetilde{G}_{ij}^{D\lambda}. \tag{4.7}$$

The viscous matrices $\widetilde{G}_{ij}^{\kappa\eta}$, $i, j \in C$, have the following structure

$$\widetilde{G}_{ij}^{\kappa\eta} = \begin{pmatrix} 0_{n_S\times n_S} & 0_{n_S\times 4} \\ 0_{4\times n_S} & \widetilde{\mathcal{K}}_{ij} \end{pmatrix}, \tag{4.8}$$

so that we only need the expressions of $\widetilde{\mathcal{K}}_{ij}$, $i,j \in C$. For sake of brevity, we only express $\widetilde{\mathcal{K}}_{11}$ and $\widetilde{\mathcal{K}}_{12}$

$$\widetilde{\mathcal{K}}_{11} = T \begin{pmatrix} (\kappa + \frac{4}{3}\eta) & 0 & 0 & (\kappa + \frac{4}{3}\eta) v_1 \\ 0 & \eta & 0 & \eta v_2 \\ 0 & 0 & \eta & \eta v_3 \\ (\kappa + \frac{4}{3}\eta) v_1 & \eta v_2 & \eta v_3 & (\kappa + \frac{1}{3}\eta) v_1^2 + \eta v \cdot v \end{pmatrix},$$

$$\widetilde{\mathcal{K}}_{12} = T \begin{pmatrix} 0 & (\kappa - \frac{2}{3}\eta) & 0 & (\kappa - \frac{2}{3}\eta) v_2 \\ \eta & 0 & 0 & \eta v_1 \\ 0 & 0 & 0 & 0 \\ \eta v_2 & (\kappa - \frac{2}{3}\eta) v_1 & 0 & (\kappa + \frac{1}{3}\eta) v_1 v_2 \end{pmatrix},$$

the other matrices being obtained by circular permutation and using the relations

$$\widetilde{\mathcal{K}}_{12} = \widetilde{\mathcal{K}}_{21}^t, \qquad \widetilde{\mathcal{K}}_{13} = \widetilde{\mathcal{K}}_{31}^t, \qquad \widetilde{\mathcal{K}}_{23} = \widetilde{\mathcal{K}}_{32}^t.$$

On the other hand, the heat and mass diffusion matrices $\widetilde{G}_{ij}^{D\lambda}$, $i,j \in C$, satisfy

$$\widetilde{G}_{11}^{D\lambda} = \widetilde{G}_{22}^{D\lambda} = \widetilde{G}_{33}^{D\lambda} = \widetilde{G}^{D\lambda}, \qquad \widetilde{G}_{ij}^{D\lambda} = 0, \quad i \neq j,$$

where

$$\widetilde{G}^{D\lambda} = \begin{pmatrix} (\mathcal{D}_{kl})_{k,l\in S} & & Sym \\ 0_{3\times n_S} & 0_{3\times 3} & \\ \Big(\sum_{k\in S} \mathcal{D}_{kl} h_k + \rho_l \theta_l T \Big)_{l\in S} & 0_{1\times 3} & \lambda' T^2 + 2 \sum_{k\in S} \rho_k \theta_k h_k T + \sum_{k,l\in S} \mathcal{D}_{kl} h_k h_l \end{pmatrix}, \tag{4.9}$$

and the symmetric matrix $\mathcal{D}$ has been defined from the multicomponent diffusion matrix D by

$$\mathcal{D}_{kl} = \rho_k \rho_l D_{kl} / r\rho. \tag{4.10}$$

Finally, the source term $\widetilde{\Omega}$ is given by

$$\widetilde{\Omega} = \Omega. \tag{4.11}$$

Proof. The calculation of the matrices $\widetilde{A}_0$, $\widetilde{A}_i$, $i \in C$, and $\widetilde{G}_{ij}$, $i,j \in C$, is lengthy but straighforward and, therefore, is omitted [Ma96]. This calculation is easily conducted by using the natural variable Y as an intermediate variable. The symmetry properties of $\widetilde{A}_0$, $\widetilde{A}_i$, $i \in C$, and $\widetilde{G}_{ij}$, $i,j \in C$, required in (S_1)–(S_4), are then obtained. We also have the identity $\widetilde{\Omega} = \Omega$, since (4.4) is derived by a change of variable.

Consider then a vector $x \in \mathbb{R}^{n_S+4}$, with components $(x_1, \dots, x_{n_S+4})$. After a little algebra, we obtain that

$$x^t \widetilde{A}_0 x = \rho T \left((x_{n_S+1} + v_1 x_{n_S+4})^2 + (x_{n_S+2} + v_1 x_{n_S+4})^2 + (x_{n_S+3} + v_1 x_{n_S+4})^2 \right) +$$

$$\sum_{k \in S} \frac{\rho_k}{r_k} \left(x_k + v_1 x_{n_S+1} + v_2 x_{n_S+2} + v_3 x_{n_S+3} + e_k^{\mathrm{tot}} x_{n_S+4} \right)^2 + \rho c_v T^2 x_{n_S+4}^2,$$

so that from (H_2) and the positivity of ρ_k, $k \in S$, and T, we deduce that $\widetilde{A}_0$ is positive definite.

On the other hand, by using (H_6)–(H_8), one can establish that

$$x^t \widetilde{G}^{D\lambda} x = \sum_{k,l \in S} \mathcal{D}_{kl} \big(x_k + (h_k + \frac{p\,\chi_k}{\rho_k}) x_{n_S+4} \big) \big(x_l + (h_l + \frac{p\,\chi_l}{\rho_l}) x_{n_S+4} \big) + \lambda T^2 x_{n_S+4}^2, \tag{4.12}$$

which shows that $\widetilde{G}^{D\lambda}$ is positive semi-definite thanks to (H_6).

Furthermore, a straightforward calculation leads to the following expression for the quadratic form associated with $\widetilde{B}(V, w)$

$$x^t \widetilde{B}(V, w) x = T(\kappa + \tfrac{1}{3}\eta)(o_1 w_1 + o_2 w_2 + o_3 w_3)^2 + T\eta(o_1^2 + o_2^2 + o_3^2) + x^t \widetilde{G}^{D\lambda} x, \tag{4.13}$$

where $o_i = x_{n_S+i} + v_i x_{n_S+4}$, $i = 1, 2, 3$, and where $w_1^2 + w_2^2 + w_3^2 = 1$. We thus obtain that the matrix $\widetilde{B}$ is symmetric, because it is the sum of symmetric matrices, and is positive semi-definite for $V \in \mathcal{O}_V$ and $w \in \mathcal{S}^2$, thanks to Assumptions (H_5) and (H_6). Finally, $\mathcal{H}$ also satisfies (E_1)–(E_4) as is easily checked and is strictly convex since $\widetilde{A}_0$ is positive definite over the open convex set $\mathcal{O}_U$. $\square$

4.2. Normal forms for multicomponent flows

In this section we establish that system (4.4) satisfies Condition N and we investigate normal forms.

Proposition 4.3. The nullspace of the matrix $\widetilde{B}$ associated with system (4.4) is one dimensional and is given by

$$N(\widetilde{B}) = \mathrm{span}(1, \dots, 1, 0, 0, 0, 0)^t. \tag{4.14}$$

Proof. According to Eq. (4.12) (4.13), the matrix $\widetilde{B}$ is positive semi-definite, so that its nullspace is constituted by the vectors x of $\mathbb{R}^{n_S+4}$ such that $x^t \widetilde{B} x = 0$. On the other hand, we have

$$x^t \widetilde{B}(V, w) x = \left(\kappa + \tfrac{1}{3}\eta\right) T (o_1 w_1 + o_2 w_2 + o_3 w_3)^2 + \eta T (o_1^2 + o_2^2 + o_3^2) +$$

$$\sum_{k,l\in S} \mathcal{D}_{kl}\big(x_k + (h_k + \frac{p\chi_k}{\rho_k})x_{n_S+4}\big)\big(x_l + (h_l + \frac{p\chi_l}{\rho_l})x_{n_S+4}\big) + \lambda T^2 x_{n_S+4}^2,$$

where $o_i = x_{n_S+i} + v_i x_{n_S+4}$, $i = 1,2,3$. As a consequence, $x^t \widetilde{B} x = 0$ implies that $x_{n_S+4} = 0$ and that $x_{n_S+i} = 0$, $i = 1,2,3$, thanks to (H_5). Therefore, x is in the nullspace of $\widetilde{B}(V,w)$ if and only if we have

$$\sum_{k,l\in S} \mathcal{D}_{kl} x_k x_l = 0. \tag{4.15}$$

Using (H_6) and (4.10) we then obtain that the nullspace of $\widetilde{B}(V,w)$ is one dimensional and spanned by $(1,\dots,1,0,0,0,0)^t$ and is thus independent of $V \in \mathcal{O}_V$ and $w \in \mathcal{S}^2$. □

Since the system of equations governing multicomponent reacting flows satisfies Condition N, we can now obtain from Lemma 3.7 the auxiliary variables U' and V'. From Lemma 3.7 and Proposition 4.3 the matrix P can be taken to be

$$P = \begin{pmatrix}
1 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & \ddots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & 0 & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 1 & 0 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 1 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 1 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{4.16}$$

in such a way that

$$U' = P^t U = \Big(\rho,\, \rho_2, \dots, \rho_{n_S},\, \rho v_1,\, \rho v_2,\, \rho v_3,\, \rho e^{\mathrm{tot}}\Big)^t. \tag{4.17}$$

The associated entropic variable is then $V' = P^{-1} V$ where V is given by (4.3), and the corresponding symmetric system is easily obtained from (4.4).

Proposition 4.4. The system in the new dependent variable V',

$$V' = \frac{1}{T}\Big(\mu_1 - \tfrac{1}{2} v\cdot v,\, \mu_2 - \mu_1, \dots, \mu_{n_S} - \mu_1,\, v_1,\, v_2,\, v_3,\, -1\Big)^t, \tag{4.18}$$

can be written

$$\widetilde{A}'_0 \partial_t V' + \sum_{i\in C} \widetilde{A}'_i \partial_i V' = \sum_{i,j\in C} \partial_i \Big(\widetilde{G}'_{ij} \partial_j V'\Big) + \widetilde{\Omega}', \tag{4.19}$$

where $V' \in \mathcal{O}_{V'} = \mathbb{R}^{n_S+3} \times (-\infty, -1/T_0)$, $\widetilde{A}'_0 = P^t \widetilde{A}_0 P$, $\widetilde{A}'_i = P^t \widetilde{A}_i P$, $i = 1,2,3$, $\widetilde{G}'_{ij} = P^t \widetilde{G}_{ij} P$, $i,j = 1,2,3$, and where $\widetilde{\Omega}' = P^t \Omega = (0, \Omega_{II})^t$.

In particular, Properties (S_1)–(S_4) of Definition 3.1 are satisfied and the dissipation matrices are given by

$$\widetilde{G}'_{ij} = \begin{pmatrix} 0 & 0_{1\times(n_S+3)} \\ 0_{(n_S+3)\times 1} & \widetilde{G}^{II,II}_{ij} \end{pmatrix}, \tag{4.20}$$

where $\widetilde{G}^{II,II}_{ij}$ is the lower right block of size n_S+3 of $\widetilde{G}$.

We now investigate normal forms for system (4.4), or, equivalently, for system (4.19). We first use the possibility of mixing parabolic components—the V'_{II} components—established in Theorem 3.8, in order to simplify the analytic expression of the normal variable, and, consequently, of the matrix coefficients appearing in the normal form. More specifically, we consider the variable

$$W = \Big(\rho,\ \log(\rho_2^{r_2}/\rho_1^{r_1}), \ldots,\ \log(\rho_{n_S}^{r_{n_S}}/\rho_1^{r_1}),\ v_1,\ v_2,\ v_3,\ T\Big)^t,$$

easily obtained by combining the V'_{II} components, and derive the corresponding normal form of the governing equations. When there is only one gas, this normal form is identical with the one previously obtained by Kawashima and Shizuta in [KS88]. We also investigate, in the Appendix, the normal form associated with the "natural" normal variable $\widehat{W} = (U'_I, V'_{II})^t$ which guarantees a conservative form for the dissipative terms of the system, leaves invariant the source term Ω_{II}, but has a more complex expression.

Theorem 4.5. The system in the variable $W = (W_I, W_{II})^t$, on the open convex set $\mathcal{O}_W = (0,\infty)\times\mathbb{R}^{n_S-1}\times\mathbb{R}^3\times(T_0,\infty)$, with hyperbolic variable $W_I = \rho$ and parabolic variable $W_{II} = \big(\log(\rho_2^{r_2}/\rho_1^{r_1}), \ldots, \log(\rho_{n_S}^{r_{n_S}}/\rho_1^{r_1}),\ v_1,\ v_2,\ v_3,\ T\big)^t$, can be written

$$\overline{A}_0^{I,I}\partial_t W_I \ + \textstyle\sum_{i\in C} \overline{A}_i^{I,I}\partial_i W_I \ + \sum_{i\in C}\overline{A}_i^{I,II}\partial_i W_{II} \ = \ 0, \tag{4.21}$$

$$\begin{aligned}\overline{A}_0^{II,II}\partial_t W_{II} + \textstyle\sum_{i\in C}\overline{A}_i^{II,I}\partial_i W_I + \sum_{i\in C}\overline{A}_i^{II,II}\partial_i W_{II} = \\ \sum_{i,j\in C}\partial_i\Big(\overline{G}_{ij}^{II,II}\partial_j W_{II}\Big) + \overline{H}_{II} + \overline{\Omega}_{II},\end{aligned} \tag{4.22}$$

and is of the normal form. The matrix $\overline{A}_0$ is given by

$$\overline{A}_0 = \begin{pmatrix} \dfrac{1}{\Sigma_\rho} & & & 0 \\ & \mathcal{X} & & \\ & & \dfrac{\rho}{T} I_3 & \\ 0 & & & \dfrac{\rho c_v}{T^2} \end{pmatrix},$$

where $\mathcal{X}$ is a square matrix of dimension $n_S - 1$ given by

$$\mathcal{X}_{kl} = \delta_{kl}\frac{\rho_k}{r_k} - \frac{\rho_k\rho_l}{r_k r_l}\frac{1}{\Sigma_\rho}, \qquad k,l \in [2, n_S]. \tag{4.23}$$

Denoting by $\xi = (\xi_1, \xi_2, \xi_3)^t$ an arbitrary vector of $\mathbb{R}^3$, the matrices $\overline{A}_i$, $i = 1,2,3$, are given by

$$\sum_{i\in C}\overline{A}_i\xi_i = \begin{pmatrix} \dfrac{v\cdot\xi}{\Sigma_\rho} & & & Sym \\ 0_{(n_S-1)\times 1} & \mathcal{X}\, v\cdot\xi & & \\ \dfrac{\rho}{\Sigma_\rho}\xi & \xi\otimes\mathcal{Z} & \dfrac{\rho}{T}v\cdot\xi\, I_3 & \\ 0 & 0_{1\times(n_S-1)} & \dfrac{\rho r}{T}\xi^t & \dfrac{\rho c_v}{T^2}v\cdot\xi \end{pmatrix},$$

where $\mathcal{Z}$ is a vector of dimension $n_S - 1$ given by

$$\mathcal{Z}_l = \rho_l - \frac{\rho_l\,\rho}{r_l\,\Sigma_\rho}, \qquad l \in [2, n_S].$$

For the heat and mass diffusion matrices, $\overline{G}_{ij}^{D\lambda}$, $i,j = 1,2,3$, we have

$$\overline{G}_{11}^{D\lambda} = \overline{G}_{22}^{D\lambda} = \overline{G}_{33}^{D\lambda} = \overline{G}^{D\lambda} =$$

$$\begin{pmatrix} 0 & & & Sym \\ 0_{(n_S-1)\times 1} & (\mathcal{D}_{kl})_{k,l\geq 2} & & \\ 0_{3\times 1} & 0_{3\times(n_S-1)} & 0_{3\times 3} & \\ 0 & \frac{1}{T}\Big(\sum\limits_{k\in S}\mathcal{D}_{kl}r_k + \rho_l\theta_l\Big)_{l\geq 2} & 0_{1\times 3} & \dfrac{\lambda'}{T^2} + 2\sum\limits_{k\in S}\dfrac{\rho_k\theta_k r_k}{T^2} + \sum\limits_{k,l\in S}\dfrac{\mathcal{D}_{kl}r_k r_l}{T^2} \end{pmatrix},$$

whereas the nondiagonal terms vanish $\overline{G}_{ij}^{D\lambda} = 0, \quad i \neq j$. The dissipation matrices due to the viscous effects $\overline{G}_{ij}^{\kappa\eta}$, $i,j = 1,2,3$, still have the structure (4.8) and the corresponding matrices $\overline{\mathcal{K}}_{ij}$, $i,j = 1,2,3$, are given by

$$\overline{\mathcal{K}}_{11} = \frac{1}{T}\begin{pmatrix} \kappa + \frac{4}{3}\eta & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \eta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \overline{\mathcal{K}}_{12} = \frac{1}{T}\begin{pmatrix} 0 & \kappa - \frac{2}{3}\eta & 0 & 0 \\ \eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

with the other ones deduced by circular permutation and from the relations

$$\overline{\mathcal{K}}_{12} = \overline{\mathcal{K}}_{21}^t, \qquad \overline{\mathcal{K}}_{13} = \overline{\mathcal{K}}_{31}^t, \qquad \overline{\mathcal{K}}_{23} = \overline{\mathcal{K}}_{32}^t.$$

Finally, the terms $\overline{H}_{II}$ is easily computed from (3.18) whereas the corresponding source term $\overline{\Omega} = (\partial_{W^*} V^*)^t\, \Omega$ is given by

$$\overline{\Omega} = \Big(0,\, m_2\omega_2, \ldots,\, m_{n_S}\omega_{n_S},\, \frac{g_1}{T},\, \frac{g_2}{T},\, \frac{g_3}{T},\, -\frac{1}{T^2}\big(\sum_{k\in S} e_k m_k \omega_k + g{\cdot}v\big)\Big)^t.$$

Proof. The calculations are lenghty but straightforward and make use of Theorem 4.2, Proposition 4.4 and Assumptions (H_1)–(H_9). □

Remark 4.6. Note that if $g = 0$ and if the source term Ω remains in a fixed subspace of $\mathbb{R}^{n_S} \times 0_{\mathbb{R}^4}$, the source term $\overline{\Omega}$ is no longer in a *fixed* subspace of $\mathbb{R}^{n_S+4}$ of the same dimension because of the coefficients e_k/T^2 in the term $\sum_{k\in S} e_k m_k \omega_k / T^2$ which introduce an explicit dependence on the state variables.

5. Global Existence and Asymptotic Stability for an Abstract System

In this section, we further consider the abstract quasilinear second order system (3.1). We assume that the system admits an entropy function and that Condition N holds, so that the system can be recast into a normal form. Under stability conditions on the source term, and assuming conditions which guarantee the dissipative structure of the linearized normal system around the constant equilibrium state, we prove global existence, asymptotic stability and decay estimates for the Cauchy problem.

5.1. Equilibrium point and conservation of entropy

We consider an abstract system of conservation laws (3.1) satisfying Assumptions (A_1)–(A_3) of Section 3. This system can then be written in the symmetric form (3.2) and in a normal form (3.7). We further assume in the following that system (3.1) possesses a constant equilibrium state.

(A_4) There exists a constant equilibrium state $U^{*e} \in \mathcal{O}_{U^*}$ such that

$$\Omega(U^{*e}) = 0.$$

The equilibrium states corresponding to the various variables are also denoted with the superscript ^{e}, so that the equilibrium states in the variables V^* and W^*, for instance, are denoted by V^{*e} and W^{*e}, respectively.

In order to establish existence theorems, we will need a priori estimates for the solution. To this purpose, we establish a conservation equation for the generalized entropy function $\mathcal{H}^*$. This equation is easily obtained by taking the scalar product of (3.2) with the vector V^* and reads

$$\partial_t \mathcal{H}^* + \sum_{i\in C^*} \partial_i \sigma_i^* = \sum_{i,j\in C^*} \partial_i \big\langle V^*, \widetilde{G}^*_{ij}\partial_j V^*\big\rangle - \sum_{i,j\in C^*} \big\langle \partial_i V^*, \widetilde{G}^*_{ij}\partial_j V^*\big\rangle + \big\langle V^*, \widetilde{\Omega}(V^*)\big\rangle, \tag{5.1}$$

where σ_i^*, $i \in C^*$, are the entropy fluxes. We also introduce the associated function $\mathfrak{h}(U^*, U^{*e})$ defined by

$$\mathfrak{h}(U^*, U^{*e}) = \mathcal{H}^*(U^*) - \mathcal{H}^*(U^{*e}) - \langle V^{*e}, U^* - U^{*e}\rangle, \qquad U^* \in \mathcal{O}_{U^*}. \tag{5.2}$$

Thanks to the strict convexity of $\mathcal{H}^*$, this function plays the role of a distance between U^* and the stationary state U^{*e} and the following proposition is easily established.

Proposition 5.1. The function $\mathfrak{h}$ is a positive, smooth, strictly convex function of $U^* \in \mathcal{O}_{U^*}$, satisfying $\mathfrak{h}(U^*, U^{*e}) = 0$ if and only if $U^* = U^{*e}$. There exists also a neighborhood $\mathfrak{O} = \{\, z \in \mathbb{R}^n;\ |z - W^{*e}| < r \,\}$ of W^{*e} with r small enough such that for any W^* in $\mathfrak{O}$ and $U^* = U^*(W^*)$

$$c|W^* - W^{*e}|^2 \leq \mathfrak{h}(U^*, U^{*e}) \leq C|W^* - W^{*e}|^2, \tag{5.3}$$

where c and C are positive constants. In addition, the function $\mathfrak{h}$ satisfies the conservation equation

$$\begin{aligned} &\partial_t \mathfrak{h} + \sum_{i\in C^*} \partial_i \Big(\sigma_i^*(U^*) - \sigma_i^*(U^{*e}) - \big\langle V^{*e}, F_i^*(U^*) - F_i^*(U^{*e})\big\rangle\Big) = \\ &\sum_{i,j\in C^*} \partial_i \big\langle V^* - V^{*e}, \widetilde{G}^*_{ij}\partial_j V^*\big\rangle - \sum_{i,j\in C^*} \big\langle \partial_i V^*, \widetilde{G}^*_{ij}\partial_j V^*\big\rangle + \big\langle V^* - V^{*e}, \widetilde{\Omega}^*(V^*)\big\rangle. \end{aligned} \tag{5.4}$$

5.2. Local dissipative structure

In order to establish global existence in time and asymptotic stability of a constant stationary state, decay estimates for linearized equations are needed. A condition which guarantees decay properties for the linearized system is the local dissipative structure introduced by Kawashima [Ka84]. This dissipative structure is completed here by stability properties of the source term.

5.2.1. *Linearized equations*

If we linearize system (3.7) around the constant stationary state W^{*e}, we obtain the following linear system in terms of the variable $z = W^* - W^{*e}$

$$\overline{A}_0^*(W^{*e})\partial_t z + \sum_{i\in C^*} \overline{A}_i^*(W^{*e})\partial_i z = \sum_{i,j\in C^*} \overline{G}_{ij}^*(W^{*e})\partial_i\partial_j z - \overline{L}^*(W^{*e}) z, \tag{5.5}$$

where $\overline{L}^*$ is defined by $\overline{L}^* = -\partial_{W^*}\overline{\Omega}^*$. We assume that this linearized system has a dissipative structure in the sense of Kawashima.

- (D_1) The matrix $\overline{A}^*_0(W^{*e})$ is symmetric and positive definite, the matrices $\overline{A}^*_i(W^{*e})$, $i \in C^*$ are symmetric, we have $\overline{G}^*_{ij}(W^{*e})^t = \overline{G}^*_{ji}(W^{*e})$ for $i, j \in C^*$ and the matrix $\overline{L}^*(W^{*e})$ is symmetric positive semi-definite,
- (D_2) There exist compensating matrices K^j, $j \in C^*$, such that the products $K^j\overline{A}^*_0(W^{*e})$, $j \in C^*$, are skew-symmetric, and such that the matrix

  $$\sum_{i,j\in C^*} \tfrac{1}{2}\Big(K^j\overline{A}^*_i(W^{*e}) + \big(K^j\overline{A}^*_i(W^{*e})\big)^t\Big) w_i w_j + \overline{B}^*(W^{*e}, w) + \overline{L}^*(W^{*e}),$$

  is positive definite for $w \in \mathcal{S}^{d-1}$.

Remark 5.2. The existence of the compensating matrices K^j, $j \in C^*$, implies that the linearized normal form is strictly dissipative in the sense that the eigenvalues $\lambda^*(\zeta, w)$ of the problem

$$\lambda^*\overline{A}^*_0(W^{*e})\phi + [\zeta \sum_{i\in C^*} \overline{A}^*_i(W^{*e})w_i - \zeta^2\overline{B}^*(W^{*e}, w) + \overline{L}^*(W^{*e})]\phi = 0,$$

for $\zeta \in \mathrm{i}\,\mathbb{R}$ and $w \in \mathcal{S}^{d-1}$, have a negative real part [SK85]. However, the converse is not known to be true [SK85]. The latter property only implies the existence of a combined compensating matrix $K(w)$, for $w \in \mathcal{S}^{d-1}$, such that $w \to K(w)$ is C^∞ on $\mathcal{S}^{d-1}$, $K(w)$ is real, the product $K(w)\overline{A}^*_0(W^{*e})$ is skew-symmetric, and $K(-w) = -K(w)$, for $w \in \mathcal{S}^{d-1}$, and such that the matrix

$$\sum_{i\in C^*} \tfrac{1}{2}\Big(K(w)\overline{A}^*_i(W^{*e}) + \big(K(w)\overline{A}^*_i(W^{*e})\big)^t\Big) w_j + \overline{B}^*(W^{*e}, w) + \overline{L}^*(W^{*e}),$$

is positive definite for $w \in \mathcal{S}^{d-1}$. It is not known, however, if the matrix $K(w)$ is of the form $\sum_{j\in C^*} K^j w_j$ [SK85]. Nevertheless, all the results obtained in this paper can be proved without the existence of matrices K^j, $j \in C$, by only using the combined compensating matrix $K(w)$, $w \in \mathcal{S}^{d-1}$, that is, by only using the strict dissipativity of the system. Nevertheless, in practical applications, it is generally possible to obtain compensating matrices K^j, $j \in C^*$, and to set $K(w) = \sum_{j\in C} K^j w_j$.

5.2.2. *Locally stable source terms*

We have already assumed that the matrix $\overline{L}^*(W^{*e})$ is symmetric in (D_1). We now further introduce local stability assumptions concerning the source term.

- (D_3) The smallest subspace containing the source term $\widetilde{\Omega}^*(V^*) = \Omega^*\big(U^*(V^*)\big)$, for all $V^* \in \mathcal{O}_{V^*}$, is included in the range of $\widetilde{L}^*(V^{*e}) = -(\partial_{V^*}\widetilde{\Omega}^*)(V^{*e})$.
- (D_4) There exists a neighborhood of V^{*e} in $\mathcal{O}_{V^*}$ and a positive constant c such that for any V^* in this neighborhood, we have

  $$c\,\big|\,\widetilde{\Omega}^*(V^*)\big|^2 \;\leq\; -\big\langle V^* - V^{e*},\, \widetilde{\Omega}^*(V^*)\big\rangle.$$

We can choose r small enough such that this inequality holds with $V^* = V^*(W^*)$ and W^* in $\mathfrak{O} = \{ z \in \mathbb{R}^n;\ |z - W^{*e}| < r \}$.

Note that Properties (D_3) and (D_4) only concern the source term $\widetilde{\Omega}^*$. Property (D_3) will be used for decay estimates whereas Property (D_4) will be needed for the existence theorem.

5.2.3. *Global dissipative structure*

The physical meaning of the entropy conservation equation (5.1) is that when

$$\sum_{i,j \in C^*} \langle \partial_i V^*, \widetilde{G}^*_{ij} \partial_j V^* \rangle \geq 0, \tag{5.6}$$

and

$$\langle V^*, \widetilde{\Omega}(V^*) \rangle \leq 0 \tag{5.7}$$

then the integral $\int_{\mathbb{R}^d} \mathcal{H}^*\, dx$ is nonincreasing in time, which corresponds to the second principle of thermodynamics.

Note however, that Property (S_4) does not imply the stronger condition (5.6). Similarly, we have only assumed, with Property (D_4), that the source term is locally stable, and it does not implies (5.7) globally. However, for multicomponent reactive flows, Properties (5.6) and (5.7) are globally satisfied, as they should for any physically reasonable model. This suggests the following definition of a strong entropy which could be used in order to obtain global estimates, not necessarily in the neighborhood of a constant state.

Definition 5.3. A function $\mathcal{H}^*$ is be said to be a strong entropy for system (3.1) if it is an entropy according to Definition 3.2 of Section 3 and if the inequalities (5.6) and (5.7) hold over $\mathcal{O}_{V^*}$.

5.3. A global existence theorem

Now that we have stated Assumptions (A_1)–(A_4) and Dissipative conditions (D_1)–(D_4), concerning the various forms of the governing equations, we investigate global existence of solutions around the stationary state W^{*e} and its asymptotic stability. Of course, Assumptions (A_1)–(A_4) imply Properties (S_1)–(S_4), $(\overline{S}_1)$–$(\overline{S}_4)$, and (N_1)–(N_3) of Section 3. Asymptotic decay towards the equilibrium state W^{*e} will be investigated in Section 5.4.

5.3.1. *The main result*

Theorem 5.4. Consider the quasilinear system (3.1) and assume that (A_1)–(A_4) and (D_1)–(D_4) hold. Let $d \geq 1$, $l \geq [d/2] + 2$, and let $W^{*0}(x)$ such that

$$W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d). \tag{5.8}$$

Then if $\|W^{*0} - W^{*e}\|_{l,2}$ is small enough, there exists a unique global solution to the Cauchy problem

$$\overline{A}_0^* \partial_t W^* + \sum_{i \in C^*} \overline{A}_i^* \partial_i W^* = \sum_{i,j \in C^*} \partial_i \Big(\overline{G}_{ij}^* \partial_j W^* \Big) + \overline{H}^* + \overline{\Omega}^*, \tag{5.9}$$

with initial conditions

$$W^*(0,x) = W^{*0}(x), \tag{5.10}$$

such that

$$\begin{cases} W_{I}^* - W_{I}^{*e} \in C^0\big([0,\infty); W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,\infty); W_2^{l-1}(\mathbb{R}^d)\big) \cap L^2\big(0,\infty; W_2^l(\mathbb{R}^d)\big), \\ W_{II}^* - W_{II}^{*e} \in C^0\big([0,\infty); W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,\infty); W_2^{l-2}(\mathbb{R}^d)\big) \cap L^2\big(0,\infty; W_2^{l+1}(\mathbb{R}^d)\big). \end{cases} \tag{5.11}$$

This solution W^* satisfies the following estimate

$$\|W^*(t) - W^{*e}\|_{l,2}^2 + \int_0^t \Big(\|\partial_x W_{I}^*(\tau)\|_{l-1,2}^2 + \|\partial_x W_{II}^*(\tau)\|_{l,2}^2 \Big)\, d\tau \;\leq\; C \|W^{*0} - W^{*e}\|_{l,2}^2, \tag{5.12}$$

and $\sup_{\mathbb{R}^d} |W^*(t) - W^{*e}|$ goes to zero as $t \to \infty$.

In order to establish this result, following Kawashima [Ka84], we restate a local existence theorem, we derive a priori estimates for the local solution of the Cauchy problem and we show that this local solution can be extended indefinitely.

5.3.2. *Local existence*

In this section we restate a local existence theorem due to Kawashima [Ka84]. Local existence is proved for an initial data near the stationary state with a control on the distance between the solution and the constant state.

Theorem 5.5. Assume that Properties (A_1)–(A_4) are satisfied. Let $d \geq 1$ and $l \geq [d/2]+2$ be integers. Let $\mathcal{O}_0$ be a bounded convex open set such that $\overline{\mathcal{O}}_0 \subset \mathcal{O}_{W^*}$, let $d_1 < d\big(\mathcal{O}_0, \partial\mathcal{O}_{W^*}\big)$, let b_0 be positive and let $\mathcal{O}_1 = \{\, z \in \mathbb{R}^n;\ d(z, \mathcal{O}_0) < d_1 \,\}$. Then there exists a positive constant T_1, depending only on $\mathcal{O}_0$, d_1 and b_0, such that for any $W^{*0}(x)$ with $W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d)$ and

$$\|W^{*0} - W^{*e}\|_{l,2} \leq b_0, \qquad W^{*0}(x) \in \mathcal{O}_0, \quad x \in \mathbb{R}^d, \tag{5.13}$$

the system (5.9) with initial condition W^{*0} has a unique solution W^* satisfying

$$\begin{cases} W_{I}^* - W_{I}^{*e} \in C^0\big([0,T_1]; W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,T_1]; W_2^{l-1}(\mathbb{R}^d)\big) \cap L^2\big(0,T_1; W_2^l(\mathbb{R}^d)\big), \\ W_{II}^* - W_{II}^{*e} \in C^0\big([0,T_1]; W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,T_1]; W_2^{l-2}(\mathbb{R}^d)\big) \cap L^2\big(0,T_1; W_2^{l+1}(\mathbb{R}^d)\big), \end{cases} \tag{5.14}$$

and $W^*(t,x) \in \mathcal{O}_1$, for $(t,x) \in [0,T_1] \times \mathbb{R}^d$. This solution also satisfies the estimate, for $t \in [0,T_1]$

$$N_l(t) \leq C_l \|W^{*0} - W^{*e}\|_{l,2}, \tag{5.15}$$

where $C_l > 1$ is a constant depending only on $\mathcal{O}_0$, d_1, and b_0, where we have defined $N_l(t) = N_l(0,t)$, and

$$N_l(t_1,t_2)^2 = \sup_{t_1\leq\tau\leq t_2} \|W^*(\tau) - W^{*e}\|_{l,2}^2 + \int_{t_1}^{t_2} \Big(\|\partial_x W_I^*(\tau)\|_{l-1,2}^2 + \|\partial_x W_{II}^*(\tau)\|_{l,2}^2 \Big)\, d\tau. \tag{5.16}$$

This theorem is essentially obtained by first considering linear equations and establishing a priori estimates. Successive iterations are then shown to be convergent and we refer to Kawashima [Ka84] for more details.

5.3.3. *A priori estimates*

We first remark that the norm $N_l(T)$ can be used is order to control the solution in a given neighborhood of W^{*e} in $\mathcal{O}_{W^*}$.

Lemma 5.6. Let $l \geq [d/2]+1$ and let $\mathcal{B}$ be a bounded neighborhood of W^{*e} in $\mathcal{O}_{W^*}$. Then there exists a constant $\beta_0(\mathcal{B})$ such that if the solution W^* satisfies $N_l(T) \leq \beta_0(\mathcal{B})$, then $W^*(t,x) \in \mathcal{B}$ for any $(t,x) \in [0,T]\times\mathbb{R}^d$. In particular, when $N_l(T) \leq \beta_0(\mathfrak{D})$, then inequality (5.3) of Proposition 5.1 and the estimate of (D_4) hold.

We now want to estimate the quantity $N_l(T)$ when it satisfies a smallness assumption. We first restate a result of Kawashima concerning the norms of higher derivatives of W_I^* and W_{II}^*. In the next lemmas, we complete this estimate by considering lower derivatives. These estimates involve the orthogonal projector $\mathcal{P}$ onto the range of $\overline{L}^*(W^{*e})$. The following result has been established by Kawashima [Ka84] and is given here without proof.

Lemma 5.7. Assume that Properties (A_1)–(A_4) and (D_1)–(D_2) hold. Let $d \geq 1$ and $l \geq [d/2]+2$ be integers and let the initial data satisfy $W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d)$. Let b be positive and W^* be a solution of (5.9) (5.10) satisfying

$$\begin{cases} W_I^* - W_I^{*e} \in C^0\big([0,T]; W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,T]; W_2^{l-1}(\mathbb{R}^d)\big) \cap L^2\big(0,T; W_2^l(\mathbb{R}^d)\big), \\ W_{II}^* - W_{II}^{*e} \in C^0\big([0,T]; W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,T]; W_2^{l-2}(\mathbb{R}^d)\big) \cap L^2\big(0,T; W_2^{l+1}(\mathbb{R}^d)\big), \end{cases}$$

and $N_l(T) \leq b$. Then there exists a constant $C_{g1} = C_{g1}(b)$ such that the following estimate holds for $t \in [0,T]$

$$\|\partial_x W^*(t)\|_{l-1,2}^2 + \int_0^t \Big(\|\mathcal{P}\partial_x W^*(\tau)\|_{l-1,2}^2 + \|\partial_x^2 W_{II}^*(\tau)\|_{l-1,2}^2 + \|\partial_x^2 W_I^*(\tau)\|_{l-2,2}^2 \Big)\, d\tau$$
$$\leq C_{g1} \Big(\|\partial_x W^{*0}\|_{l-1,2}^2 + N_l(T)^3 \Big). \tag{5.17}$$

We next estimate $\|W^* - W^{*e}\|_{0,2}$ and $\int_0^t (\|\partial_x W_{II}^*(\tau)\|_{0,2}^2 + \|\overline{\Omega}^*(\tau)\|_{0,2}^2)\, d\tau$ by means of the modified entropy function $\mathfrak{h}$.

Proposition 5.8. Keep the hypotheses of Lemma 5.7 and assume also that (D_4) holds and that $b \leq \beta_0(\mathfrak{D})$. Then there exists a positive constant $C_{g2} = C_{g2}(b)$ such that the following estimate holds for $t \in [0, T]$

$$\|W^*(t) - W^{*e}\|_{0,2}^2 + \int_0^t \left(\|\partial_x W_{II}^*(\tau)\|_{0,2}^2 + \|\,\overline{\Omega}^*(\tau)\|_{0,2}^2 \right) d\tau$$

$$\leq C_{g2} \left(\|W^{*0} - W^{*e}\|_{0,2}^2 + N_l(T)^3 \right). \tag{5.18}$$

Proof. Integrating the conservation equation (5.4) for $\mathfrak{h}$ over $[0,T] \times \mathbb{R}^d$ using inequality (5.3) and (D_4) we easily obtain that

$$\|W^*(t) - W^{*e}\|_{0,2}^2 + \sum_{i,j \in C^*} \int_0^t \int_{\mathbb{R}^d} \langle \partial_i V^*, \widetilde{G}_{ij}^* \partial_j V^* \rangle \, dx d\tau$$

$$+ \int_0^t \int_{\mathbb{R}^d} \left| \widetilde{\Omega}^*(V^*) \right|^2 dx d\tau \;\leq\; C \|W^{*0} - W^{*e}\|_{0,2}^2. \tag{5.19}$$

From Property (N_1) we also have $\langle \partial_i V^*, \widetilde{G}_{ij}^* \partial_j V^* \rangle = \langle \partial_i V_{II}^{*\prime}, \widetilde{G}_{ij}^{*\prime II,II} \partial_j V_{II}^{*\prime} \rangle$ since $V^{*\prime} = P^{-1} V^*$ and $\widetilde{G}_{ij}^{*\prime} = P^t \widetilde{G}_{ij}^* P$, and there exists constants c and C such that

$$\sum_{i,j \in C^*} \int_0^t \int_{\mathbb{R}^d} \langle \partial_i V_{II}^{*\prime}, \widetilde{G}_{ij}^{*\prime II,II} \partial_j V_{II}^{*\prime} \rangle \, dx d\tau \;\geq\; c \int_0^t \int_{\mathbb{R}^d} |\partial_x V_{II}^{*\prime}|^2 \, dx d\tau \;-\; C\, N_l(t)^3,$$

since $\sum_{i,j \in C^*} \widetilde{G}_{ij}^{*\prime II,II}(W^{*e}) w_i w_j$ is positive definite. The proof is then complete since $W_{II}^* = \varphi_{II}(V_{II}^{*\prime})$ where φ_{II} is a diffeomorphism of $\mathbb{R}^{n-n_0}$ and $\overline{\Omega}^* = (\partial_{W^*} V^*)^t \, \widetilde{\Omega}^*$. ☐

5.3.4. *More a priori estimates*

We now focus on a priori estimates concerning $\int_0^t \|\partial_x W_I^*(\tau)\|_{0,2}^2 d\tau$.

Lemma 5.9. Keep the assumptions of Lemma 5.7. Then there exists a positive constant $C_{g3} = C_{g3}(b)$ such that the following estimate holds for $t \in [0, T]$

$$\int_0^t \|\partial_x W_I^*(\tau)\|_{0,2}^2 d\tau - C_{g3} \left(\|W^*(t) - W^{*e}\|_{1,2}^2 + \int_0^t \|\partial_x \mathcal{P} W^*(\tau)\|_{0,2}^2 \, d\tau \right.$$

$$\left. + \int_0^t \|\partial_x W_{II}^*(\tau)\|_{1,2}^2 \, d\tau + \int_0^t \|\overline{\Omega}^*(\tau)\|_{0,2}^2 \, d\tau \right) \leq C_{g3} \left(\|W^{*0} - W^{*e}\|_{1,2}^2 + N_l(T)^3 \right). \tag{5.20}$$

Proof. We rewrite the system (5.9) in the form

$$\overline{A}_0(W^{*e}) \partial_t W^* + \sum_{i \in C} \overline{A}_i^*(W^{*e}) \partial_i W^* - \sum_{i,j \in C} \overline{G}_{ij}^*(W^{*e}) \partial_i \partial_j W^*$$

$$- \overline{A}_0^*(W^{*e})\big(\overline{A}_0^*(W^*)\big)^{-1}\overline{\Omega}^* = h, \tag{5.21}$$

where

$$\begin{aligned} h = & \sum_{i\in C^*} \overline{A}_0^*(W^{*e})\Big[\big(\overline{A}_0^*(W^{*e})\big)^{-1}\overline{A}_i(W^{*e}) - \big(\overline{A}_0^*(W^*)\big)^{-1}\overline{A}_i^*(W^*)\Big]\partial_i W^* \\ & - \sum_{i,j\in C^*} \overline{A}_0^*(W^{*e})\Big[\big(\overline{A}_0^*(W^{*e})\big)^{-1}\overline{G}_{ij}^*(W^{*e}) - \big(\overline{A}_0^*(W^*)\big)^{-1}\overline{G}_{ij}(W^*)\Big]\partial_i\partial_j W^* \\ & + \overline{A}_0^*(W^{*e})\big(\overline{A}_0^*(W^*)\big)^{-1}\Big(\sum_{i,j\in C} \partial_{W^*}\overline{G}_{ij}^*(W^*)\partial_i W^*\partial_j W^* + \overline{H}^*\Big) \end{aligned}$$

Multiplying this equation by K^j, taking the scalar product with $\partial_j W^*$, summing with respect to j, and integrating over $[0,t]\times\mathbb{R}^d$ yields the estimate (5.20) up to the rightest term which reads $\int_0^t \mathcal{T}(\tau)\,d\tau$ with $\mathcal{T}$ given by

$$\mathcal{T}(t) = \sum_{i\in C^*}\int_{\mathbb{R}^d} \langle \partial_j W^*, K^j h\rangle\, dx. \tag{5.22}$$

This results form the skew-symmetry of $K^j\overline{A}_0^*(W^{*e})$ which implies that

$$\begin{aligned} \Big\langle \partial_j W^*,\, K^j\overline{A}_0^*(W^{*e})\partial_t W^*\Big\rangle = {} & \tfrac{1}{2}\partial_t\Big\langle \partial_j W^*,\, K^j\overline{A}_0^*(W^{*e})(W^* - W^{*e})\Big\rangle \\ & + \ \tfrac{1}{2}\partial_j\Big\langle (W^* - W^{*e}),\, K^j\overline{A}_0^*(W^{*e})\partial_t W^*\Big\rangle, \end{aligned}$$

from Property (D_2) and from the structure of $\overline{G}^*(W^{*e})_{ij}$, $i,j\in C^*$, and $\overline{L}^*(W^{*e})$ which yield that

$$\sum_{i,j\in C^*} \Big\langle \partial_j W^*, K^j\overline{A}_i^*(W^{*e})\partial_i W^*\Big\rangle \geq c|\partial_x W^*|^2 - C\Big(|\partial_x W_{II}^*|^2 + |\partial_x \mathcal{P}W^*|^2\Big).$$

We further deduce from expression (5.22) that

$$\mathcal{T}(t) = O\Big(\int_{\mathbb{R}^d} \big(|\partial_x W^*|^3 + |W^* - W^{*e}|\,|\partial_x W^*|^2 + |W^* - W^{*e}|\,|\partial_x W^*|\,|\partial_x^2 W_{II}^*|\big)\,dx\Big),$$

and, therefore, we have $\int_0^t \mathcal{T}(t)\,dt \leq C N_l(t)^3$ and the proof is complete. □

We can now combine the preceding lemmas in order to obtain a priori estimates for $N_l(T)$.

Proposition 5.10. Keep the hypotheses of Lemma 5.7 and assume also that (D_4) holds. Then there exists a positive constant $C_g = C_g(\beta_0(\mathfrak{O}))$, depending only on $\beta_0(\mathfrak{O})$, such that the following a priori estimate holds for $t\in[0,T]$

$$N_l(t) \leq C_g \|W^{*0} - W^{*e}\|_{l,2}. \tag{5.23}$$

Proof. Letting $b = \beta_0(\mathfrak{O})$ in Lemma 5.7, Proposition 5.8 and Lemma 5.9 yields the inequality $N_l(T)^2 \leq C_{g4}\big(\|W^{*0} - W^{*e}\|_{l,2}^2 + N_l(T)^3\big)$. As a consequence, for $N_l(T) < 1/(2C_{g4})$, we obtain the estimate (5.23). □

5.3.5. *Global existence proof*

We now use repeatedly the local existence theorem and the a priori estimates [Ka84]. We apply Theorem 5.5 with $\mathcal{O}_0 = \mathfrak{O}$, $d_1 < d(\mathfrak{O}, \partial\mathcal{O}_{W^*})$, and $b_0 = \beta_0(\mathfrak{O})$, and we assume that W^{*0} satisfies

$$\|W^{*0} - W^{*e}\|_{l,2} \leq b_1 = \min\left(\frac{\beta_0(\mathfrak{O})}{C_l}, \frac{\beta_0(\mathfrak{O})}{C_g(1+C_l^2)^{\frac{1}{2}}}\right). \tag{5.24}$$

The solution is then defined on $[0, T_1]$ and from (5.15) we obtain that $N_l(t) \leq C_l b_1 \leq \beta_0(\mathfrak{O})$ over $[0, T_1]$. As a consequence, the estimates obtained in Proposition 5.10 hold and $N_l(T_1) \leq C_g b_1$. We then apply again Theorem 5.5 on $[T_1, 2T_1]$, with T_1 as a new initial time, still with $\mathcal{O}_0 = \mathfrak{O}$, and with the same d_1 and b_0. The solution on $[T_1, 2T_1]$ then satisfies the estimate $N_l(T_1, 2T_1) \leq C_l N_l(T_1)$. As a consequence, we obtain $N_l(2T_1) \leq \big(N_l(T_1)^2 + N_l(T_1, 2T_1)^2\big)^{\frac{1}{2}} \leq (1 + C_l^2)^{\frac{1}{2}} N_l(T_1)$ so that

$$N_l(2T_1) \leq (1 + C_l^2)^{\frac{1}{2}} C_g b_1 \leq \beta_0(\mathfrak{O}).$$

The estimates of Proposition 5.10 are therefore again valid on $[0, 2T_1]$ and we can again use Theorem 5.5 on $[2T_1, 3T_1]$. An easy induction then yields global existence and $N_l(t) \leq C_g \|W^{*0} - W^{*e}\|_{l,2}$ for $t \geq 0$.

Defining then $\Phi(t) = \|\partial_x W^*(t)\|_{l-2,2}^2$, it follows from the a priori estimates and the relations (5.9) that

$$\int_0^\infty |\Phi(t)|\, dt + \int_0^\infty |\partial_t \Phi(t)|\, dt \;\leq\; C\|W^{*0} - W^{*e}\|_{l,2}^2,$$

which shows that $\Phi(t) \to 0$ as $t \to \infty$. From the interpolation inequality

$$\sup_{\mathbb{R}^d} |W^*(t) - W^{*e}| \;\leq\; C\,\|\partial_x^{l-1} W^*(t)\|_{0,2}^a\, \|W^*(t)\|_{0,2}^{1-a},$$

with $a = d/\big(2(l-1)\big) > 0$, we then deduce that $\sup_{\mathbb{R}^d} |W^*(t) - W^{*e}| \to 0$ as $t \to \infty$.

5.4. Asymptotic decay

In this section we establish decay estimates towards the equilibrium state.

Theorem 5.11. Assume that Properties (A_1)–(A_4) and (D_1)–(D_4) hold. Let $d \geq 1$ and $l \geq [d/2] + 3$ be integers and let $W^{*0}(x)$ be such that

$$W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d). \tag{5.25}$$

Assume that $\|W^{*0} - W^{*e}\|_{l,2}$ is small enough so that there exists a unique global solution to the Cauchy problem from Theorem 5.4. Assume further that $W^{*0} - W^{*e} \in W_2^l(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, where $p = 1$, if $d = 1$, and $p \in [1, 2)$, if $d \geq 2$. Defining for $\tau \leq l$

$$\|W^{*0} - W^{*e}\|_{\tau,2|p} = \|W^{*0} - W^{*e}\|_{\tau,2} + \|W^{*0} - W^{*e}\|_{0,p}, \tag{5.26}$$

then, if $\|W^{*0} - W^{*e}\|_{l,2|p}$ is small enough, the unique global solution to the Cauchy problem satisfies the decay estimate

$$\|W^*(t) - W^{*e}\|_{l-2,2} \leq C(1+t)^{-\gamma} \|W^{*0} - W^{*e}\|_{l-2,2|p}, \tag{5.27}$$

for $t \in [0, \infty)$, where C a positive constant and where $\gamma = d \times (1/2p - 1/4)$.

Proof. Introducing the new variable $z(U^*) = \big(\partial_{U^*} W^*(U^{*e})\big)\,(U^* - U^{*e})$ we obtain from (3.1) that

$$\overline{A}_0^*(W^{*e})\partial_t z + \sum_{i \in C^*} \overline{A}_i^*(W^{*e})\partial_i z - \sum_{i,j \in C^*} \overline{G}_{ij}^*(W^{*e})\partial_i\partial_j z + \overline{L}^* z \;=\; h^1 + h^2, \tag{5.28}$$

and $z(0, x) = z(U^{*0})$. The first nonlinear term h^1 is in conservative form and reads

$$h^1 = \sum_{i \in C^*} \partial_i \,(b_i + b_i')$$

where

$$\begin{cases} b_i = \big(\partial_{W^*} V^*(W^{*e})\big)^t \,\big(F_i^*(W^{*e}) - F_i^*(W^*)\big) + \overline{A}_i^*(W^{*e})\, z, \\ b_i' = \displaystyle\sum_{j \in C^*} \Big(\big(\partial_{W^*} V^*(W^{*e})\big)^t \, G_{ij}^*(U^*)\, \partial_{W^*} U^*(W^{*e}) - \overline{G}_{ij}^*(W^{*e}) \Big)\, \partial_j z. \end{cases}$$

On the other hand, the second term h^2 reads

$$h^2 = \big(\partial_{W^*} V^*(W^{*e})\big)^t \,\widetilde{\Omega}^* \;+\; \overline{L}^*(W^{*e})\, z,$$

and lies in the subspace $R\big(\overline{L}^*(W^{*e})\big)$ thanks to (D_3), so that $(I - \mathcal{P})\, h^2 = 0$.

We now introduce the symbol associated with the linear part of (5.28)

$$\overline{S}(\xi) = \overline{A}_0^*(W^{*e})^{-\frac{1}{2}} \Big\{ \mathrm{i} \sum_{i \in C^*} \overline{A}_i^*(W^{*e})\xi_i + \sum_{i,j \in C^*} \overline{G}_{ij}^*(W^{*e})\xi_i\xi_j + \overline{L}^*(W^{*e}) \Big\} \overline{A}_0^*(W^{*e})^{-\frac{1}{2}},$$

and we define for $\phi \in L^2(\mathbb{R}^d)$

$$\exp(-t\overline{S})\phi(x) \;=\; \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \exp(\mathrm{i}x \cdot \xi)\, \exp\big(-t\overline{S}(\xi)\big)\, \hat{\phi}(\xi)\, d\xi,$$

where $\hat{\phi}$ is the Fourier transform of ϕ. We can then express the solution of system (5.28) by the relation [Ka84]

$$z(t) = \exp(-t\overline{S})z(0) + \int_0^t \exp(-(t-\tau)\overline{S})h^1(\tau)\,d\tau + \int_0^t \exp(-(t-\tau)\overline{S})h^2(\tau)\,d\tau, \tag{5.29}$$

and we estimate $\|z(t)\|_{l-2,2}$ in two steps. We first deduce from Proposition 3.12 of [Ka84] the estimate

$$\|\exp(-t\overline{S})z(0) + \int_0^t \exp(-(t-\tau)\overline{S})h^2(\tau)d\tau\|_{l-2,2} \;\leq\; C(1+t)^{-\gamma}\|z(0)\|_{l-2,2|p}, \tag{5.30}$$

where $\gamma = d \times (1/2p - 1/4)$ and C denotes a positive constant, since the nonlinear term h^2 satisfies $(I - \mathcal{P})\,h^2 = 0$.

On the other hand, by using the estimate 3.A.14 of [Ka84] or Theorem 1.2 of [SK85], we also obtain that

$$\begin{aligned}\|\int_0^t \exp(-(t-\tau)\overline{S})h^1(\tau)\,d\tau\|_{l-2,2} \;\leq\; & C\int_0^t \exp(-c(t-\tau))\|h^1(\tau)\|_{l-2,2}\,d\tau \\ & + C\int_0^t (1+t-\tau)^{-d/4-1/2}\|\sum_{i\in C^*}(b_i + b_i')\,(\tau)\|_{0,1}\,d\tau, \end{aligned} \tag{5.31}$$

where c and C are positive constants. Since the initial data (5.10) is supposed to be sufficiently near W^{*e} in $W_2^l(\mathbb{R}^d) \cap L^p(X)$, the a priori estimates show that $\|z\|_{l,2}$ remains small for any time and we can write that $b_i = O(|z|^2)$ and $b_i' = (\partial_i z)\,O(|z|)$. We thus obtain that for $l \geq [d/2] + 3$, and for $\|z\|_{l,2}$ sufficiently small, we have

$$\sum_{i\in C^*}\Big(\|b_i\|_{0,1} + \|b_i'\|_{0,1}\Big) \leq C\|z\|_{1,2}^2, \tag{5.32}$$

$$\sum_{i\in C^*}\Big(\|\partial_i b_i\|_{l-2,2} + \|\partial_i b_i'\|_{l-2,2}\Big) \leq C\|z\|_{l-2,2}\|z\|_{l,2}. \tag{5.33}$$

Combining (5.29) with inequalities (5.30) (5.31) (5.32) and (5.33) finally yields

$$\begin{aligned}\|z(t)\|_{l-2,2} \;\leq\; & C(1+t)^{-\gamma}\|z(0)\|_{l-2,2|p} + C\int_0^t \exp\big(-c(t-\tau)\big)\|z(\tau)\|_{l,2}\|z(\tau)\|_{l-2,2}\,d\tau \\ & + C\int_0^t (1+t-\tau)^{-d/4-1/2}\|z(\tau)\|_{1,2}^2\,d\tau. \end{aligned} \tag{5.34}$$

Defining then $|||z(t)|||_{l-2,\gamma} = \sup_{0\leq\tau\leq t}(1+\tau)^{\gamma}\|z(\tau)\|_{l-2,2}$, we obtain

$$|||z(t)|||_{l-2,\gamma} \leq C\|z(0)\|_{l-2,2|p} + C\mu_3(t)\|z(0)\|_{l,2}\,|||z(t)|||_{l-2,\gamma} + C\mu_4(t)\,|||z(t)|||_{l-2,\gamma}^2,$$

where

$$\begin{cases} \mu_3(t) = \sup\limits_{0\leq\tau\leq t} (1+\tau)^\gamma \displaystyle\int_0^\tau \exp\bigl(-c(\tau-\tau_1)\bigr)(1+\tau_1)^{-\gamma}\,d\tau_1, \\ \mu_4(t) = \sup\limits_{0\leq\tau\leq t} (1+\tau)^\gamma \displaystyle\int_0^\tau (1+\tau-\tau_1)^{-d/4-1/2}(1+\tau_1)^{-2\gamma}\,d\tau_1. \end{cases}$$

Since $\mu_3(t)$ and $\mu_4(t)$ are uniformly bounded with respect to t, we obtain the desired estimate for $|\!|\!| z(0) |\!|\!|_{l-2,\gamma}$ small enough and this completes the proof. $\square$

6. Equilibrium Points for Maxwellian Chemistry

In order to apply Theorem 5.4 and 5.11 to the multicomponent reactive flow governing equations (2.32), it is first necessary to establish the existence of equilibrium states and to investigate the corresponding linearized source terms. This is the purpose of this section where we introduce the detailed structure of the source term ω_k, $k \in S$.

6.1. Maxwellian chemistry

In the following Sections, in order to investigate asymptotic stability of constant equilibrium states, the source term is taken to be

$$\Omega = (m_1\omega_1, \ldots, m_{n_S}\omega_{n_S}, 0, 0, 0, 0)^t, \tag{6.1}$$

where m_k is the molar mass of the k^{th} species and ω_k the molar production rate of the k^{th} species. More specifically, we no longer consider force terms in Ω and only chemistry effects are included in the model. The molar production rates that we consider are the Maxwellian production rates obtained from the kinetic theory [EG94]. These rates are obtained in a reactive kinetic framework when the chemistry characteristic times are larger than the mean free times of the molecules [EG94].

Therefore, we consider a system of n_R reversible reactions for n_S species

$$\sum_{k\in S} \nu'_{ki}\mathcal{S}_k \rightleftharpoons \sum_{k\in S} \nu''_{ki}\mathcal{S}_k, \qquad i \in R, \tag{6.2}$$

where $\mathcal{S}_k$ is the chemical symbol of the k^{th} species, ν'_{ki} and ν''_{ki} the stoichiometric coefficients of the k^{th} species in the i^{th} reaction, and $R = [1, n_R]$ the set of reaction indices. The chemical species are assumed to be constituted by elements and we denote by $\mathcal{E}_{kl}$ the number of l^{th} element in the k^{th} species. We also denote by $E = [1, n_E]$ the set of element indices and by n_E the number of elements.

The molar production rate of the k^{th} species is then given by

$$\omega_k = \sum_{i \in R} \nu_{ki} \tau_i, \tag{6.3}$$

where

$$\nu_{ki} = \nu''_{ki} - \nu'_{ki}, \tag{6.4}$$

and where τ_i is the rate of progress of the i^{th} reaction. This rate τ_i is given by

$$\tau_i = K_{fi} \prod_{k \in S} \gamma_k^{\nu'_{ki}} - K_{ri} \prod_{k \in S} \gamma_k^{\nu''_{ki}}, \tag{6.5}$$

where $\gamma_k = \rho_k / m_k$ is the molar concentration of the k^{th} species, K_{fi} and K_{ri} are the direct and reverse rate constants of the i^{th} reaction. The quantities K_{fi} and K_{ri} are functions of the temperature and their ratio is the equilibrium constant K_{ei} of the i^{th} reaction

$$K_{ei}(T) = \frac{K_{fi}(T)}{K_{ri}(T)}. \tag{6.6}$$

The equilibrium constant $K_{ei}(T)$ is given by

$$\log K_{ei}(T) = \sum_{k \in S} \frac{\nu_{ki} m_k}{R_{\mathrm{g}}} \big(s_k(m_k, T) - \frac{h_k(T)}{T} \big), \tag{6.7}$$

where h_k is the enthalpy per unit mass of the k^{th} species , R_{g} the universal gas constant, and $s_k(m_k, T)$ is the specific entropy of the k^{th} species at unit concentration, that is at $\rho_k = m_k$.

Remark 6.1. The direct rate constant is usually approximated by using an Arrhenius empirical relation

$$K_{fi}(T) = \mathfrak{A}_i T^{\mathfrak{b}_i} \exp\Big(- \frac{\mathfrak{T}_i}{R_{\mathrm{g}} T} \Big),$$

where $\mathfrak{A}_i$ is the pre-exponential factor, $\mathfrak{b}_i$ the pre-exponential exponent and $\mathfrak{T}_i$ the activation energy of the i^{th} reaction, but the exact expression of $K_{fi}(T)$ will not be needed in the following.

6.2. Mathematical assumptions for chemistry terms

The mathematical assumptions concerning the chemical productions rates are the following.

(H_{10}) The rate constants $\big(K_{fi}(T)\big)_{i \in R}$ are C^∞ positive bounded functions of $T \in [T_0, \infty)$ and the reverse constants are given by $K_{fi}(T)/K_{ei}(T)$ where $K_{ei}(T)$ is the equilibrium constant of the i^{th} reaction.

(H_{11}) The element vectors $\mathcal{E}_l$, $l \in E$, defined by $\mathcal{E}_l = (\mathcal{E}_{1l}, \ldots, \mathcal{E}_{n_S l})^t$, and the reaction vectors ν_i, $i \in R$, defined by $\nu_i = (\nu_{1i}, \ldots, \nu_{n_S i})^t$, satisfy the element conservation relations

$$\langle \nu_i, \mathcal{E}_l \rangle = 0, \qquad i \in R, \quad l \in E,$$

where $\langle x, y \rangle$ denotes the scalar product between vectors x and y. The space spanned by the reaction vectors is denoted by $\mathcal{R}$ and the space spanned by the element vectors is denoted by $\mathcal{E}$

$$\mathcal{R} = \mathrm{span}\{\ \nu_i,\ l \in R\ \},$$

$$\mathcal{E} = \mathrm{span}\{\ \mathcal{E}_l,\ l \in E\ \},$$

in such a way that

$$\mathcal{R} \subset \mathcal{E}^-, \qquad \mathcal{E} \subset \mathcal{R}^-.$$

(H_{12}) The vector of chemical production rates $\omega = (\omega_1, \ldots, \omega_{n_S})^t$ is given by (6.3) (6.4) which can be written in vector form

$$\omega = \sum_{i \in R} \tau_i\ \nu_i, \tag{6.8}$$

with the rate of progress τ_i given by (6.5). In particular, we always have $\omega(U) \in \mathcal{R}$ for $U \in \mathcal{O}_U$.

(H_{13}) The species molar mass m_k, $k \in S$, are related to the elemental masses a_l, $l \in E$, by the relations

$$m_k = \sum_{l \in E} a_l\, \mathcal{E}_{kl}.$$

Denoting by m the mass vector $m = (m_1, \ldots, m_{n_S})^t$, these relations can be written in vector form

$$m = \sum_{l \in E} a_l\, \mathcal{E}_l. \tag{6.9}$$

With respect to these assumptions, we make the following comments. We first note that Properties (H_{10}) and (H_{12}) imply the smoothness Property (H_4). Similarly, since the force term g is taken to be zero in the following sections, the smoothness Property (H_3) is also trivially satisfied. Properties (H_{12}) and (H_{13}) also imply the mass conservation relation (H_9) since

$$\sum_{k \in S} m_k \nu_{ki} = \langle m, \nu_i \rangle = \sum_{l \in E} a_l \langle \mathcal{E}_l, \nu_i \rangle = 0,$$

and therefore

$$\sum_{k \in S} m_k \omega_k = \langle m, \omega \rangle = \sum_{i \in R} \tau_i \langle m, \nu_i \rangle = 0.$$

The element vectors may taken to be the atomic elements provided that the corresponding vectors are independent. When this is not the case, it is first necessary to eliminate linearly dependent atomic elements. Finally, the fundamental relation between the direct and reverse rate constants $K_{ri}(T) = K_{fi}(T)/K_{ei}(T)$ is a direct consequence of the kinetic theory [EG94].

6.3. Miscellaneous

In the following, we will have to manipulate the mass weighted production rates $m_k \omega_k$, $k \in S$. To this purpose, we introduce the mass weights matrix M, of order n_S, defined by

$$M = \mathrm{diag}(m_1, \dots, m_{n_S}),$$

which acts on vectors of length n_S. The mass weighted stoichiometric coefficients are then the vectors $M\nu_i$, $i \in R$, and the specific elemental compositions are the vectors $M^{-1}\mathcal{E}_l$, $l \in E$. The corresponding spaces $M\mathcal{R}$ and $M^{-1}\mathcal{E}$, spanned by these mass weighted vectors, are also such that $M\mathcal{R} \subset (M^{-1}\mathcal{E})^-$ and $M^{-1}\mathcal{E} \subset (M\mathcal{R})^-$ in the composition space $\mathbb{R}^{n_S}$. In particular, we have $M\omega \in M\mathcal{R}$ and the mass relation (6.9) can also be written

$$\mathcal{U} = \sum_{l \in E} a_l M^{-1} \mathcal{E}_l,$$

where $\mathcal{U} \in \mathbb{R}^{n_S}$ is the unity vector defined by $\mathcal{U} = (1, \dots, 1)^t$, which implies that $\mathcal{U} \in (M\mathcal{R})^-$.

Various forms of the multicomponent reactive flow governing equations have been investigated in Section 4. We present here some additional properties of the corresponding source terms. We first observe that the source term $\widetilde{\Omega}$ of the symmetric form (4.4) is identical to that of the original formulation, that is, $\widetilde{\Omega} = \Omega$. As a consequence, we have

$$\Omega = \widetilde{\Omega} = \left(m_1\omega_1, \dots, m_{n_S}\omega_{n_S}, 0, 0, 0, 0\right)^t \in M\mathcal{R} \times 0_{\mathbb{R}^4}. \tag{6.10}$$

On the other hand, the source terms for the auxiliary variables $U' = P^t U$ and $V' = P^{-1}V$, where the constant matrix P is given by Eq. (4.16), are given by $\Omega' = \widetilde{\Omega}' = P^t \Omega$, that is to say

$$\Omega' = \widetilde{\Omega}' = (0, m_2\omega_2, \dots, m_{n_S}\omega_{n_S}, 0, 0, 0, 0)^t. \tag{6.11}$$

In particular, Ω' is also in a fixed subspace of $\mathbb{R}^{n_S+4}$ of dimension $\dim(\mathcal{R})$ since

$$\Omega' \in P^t(M\mathcal{R} \times 0_{\mathbb{R}^4}). \tag{6.12}$$

Finally, the source terms $\overline{\Omega}$ corresponding to the variable W is given by

$$\overline{\Omega} = (0,\ m_2\omega_2, \dots,\ m_{n_S}\omega_{n_S},\ 0,\ 0,\ 0,\ -\frac{1}{T^2}\sum_{k \in S} e_k m_k \omega_k)^t. \tag{6.13}$$

6.4. Equilibrium points

In this section we establish the existence of constant equilibrium states when the source terms are taken as in (H_{10})–(H_{13}). We restate existence and uniqueness of an equilibrium density vector at a fixed temperature in a given affine submanifold of element conservation, as well as detailed balance at equilibrium [Kr70] [SS65].

Let us introduce the vector ϱ of species densities

$$\varrho = (\rho_1, \ldots, \rho_{n_S})^t, \tag{6.14}$$

and the vector $\mathcal{Y}$ of reduced chemical potential

$$\mathcal{Y} = \frac{1}{R_{\mathrm{g}} T} (\mu_1, \ldots, \mu_{n_S})^t, \tag{6.15}$$

which coincides with the first n_S components of V up to a scaling factor and up to a term proportional to $\mathcal{U} = (1, \ldots, 1)^t$.

Proposition 6.2. The reduced entropy production due to chemical reactions

$$\sigma(\rho_1 \ldots, \rho_{n_S}, T) = \langle \mathcal{Y}, M\omega \rangle = \frac{1}{R_{\mathrm{g}} T} \sum_{k \in S} \mu_k m_k \omega_k,$$

defined on $(0, \infty)^{n_S} \times (T_0, \infty)$, is nonpositive and admits 0 as a maximum at equilibrium points. Any equilibrium point $(\rho_1^e, \ldots, \rho_{n_S}^e, T^e)^t$ in $(0, \infty)^{n_S} \times (T_0, \infty)$, that is, any point $(\rho_1^e, \ldots, \rho_{n_S}^e, T^e)^t$ where the source terms vanish

$$\omega_k(\rho_1^e, \ldots, \rho_{n_S}^e, T^e) = 0, \qquad k \in S, \tag{6.16}$$

is also such that the rates of progress of each reaction vanish

$$\tau_i(\rho_1^e, \ldots, \rho_{n_S}^e, T^e) = 0, \qquad i \in R, \tag{6.17}$$

which can also be written in the form

$$\Big\langle \mathcal{Y}(\rho_1^e, \ldots, \rho_{n_S}^e, T^e), M\nu_i \Big\rangle = 0, \qquad i \in R. \tag{6.18}$$

Proof. Some straightforward calculation yields that

$$\tau_i = \Delta_i \Big(1 - \exp\langle \mathcal{Y}, M\nu_i \rangle \Big), \tag{6.19}$$

where

$$\Delta_i = K_{fi} \prod_{k \in S} \gamma_k^{\nu'_{ki}}, \qquad i \in R, \tag{6.20}$$

thanks to the expression of the equilibrium constant K_{ei}. As a consequence, we have

$$\sigma = \sum_{i \in R} \langle \mathcal{Y}, M\nu_i \rangle \, \tau_i = \sum_{i \in R} \Delta_i \langle \mathcal{Y}, M\nu_i \rangle \Big(1 - \exp\langle \mathcal{Y}, M\nu_i \rangle \Big), \tag{6.21}$$

and, since Δ_i is positive, we deduce that $\sigma(\varrho, T) \le 0$ and that $\sigma(\varrho, T) = 0$ if and only if $\langle \mathcal{Y}, M\nu_i \rangle = 0$, $i \in R$, that is to say, if and only if $\tau_i = 0$, $i \in R$. In addition, $\tau_i = 0$, $i \in R$ if and only if $\omega_k = 0$, $k \in S$, from the expression of σ. □

Proposition 6.3. Consider a temperature $T^e \in (T_0, \infty)$, a point ϱ_c of $(0,\infty)^{n_S}$ and the associated reaction simplex $(\varrho_c + M\mathcal{R}) \cap (0,\infty)^{n_S}$. Assume that Properties (H_{10})–(H_{13}) hold. Then there exists a unique equilibrium state ϱ^e in the reaction simplex $(\varrho_c + M\mathcal{R}) \cap (0,\infty)^{n_S}$ where the source term ω vanishes, and, therefore, where the reaction rates of progress also vanish. Finally, as a function of ϱ, the entropy production due to chemical reactions admits 0 as a strict maximum at ϱ^e over the reaction simplex $(\varrho_c + M\mathcal{R}) \cap (0,\infty)^{n_S}$.

Proof. We characterize the equilibrium point as the only extremum of the reduced Helmholtz free energy function [Kr70][SS65]

$$\mathfrak{H}(\varrho, T) = \langle \varrho, \mathcal{Y} - M^{-1}\mathcal{U} \rangle,$$

where $\mathcal{U} = (1, \ldots, 1)^t$. The function $\mathfrak{H}$ is a C^∞ strictly convex function of ϱ in $(0,\infty)^{n_S}$ at a fixed temperature $T = T^e$, and is given by

$$\mathfrak{H}(\varrho, T) = \frac{1}{R_{\mathrm{g}}} \sum_{k \in S} r_k \rho_k \big(\log \rho_k - 1 + a_k(T^e)\big),$$

where $a_k(T) = \mu_k(1, T)/r_k T$. The partial derivative with respect to the mass density vector ϱ is also given by

$$\partial_\varrho \mathfrak{H} = \mathcal{Y}.$$

The function $\mathfrak{H}$ is easily extended over the closure of $(\varrho_c + M\mathcal{R}) \cap (0,\infty)^{n_S}$ into a continuous function by using $0 \log 0 = 0$, and, therefore, $\mathfrak{H}$ admits a minimum on this convex compact set. Note that the boundedness of the reaction simplex is a direct consequence of $\mathcal{U} \in (M\mathcal{R})^-$, that is, of mass conservation. This minimum cannot be reached at the boundaries as easily checked by inspecting the sign of the normal derivative. As a consequence, it is reached in the interior, and, thanks to the strict convexity of $\mathfrak{H}$, this mimimum is unique. Since the minimum is reached in the interior of the reaction simplex, we also have

$$\mathcal{Y}^e = \mathcal{Y}(\varrho^e, T^e) \in (M\mathcal{R})^-. \tag{6.22}$$

As a consequence, $\sigma = \langle \mathcal{Y}, M\omega \rangle$ vanishes at (ϱ^e, T^e) which is therefore an equilibrium point. Conversely, from (6.21), any equilibrium point on the reaction simplex is such that the quantities $\langle \mathcal{Y}, M\nu_i \rangle$, $i \in R$, vanish so that the partial derivatives of $\mathfrak{H}$ along the reaction simplex are zero. Since $\mathfrak{H}$ is a strictly convex function, it reaches a minimum at this point. Therefore, this point coincides with the unique minimum of $\mathfrak{H}$ and the proof is complete. □

As a direct consequence of the preceding results, we obtain the following proposition, which implies Property (A_4).

Proposition 6.4. Given a temperature T^e and a mass density vector ϱ_c, and under Assumptions (H_{10})–(H_{13}), there exists a unique constant stationary state $U^e = U(Y^e)$, with $Y^e = (\rho_1^e, \dots, \rho_{n_S}^e, 0, 0, 0, T^e)^t$, such that

$$\Omega(U^e) = 0, \tag{6.23}$$

and $\varrho^e \in (\varrho_c + M\mathcal{R}) \cap (0, \infty)^{n_S}$, $v^e = 0$, and $\mathcal{Y}^e \in (M\mathcal{R})^-$.

We will denote by V^e and W^e the equilibrium states in the variables V and W, respectively. In the following proposition, we establish Property (D_4).

Proposition 6.5. Let $U^e = U(Y^e)$, with $Y^e = (\rho_1^e, \dots, \rho_{n_S}^e, 0, 0, 0, T^e)^t$, be a constant equilibrium state in $\mathcal{O}_U$. Then we have $V^e \in (M\mathcal{R} \times 0_{\mathbb{R}^4})^-$ and there exists a neighborhood $\mathfrak{V}$ of V^e such that

$$c\big|\,\widetilde{\Omega}(V)\big|^2 \;\le\; -\langle V - V^e, \widetilde{\Omega}(V)\rangle, \qquad V \in \mathfrak{V}, \tag{6.24}$$

where c is a positive constant.

Proof. We first note that $\mathcal{Y}^e = (1/R_{\mathrm{g}})V^e$ from (4.3) since $v^e = 0$. This implies that $V^e \in (M\mathcal{R} \times 0_{\mathbb{R}^4})^-$ from Proposition 6.4. Combining this property with (6.10) and noting that

$$(V_1, \dots, V_{n_S})^t = R_{\mathrm{g}}\mathcal{Y} + \tfrac{1}{2}(v{\cdot}v/T)\,\mathcal{U},$$

where $\mathcal{U} = (1, \dots, 1)^t \in (M\mathcal{R})^-$, we obtain

$$\langle V - V^e, \widetilde{\Omega}(V)\rangle = \frac{1}{T}\sum_{k\in S} \mu_k m_k \omega_k = R_{\mathrm{g}}\langle \mathcal{Y}, M\omega\rangle. \tag{6.25}$$

From (6.21) we also have

$$\langle \mathcal{Y}, M\omega\rangle = \sum_{i\in R} \Delta_i \langle \mathcal{Y}, M\nu_i\rangle \Big(1 - \exp\big\langle \mathcal{Y}, M\nu_i\big\rangle\Big), \tag{6.26}$$

where $\Delta_i = K_{fi}\prod_{k\in S}\gamma_k^{\nu'_{ki}}$, $i \in R$, is a positive function. Denoting by $\mathfrak{V}$ any closed ball centered on V^e and included in $\mathcal{O}_V$, we deduce that for any V in $\mathfrak{V}$

$$c\sum_{i\in R}\langle \mathcal{Y}, M\nu_i\rangle^2 \;\le\; -\langle \mathcal{Y}, M\omega\rangle \;\le\; C\sum_{i\in R}\langle \mathcal{Y}, M\nu_i\rangle^2, \tag{6.27}$$

for positive constants c and C depending on $\mathfrak{V}$. On the other hand the relations $M\omega = \sum_{i\in R}\tau_i M\nu_i$ and (6.19) also imply that

$$|M\omega|^2 \;\le\; C'\sum_{i\in R}\tau_i^2 \;\le\; C''\sum_{i\in R}\langle \mathcal{Y}, M\nu_i\rangle^2, \tag{6.28}$$

where C' and C'' are positive constants, and combining (6.25), (6.27) and (6.28) completes the proof. □

6.5. Linearized source term

In this section we describe the linearized chemical source term around an equilibrium state and we establish Property (D_3).

Proposition 6.6. The linearized source term $\widetilde{L}(V^e) = -(\partial_V \widetilde{\Omega})(V^e)$ around the stationary state V^e is given by

$$\widetilde{L}(V^e) = \begin{pmatrix} (\mathcal{C}_{kl})_{k,l\in S} & 0_{n_S\times 4} \\ & \\ 0_{4\times n_S} & 0_{4\times 4} \end{pmatrix}, \tag{6.29}$$

where

$$\mathcal{C}_{kl} = \sum_{i\in R} \Lambda_i \; m_k \nu_{ki} \; m_l \nu_{li}, \tag{6.30}$$

and

$$\Lambda_i = \frac{K_{fi}(T^e)}{R_{\mathrm{g}}} \prod_{k\in S} (\gamma_k^e)^{\nu'_{ki}} = \frac{K_{ri}(T^e)}{R_{\mathrm{g}}} \prod_{k\in S} (\gamma_k^e)^{\nu''_{ki}}. \tag{6.31}$$

The matrix $\widetilde{L}(V^e)$ can also we written in the form

$$\widetilde{L}(V^e) = \sum_{i\in R} \Lambda_i \; \mathfrak{n}_i \otimes \mathfrak{n}_i,$$

where $\mathfrak{n}_i$ denotes the vector $\mathfrak{n}_i = (M\nu_i, 0_{\mathbb{R}^4})^t$. This matrix $\widetilde{L}(V^e)$ is symmetric positive semi-definite and satisfies

$$R\big(\widetilde{L}(V^e)\big) = M\mathcal{R} \times 0_{\mathbb{R}^4},$$

in such a way that we have $\Omega\big(U(V)\big) = \widetilde{\Omega}(V) \in R\big(\widetilde{L}(V^e)\big)$, for all $V \in \mathcal{O}_V$.

Proof. Evaluating the matrix $\widetilde{L}(V^e)$ is straightforward from (H_{10})–(H_{13}). The expression (6.30) shows that $\widetilde{L}(V^e)$ is symmetric and yields

$$\big\langle x, \widetilde{L}(V^e) x \big\rangle = \sum_{i\in R} \Lambda_i \big\langle \mathfrak{n}_i, \, x \big\rangle^2 = \sum_{i\in R} \Lambda_i \Big(\sum_{k\in S} x_k m_k \nu_{ki} \Big)^2, \tag{6.32}$$

where $x = (x_1, \dots, x_{n_S+4})^t$, so that $\widetilde{L}(V^e)$ is positive semi-definite. Furthermore, the nullspace of $\widetilde{L}(V^e)$ is also constituted by the vectors orthogonal to the vectors $\mathfrak{n}_i$, $i \in R$, and we thus conclude that $R\big(\widetilde{L}(V^e)\big) = N\big(\widetilde{L}(V^e)\big)^{-} = M\mathcal{R} \times 0_{\mathbb{R}^4}$. □

Property (D_3) is now a direct consequence of Proposition 6.6.

Remark 6.7. The fact that the rank of $\widetilde{L}$ is maximal at V^e shows that the equilibrium point is regular in the sense of Krambeck [Kr70].

7. Asymptotic Stability for Multicomponent Reactive Flows

In this section we apply theorems 5.4 and Theorem 5.11 to the multicomponent reactive flow governing equations (2.32). To this purpose, we have to establish that Properties (A_1)–(A_4) and (D_1)–(D_4) are satisfied. In the previous sections, we have already established Properties (A_1)–(A_4) and (D_3)–(D_4). Therefore, we now only have to investigate the dissipative structure of the linearized normal form and to establish (D_1)–(D_2). We will use the normal variable W introduced in Theorem 4.5 but the natural normal variable $\widehat{W}$ could be used as well.

7.1. The linearized normal form

If we linearize the symmetric hyperbolic-parabolic system (4.21) (4.22) around this constant stationary state W^e, we obtain the linear symmetric system

$$\overline{A}_0(W^e)\partial_t z + \sum_{i\in C} \overline{A}_i(W^e)\partial_i z = \sum_{i,j\in C} \overline{G}_{ij}(W^e)\partial_i\partial_j z - \overline{L}(W^e)z. \tag{7.1}$$

The relation $\overline{\Omega} = (\partial_W V)^t \widetilde{\Omega}$ first yields by linearization that

$$\overline{L}(W^e) = \big(\partial_W V\,(W^e)\big)^t\ \widetilde{L}(V^e)\ \partial_W V(W^e), \tag{7.2}$$

taking into account that $\widetilde{\Omega}(V^e) = 0$. As a consequence, $\overline{L}(W^e)$ is also symmetric positive semi-definite. Taking into account that (4.21) (4.22) is a normal form, we thus obtain that Property (D_1) is satisfied. We now investigate the existence of compensating matrices K^j, $j \in C$.

Proposition 7.1. For a sufficiently small positive α, the matrices K^j, $j \in C$, defined by

$$\sum_{j\in C} \xi_j K^j = \alpha \begin{pmatrix} 0 & 0 & \xi^t & 0 \\ 0 & 0 & 0 & 0 \\ -\xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \big(\overline{A}_0(W^e)\big)^{-1}, \tag{7.3}$$

where $\xi = (\xi_1, \xi_2, \xi_3)^t$, are compensating matrices. In particular, the products $K^j \overline{A}_0$ are skew-symmetric and the matrix

$$\sum_{i,j\in C} \tfrac{1}{2}\Big(K^j\overline{A}_i(W^e) + \big(K^j\overline{A}_i(W^e)\big)^t\Big) w_i w_j + \overline{B}(W^e, w) + \overline{L}(W^e),$$

is positive definite for $w \in \mathcal{S}^2$.

Proof. It is obvious by construction that the products $K^j \overline{A}_0$, $j \in C$, are skew-symmetric. On the other hand, a direct calculation yields

$$\sum_{i,j\in C} \xi_j K^j \overline{A}_i(W^e)\xi_i = \begin{pmatrix} \dfrac{\alpha T^e}{\Sigma_\rho^e}|\xi|^2 & \dfrac{\alpha T^e}{\rho^e}\mathcal{Z}^t|\xi|^2 & 0 & \alpha r^e|\xi|^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha\rho^e T^e \xi\otimes\xi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{7.4}$$

where $\mathcal{Z}$ is a vector of $\mathbb{R}^{n_S-1}$ given by

$$\mathcal{Z}_l = \rho_l^e - \frac{\rho_l^e \, \rho^e}{r_l \, \Sigma_\rho^e}, \qquad l \in [2, n_S],$$

and where the superscript ^{e} indicates that the corresponding quantity is evaluated at W^e. As a consequence, for $\xi \in \mathcal{S}^2$, we have $|\xi| = 1$, and there exists $C \in (0, \infty)$ such that

$$\langle x^t, \sum_{ij \in C} \xi_j K^j \overline{A}_i(W^e) \xi_i x \rangle \ \geq \ \frac{\alpha T^e}{2 \Sigma_\rho^e} \Big(x_1^2 - C \sum_{l \in [2, n_S+4]} x_l^2 \Big).$$

Using now Property (N_2), the matrix

$$\sum_{ij \in C} \tfrac{1}{2} \Big(K^j \overline{A}_i(W^e) + \big(K^j \overline{A}_i(W^e) \big)^t \Big) w_i w_j + \overline{B}(W^e, w),$$

is positive definite for $w \in \mathcal{S}^2$ and α sufficiently small. $\square$

Remark 7.2. The particular normal form in the W variable has additional properties. More specifically, we have the relations

$$K^j \overline{A}_0(W^e) \, \big(\overline{A}_0(W) \big)^{-1} \overline{\Omega} = 0, \qquad j \in C, \tag{7.5}$$

which can be established by a direct calculation. Indeed, from (6.13) and the expression of $\overline{A}_0$ in Theorem 4.5 we easily get that

$$\big(\overline{A}_0(W) \big)^{-1} \overline{\Omega} = R_{\mathrm{g}} \Big(0, (\frac{\omega_2}{\rho_2} - \frac{\omega_1}{\rho_1}), \ldots, (\frac{\omega_{n_S}}{\rho_{n_S}} - \frac{\omega_1}{\rho_1}), 0, 0, 0, -\frac{1}{\rho c_v} \sum_{k \in S} e_k m_k \omega_k \Big)^t,$$

and from the sparse structure of $\sum_{j \in C} \xi_j K^j \overline{A}_0(W^e)$ we immediately obtain (7.5). As a consequence, for this particular formulation, the term involving $\|\overline{\Omega}\|_{0,2}$ is not needed in the estimate (5.20).

Remark 7.3. Similarly, with the "natural" $\widehat{W}$ formulation of the Appendix, one can establish the particular estimate

$$c |\mathcal{P}(\widehat{W} - \widehat{W}^e)| \leq |\widehat{\Omega}| \leq C |\mathcal{P}(\widehat{W} - \widehat{W}^e)|,$$

where c and C are positive constants. As a consequence, the terms involving the norm $\|\widehat{\Omega}\|_{0,2}$ in a priori estimates can be replaced by the corresponding terms involving the norm $\|\mathcal{P}(\widehat{W} - \widehat{W}^e)\|_{0,2}$.

7.2. Global existence and asymptotic stability for reactive flows

In the previous sections, we have established that Properties (A_1)–(A_4) and (D_1)–(D_4) are satisfied. Therefore, Theorem 5.4 and Theorem 5.11 can be applied to the system (4.21) (4.22) governing multicomponent reactive flows, written in the $W = (W_I, W_{II})^t$ variable, with the hyperbolic variable

$$W_I = \rho, \tag{7.6}$$

and the parabolic variable

$$W_{II} = \big(\log(\rho_2^{r_2}/\rho_1^{r_1}), \ldots, \log(\rho_{n_S}^{r_{n_S}}/\rho_1^{r_1}),\, v_1,\, v_2,\, v_3,\, T\big)^t. \tag{7.7}$$

Theorem 7.4. Consider system (4.1) (4.2), let $d \in \{1, 2, 3\}$, $l \geq [d/2] + 2$ and let $W^0(x)$ such that

$$W^0 - W^e \in W_2^l(\mathbb{R}^d).$$

Then, if $\|W^0 - W^e\|_{l,2}$ is small enough, there exists a unique global solution to the the Cauchy problem

$$\overline{A}_0 \partial_t W + \sum_{i \in C} \overline{A}_i \partial_i W = \sum_{i,j \in C} \partial_i \Big(\overline{G}_{ij} \partial_j W \Big) \, + \, \overline{H} \, + \, \overline{\Omega},$$

with initial condition

$$W(0, x) = W^0(x),$$

such that

$$\begin{cases} W_I - W_I^e \in C^0\big([0,\infty); W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,\infty); W_2^{l-1}(\mathbb{R}^d)\big) \cap L^2(0,\infty; W_2^l(\mathbb{R}^d)), \\ W_{II} - W_{II}^e \in C^0\big([0,\infty); W_2^l(\mathbb{R}^d)\big) \cap C^1\big([0,\infty); W_2^{l-2}(\mathbb{R}^d)\big) \cap L^2(0,\infty; W_2^{l+1}(\mathbb{R}^d)). \end{cases}$$

Furthermore, W satisfies the estimate

$$\begin{aligned} \|W(t) - W^e\|_{l,2}^2 + \int_0^t \Big(\|\partial_x \rho(\tau)\|_{l-1,2}^2 &+ \sum_{k \geq 2} \|\partial_x \log(\rho_k^{r_k}/\rho_1^{r_1})(\tau)\|_{l,2}^2 \\ &+ \|\partial_x v(\tau)\|_{l,2}^2 + \|\partial_x T(\tau)\|_{l,2}^2 \Big)\, d\tau \;\leq\; C \|W^0 - W^e\|_{l,2}^2, \end{aligned}$$

and $\sup_{x \in \mathbb{R}^d} |W(t) - W^e|$ goes to zero as $t \to \infty$.

In addition, assume that $l \geq [d/2] + 3$ and $W^0 - W^e \in W_2^l(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, with $p = 1$, if $d = 1$, and $p \in [1, 2)$, if $d \geq 2$. Then, if $\|W^0 - W^e\|_{l,2|p}$ is small enough, the unique global solution to Cauchy problem also satisfies the decay estimate

$$\|W(t) - W^e\|_{l-2,2} \leq C(1+t)^{-\gamma} \|W^0 - W^e\|_{l-2,2|p}, \qquad t \in [0, \infty),$$

where C is a positive constant and $\gamma = d \times (1/2p - 1/4)$.

Appendix

We consider here the normal form of the multicomponent reactive flows system of equations obtained with the "natural" normal variable $\widehat{W} = (U'_I, V'_{II})^t$ suggested by Theorem 3.8. This normal form has dissipative terms in conservative form and leaves unchanged the parabolic components of the source term Ω_{II}. This form, however, has a more complex expression than the one obtained in Theorem 4.5. Theorem 7.4 can also be applied to this normal form of the system.

The system in the new variable

$$\widehat{W} = \frac{1}{T}\Big(\rho T,\, \mu_2 - \mu_1, \dots, \mu_{n_S} - \mu_1,\, v_1,\, v_2,\, v_3,\, -1\Big)^t, \tag{A.1}$$

on the open convex set $\mathcal{O}_{\widehat{W}} = (0,\infty) \times \mathbb{R}^{n_S - 1} \times \mathbb{R}^3 \times (-\infty, -1/T_0)$ can be written in the form

$$\widehat{A}_0^{I,I}\,\partial_t \widehat{W}_I \;+ \textstyle\sum_{i\in C} \widehat{A}_i^{I,I}\,\partial_i \widehat{W}_I \;+ \textstyle\sum_{i\in C} \widehat{A}_i^{I,II}\,\partial_i \widehat{W}_{II} \;=\; 0, \tag{A.2}$$

$$\widehat{A}_0^{II,II}\,\partial_t \widehat{W}_{II} + \textstyle\sum_{i\in C} \widehat{A}_i^{II,I}\,\partial_i \widehat{W}_I + \textstyle\sum_{i\in C} \widehat{A}_i^{II,II}\,\partial_i \widehat{W}_{II} =$$

$$\sum_{i,j\in C} \partial_i\Big(\widehat{G}_{ij}^{II,II}\,\partial_j \widehat{W}_{II}\Big) + \widehat{\Omega}_{II}. \tag{A.3}$$

and is of the normal form. The matrix $\widehat{A}_0$ is given by

$$\widehat{A}_0 = \begin{pmatrix} \frac{1}{\Sigma_\rho} & & & Sym \\ 0 & \mathcal{X} & & \\ 0 & 0 & \rho T I_3 & \\ 0 & (\Xi_e)^t & \rho T v^t & \Upsilon'_e \end{pmatrix}, \tag{A.4}$$

where Ξ_e is a vector given by given by

$$\Xi_{el} = \frac{\rho_l}{r_l}\left(e_l^{\mathrm{tot}} - \frac{\Sigma_e}{\Sigma_\rho}\right), \qquad l \in [2, n_S],$$

and where

$$\Upsilon'_e = \sum_{k\in S} \frac{\rho_k}{r_k} {e_k^{\mathrm{tot}}}^2 + \rho T\,(v\cdot v + c_v T) - \frac{\Sigma_e^2}{\Sigma_\rho}.$$

The matrices $\widehat{A}_i$, $i = 1, 2, 3$, are also given by

$$\sum_{i\in C} \widehat{A}_i \xi_i = \begin{pmatrix} \dfrac{v\cdot\xi}{\Sigma_\rho} & & & Sym \\ 0_{(n_S - 1)\times 1} & \mathcal{X}\, v\cdot\xi & & \\ \dfrac{\rho T}{\Sigma_\rho}\xi & T\xi \otimes \mathcal{Z} & \rho T v\cdot\xi\, I_3 & \\ \dfrac{\rho T}{\Sigma_\rho} v\cdot\xi & (v\cdot\xi)\Xi_h^t & \rho T (v\cdot\xi) v^t + \rho T \Xi_{h0}\xi^t & \Upsilon'_h\, v\cdot\xi \end{pmatrix}, \tag{A.5}$$

where Ξ_{h0} and Ξ_{hl} are defined by

$$\Xi_{h0} = h^{\rm tot} - \frac{\Sigma_h}{\Sigma_\rho} + \frac{\rho T}{\Sigma_\rho}, \qquad \Xi_{hl} = \frac{\rho_l}{r_l}\left(h_l^{\rm tot} - \frac{\Sigma_h}{\Sigma_\rho}\right), \quad l \in [2, n_S],$$

and where

$$\Upsilon_h' = \sum_{k\in S} \frac{\rho_k}{r_k} h_k^{{\rm tot}\,2} + \rho T\,(v\cdot v + (c_v + r)T) - \frac{\Sigma_h^2}{\Sigma_\rho} + \frac{(\rho T)^2}{\Sigma_\rho}.$$

For the dissipative part, we also have

$$\widehat{G}_{ij}^{II,II} = \widetilde{G}_{ij}^{II,II}, \qquad (A.6)$$

and an explicit expression for $\widetilde{G}_{ij}^{II,II}$ is given by (4.7)–(4.10). Finally, concerning the source term, we have

$$\widehat{\Omega}_I = \widetilde{\Omega}_I' = \Omega_I' = 0, \qquad \widehat{\Omega}_{II} = \widetilde{\Omega}_{II}' = \Omega_{II}' = \Omega_{II}. \qquad (A.7)$$

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