Reinforcement Learning
Book of Proofs

E. Le Pennec

November 8, 2024

Contents

1 History Dependent or Markov Policies

Proposition 1.1 Equivalence of History Dependent and Markov Policies

Let π be a stochastic history dependent policy. For each state $s_0 \in S$, there exists a Markov stochastic policy π' such that $V^{\pi'}(s_0) = V^{\pi}(s_0)$.

Proof. Let $\pi'(a_t|s_t) = \mathbb{E}[\pi(a_t|H_t)|S_t = s_t, S_0 = s_0]$, we can prove by recursion that

$$\mathbb{P}_{\pi'}(S_t = s_t, A_t = a_t|S_0 = s_0) = \mathbb{P}_{\pi}(S_t = s_t, A_t = a_t|S_0 = s_0)\,.$$

This holds by definition for $t = 0$. Now assume the property is true for $t' \leq t - 1$. By construction,

$$\begin{aligned}\mathbb{P}_{\pi}(S_t = s_t|S_0 = s_0) &= \sum_{s_{t-1}} \sum_{a_{t-1}} p(s_t|s_{t-1}, A_{t-1})\mathbb{P}_{\pi}(S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1}|S_0 = s_0) \\ &= \sum_{s_{t-1}} \sum_{a_{t-1}} p(s_t|s_{t-1}, a_{t-1})\mathbb{P}_{\pi'}(S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1}|S_0 = s_0) \\ &= \mathbb{P}_{\pi'}(S_t = s_t|S_0 = s_0)\,.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{P}_{\pi'}(S_t = s_t, A_t = a_t|s_0) &= \pi'(a_t|s_t)\mathbb{P}_{\pi'}(S_t = s_t|S_0 = s_0) \\ &= \mathbb{E}_{\pi}[\mathbb{P}_{\pi}(A_t = a_t|H_t, S_t = s_t, S_0 = s_0)]\,\mathbb{P}_{\pi}(S_T = s_t|S_0 = s_0) \\ &= \mathbb{E}_{\pi}[\mathbb{P}_{\pi}(S_t = s_t, A_T = a_t, H_t|S_0 = s_0)]\,.\end{aligned}$$

It suffices then to notice that the quality criterion of π and π' depends on π only through respectively $\mathbb{E}_{\pi}[r(S_t, A_t)|S_0 = s_0]$ or $\mathbb{E}_{\pi'}[r(S_t, A_t)|S_0 = s_0]$ which are equals. □

2 Discounted Reward

2.1 Evaluation of a policy

Definition 2.1.1 **Value Function**

$$\begin{aligned} v_\pi(s) &= \mathbb{E}_\pi\left[\sum_{t=0}^{+\infty} \gamma^t R_{t+1} \middle| S_0 = s\right] \\ &= \sum_{t=0}^{+\infty} \gamma^t \mathbb{E}_\pi[R_{t+1}|S_0 = s] \end{aligned}$$

Definition 2.1.2 **Bellman Operator**

$$\begin{aligned} \mathcal{T}^\pi v(s) &= \mathbb{E}_\pi[R|s] + \gamma \sum_{s'} \mathbb{P}_\pi(s'|s)\, v(s') \\ \mathcal{T}^\pi v &= r_\pi + \gamma P_\pi v \end{aligned}$$

Proposition 2.1.3 **Value Function Characterization**

Let π be a stationary Markov policy, if $0 < \gamma < 1$ then v_π is the only solution of $v = \mathcal{T}^\pi v$,

$$v = r_\pi + \gamma P_\pi v,$$

and $v_\pi = (\mathrm{Id} - \gamma P_\pi)^{-1} r_\pi$.

Proof. By definition, if v is a solution of $v = \mathcal{T}^\pi v$ then $(\mathrm{Id} - \gamma P_\pi)v = r_\pi$. As P_π is a stochastic matrix, $|||P_\pi||| \leq 1$ and thus

$$\sum_{k=0}^{\infty} \gamma^k P_\pi^k$$

is well defined. One verify easily that this is an inverse of $I - \gamma P_\pi$ and such a v exists, is unique and equal to

$$\sum_{k=0}^{\infty} \gamma^k P_\pi^k r_\pi.$$

Now,

$$\begin{aligned}
v_\pi(s) &= \sum_{t=0}^{+\infty} \gamma^t \mathbb{E}_\pi[R_{t+1}|S_0 = s] \\
&= \sum_{t=0}^{+\infty} \gamma^t \sum_{s'} \mathbb{P}_\pi(S_t = s'|S_0 = s)\, \mathbb{E}_\pi[R|S = s'] \\
&= \sum_{t=0}^{+\infty} \gamma^t \sum_{s'} (P_\pi^t)_{s,s'} r_\pi(s') \\
&= \sum_{t=0}^{+\infty} \gamma^t (P_\pi^t r_\pi)(s)
\end{aligned}$$

and thus $v = v_\pi$. □

Proposition 2.1.4 **Bellman Operator Property**

The operator $\mathcal{T}^\pi$ satisfies the following contraction property

$$\|\mathcal{T}^\pi v - \mathcal{T}^\pi v'\|_\infty \leq \gamma \|v - v'\|_\infty$$

Furthermore, $v \leq v'$ implies $\mathcal{T}^\pi v \leq \mathcal{T}^\pi v'$ and $\mathcal{T}^\pi(v + \delta \mathbb{1}) = \mathcal{T}^\pi v + \gamma \delta \mathbb{1}$

Proof. For any s,

$$\begin{aligned}
|\mathcal{T}^\pi(v) - \mathcal{T}^\pi(v')(s)| &= |\gamma P_\pi(v - v')(s)| \\
&\leq \gamma \|v - v'\|_\infty
\end{aligned}$$

because P_π is a stochastic matrix.

It suffices to use the positivity of a stochastic matrix and the fact that $\mathbb{1}$ is a eigenvector for the eigenvalue 1 to obtain the two remaining properties. □

Proposition 2.1.5 **Policy Prediction**

For any v_0, define $v_{n+1} = \mathcal{T}^\pi v_n$ then

$$\lim_{n \to \infty} v_n = v_\pi$$

and

$$\|v_n - v_\pi\|_\infty \leq \gamma^n \|v_0 - v_\pi\|_\infty$$

Furthermore,

$$\|v_n - v_\pi\|_\infty \leq \frac{\gamma}{1 - \gamma} \|v_n - v_{n-1}\|_\infty$$

Finally, if $v_0 \geq \mathcal{T}^\pi v_0$ (respectively $v_0 \leq \mathcal{T}^\pi v_0$) then $v_0 \geq v_\pi$ (respectively $v_0 \leq v_\pi$) and v_n converges monotonously to v_π.

Proof. For the first part of the proposition, we notice that v_π is the only fixed point of $\mathcal{T}^\pi$ which is a contraction. Hence, by the fixed point theorem, for any v_0, the sequence defined by $v_{n+1} = \mathcal{T}^\pi v_n$ converges toward v_π.

A straightforward computation shows that

$$\|v_n - v_\pi\|_\infty \leq \gamma \|v_{n-1} - v_\pi\|_\infty \leq \gamma^n \|v_0 - v_\pi\|_\infty.$$

Along the same line,

$$\|v_{n+k} - v_{n+k+1}\|_\infty \leq \gamma^{k+1} \|v_n - v_{n-1}\|_\infty.$$

This implies that

$$\begin{aligned}\|v_n - v_\pi\|_\infty &\leq \sum_{i=0}^{k} \|v_{n+i} - v_{n+i+1}\|_\infty + \|v_{n+k+1} - v_\infty\|_\infty \\ &\leq \frac{\gamma - \gamma^{k+2}}{1-\gamma} \|v_n - v_{n-1}\|_\infty + \gamma^{n+k+1} \|v_0 - v_\pi\|_\infty\end{aligned}$$

which yields the result by taking the limit in k.

Finally, note that as

$$v_{n+2} = r_\pi + \gamma P_\pi v_{n+1}$$

and P_π is made of non negative elements, $v_{n+1} \leq v_n$ implies

$$v_{n+2} \leq r_\pi + \gamma P_\pi v_n = v_{n+1}.$$

Thus $v_1 = \mathcal{T}^\pi v_0 \leq v_0$ implies that v_n is a decreasing sequence whose limit is v_*, yielding the result. The increasing case is obtained with a similar proof. □

2.2 Optimal Policy

2.2.1 Characterization

Definition 2.2.1 **Optimal Reward**

$$v_\star(s) = \max_\pi v_\pi(s)$$

where the maximum can be taken indifferently in the set of history dependent policies or Markov policies.

Definition 2.2.2 **Optimal Bellman Operator**

$$\begin{aligned}\mathcal{T}^* v(s) &= \max_a \mathbb{E}[R|S=s, A=a] + \gamma \sum_{s'} \mathbb{P}(S'=s'|S=s, A=a)\, v(s') \\ &= \max_a r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s')\end{aligned}$$

Proposition 2.2.3 **Optimal Bellman Operator and Markov Policies**

$$\mathcal{T}^* v(s) = \max_{\pi \in \mathcal{S}} \mathcal{T}^\pi v(s)$$

or $\mathcal{T}^* v = \max_{\pi\in\mathcal{S}} r_\pi + \gamma P_\pi v$ *where* $\mathcal{S}$ *is the set of deterministic Markov policies and the* $\max$ *is componentwise.*

Proof. $\pi_a = e_a$ is such that $\mathcal{T}^{\pi_a}(s) = \mathbb{E}[R|s,a] + \gamma \sum_{s'} p(s'|s,a)v(s')$ so that $\max_\pi \mathcal{T}^\pi(s) \geq \mathcal{T}^*(s)$.

Now, for any π,

$$\begin{aligned}\mathcal{T}^\pi(s) &= \sum_a \pi(a|s) \left(\mathbb{E}[R|S=s, A=a] + \gamma \sum_{s'} p(s'|s,a) v(s') \right) \\ &\leq \max_a \mathbb{E}[R|S=s, A=a] + \gamma \sum_{s'} p(s'|s,a) v(s') \\ &\leq \mathcal{T}^*(s)\end{aligned}$$

□

Proposition 2.2.4 **Bellman Operator Property**

The operator $\mathcal{T}^*$ *satisfies the following contraction property*

$$\|\mathcal{T}^* v - \mathcal{T}^* v'\|_\infty \leq \gamma \|v - v'\|_\infty$$

Furthermore, $v \leq v'$ *implies* $\mathcal{T}^* v \leq \mathcal{T}^* v'$ *and* $\mathcal{T}^*(v + \delta\mathbb{1}) = \mathcal{T} v + \gamma\delta\mathbb{1}$

Proof. For any s, if $\mathcal{T}^*v(s) \geq \mathcal{T}^*v'(s)$

$$
\begin{aligned}
|\mathcal{T}^*v - \mathcal{T}^*v'(s)| &= \mathcal{T}^*v(s) - \mathcal{T}^*v'(s) \\
&= \max_a r(s,a) + \gamma \sum_{s'} p(s'|s,a)v(s') - \left(\max_a r(s,a) + \gamma \sum_{s'} p(s'|s,a)v'(s')\right) \\
&\leq \max_a \left(r(s,a) + \gamma \sum_{s'} p(s'|s,a)v(s') - \left(r(s,a) + \gamma \sum_{s'} p(s'|s,a)v'(s')\right)\right) \\
&\leq \gamma \max_a \sum_{s'|s,a} p(s'|s,'a)(v(s') - v'(s')) \\
&\leq \gamma \|v - v'\|_\infty
\end{aligned}
$$

Now, if $v \leq v'$, for any a'

$$
\begin{aligned}
r(s,a') + \gamma \sum_{s'} p(s'|s,a')v(s') &\leq r(s,a') + \gamma \sum_{s'} p(s'|s,a')v'(s') \\
&\leq \mathcal{T}^*v'(s)
\end{aligned}
$$

hence $\mathcal{T}^*v \leq \mathcal{T}^*v'$. □

Finally,

$$
\begin{aligned}
\mathcal{T}^*(v + \delta\mathbb{1})(s) &= \max_a r(s,a) + \gamma \sum_{s'} p(s'|s,a)(v(s') + \delta) \\
&= \max_a r(s,a) + \gamma \sum_{s'} p(s'|s,a)v(s') + \delta \\
&= \mathcal{T}^*(v)(s) + \delta.
\end{aligned}
$$

Proposition 2.2.5 **Optimal Reward Characterization**

$v_\star$ is the unique solution of $V = \mathcal{T}^*V$.

Proof. Assume $v \geq \mathcal{T}^*v$ so that

$$v \geq \max_\pi r_\pi + \gamma P_\pi v.$$

Let $\pi = (\pi_0, \pi_1, \ldots)$ be a sequence of Markov policies,

$$
\begin{aligned}
v &\geq r_{\pi_0} + \gamma P_{\pi_0} v \\
v &\geq r_{\pi_0} + \gamma P_{\pi_0}(r_{\pi_1} + \gamma P_{\pi_1} v) \\
v &\geq \sum_{k=0}^{n} \gamma^k P_\pi^t r_{\pi_k} + \gamma^{n+1} P_\pi^{n+1} v
\end{aligned}
$$

where $P_\pi^k = \prod_{k'<k} P_{\pi_{k'}}$. As $v_\pi = \sum_{k=0}^\infty \gamma^k P_\pi^k r_{\pi_k}$, we verify that

$$v - v_\pi \geq \gamma^{n+1} P_\pi^{n+1} v - \sum_{k=n+1}^{\infty} \gamma^k P_\pi^k r_{\pi_k}.$$

Taking the limit in n yields $v \geq v_\pi$ and thus $v \geq v_*$.

Now, if $v \leq \mathcal{T}^* v = \max_\pi r_\pi + \gamma P_\pi v$ then assuming the max is reached at $\tilde{\pi}$

$$v \leq r_{\tilde{\pi}} + \gamma P_{\tilde{\pi}} v \leq \sum_{k=0}^{n} \gamma^k P_{\tilde{\pi}}^t r_{\tilde{\pi}} + \gamma^{n+1} P_{\tilde{\pi}}^{n+1} v$$

and thus $v \leq v_{\tilde{\pi}} \leq v_*$.

We deduce thus that $v = \mathcal{T}^* v$ implies $v = v_*$. It remains to prove that such a solution exists. This is a direct application of the fixed point theorem for the operator $\mathcal{T}^*$. □

Proposition 2.2.6

Any policy π_* such that $v_{\pi_*} = v_*$ is optimal.

Proof. This is a direct consequence of the previous theorem. □

Proposition 2.2.7

Any stationary policy π_* verifying $\pi_* \in \operatorname{argmax} r_\pi + \gamma P_\pi v_*$ is optimal.

Proof. By definition,

$$\begin{aligned} \mathcal{T}^{\pi_*} v_* &= r_{\pi_*} + P_{\pi_*} v_* \\ &= \max_\pi r_\pi + P_\pi v_* \\ &= \mathcal{T}^* v_* = v_*. \end{aligned}$$

Hence $v_{\pi_*} = v_*$ and the policy is optimal. □

2.2.2 Policy Improvement and Policy Iteration

Proposition 2.2.8 **One step look-head policy improvement**

For any π, π_+ define by

$$\pi_+ \in \operatorname*{argmax}_{\pi'} r_{\pi'} + \gamma P_{\pi'} v_\pi$$

satisfies

$$v_{\pi_+} \geq v_\pi$$

Proof. By construction,

$$r_{\pi_+} + \gamma P_{\pi_+} v_\pi \geq r_\pi + \gamma P_\pi v_\pi = v_\pi$$

and thus

$$r_{\pi_+} - (I - \gamma P_{\pi_+}) v_\pi \geq 0.$$

It suffices to notice that $v_{\pi_+} = (I - \gamma P_{\pi_+})^{-1} r_{\pi_+}$ so that

$$v_{\pi_+} - v_\pi = (I - \gamma P_{\pi_+})^{-1} \left(r_{\pi_+} - (I - \gamma P_{\pi_+}) v_\pi \right) \qquad \geq 0$$

where we have used the positivity of $(I - \gamma P_{\pi_+})^{-1} = \sum \gamma^k P_{\pi_+}^k$. □

Proposition 2.2.9

Let $\Delta = \mathcal{T}^* - \mathrm{Id}$, *the policy iteration scheme satisfies*

$$v_{n+1} = v_n + \sum_{k=0}^{\infty} \gamma^k P_{\pi_{n+1}}^k \Delta v_n.$$

Proof. As proved before,

$$v_{n+1} = (\mathrm{Id} - \gamma P_{\pi_{n+1}})^{-1} r_{\pi_{n+1}}.$$

Now by construction,

$$\mathcal{T}^* v_n = \mathcal{T}^{\pi_{n+1}} v_n = r_{\pi_{n+1}} + \gamma P_{\pi_{n+1}} v_n$$

and thus

$$r_{\pi_{n+1}} = \Delta v_n + (\mathrm{Id} - \gamma P_{\pi_{n+1}}) v_n.$$

This implies immediately

$$\begin{aligned} v_{n+1} &= v_n + (\mathrm{Id} - \gamma P_{\pi_{n+1}})^{-1} \Delta v_n \\ &= v_n + \sum_{k=0}^{\infty} \gamma^k P_{\pi_{n+1}}^k \Delta v_n \end{aligned}$$

□

2.2.3 Value Iteration

Proposition 2.2.10

For any v_0, *define* $v_{n+1} = \mathcal{T}^* v_n$ *then*

$$\lim_{n \to \infty} v_n = v_*$$

and

$$\|v_n - v_*\|_\infty \leq \gamma^n \|v_0 - v_*\|_\infty$$

Furthermore,

$$\|v_n - v_*\|_\infty \leq \frac{\gamma}{1 - \gamma} \|v_n - v_{n-1}\|_\infty$$

Finally, if $v_0 \geq \mathcal{T}^* v_0$ *(respectively* $v_0 \leq \mathcal{T}^* v_0$*) then* $v_0 \geq v_*$ *(respectively* $v_0 \leq v_*$*) and* v_n *converges monotonously to* v_*.

Proof. For the first part of the proposition, we notice that v_* is the only fixed point of $\mathcal{T}^*$ which is a contraction. Hence, by the fixed point theorem, for any v_0, the sequence defined by $v_{n+1} = \mathcal{T}^* v_n$ converges toward v_*.

A straightforward computation shows that

$$\|v_n - v_*\|_\infty \leq \gamma \|v_{n-1} - v_*\|_\infty \leq \gamma^n \|v_0 - v_*\|_\infty.$$

Along the same line,

$$\|v_{n+k} - v_{n+k+1}\|_\infty \leq \gamma^{k+1} \|v_n - v_{n-1}\|_\infty.$$

This implies that

$$\begin{aligned}\|v_n - v_*\|_\infty &\leq \sum_{i=0}^{k} \|v_{n+i} - v_{n+i+1}\|_\infty + \|v_{n+k+1} - v_*\|_\infty \\ &\leq \frac{\gamma - \gamma^{k+2}}{1-\gamma} \|v_n - v_{n-1}\|_\infty + \gamma^{n+k+1} \|v_0 - v_*\|_\infty\end{aligned}$$

which yields the result by taking the limit in k. □

Proposition 2.2.11

For any v and any $\pi \in \operatorname{argmax}_\pi \mathcal{T}^\pi v$,

$$\|v_\pi - v_*\|_\infty \leq \frac{2\gamma}{1-\gamma} \|v - v_*\|_\infty$$

If $v = \mathcal{T}^* v'$ then

$$\|v_\pi - v_*\|_\infty \leq \frac{2\gamma}{1-\gamma} \|v - v'\|_\infty$$

Proof. By definition of π, $\mathcal{T}^\pi v = \mathcal{T}^* v$, hence

$$\begin{aligned}\|v_\pi - v_*\|_\infty &\leq \|v_\pi - \mathcal{T}^\pi v\|_\infty + \|\mathcal{T}^* v - v_*\|_\infty \\ &\leq \|\mathcal{T}^\pi v_\pi - \mathcal{T}^\pi v\|_\infty + \|\mathcal{T}^* v - \mathcal{T}^* v_*\|_\infty \\ &\leq \gamma \|v_\pi - v\|_\infty + \gamma \|v - v_*\|_\infty \\ &\leq \gamma \|v_\pi - v_*\|_\infty + 2\gamma \|v - v_*\|_\infty\end{aligned}$$

and thus

$$\|v_\pi - v_*\|_\infty \leq \frac{2\gamma}{1-\gamma} \|v - v_*\|_\infty$$

For the second inequality,

$$\|v_\pi - v_*\|_\infty \leq \|v_\pi - v\|_\infty + \|v - v_*\|_\infty$$

Now

$$\begin{aligned}\|v_\pi - v\|_\infty &\leq \|\mathcal{T}^\pi v_\pi - \mathcal{T}^\pi v\|_\infty + \|\mathcal{T}^* v - \mathcal{T}^* v'\|_\infty \\ &\leq \gamma \|v_\pi - v\|_\infty + \gamma \|v - v'\|_\infty\end{aligned}$$

and thus

$$\|v_\pi - v\|_\infty \leq \frac{\gamma}{1-\gamma} \|v - v'\|_\infty$$

Along the same line,

$$\begin{aligned}\|v - v_*\|_\infty &\leq \|v - \mathcal{T}^* v\|_\infty + \|\mathcal{T}^* v - v_*\|_\infty \\ &\leq \|\mathcal{T}^* v' - \mathcal{T}^* v\|_\infty + \|\mathcal{T}^* v - \mathcal{T}^* v_*\|_\infty \\ &\leq \gamma \|v - v'\|_\infty + \gamma \|v - v_*\|_\infty\end{aligned}$$

and thus

$$\|v - v_*\|_\infty \leq \frac{\gamma}{1-\gamma} \|v - v'\|_\infty$$

. Combining those two bounds yields the result. □

2.2.4 Modified Policy Iteration

Proposition 2.2.12 **MPI**

Let v_0 such that $\mathcal{T}^* v_0 \geq v_0$, define for any n and any m_n

- $\pi_{n+1} \in \operatorname{argmax} r_\pi + P_\pi v_n$
- $v_{n,0} = \mathcal{T}^* v_n = \mathcal{T}^{\pi_{n+1}} v_n$
- $v_{n,m} = \mathcal{T}^{\pi_{n+1}} v_{n,m-1}$
- $v_{n+1} = v_{m_n}$

then $v_{n+1} \geq v_n$ and

$$\lim_{n\to\infty} v_n = v_*.$$

At any step,

$$\|v_{\pi_{n+1}} - v_*\|_\infty \leq \frac{2}{1-\gamma} \|v_n - v_{n,0}\|_\infty$$

Furthermore,

$$\|v_{n+1} - v_*\|_\infty \leq \left(\frac{\gamma - \gamma^{m_n+1}}{1-\gamma} |||P_{\pi_{n+1}} - P_{\pi_*}||| + \gamma^{m_n+1} \right) \|v_n - v_*\|_\infty$$

Proposition 2.2.13

Let $\Delta = \mathcal{T}^* - \mathrm{Id}$*, let* $W_\pi^{(m)} v = (\mathcal{T}^\pi)^{m+1} v$*,*

$$\begin{aligned} W_\pi^{(m)} v &= \sum_{k=0}^{m} \gamma^k P_\pi^k r_\pi + \gamma^{m+1} P_\pi^{m_n+1} v \\ &= v_n + \sum_{k=0}^{m} \gamma^k P_\pi^k \Delta v \end{aligned}$$

Proof. By definition,

$$\begin{aligned} W_\pi^{(m)} v &= (\mathcal{T}^\pi)^{m_n+1} v \\ &= r_\pi + \gamma P_\pi (\mathcal{T}^\pi)^m v \\ &= \sum_{k=0}^{m} \gamma^k P_\pi^k r_\pi + \gamma^{m+1} P_\pi^{m+1} v \\ &= \sum_{k=0}^{m} \gamma^k P_\pi^k \left(r_\pi + \gamma P_\pi v - v \right) + v \\ &= v + \sum_{k=0}^{m} \gamma^k P_\pi^k \Delta v \end{aligned}$$

□

Proposition 2.2.14

Define $W_*^{(m_n)}$ *by*

$$W_*^{(m_n)} v(s) = \max_\pi W_\pi^{(m_n)} v(s).$$

then $W_*^{(m_n)}$ *is a contraction:*

$$\|W_*^{(m_n)} v - W_*^{(m_n)} v'\|_\infty \le \gamma^{m_n+1} \|v - v'\|_\infty.$$

Furthermore, $W_*^{(m_n)} v_* = v_*$*.*

Proof. Assume without loss of generality that $W_*^{(m_n)} v(s) - W_*^{(m_n)} v'(s) \ge 0$ and let $\tilde{\pi} \in \operatorname{argmax} W_\pi^{(m_n)} v(s)$,

$$\begin{aligned} W_*^{(m_n)} v(s) - W_*^{(m_n)} v'(s) &= \max_\pi W_*^{(m_n)} v(s) - \max_\pi W_*^{(m_n)} v'(s) \\ &\le W_{\tilde{\pi}}^{(m_n)} v(s) - W_{\tilde{\pi}}^{(m_n)} v'(s) \\ &\le \gamma^{m_n+1} P_{\tilde{\pi}}^{m_n+1} (v - v')(s) \\ &\le \gamma^{m_n+1} \|v - v'\|_\infty \end{aligned}$$

By construction $\Delta v_* = \mathcal{T}^* v_* - v_* = 0$ and thus $W_\pi^{(m_n)} v_* = v_*$. We deduce immediately that $W_*^{(m_n)} v_* = \sup_\pi W_\pi^{(m_n)} v_* = v_*$ □

Proposition 2.2.15

If $u \geq v$ then for any π, $W_*^m u \geq W_\pi^m v$

If $u \geq v$ and $\Delta u \geq 0$ then for any π $W_\pi u \geq \mathcal{T}^* v$.

If $\Delta u \geq 0$ and π_u such that $\mathcal{T}^* u = \mathcal{T}^{\pi_u} u$ then $W_{\pi_u}^{(m)} u \geq 0$

Proof. By definition,

$$\begin{aligned} W_*^m u - W_\pi^m v &\geq W_\pi^m u - W_\pi^m v \\ &\geq W_\pi^m (u - v) \\ &\geq \gamma^{m_n+1} P_\pi^{m_n+1} (u - v) \geq 0 \end{aligned}$$

Now,

$$\begin{aligned} W_\pi^{(m)} u &= u + \sum_{k=0}^{m} \gamma^k P_\pi^k \Delta u \\ &\geq u + \Delta u = \mathcal{T}^* u \\ &\geq \mathcal{T}^* v \end{aligned}$$

By construction

$$\begin{aligned} \Delta W_{\pi_u}^{(m)} u &= \mathcal{T}^* W_{\pi_u}^{(m)} u - W_{\pi_u}^{(m)} u \\ &\geq \mathcal{T}^{\pi_u} W_{\pi_u}^{(m)} u - W_{\pi_u}^{(m)} u \\ &\geq \Delta u - \mathcal{T}^{\pi_u} u + u \\ &\geq \Delta u + (\gamma P_{\pi_u} - \mathrm{Id}) \left(W_{\pi_u}^{(m)} u - u \right) \qquad \geq \Delta u + (\gamma P_{\pi_u} - \mathrm{Id}) \sum_{k=0}^{m} \gamma^k P_{\pi_u}^k \Delta u \\ &\geq \gamma^m P_{\pi_u}^m \Delta u \geq 0 \end{aligned}$$

□

Proof of MPI. Let $u_0 = v_0 = w_0$.

By construction $\mathcal{T}^{\pi_{n+1}} v_n = \mathcal{T}^* v_n$ and one verify easily that $v_{n+1} = (\mathcal{T}^{\pi_{n+1}})^{m_n+1} v_n = W_{\pi_{n+1}}^{(m_n)} v_n$.

Define now, $u_{n+1} = \mathcal{T}^* u_n$ and $w_{n+1} = W_*^{(m_n)} w_n$. We can prove by recursion that $\Delta v_n \geq 0$, $v_{n+1} \geq v_n$ and $u_n \leq v_n \leq w_n$.

By assumption, $\Delta v_0 \geq 0$ so that $v_1 = W_{\pi_1}^{(m_n)} v_0 \geq \mathcal{T}^* v_0 \geq v_0$.

Assume the property holds for $n - 1$ then using the previous lemmas one obtains immediately $\Delta v_n \geq 0$ and

$$u_n = \mathcal{T}^* u_{n-1} \leq v_n = W_{\pi_n}^{(m_{n-1})} v_{n-1} \leq w_n = W_*^{(m_{n-1})} w_{n-1}$$

Finally,

$$\begin{aligned} v_n &= W_{\pi_n}^{(m_{n-1})} v_{n-1} \\ &= v_{n-1} + \sum_{k=0} m_{n-1} \gamma^k P_{\pi_n} \Delta v_{n-1} \\ &\geq v_{n-1}. \end{aligned}$$

Now, we have already proved that $u_n = \mathcal{T}^* u_0$ tends to v_* with

$$\|u_n - v_*\|_\infty \leq \gamma^n \|v_0 - v_*\|_\infty$$

It suffices now to prove that w_n also converges toward v_* to obtain the convergence of v_n. We verify that

$$\begin{aligned} \|w_n - v_*\|_\infty &= \|W_*^{(m_{n-1})} w_{n-1} - W_*^{(m_{n-1})} v_*\|_\infty \\ & \gamma^{m_{n-1}} \|w_{n-1} - v_*\|_\infty \\ & \gamma^{\sum_{k=0}^{n-1} m_k} \|v_0 - v_*\|_\infty \end{aligned}$$

which implies the convergence of w_n.

We have

$$\|v_{\pi_{n+1}} - v_*\|_\infty \leq \|v_{\pi_{n+1}} - v_n\|_\infty + \|v_n - v_*\|_\infty$$

Notice that $v_{n,0} = \mathcal{T}^{\pi_{n+1}} v_n = \mathcal{T}^* v_n$ so that

$$\begin{aligned} \|v_{\pi_{n+1}} - v_n\|_\infty &\leq \|v_{\pi_{n+1}} - v_{n,0}\|_\infty + \|v_{n,0} - v_n\|_\infty \\ &\leq \|\mathcal{T}^{\pi_{n+1}} v_{\pi_{n+1}} - \mathcal{T}^{\pi_{n+1}} v_n\|_\infty + \|v_{n,0} - v_n\|_\infty \\ &\leq \gamma \|v_{\pi_{n+1}} - v_n\|_\infty + \|v_{n,0} - v_n\|_\infty \end{aligned}$$

Along the same line,

$$\begin{aligned} \|v_* - v_n\|_\infty &\leq \|v_* - v_{n,0}\|_\infty + \|v_{n,0} - v_n\|_\infty \\ &\leq \|\mathcal{T}^* v_* - \mathcal{T}^* v_n\|_\infty + \|v_{n,0} - v_n\|_\infty \\ &\leq \gamma \|v_* - v_n\|_\infty + \|v_{n,0} - v_n\|_\infty \end{aligned}$$

Combining those two inequalities yields

$$\|v_{\pi_{n+1}} - v_*\|_\infty \leq \frac{2}{1-\gamma} \|v_n - v_{0,n}\|_\infty$$

As show before,

$$0 \leq v_* - v_{n+1} \leq v_* - v_n - \sum_{k=0}^{m_n} \gamma^k P_{\pi_{n+1}}^k \Delta v_n$$

Now, let π_* such that $\mathcal{T}^{\pi_*}v_* = Bv_*$,

$$\begin{aligned}\Delta_n = \Delta v_n - \Delta v_* = \mathcal{T}^* v_n - v_n - (\mathcal{T}^* v_* - v_*) \\ &\leq \mathcal{T}^{\pi_*} v_n - v_n - (\mathcal{T}^{\pi_*} v_* - v_*) \\ &\leq (\gamma P_{\pi_*} - \mathrm{Id})(v_n - v_*)\end{aligned}$$

Thus

$$\begin{aligned}0 \leq v_* - v_{n+1} \leq v_* - v_n - \sum_{k=0}^{m_n} \gamma^k P^k_{\pi_{n+1}}(\gamma P_{\pi_*} - \mathrm{Id})(v_n - v_*) \\ &\leq \sum_{k=1}^{m_n} \gamma^k P^k_{\pi_{n+1}}(v_n - v_*) - \sum_{k=0}^{m_n} \gamma^{k+1} P_{\pi_{n+1}} P_{\pi_*}(v_n - v_*) \\ &\leq \sum_{k=0}^{m_n} \gamma^{k+1} P^k_{\pi_{n+1}}(P_{\pi_{n+1}} - P_{\pi_*})(v_n - v_*) - \gamma^{m_n+1} P^{m_n+1}_{\pi_{n+1}}(v_n - v_*) \\ &\leq \sum_{k=0}^{m_n} \gamma^{k+1} |||P_{\pi_{n+1}} - P_{\pi_*}||| \|v_n - v_*\|_\infty + \gamma^{m_n+1} \|v_n - v_*\|_\infty \\ &\leq \left(\frac{\gamma - \gamma^{m_n+1}}{1-\gamma} |||P_{\pi_{n+1}} - P_{\pi_*}||| + \gamma^{m_n+1}\right) |||v_n - v_*\|_\infty\end{aligned}$$

□

2.3 Asynchronous Dynamic Programming

Proposition 2.3.1

Assume $\mathcal{T}^{\pi_0}v_0 \geq v_0$ and at any step n

- *Define a subset S_n of the states and*
- *Either*
  - *keep the policy $\pi_{n+1} = \pi_n$ and update the value function following*

$$v_{n+1}(s) = \begin{cases} \mathcal{T}^{\pi_n} v_n(s) & \textit{if } s \in S_n \\ v_n(s) & \textit{otherwise} \end{cases}$$

  - *keep the value function $s_{n+1} = s_n$ and update the policy following*

$$\pi_{n+1}(s) = \begin{cases} \operatorname{argmax}_a r(s,a) + \gamma P_{\pi_a} v_n(s) & \textit{if } s \in S_n \\ \pi_n(s) & \textit{otherwise} \end{cases}$$

Assume that for any state s and any n there exist $n' > n$ such that $s \in S_{n'}$ and one performs a value update at step n' and $n'' > n$ such that $s \in S_{n''}$ and one performs a policy update at step n'' then s_n tends monotonously to s_*.

Proof. We start by proving by recursion that $\mathcal{T}^{\pi_n} v_n \geq v_n$ implies

$$\mathcal{T}^{\pi_{n+1}} v_{n+1} \geq v_{n+1} \geq v_n \quad \text{and} \quad \mathcal{T}^{\pi_n} v_n$$

Note that that $\mathcal{T}^{\pi_0} v_0 \geq v_0$ is an assumption.

Assume now that $\mathcal{T}^{\pi_n} v_n \geq v_n$, then either at step n we update the value function or the policy.

If we update the value function, $\pi_{n+1} = \pi_n$ and thus

$$v_{n+1}(s) = \begin{cases} \mathcal{T}^{\pi_n} v_n(s) & \text{if } s \in S_n \\ v_n(s) & \text{otherwise} \end{cases}$$

As $\mathcal{T}^{\pi_n} v_n(s) \geq v_n(s)$, we deduce $\mathcal{T}^{\pi_n} v_n \geq v_{n+1} \geq v_n$. It suffices to notice that $v_{n+1} \geq v_n$ implies

$$\mathcal{T}^{\pi_{n+1}} v_{n+1} = \mathcal{T}^{\pi_n} v_{n+1} \geq \mathcal{T}^{\pi_n} v_n$$

to obtain

$$\mathcal{T}^{\pi_{n+1}} v_{n+1} \geq v_{n+1} \geq v_n.$$

Now, if we update the policy, $v_{n+1} = v_n$ and

$$\mathcal{T}^{\pi_{n+1}} v_n(s) = \begin{cases} \mathcal{T}^* v_n(s) & \text{if } s \in S_n \\ \mathcal{T}^{\pi_n} v_n(s) & \text{otherwise} \end{cases}$$

which implies $\mathcal{T}^{\pi_{n+1}} v_n \geq \mathcal{T}^{\pi_n} v_n$ and thus as $v_{n+1} = v_n$

$$\mathcal{T}^{\pi_{n+1}} v_{n+1} \geq \mathcal{T}^{\pi_n} v_n \geq v_n = v_{n+1}.$$

We deduce thus that

$$\mathcal{T}^* v_{n+1} \geq \mathcal{T}^{\pi_{n+1}} v_{n+1} \geq v_{n+1} \geq v_n.$$

which implies if we take the limit in k

$$v_* \geq v_{n+1} \geq v_n.$$

Hence v_n converges toward a limit $\tilde{v}$ satisfying

$$v_n \leq \tilde{v} \leq \mathcal{T}^* \tilde{v} \leq v_*.$$

Assume now that there exists s such that $\tilde{v}(s) < \mathcal{T}^* \tilde{v}(s)$. By continuity of $\mathcal{T}^*$, there exists n such that for all $n' \geq n$

$$\tilde{v}(s) < \mathcal{T}^* v_{n'}(s)$$

Let $n' \geq n$ such that one updates the policy of s and n'' the smallest integer larger than n'' where one updates the value of s.

$$\begin{aligned} v_{n''+1}(s) &= \mathcal{T}^{\pi_{n''}} v_{n''}(s) \\ &\geq \mathcal{T}^{\pi_{n'+1}} v_{n'+1}(s) \\ &\geq \mathcal{T}^{\pi_{n'+1}} v_{n'}(s) \\ &\geq \mathcal{T}^* v_{n'}(s) > \tilde{v}(s) \end{aligned}$$

which is impossible. □

2.4 Approximate Dynamic Programming

Proposition 2.4.1

If in a Generalized Policy Improvement, for all k

$$\|v_k - v_{\pi_k}\|_\infty \leq \epsilon$$

and

$$\|\mathcal{T}^{\pi_{k+1}} v_k - \mathcal{T}^* v_k\|_\infty \leq \delta$$

then

$$\limsup \max_s \left(v_*(s) - v_{\pi_k}(s)\right) \leq \frac{\delta + 2\gamma\epsilon}{(1-\gamma)^2}$$

Proof. By construction,

$$\begin{aligned}
v_{\pi_k}(s) - v_{\pi_{k+1}}(s) &= \mathcal{T}^{\pi_k} v_{\pi_k}(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_{k+1}} \\
&= \mathcal{T}^{\pi_k} v_{\pi_k}(s) - \mathcal{T}^{\pi_k} v_k(s) + \mathcal{T}^{\pi_k} v_k(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_{k+1}} \\
&\leq \gamma\epsilon + \mathcal{T}^* v_k(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_{k+1}} \\
&\leq \gamma\epsilon + \mathcal{T}^{\pi_{k+1}} v_k(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_{k+1}} + \delta \\
&\leq \gamma\epsilon + \mathcal{T}^{\pi_{k+1}} v_k(s) + \mathcal{T}^{\pi_{k+1}} v_{\pi_k}(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_k}(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_{k+1}} + \delta \\
&\leq 2\gamma\epsilon + \delta + \gamma \max_{s'} \left(v_{\pi_k}(s') - v_{\pi_{k+1}}(s')\right)
\end{aligned}$$

and thus

$$\max_{s'} \left(v_{\pi_k}(s') - v_{\pi_{k+1}}(s')\right) \leq \frac{2\gamma\epsilon + \delta}{1-\gamma}.$$

Now,

$$\begin{aligned}
v_*(s) - v_{\pi_{k+1}}(s) &= v_*(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_{k+1}}(s) \\
&= v_*(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_k}(s) + \mathcal{T}^{\pi_{k+1}} v_{\pi_k}(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_{k+1}}(s) \\
&\leq v_*(s) - \mathcal{T}^{\pi_{k+1}} v_{\pi_k}(s) + \gamma \frac{2\gamma\epsilon + \delta}{1-\gamma} \\
&\leq v_*(s) - \mathcal{T}^{\pi_{k+1}} v_k(s) + \gamma\epsilon + \gamma \frac{2\gamma\epsilon + \delta}{1-\gamma} \\
&\leq v_*(s) - \mathcal{T}^* v_k(s) + \gamma\epsilon + \delta + \gamma \frac{2\gamma\epsilon + \delta}{1-\gamma} \\
&\leq \mathcal{T}^* v_*(s) - \mathcal{T}^* v_{\pi_k}(s) + 2\gamma\epsilon + \delta + \gamma \frac{2\gamma\epsilon + \delta}{1-\gamma} \\
&\leq \gamma \max_s \left(v_*(s) - v_{\pi_k}(s)\right) + 2\gamma\epsilon + \delta + \gamma \frac{2\gamma\epsilon + \delta}{1-\gamma}
\end{aligned}$$

thus

$$\max_s \left(v_*(s) - v_{\pi_{k+1}}(s)\right) \leq \gamma \max_s \left(v_*(s) - v_{\pi_k}(s)\right) + 2\gamma\epsilon + \delta\gamma\frac{2\gamma\epsilon + \delta}{1-\gamma}$$

and

$$\limsup \max_s \left(v_*(s) - v_{\pi_k}(s)\right) \leq \limsup \gamma \max_s \left(v_*(s) - v_{\pi_k}(s)\right) + 2\gamma\epsilon + \delta + \gamma\frac{2\gamma\epsilon + \delta}{1-\gamma}$$

which implies

$$\limsup \max_s \left(v_*(s) - v_{\pi_k}(s)\right) \leq \frac{2\gamma\epsilon + \delta}{(1-\gamma)^2}$$

□

3 Finite Horizon

Proposition 3.1

If $v_0 = r_{\pi,T-1}$ *and* $v_n = \mathcal{T}^{\pi,T-n} v_{n-1} = r_{\pi,T-n} + P_{\pi,T-n} v_{n-1}$ *then*

$$v_n(s) = \mathbb{E}_\pi \left[\sum_{t=T-n-1}^{T-1} R_{t+1} | S_{t-n-1} = s \right] = v_{\pi,T-n}(s)$$

If $v_0 = r_*$ *and* $v_{n+1} = \mathcal{T}^* v_n$ *then*

$$v_n(s) = \max_\pi \mathbb{E}_\pi \left[\sum_{t=T-n-1}^{T-1} R_{t+1} | S_{t-n-1} = s \right] = v_{*,T-n}(s)$$

Proof. If $n = 0$ then by definition $v_{\pi,T}(s) = \mathbb{E}_\pi[R_T | S_{T-1} = s] = r_{\pi,T-1}(s)$.

Now,

$$\begin{aligned}
v_{\pi,T-n}(s) &= \mathbb{E}_\pi \left[\sum_{t=T-n-1}^{T-1} R_{t+1} \middle| S_{T-n-1} = s \right] \\
&= r_{\pi,T-n-1}(s) + \mathbb{E}_\pi \left[\sum_{t=T-n}^{T-1} R_{t+1} \middle| S_{T-n-1} = s \right] \\
&= r_{\pi,T-n-1}(s) + \sum \sum_a p(s'|s,a)\pi(a|s) \mathbb{E}_\pi \left[\sum_{t=T-n}^{T-1} R_{t+1} \middle| S_{t-n} = s' \right] \\
&= r_{\pi,T-n-1}(s) + P_{\pi,T-n-1} v_{\pi,T-n-1}(s)
\end{aligned}$$

Along the same line, if $n = 0$ then by definition $v_{*,T}(s) = \max_\pi \mathbb{E}_\pi[R_T | S_{T-1} = s] = \max_\pi v_{\pi,T}(s) = r_*(s)$.

Now,

$$\begin{aligned}
v_{*,T-n}(s) &= \max_{\pi} \mathbb{E}_{\pi}\left[\sum_{t=T-n-1}^{T-1} R_{t+1} \middle| S_{T-n-1} = s\right] \\
&= \max_{\pi}\left(r_{\pi}(s) + \mathbb{E}\left[\sum_{t=T-n}^{T-1} R_{t+1} \middle| S_{T-n-1} = s\right]\right) \\
&= \max_{\pi}\left(r_{\pi,T-n-1}(s) + \sum \sum_{a} p(s'|s,a)\pi(a|s)\mathbb{E}\left[\sum_{t=T-n}^{T-1} R_{t+1} \middle| S_{t-n} = s'\right]\right) \\
&= \max r_{\pi,T-n-1}(s) + P_{\pi,T-n-1} \max_{\pi} v_{\pi,T-n-1}(s) \\
&= \mathcal{T}^{*} v_{*,T-n-1}(s)
\end{aligned}$$

□

4 Non Discounted Total Reward

Definition 4.1

Let s_{abs} be the absorbing state, we define the expected absorption time starting from s $\tau_\pi(s)$ by

$$\tau_\pi(s) = \mathbb{E}_\pi\left[\inf_{S_t=\tilde{s}} t \middle| S_0 = s\right].$$

If τ_π is finite, we say that π is proper.

Definition 4.2

We define the maximum expected absorption time starting from s by $\tau_*(s)$ by

$$\tau_*(s) = \max_\pi \tau_\pi(s)$$

Proposition 4.3

If $\tau_\pi < +\infty$ then

$$\tau_\pi = 1 + P_\pi \tau_\pi. = \mathcal{T}^\pi \tau_\pi$$

If $\tau_* < +\infty$ then

$$\tau_* = \max_\pi 1 + P_\pi \tau_*. = \mathcal{T} \tau_\pi$$

Proof. It suffices to notice that $\tau_\pi(s) = \mathbb{E}_\pi\left[\sum_{t=0}^{+\infty} R_{t+1}\right]$ with $R_t = 0$ if $s_t = \tilde{s}$ and 1 otherwise. □

Proposition 4.4

$\mathcal{T}^\pi$ is a contraction of factor $\max \frac{\tau_\pi(s)-1}{\tau_\pi(s)}$ with respect to the norm $\|\cdot\|_{\infty,1/\tau_\pi}$

$\mathcal{T}^\pi$ and $\mathcal{T}^*$ are contraction of factor $\max \frac{\tau_*(s)-1}{\tau_*(s)}$ with respect to the norm $\|\cdot\|_{\infty,1/\tau_*}$.

Proof.

$$\begin{aligned}
|\mathcal{T}^{\pi} v(s) - \mathcal{T}^{\pi} v'(s)| &\le |P_{\pi}(v - v')(s)| \\
&\le P_{\pi}(\tau \times \frac{|v - v'|}{\tau})(s) \\
&\le P_{\pi}\tau(s)\|v - v'\|_{\infty,1/\tau} \\
&\le \tau(s)\frac{1 + P_{\pi}\tau(s) - 1}{\tau(s)}\|v - v'\|_{\infty,1/\tau} \\
&\le \tau(s)\frac{1 + P_{*}\tau(s) - 1}{\tau(s)}\|v - v'\|_{\infty,1/\tau}
\end{aligned}$$

which yields the result for both $\tau = \tau_{\pi}$ and $\tau = \tau_{*}$.

Now, assume without loss of generality that $\mathcal{T}^{*}v(s) \ge \mathcal{T}^{*}v'(s)$,

$$\begin{aligned}
&|\mathcal{T}^{*} v(s) - \mathcal{T}^{*} v'(s)| \\
&\qquad = \max_{\pi} \mathcal{T}^{\pi} v(s) - \max_{\pi} \mathcal{T}^{\pi} v'(s) \\
&\qquad \le \max_{\pi} \left(\mathcal{T}^{\pi} v(s) - \mathcal{T}^{\pi} v'(s)\right) \\
&\qquad \le \tau(s)\frac{1 + P_{*}\tau(s) - 1}{\tau(s)}\|v - v'\|_{\infty,1/\tau}
\end{aligned}$$

which yields the result for $\tau = \tau_{*}$. □

5 Bandits

5.1 Regret

Definition 5.1.1

A k-armed bandit is defined by a collection of k random variable $R(a)$, $a \in \{1, \dots, k\}$.

The best arm is a_* is such that $\mathbb{E}[R(a_*)] \geq \max_a \mathbb{E}[R(a)]$.

For any policy π, the regret is defined by

$$r_{T,\pi} = T\mathbb{E}[R(a_*)] - \mathbb{E}\left[\sum_{t=1}^{T} R(A_t)\right]$$

where A_t is the arm chosen at time t following the policy π.

Proposition 5.1.2

Let $T_t(a) = \sum_{s=1}^{t} \mathbf{1}_{A_s=i}$ and $\Delta(a) = \mathbb{E}[R(a_*)] - \mathbb{E}[R(a)]$ then

$$r_{n,\pi} = \sum_{a=1}^{k} \Delta(a)\mathbb{E}[T_t(a)]$$

Proof. By definition,

$$\begin{aligned}
r_{T,\pi} &= n\mathbb{E}[R(a_*)] - \mathbb{E}\left[\sum_{t=1}^{T} R(A_t)\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{T} (\mathbb{E}[R(a_*)] - R(A_t))\right] \\
&= \mathbb{E}\left[\sum_{t=1}^{T}\sum_{a=1}^{k} \mathbf{1}_{A_t=a} (\mathbb{E}[R(a_*)] - R(a))\right] \\
&= \sum_{a=1}^{k} \mathbb{E}\left[\sum_{t=1}^{T} \mathbf{1}_{A_t=a} (\mathbb{E}[R(a_*)] - R(a))\right] \\
&= \sum_{a=1}^{k} \mathbb{E}\left[\sum_{t=1}^{T} \mathbf{1}_{A_t=a} \Delta(a)\right] \\
&= \sum_{a=1}^{k} \mathbb{E}[T_t(a)]\,\Delta(a)
\end{aligned}$$

□

5.2 Concentration of subgaussian random variables

Definition 5.2.1

A random variable X is said to be σ-subgaussian if

$$\mathbb{E}[\exp \lambda X] \leq \exp(\lambda^2\sigma^2/2)$$

Proposition 5.2.2

If X is σ-subgaussian then for any $\epsilon > 0$

$$\mathbb{P}(X \geq \epsilon) \leq \exp\left(\frac{-\epsilon^2}{2\sigma^2}\right)$$

Proof.

$$\begin{aligned}\mathbb{P}(X \geq \epsilon) &= \mathbb{P}(\exp(\lambda X) \geq \exp(\lambda\epsilon)) \\ &\leq \frac{\mathbb{E}[\exp(\lambda X)]}{\exp(\lambda\epsilon)} \\ &\leq \exp(\lambda^2\sigma^2/2 - \lambda\epsilon) \\ &\leq \exp\left(\frac{-\epsilon^2}{2\sigma^2}\right)\end{aligned}$$

where the last inequality is obtained by optimizing in λ. □

Proposition 5.2.3

If X is σ-subgaussian and Y is σ'-subgaussian conditionnaly to X then

- $\mathbb{E}[X] = 0$ *and* $\mathbb{V}\text{ar}\,[X] \leq \sigma^2$
- cX *is* $|c|\sigma$*-subgaussian.*
- $X + Y$ *is* $\sqrt{\sigma^2 + (\sigma')^2}$*-subgaussian.*

Proof.

$$\mathbb{E}[\exp \lambda X] = \sum_k \frac{\lambda^k}{k!}\mathbb{E}\left[X^k\right]$$

while

$$\exp(\lambda^2\sigma^2/2) = \sum_k \frac{\lambda^{2k}\sigma^{2k}}{2^k k!}$$

By looking at the term in front of λ^1 and λ^2, we obtain

$$\lambda\mathbb{E}[X] \leq 0 \quad \text{and} \quad \frac{\lambda^2}{2!}\mathbb{E}\left[X^2\right] \leq \frac{\lambda^2\sigma^2}{2 \times 1!}$$

which implies

$$\mathbb{E}[X] = 0 \quad \text{and} \quad \mathbb{V}\mathrm{ar}\,[X] \leq \sigma^2.$$

By definition,

$$\mathbb{E}[\exp(\lambda c X)] \leq \exp(\lambda^2 c^2 \sigma^2/2)$$

hence the $|c|\sigma$-subgaussianity of cX.

Now,

$$\begin{aligned} \mathbb{E}[\exp(\lambda(X+Y))] &\leq \mathbb{E}[\mathbb{E}[\exp(\lambda(X+Y))|X]] \\ &\leq \mathbb{E}[\mathbb{E}[\exp(\lambda X)\exp(\lambda Y))|X]] \\ &\leq Esp\exp(\lambda X)\exp(\lambda^2(\sigma')^2/2) \\ &\leq \exp\left(\lambda^2(\sigma^2+(\sigma')^2)/2\right) \end{aligned}$$

□

Proposition 5.2.4

If $X_i - \mu$ are iid σ-subgaussian variable,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geq \mu + \epsilon\right) \leq \exp\left(-\frac{n\epsilon^2}{2\sigma^2}\right) \quad \textit{and} \quad \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \leq \mu - \epsilon\right) \leq \exp\left(-\frac{n\epsilon^2}{2\sigma^2}\right)$$

Proof. It suffices to notice that $\frac{1}{n}\sum_{i=1}^n X_i - \mu$ and $\mu - \frac{1}{n}\sum_{i=1}^n X_i$ are $\sigma/\sqrt{n}$-subgaussian. □

5.3 Explore Then Commit strategy

Definition 5.3.1

The simple current mean estimate $Q_t(a)$ is defined by

$$Q_t(a) = \frac{1}{T_t(a)}\sum_{s=1}^t \mathbf{1}_{A_s=a} R_s(a)$$

Proposition 5.3.2

Assume we play the arm successively during Km steps and then play the arm which maximize the current mean estimate $Q_t(a)$ then if the $R(a) - \mathbb{E}[R(a)]$ is 1-subgaussian

$$r_{T,\pi} \leq \min(m, T/K) \sum_{a=1}^{k} \Delta(a) + \max(T - mK, 0) \sum_{a=1}^{k} \Delta(a) \exp(-m\Delta(a)^2/4)$$

Furthermore,

$$\mathbb{P}(a_T = a_*) \geq 1 - \sum_{a \neq a_*} \exp(-m\Delta(a)^2/4)$$

Proof. We have

$$r_{T,\pi} = \sum_{a=1}^{k} \Delta(a) \mathbb{E}[T_T(a)]\,,$$

we can thus focus on $\mathbb{E}[T_T(a)]$.

Now

$$\begin{aligned}
\mathbb{E}[T_T(a)] &\leq \min(m, n/K) + \max(n - mK, 0)\mathbb{P}(a_{mK+1} = a) \\
&\leq \min(m, n/K) + \max(n - mK, 0)\mathbb{P}\Big(Q_t(a) \geq \max_{a' \neq a} Q_t(a')\Big) \\
&\leq \min(m, n/K) + \max(n - mK, 0)\mathbb{P}(a_{mK+1} = a) \\
&\leq \min(m, n/K) + \max(n - mK, 0)\mathbb{P}(Q_m(a) \geq Q_m(a_*)) \\
&\leq \min(m, n/K) + \max(n - mK, 0)\mathbb{P}(Q_{mK+1}(a) - \mathbb{E}[R(a)] - (Q_{mK+1}(a_*) - \mathbb{E}[R(a_*)]) \geq \Delta(a
\end{aligned}$$

It suffices then to notice that $Q_{mK+1}(a) - \mathbb{E}[R(a)] - (Q_{mK+1}(a_*) - \mathbb{E}[R(a_*)])$ is $\sqrt{2/m}$-subgaussian to obtain

$$\begin{aligned}
\mathbb{E}[T_T(a)] &\leq \min(m, n/K) + \max(n - mK, 0)\mathbb{P}(Q_{mK+1}(a) \geq Q_{mK+1}(a_*)) \\
&\leq \min(m, n/K) + \max(n - mK, 0) \exp(-m\Delta(a)^2/4)
\end{aligned}$$

.

Now

$$\begin{aligned}
\mathbb{P}(a_T = a_*) &= 1 - \sum a \neq a_* \mathbb{P}(a_T = a) \\
&\leq 1 - \sum_{a \neq a_*} \exp(-m\Delta(a)^2/4)
\end{aligned}$$

□

5.4 ϵ-greedy strategy

Proposition 5.4.1

Let π be an ϵ_t-greedy strategy,

$$r_{T,\pi} \geq \sum_{t=1}^{T} \frac{\epsilon_t}{k} \sum_{a=1}^{k} \Delta(a)$$

Proof. By definition of an ϵ-greedy strategy

$$\mathbb{E}[T_t(a)] \geq \sum_{t=1}^{T} \frac{\epsilon_t}{k}$$

hence the first result. □

Proposition 5.4.2

Let π be an ϵ_t-greedy strategy,

$$\mathbb{P}(A_T = a_*) \geq 1 - \epsilon_T - \Sigma_t \exp(-\Sigma_T/(6k)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4k)}$$

with $\Sigma_T = \sum_{s=1}^{T} \epsilon_s$.

Furthermore,

$$\mathbb{P}(a_* = \operatorname{argmax} Q_{T,a}) \geq 1 - \Sigma_t \exp(-\Sigma_T/(6k)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4k)}$$

If $\epsilon_t = c/t$,

$$r_{T,\pi} \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T) + 1}{k} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

as soon as $c/(6k) > 1$ and $c \min_{a \neq a_*} \Delta(a)/4k < 1$.

If $\epsilon_t = c \log(t)/t$ then

$$r_{T,\pi} \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T)(\log(T) + 1)}{k} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

Proof. By definition of π,

$$\mathbb{P}(A_T = a) \leq \frac{\epsilon_t}{k} + (1 - \frac{\epsilon_t}{k} \mathbb{P}(Q_T(a) \geq Q_T(a_*))$$

and

$$\mathbb{P}(Q_T(a) \geq Q_T(a_*)) \leq \mathbb{P}(Q_T(a) \geq \mu(a) + \Delta(a)/2) + \mathbb{P}(Q_T(a_*) \leq \mu(a_*) - \Delta(a)/2) \,.$$

By symmetry, it suffices to bound

$$\begin{aligned}
\mathbb{P}(Q_T(a) \geq \mu(a) + \Delta/2) &\leq \sum_{t=1}^{T} \mathbb{P}(T_t(a) = t, Q_T(a) \geq \mu(a) + \Delta/2) \\
&\leq \sum_{t=1}^{T} \mathbb{P}\left(T_T(a) = t, \frac{1}{t}\sum_{k=1}^{t} R_k(a) \geq \mu(a) + \Delta/2\right) \\
&\leq \sum_{t=1}^{T} \mathbb{P}\left(T_T(a) = t \middle| \frac{1}{t}\sum_{k=1}^{t} R_k(a) \geq \mu(a) + \Delta/2\right) \mathbb{P}\left(\frac{1}{t}\sum_{k=1}^{t} R_k(a) \geq \mu(a) + \Delta/2\right) \\
&\leq \sum_{t=1}^{T} \mathbb{P}\left(T_T(a) = t \middle| \frac{1}{t}\sum_{k=1}^{t} R_k(a) \geq \mu(a) + \Delta/2\right) e^{-\Delta^2 t/2} \\
&\leq \sum_{t=1}^{T_0} \mathbb{P}\left(T_T(a) = t \middle| \frac{1}{t}\sum_{k=1}^{t} R_k(a) \geq \mu(a) + \Delta/2\right) + \sum_{t=T_0+1}^{T} e^{-\Delta^2 t/2}
\end{aligned}$$

Let $T_T^R(a)$ be the number of time the arm a has been chosen at random before time T

$$\begin{aligned}
&\leq \sum_{t=1}^{T_0} \mathbb{P}\left(T_T^R(a) \leq t \middle| \frac{1}{t}\sum_{k=1}^{t} R_k(a) \geq \mu(a) + \Delta/2\right) + \frac{2}{\Delta^2} e^{-\Delta^2 T_0/2} \\
&\leq \sum_{t=1}^{T_0} \mathbb{P}\Big(T_T^R(a) \leq t\Big) + \frac{2}{\Delta^2} e^{-\Delta^2 T_0/2}
\end{aligned}$$

Now the Bernstein inequality yields

$$\mathbb{P}\Big(T_t^R(a) \leq \mathbb{E}\Big[T_t^R(a)\Big] - \lambda\Big) \leq \exp\left(-\frac{\lambda^2/2}{\mathbb{V}\mathrm{ar}\ [T_t^R(a)] + \lambda/2}\right)$$

with

$$\begin{aligned}
\mathbb{E}\Big[T_t^R(a)\Big] &= \sum_{s=1}^{t} \frac{\epsilon_s}{k} \\
\mathbb{V}\mathrm{ar}\ \Big[T_t^R(a)\Big] &= \sum_{s=1}^{t} \frac{\epsilon_s}{k}(1 - \frac{\epsilon_s}{k}) \\
&\leq \sum_{s=1}^{t} \frac{\epsilon_s}{k},
\end{aligned}$$

. Choosing $T_0 = \frac{1}{2}\frac{\Sigma_T}{k} = \frac{1}{2}\sum_{s=1}^T \frac{\epsilon_s}{k} = \frac{1}{2}\mathbb{E}\left[T_T^R(a)\right] \leq \frac{1}{2}\,\mathbb{V}\mathrm{ar}\left[T_T^R(a)\right]$ leads

$$\begin{aligned}\mathbb{P}\left(T_T^R(a) \leq T_0\right) &= \mathbb{P}\left(T_T^R(a) \leq 2T_0 - T_0\right)\\ &\leq \exp\left(-\frac{T_0^2/2}{\sigma^2 + T_0/2}\right)\\ &\leq \exp\left(-\frac{T_0^2/2}{T_0 + T_0/2}\right)\\ &\leq \exp(-T_o/3)\end{aligned}$$

which implies

$$\mathbb{P}(Q_T(a) \geq \mu(a) + \Delta/2) \leq T_0 \exp(-T_o/3) + \frac{2}{\Delta^2} e^{-\Delta^2 T_0/2}$$

and thus

$$\begin{aligned}\mathbb{P}(a = \operatorname{argmax} Q_T(a)) &\leq 2(1 - \frac{\epsilon_T}{k})\left(\Sigma_T/(2k)\exp(-\Sigma_T/(6k)) + \frac{2}{\Delta(a)^2} e^{-\Delta(a)^2\Sigma_T/4}\right)\\ &\leq \frac{\epsilon_T}{k} + \frac{\Sigma_T}{k}\exp(-\Sigma_T/(6k) + \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2\Sigma_T/(4k)}\end{aligned}$$

with $\Sigma_T = \sum_{s=1}^T \epsilon_s$ which goes to 0 as soon as Σ_T tends to $+\infty$ We deduce then that

$$\mathbb{P}(A_T = a) \leq \frac{\epsilon_T}{k} + \frac{\epsilon_T}{k} + \frac{\Sigma_T}{k}\exp(-\Sigma_T/(6k) + \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2\Sigma_T/(4k)}$$

which goes to 0 if furthermore ϵ_T tends to 0

Finally,

$$\begin{aligned}\mathbb{E}[T_T(a)] &= \sum_{t=1}^T \mathbb{P}(A_t = a)\\ &\leq \sum_{t=1}^T \left(\frac{\epsilon_t}{k} + \frac{\Sigma_t}{k}\exp(-\Sigma_t/(6k) + \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2\Sigma_t/(4k)}\right)\end{aligned}$$

Hence

$$r_{T,\pi} \leq \sum_{a \neq a_*}\left(\Delta(a)\left(\frac{\Sigma_T}{k} + \sum_{t=1}^T \frac{\Sigma_t}{k} e^{-\Sigma_t/(6k)}\right) + \frac{4}{\Delta(a)}\sum_{t=1}^T e^{-\Delta(a)^2\Sigma_t/(4k)}\right)$$

Assume that $\epsilon_t = c/t$ so that $\Sigma_t \leq c(\ln(t) + 1)$ then the previous inequality becomes

$$\begin{aligned}r_{T,\pi} &\leq \sum_{a \neq a_*}\left(\Delta(a)\left(c\frac{\log(T)+1}{k} + \sum_{t=1}^T c\frac{\log(t)+1}{k} e^{-c(\log(t)+1)/(6k)}\right) + \frac{4}{\Delta(a)}\sum_{t=1}^T e^{-\Delta(a)^2 c(\log(t)+1)/(4k)}\right)\\ &\leq \sum_{a \neq a_*}\left(\Delta(a)\left(c\frac{\log(T)+1}{k} + C\right) + \frac{4}{\Delta(a)}C'\right)\end{aligned}$$

as soon as $c/(6k) > 1$ and $c \min_{a \neq a_*} \Delta(a)/4k < 1$.

If $\epsilon_t = c\log(t)/t$ then

$$r_{T,\pi} \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T)(\log(T)+1)}{k} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

□

5.5 UCB strategy

Proposition 5.5.1

Assume we use a UCB strategy with a variance term $\sqrt{\frac{c \log t}{T_t(a)}}$ *then*

$$r_n(t) \leq C_c \sum_a \Delta(a) + \sum_a \frac{4c \ln t}{\Delta(a)}.$$

with $C_c < +\infty$ *as soon as* $c > 3/2$

Furthermore

$$\mathbb{P}(A_t = a_*) \geq 1 - 2kt^{-2c+2}$$

as soon as $t \geq \max_a \frac{4c \ln t}{\Delta(a)^2}$.

Proof. By construction,

$$\begin{aligned}
T_t(a) &= \sum_{s=1}^{t} \mathbf{1}_{A_s=a} \\
&\le \sum_{s=1}^{t} \mathbf{1}_{Q_s(a)+c_s(a)=\max Q_s(a')+c_s(a')} \\
&\le T_0(a) + \sum_{s=T_0+1}^{t} \mathbf{1}_{Q_s(a)+c_s(a)=\max Q_s(a')+c_s(a'),T_s(a)\ge T_0(a)} \\
&\le T_0(a) + \sum_{s=T_0+1}^{t} \mathbf{1}_{Q_s(a)+c_s(a)\ge Q_s(a_*)+c_s(a_*),T_t(a)\ge T_0(a)} \\
&\le T_0(a) + \sum_{s=T_0+1}^{t} \mathbf{1}_{\max_{T_0(a)\le s''\le t} \frac{1}{s''} \sum j=1^{s''} R(a)_{(j)}+\sqrt{\frac{c\ln s}{s''}}\ge \min_{s'\le t} \frac{1}{s'} \sum j=1^{s'} R(a_*)_{(j)}+\sqrt{\frac{c\ln s}{s'}}} \\
&\le T_0(a) + \sum_{s=T_0+1}^{t} \sum_{s'=1}^{s-1} \sum_{s''=T_0(a)}^{s-1} \mathbf{1}_{\frac{1}{s''} \sum j=1^{s''} R(a)_{(j)}+\sqrt{\frac{c\ln s}{s''}}\ge\frac{1}{s'} \sum j=1^{s'} R(a_*)_{(j)}+\sqrt{\frac{c\ln s}{s'}}} \\
&\le T_0(a) + \sum_{s=T_0+1}^{t} \sum_{s'=1}^{s-1} \sum_{s''=T_0(a)}^{s-1} \mathbf{1}_{\mu(a_*)\le\mu(a)+2\sqrt{\frac{c\ln s}{s''}}} + \mathbf{1}_{\frac{1}{s''} \sum j=1^{s''} R(a)_{(j)}\ge\mu(a)+\sqrt{\frac{c\ln s}{s''}}} \\
&\qquad\qquad + \mathbf{1}_{\frac{1}{s'} \sum j=1^{s'} R(a_*)_{(j)}\le\mu(a_*)-\sqrt{\frac{c\ln s}{s'}}} \\
&\le T_0(a) + \sum_{s=T_0+1}^{t} \sum_{s'=1}^{s-1} \sum_{s''=T_0(a)}^{s-1} \mathbf{1}_{\mu(a_*)\le\mu(a)+2\sqrt{\frac{c\ln s}{s''}}} + 2e^{-2c\ln s} \\
\mathbb{E}[T_t(a)] &\le T_0(a) + \sum_{s=T_0+1}^{t} \sum_{s'=1}^{s-1} \sum_{s''=T_0(a)}^{s-1} \mathbf{1}_{\Delta(a)\le 2\sqrt{\frac{2c\ln t}{s''}}} + 2s^{-2c}
\end{aligned}$$

choosing $T_0(a) = \frac{4c\ln t}{\Delta(a)^2}$

$$\begin{aligned}
&\le \frac{4c\ln t}{\Delta(a)^2} + \sum_{s=T_0+1}^{t} 2s^{-2c+2} \\
&\le \frac{4c\ln t}{\Delta(a)^2} + C_c
\end{aligned}$$

as soon as $c > 3/2$.

One deduce thus

$$r_n(t) \le C_c \sum_a \Delta(a) + \sum_a \frac{4c\ln t}{\Delta(a)}.$$

Note that we have shown

$$\mathbb{P}(A_t = a) \le 2t^{-2c}$$

as soon as $t \geq \frac{4c \ln t}{\Delta(a)^2}$. Thus

$$\mathbb{P}(A_t = a_*) \geq 1 - 2kt^{-2c+2}$$

as soon as $t \geq \max_a \frac{4c \ln t}{\Delta(a)^2}$. □

6 Stochastic Approximation

6.1 Convergence of a mean

Proposition 6.1.1

Assume X_i are i.i.d. such that $\mathbb{E}[X_i|\mathcal{F}_{i-1}] = \mu$ and $\mathbb{V}\mathrm{ar}\left[X_i|\mathcal{F}_{i-1}\right] \leq \sigma^2$, let

$$M_n = M_{n-1} + \alpha_n(X_n - M_{n-1})$$

with $1 \geq \alpha_i \geq 0$ then

- *if $\sum_{i=1}^n \alpha_i \to +\infty$ and $\sum_{i=1}^n \alpha_i^2 < +\infty$, $M_n \to \mu$ in quadratic norm.*
- *$\alpha_i = \alpha$ then $\limsup \|M_n - \mu|^2 \leq \alpha\sigma^2$*

Proof. By definition,

$$\begin{aligned}M_n &= M_{n-1} + \alpha_n(X_n - M_{n-1})\\ &= (1-\alpha_n)M_{n-1} + \alpha_n X_n\\ &= \prod_{i=1}^n (1-\alpha_i)M_0 + \sum_{k=1}^n \prod_{i=k+1}^n (1-\alpha_i)\alpha_k X_k\end{aligned}$$

thus

$$\mathbb{E}\left[\|M_n - \mu\|^2\right] = \prod_{i=1}^n (1-\alpha_i)\|M_0 - \mu\|^2 + \sum_{k=1}^n \prod_{i=k+1}^n (1-\alpha_i)^2\alpha_k^2\sigma^2$$

Thus it suffices to prove that

$$\prod_{i=1}^n (1-\alpha_i) \to 0 \quad \text{and} \quad \sum_{k=1}^n \prod_{i=k+1}^n (1-\alpha_i)^2\alpha_k^2 \to 0$$

For the first part, we use $(1-x) \leq e^{-x}$ for $0 \leq x \leq 1$ to obtain

$$\prod_{i=1} (1-\alpha_i) \leq e^{-\sum_{i=1}^n \alpha_i}$$

which goes to 0 if $\sum_{i=1}^n \alpha_i \to +\infty$.

For the second one,

$$\begin{aligned}\sum_{k=1}^{n}\prod_{i=k+1}^{n}(1-\alpha_i)^2\alpha_k^2 &\leq \sum_{k=1}^{m}\prod_{i=k+1}^{n}(1-\alpha_i)^2\alpha_k^2 + \sum_{k=m+1}^{n}\prod_{i=k+1}^{n}(1-\alpha_i)^2\alpha_k^2\\ &\leq \sum_{k=1}^{m}\prod_{i=m}^{n}(1-\alpha_i)^2\alpha_k^2 + \max_{k\geq m+1}\alpha_k\sum_{k=m+1}^{n}\left(\prod_{i=k+1}^{n}(1-\alpha_i)-\prod_{i=k}^{n}(1-\alpha_i)\right)\\ &\leq e^{-2\sum_{k=m}^{n}\alpha_i}\sum_{k=1}^{m}\alpha_k^2 + \max_{k\geq m+1}\alpha_k\left(1-\prod_{i=m+1}^{n}(1-\alpha_i)\right)\\ &\leq e^{-2\sum_{k=m}^{n}\alpha_i}\sum_{k=1}^{m}\alpha_k^2 + \max_{k\geq m+1}\alpha_k\end{aligned}$$

Choosing $m = n/2$ yields

$$\mathbb{E}\left[\|M_n-\mu\|^2\right] \leq e^{-\sum_{i=1}^{n}\alpha_i}\|M_0-\mu\|^2 + e^{-2\sum_{k=n/2}^{n}\alpha_i}\sum_{k=1}^{n/2}\alpha_k^2\sigma^2 + \max_{k\geq n/2}\alpha_k\sigma^2$$

If we assume that $\sum_{k=1}^{n}\alpha_i \to +\infty$ and $\sum_{k=1}^{m}\alpha_k^2 < +\infty$ then all the term in the right hand side goes to 0.

If we assume $\alpha_k = \alpha$ then

$$\mathbb{E}\left[\|M_n-\mu\|^2\right] \leq e^{-n\alpha}\|M_0-\mu\|^2 + ne^{-n\alpha}\alpha^2\sigma^2 + \alpha\sigma^2$$

which is yields the result. □

6.2 Generic Stochastic Approximation

Definition 6.2.1 **Generic Stochastic Algorithm**

Let H_t be a sequence of approximation of an operator h, let $\alpha_i(t)$ be a set of non negative sequences, for any initial value X_0, we define the following iterative scheme

$$X_{t+1,i} = X_{t,i} + \alpha_i(t)H_t(X_t)_i.$$

Definition 6.2.2

h and H_t are compatible if

$$H_t(x) = h(x) + \epsilon_t(x) + \delta_t(x)$$

with

$$\mathbb{E}[\epsilon_t(x)|\mathcal{F}_t] = 0 \quad \textit{and} \quad \mathbb{V}\mathrm{ar}\left[\epsilon_t(x)|\mathcal{F}_t\right] \leq c_0(1+\|x\|^2)$$

and with probability 1

$$\|\delta_n(x)\|^2 \leq c_n(1+\|x\|)^2$$

with $c_n \to 0$ *and either*

- *it exists a non negative* V C^1 *with* L*-Lipschitz gradient satisfying*

$$\langle \nabla V(x), h(x)\rangle \leq -c\|\nabla V(x)\|^2$$
$$\mathbb{E}\left[\|H_t(x)\|^2\right] \leq c_0'(1+\|\nabla V(x)\|^2),$$

- *or* h *is a contraction for the norm considered.*

Proposition 6.2.3 **Generic Stochastic Approximation**

Assume that for any i*, we have almost surely*

$$\sum_{i=1}^{T} \alpha_i \to +\infty \quad \text{and} \quad \sum_{i=1}^{T} \alpha_i^2 < +\infty$$

Then providing h *and* H_t *are* compatible,

$$h(X_n) \to 0.$$

Proof. See Neuro-Dynamic programming from Bertsekas and Tsitsiklis. □

Lemma 6.2.4

$$\begin{aligned} \text{From} \quad & \theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k) \quad \text{with} \quad h_k(\theta) = H(\theta) + \epsilon_k + \eta_k \\ \text{to} \quad & \frac{d\tilde{\theta}}{dt} = H(\tilde{\theta}) \end{aligned}$$

Sketch.

- Difference between θ and a solution of the ODE with $\tilde{\theta}(t_k) = \theta_k$ at t_{k+l}:

$$\begin{aligned}
\theta(t_{k+l}) - \tilde{\theta}(t_{k+l}) &= \int_{t_k}^{t_{k+l}} \left(\theta'(u) - \tilde{\theta}'(u)\right) du \\
&= \sum_{k'=k}^{k+l-1} \int_{t_{k'}}^{t_{k'+1}} \left(H(\theta(t_k)) + \epsilon_k + \eta_k - H(\tilde{\theta}(u))\right) du \\
&= \sum_{k'=k}^{k+l-1} \int_{t_{k'}}^{t_{k'+1}} \left(H(\theta(t_k)) - H(\tilde{\theta}(u))\right) du \\
&\quad + \sum_{k'=k}^{k+l-1} \alpha_{k'}\epsilon_{k'} + \sum_{k'=k}^{k+l-1} \alpha_{k'}\eta_{k'}
\end{aligned}$$

- The last two term are going to be small by construction...
- Difference between θ and a solution of the ODE with $\tilde{\theta}(t_k) = \theta_k$ at t_{k+l}:

$$\begin{aligned}
\theta(t_{k+l}) - \tilde{\theta}(t_{k+l}) = &\sum_{k'=k}^{k+l-1} \int_{t_{k'}}^{t_{k'+1}} \left(H(\theta(t_k)) - H(\tilde{\theta}(u))\right) du \\
&+ \sum_{k'=k}^{k+l-1} \alpha_{k'}\epsilon_{k'} + \sum_{k'=k}^{k+l-1} \alpha_{k'}\eta_{k'}
\end{aligned}$$

- The last two term are going to be small by construction:

$$\mathbb{E}\left[\sum_{k'=k}^{k+l-1} \alpha_{k'}\epsilon_{k'}\right] = 0 \quad \text{and} \quad \mathbb{V}\text{ar}\left[\sum_{k'=k}^{k+l-1} \alpha_{k'}\epsilon_{k'}\right] < \sigma^2 \sum_{k'=k}^{k+l-1} \alpha_{k'}^2 \to 0$$

$$\| \sum_{k'=k}^{k+l-1} \alpha_{k'}\eta_{k'} \| \leq (t_{k+l-1} - t_k) \sup_{k' \geq k} \|\eta_{k'}\|$$

which is small if we assume that $t_{k+l-1} - t_k \leq \Delta$.

- We can now use a Lipchitz assumption on H to obtain:

$$\begin{aligned}
\left\|\int_{t_{k'}}^{t_{k'+1}} \left(H(\theta(t_{k'})) - H(\tilde{\theta}(u))\right) du\right\| &\leq L \int_{t_{k'}}^{t_{k'+1}} \|\theta(t_{k'}) - \tilde{\theta}(u)\| du \\
&\leq L\alpha_{k'}\|\theta(t_{k'}) - \tilde{\theta}(t_{k'})\| + L \int_{t_{k'}}^{t_{k'+1}} \|\theta(\tilde{t}_{k'}) - \tilde{\theta}(u) du\| \\
&\leq L\alpha_{k'}\|\theta(t_{k'}) - \tilde{\theta}(t_{k'})\| + L\|H\|_\infty \alpha_{k'}^2
\end{aligned}$$

- Combinining all the results leads to

$$\begin{aligned}
\|\theta(t_{k+l}) - \tilde{\theta}(t_{k+l})\| \leq L &\sum_{k'=k}^{k+l-1} \alpha_{k'} \|\theta(t_{k'}) - \tilde{\theta}(t_{k'})\| \\
&+ L\|H\|_\infty \sum_{k'=k}^{k+l-1} \alpha_{k'}^2 + + \left\|\sum_{k'=k}^{k+l-1} \alpha_{k'}\epsilon_{k'}\right\| + \sum_{k'=k}^{k+l-1} \alpha_{k'} \|\eta_{k'}\|
\end{aligned}$$

- Using a discrete Gronwall Lemma, $\forall l \leq l'', z_l \leq L \sum_{l'=0}^{l-1} \alpha_{l'} z_{l'} + A \Rightarrow z_{l''} \leq A e^{L \sum_{l'=0}^{l''-1} \alpha_{l'}}$, we obtain that if $t_{k+l} - t_k \leq \Delta$

$$\|\theta(t_{k+l}) - \tilde{\theta}(t_{k+l})\| \leq \underbrace{\left(L\|H\|_\infty \sum_{k'=k}^{\infty} \alpha_{k'}^2 + \sup_{l' \leq l} \left\| \sum_{k'=k}^{k+l'-1} \alpha_{k'} \epsilon_{k'} \right\| + L \sup_{k' \geq k} \|\eta_{k'}\| \right)}_{\longrightarrow 0 \text{ when } k \to \infty} e^{L\Delta}$$

□

6.3 TD(λ) and linear approximation

Proposition 6.3.1

Provided there is a unique stationary distribution μ on the states, that the basis function are linearily independent and

$$\sum_{i=1}^{T} \alpha \to +\infty \quad \textit{and} \quad \sum_{i=1}^{T} \alpha^2 < +\infty$$

For any $\lambda \in (0,1)$, the TD(λ) algorithm with linear approximation converges with probability one. The limit $\boldsymbol{w}_{*,\lambda}$ is the unique solution of

$$\Pi_\mu \mathcal{T}^{\pi,(\lambda)} \mathbb{X} \boldsymbol{w}_{*,\lambda} = \mathbb{X} \boldsymbol{w}_{*,\lambda}.$$

Furthermore,

$$\|\mathbb{X}\boldsymbol{w}_{*,\lambda} - v_\pi\|_{2,\mu} \leq \frac{1-\lambda\gamma}{1-\gamma} \|\Pi_\mu v_\pi - v_\pi\|_{2,\mu}$$

Proof. See Tsitsiklis and Van Roy. □

Proof. Assume $\boldsymbol{A}$ is invertible and let $\boldsymbol{w}_{TD} = \boldsymbol{A}^{-1}\boldsymbol{b}$

$$\begin{aligned} \mathbb{E}[\boldsymbol{w}_{t+1} - \boldsymbol{w}_{TD} | \boldsymbol{w}_t] &= \boldsymbol{w}_t + \alpha(\boldsymbol{b} - \boldsymbol{A}\boldsymbol{w}_t) - \boldsymbol{w}_{TD} \\ &= (\mathrm{Id} - \alpha\boldsymbol{A})(\boldsymbol{w}_t - \boldsymbol{w}_{TD}) \end{aligned}$$

If we prove that $\boldsymbol{A}$ is positive definite then $\boldsymbol{A}$ will be invertible and the asymptotic algorithm will converge provided α is small enough.

In the continuous task setting,

$$\begin{aligned}
\boldsymbol{A} &= \sum_s \mu(s) \sum_a \pi(a|s) \sum_{r,s'} p(r, s'|s, a) \boldsymbol{x}(s)(\boldsymbol{x}(s) - \gamma \boldsymbol{x}(s'))^t \\
&= \sum_s \mu(s) \sum_a \pi(a|s) \sum_{s'} p_\pi(s'|s) \boldsymbol{x}(s)(\boldsymbol{x}(s) - \gamma \boldsymbol{x}(s'))^t \\
&= \sum_s \mu(s) \boldsymbol{x}(s) \left(\boldsymbol{x}(s) - \gamma \sum_{s'} p_\pi(s'|s) \boldsymbol{x}(s') \right)^t \\
&= \boldsymbol{X}^t \boldsymbol{D} (\mathrm{Id} - \gamma P_\pi) \boldsymbol{X}
\end{aligned}$$

where D is a diagonal matrix having $\mu(s)$ on the diagonal.

As P_π is a stochastic matrix, the row sums of $\boldsymbol{D}(\mathrm{Id} - \gamma P_\pi)$ are non negative. Recall that μ is such that $\mu^t P_\pi = \mu^t$ and thus

$$\begin{aligned}
\mathbf{1}^t \boldsymbol{D}(\mathrm{Id} - \gamma P_\pi) &= \mu^t (\mathrm{Id} - \gamma P_\pi) \\
&= \mu^t - \gamma \mu^t P_\pi \\
&= (1 - \gamma) \mu^t > 0
\end{aligned}$$

□