Reinforcement Learning
Reinforcement Learning: Prediction and Planning in the Tabular Setting

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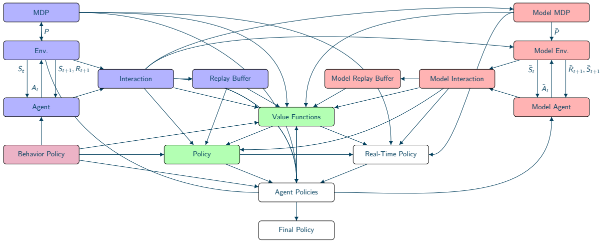
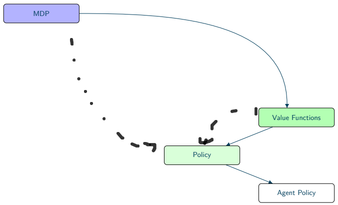
RL: What Are We Going To See?



Outline

- Operations Research and MDP.
- Reinforcement learning and interactions.
- More tabular reinforcement learning.
- Reinforcement and approximation of value functions.
- Actor/Critic: a Policy Point of View

Operations Research and MDP

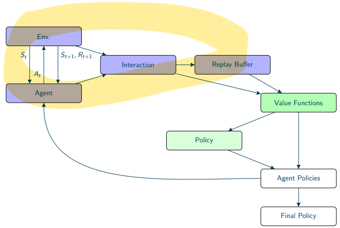


How to find the best policy knowing the MDP?

- Is there an optimal policy?
- How to estimate it numerically?

- Finite states/actions space assumption (tabular setting).
- Focus on interative methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Focus on the discounted setting.

Reinforcement Learning and Interactions



How to find the best policy not knowing the MDP?

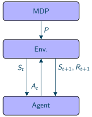
- How to interact with the environment to learn a good policy?
- Can we use a Monte Carlo strategy outside the episodic setting?
- How to update value functions after each interaction?

- Focus on stochastic methods using tabular value functions (Q learning, SARSA…)
- Policy deduced by a statewise optimization of the value function over the actions.

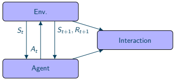
Outline

Reinforcement Learning

From



to



From Probability to Statistics?

- What to do if one has no knowledge of the underlying MDP?
- Only information through interactions!
- Prediction? Planning?
- Focus on the discounted setting

Outline

Monte Carlo, i.e. Just Play!

T time steps
N episodes

- Most simple way to evaluate a policy.

Just Play Following Policy Π

- Play N episodes following the policy.
- During each episode, compute the (discounted) gain.
- Compute the average gain.

- What is computed?

Average Gain or Value Function

$$\mathbb{E}[G_0] \quad \text{vs} \quad v_{t,\Pi}(s) = \mathbb{E}[G_t | S_t = s]$$

Prediction as Value Function Evaluation

- Not the same goal.
- By construction,

$$\mathbb{E}[G_0] = \sum_s \mu_0(s) v_{0,\Pi}(s)$$

- Much easier to compute the average gain than the value function (even if we use a stationary policy)

- Average gain is nevertheless the most classical way to evaluate a policy (with a single number).
- Implicit episodic setting if we do not want to use approximated gain.

Average Gain Estimation

Episodic: Evaluation by MC

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of episodes N
init: $V = 0$,$n = 0$
repeat
- $n \leftarrow n + 1$
- $t \leftarrow 0$
- $G \leftarrow 0$
- Pick initial state S_0 following μ_0
- **repeat**
  - Pick action A_t according to $\pi(\cdot|S_t)$
  - $G \rightarrow G + \gamma^t R_{t+1}$
  - $t \leftarrow t + 1$
- **until** *episod ends at time* T
- $V \leftarrow V + G$

until $n == N$
$V \leftarrow V/N$
output: Average gain V

Monte Carlo Prediction

- How to estimate $v_{t,\Pi}$?

Just Play Following Policy Π

- Play N episodes following the policy.
- During episode, record S_t and R_t.
- After each episode, compute recursively for each time t the gain G_t.
- Estimate $v_{t,\Pi}(s)$ by the average G_t over all trajectories such that $S_t = s$

- **May require a lot of game to have a non empty set for each state s at each time t**

Monte Carlo Prediction

- How to estimate v_Π for a stationary policy?

Just Play Following Policy Π

- Play N episodes following the policy.
- During each episode, record S_t and R_t.
- After each episode, compute recursively for each time t the gain G_t.
- Estimate $v_\Pi(s)$ by the average over all trajectories of all G_t such that $S_t = s$, whatever t.

- The same state may be reached several time during a single episode...
- First-visit variant: Use only the first visit of s for each episode.

Monte Carlo Prediction

Episodic: Prediction by MC

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of episodes N
init: $\forall s, V(s), n = 0, N(s) = 0$
repeat
- $n \leftarrow n + 1$
- $t \leftarrow 0$
- Pick initial state S_0 following μ_0
- **repeat**
  - (If First-visit) $N(S_t) \leftarrow N(S_t) + 1$
  - Pick action A_t according to $\pi(\cdot|S_t)$
  - Record R_{t+1}, S_{t+1}
  - $t \leftarrow t + 1$
- **until** *episod ends at time* T
- $G_{T+1} = 0$
- $t \rightarrow T + 1$
- **repeat**
  - $t \leftarrow t - 1$
  - Compute $G_t = R_{t+1} + \gamma G_{t+1}$
  - (If First-visit) $V(S_t) = V(S_t) + G_t$
- **until** $t = 0$

until $n == N$
for $s \in \mathcal{S}$ **do**
- $V(s) \leftarrow V(s)/N(s)$

end
output: Value function V

Monte Carlo Prediction Analysis

First-Visit Variant Analysis

- Straightforward analysis as all the used values for a given state s are independent.
- Variance of order $1/N(s)$ where $N(s)$ is the number of episods where s is visited.
- Convergence if the number of visit goes to ∞.
- Strong assumption is practice as some states may not be visited by a given policy (if we cannot play on the initial state).

- Every-visit works. . . but not necessarily better!

Outline

Monte Carlo Planning

- Can we use a MC approach to find a good policy?

A First Attempt

- Estimate $v_\pi(s)$ by $V_\pi(s)$ using MC.
- Compute $Q_\pi(s,a) = r(s,a) + \gamma \sum_{s'} p(s'|s,a) V_\pi(s)$
- Enhance the current policy by setting $\pi(s) = \operatorname{argmax}_a Q_\pi(s,a)$

- Inspired by the Operations Research results. . .
- But unusable as r and p are unknown!

Monte Carlo Planning

A Second Attempt

- Estimate $q_\pi(s, a)$ by $Q_\pi(s, a)$ using MC.
- Enhance the current policy by setting $\pi(s) = \operatorname{argmax}_a Q_\pi(s, a)$

- Requires that $N(s, a)$ the number of times that an episode contains the state s followed by action a goes to ∞.
- Impossible with a deterministic policy!

Monte Carlo Planning

Classical Exploratory Policies...

- Stochastic policies ensuring that any action can occurs at any state.
- ϵ-exploratory policy: use a determistic policy and replace it with a random action with probability ϵ.
- Gibbs policy: use a policy where $\pi(a|s) \propto e^{Q(a,s)} > 0$.

A Final Attempt

- Start from an exploratory policy.
- Estimate $q_\pi(s,a)$ by $Q_\pi(s,a)$ using MC.
- Enhance the current policy while remaining a exploratory policy.

- Last step is not straightforward...
- except for ϵ-deterministic policy for which the ϵ-exploratory policy with base policy $\pi(s) = \operatorname{argmax}_a Q_\pi(s,a)$ works.
- No convergence proof.

Outline

Advanced Implementation of Monte Carlo Prediction

$$X_1 \cdots X_n \rightarrow \overline{X}_n = \frac{1}{n}\sum X_i = \overline{X}_{n-1} + \frac{1}{n}\left(X_n - \overline{X}_{n-1}\right) = \frac{1}{n}X_n + \left(1-\frac{1}{n}\right)\overline{X}_{n-1} = \frac{1}{n}X_n + \frac{n-1}{n}\frac{1}{n-1}\sum_{i=1}^{n-1} X_i = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t))(G_t - V_\pi(S_t))$$

On-Line Monte Carlo

- Average for a given state can be updated each time we have the gain G_t for a state S_t.
- Just use $\alpha(N) = 1/N$ and increment $N(S_t)$.
- No need to record the values between episodes. . .

- We still need to wait until the end of each episode to compute G_t.
- Can we do better?

Advanced MC Prediction

Episodic: Prediction by MC

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of episodes N
init: $\forall s, V(s), n = 0, N(s) = 0$
repeat
  $n \leftarrow n + 1$
  $t \leftarrow 0$
  Pick initial state S_0 following μ_0
  repeat
    (If First-visit) $N(S_t) \leftarrow N(S_t) + 1$
    Pick action A_t according to $\pi(\cdot|S_t)$
    Record R_{t+1}, S_{t+1}
    $t \leftarrow t + 1$
  until *episod ends at time* T
  $G_{T+1} = 0$
  $t \rightarrow T + 1$
  repeat
    $t \leftarrow t - 1$
    Compute $G_t = R_{t+1} + \gamma G_{t+1}$
    (If First-visit) $V(S_t) = V(S_t) + \frac{1}{N(S_t)}(G_t - V(S_t))$
  until $t = 0$
until $n == N$
output: Value function V

- We still need to wait until the end of each episode to compute G_t.
- Can we do better?

Prediction with Temporal Differencies

From $V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t))(G_t - V_\pi(S_t))$

to $V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t)) \underbrace{(R_{t+1} + \gamma V_\pi(S_{t+1}) - V_\pi(S_t))}_{\delta_t}$

Bootstrap Strategy

- Replace G_t by an instantaneous estimate $R_{t+1} + \gamma V_\pi(S_{t+1})$.
- Amounts to replace $\gamma R_{t+2} + \gamma^2 R_{t+3}$ by an approximation of its expectation given S_{t+1}: $v_\pi(S_{t+1})$.
- Bootstrap as we use the current estimate $V_\pi(S_{t+1})$ instead of the true value.
- $\delta_t = R_{t+1} + \gamma V_\pi(S_{t+1}) - V_\pi(S_t)$ is called a temporal difference.

- No need to wait until the end of the episodes!
- Can be used in the discounted setting.

Discounted / Episodic

TD Prediction

Discounted: Prediction by TD

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, V(s), n = 0, N(s) = 0,\ t' = 0$
repeat
- $t \leftarrow 0$
- Pick initial state S_0 following μ_0
- **repeat**
  - $N(S_t) \leftarrow N(S_t) + 1$
  - Pick action A_t according to $\pi(\cdot|S_t)$
  - $V(S_t) \leftarrow V(S_t) + \alpha(N(S_t))\,(R_{t+1} + \gamma V(S_{t+1}) - V(S_t))$
  - $t \leftarrow t + 1$
- **until** *episod ends at time* T' *or* $t' = T$

until $t' = T$
output: Value function V

- **But does this work?**

Prediction with Temporal Differencies

$$\mathbb{E}[\delta_t|S_t] = \mathbb{E}[R_{t+1} + \gamma V_\pi(S_{t+1}) - V_\pi(S_t)|S_t] = (\mathcal{T}^\pi - \mathrm{Id})\, V_\pi(S_t)$$

TD and Bellman Operator

- TD as an approximate Policy Iteration:
$$\mathbb{E}[V_\pi](S_t) \leftarrow V_\pi + \alpha(N(S_t))\,(\mathcal{T}^\pi - \mathrm{Id})\, V_\pi(S_t)$$
- Proof of convergence of this algorithm to a zero of $\mathcal{T}^\pi - \mathrm{Id}$, i.e. the fixed point of $\mathcal{T}^\pi$!
- Proof requires a mild assumption of α (satisfied by $\alpha(N) = 1/N$) and the strong assumption that $N(s)$ goes to ∞.

- MC could be interpreted in a similar way (stochastic approximation) by noticing that $\mathbb{E}[G_t - V_\pi(S_t)|S_t] = v_\pi(S_t) - V_\pi(S_t)$.
- Often use with a constant α

MC vs TD

$$
\begin{aligned}
& V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t))(G_t - V_\pi(S_t)) \\
\text{or} \quad & V_\pi(S_t) \leftarrow V_\pi(S_t) + \alpha(N(S_t)) \underbrace{(R_{t+1} + \gamma V_\pi(S_{t+1}) - V_\pi(S_t))}_{\delta_t}
\end{aligned}
$$

MC vs TD

- Both are based on stochastic approximation.
- Both converges (under similar assumptions) to the correct value function.
- TD does not require to wait until the end of the episode.
- No theorical difference in the speed of convergence but often TD is better...
- Solve different approximate problems when used with a finite set of episodes:
  - MC compute the empirical gain from any state.
  - TD compute the value function of the empirical Bellman operator (the one obtained by using the empirical transition probabilities)

- If V_π is kept constant during an episode

$$G_t - V_\pi(S_t) = \sum \gamma^{t'-t} \delta_t$$

Outline

Stochastic Approximation

$$\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k) \quad \text{with} \quad h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$$
$$\implies \theta_k \to \{\theta, H(\theta) = 0\}$$

Stochastic Approximation

- Family of sequential stochastic algorithm converging to a zero of a function.
- Classical assumptions:
  - $\mathbb{E}[\epsilon_k] = 0$, $\mathbb{V}\text{ar}\,[\epsilon_k] < \sigma^2$, and $\mathbb{E}[\|\eta_k\|] \to 0$,
  - $\sum_k \alpha_k \to \infty$ and $\sum_k \alpha_k^2 < \infty$,
  - the algorithm converges if we replace h_k by H.

- Convergence toward a neighborhood if α is kept constant (as often in practice).
- Most famous example are probably Robbins-Monro and SGD.
- Proof quite technical in general.
- The convergence with H is easy to obtain for a contraction.

Stochastic Approximation and ODE

From $\theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k)$ with $h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$

to $\frac{d\tilde{\theta}}{dt} = H(\tilde{\theta})$

TD = $h = R_{t+1} + \gamma V(S_{t+1}) - V(S_t)$

$H = (\mathcal{T}^\pi - Id) V$

ODE Approach

- General proof showing that the algorithm converges provided the ODE converges.
- Rely on the rewriting the equation

$$\frac{\theta_{k+1} - \theta_k}{\alpha_k} = h_k(\theta_k) = H(\theta_k) + \epsilon_k + \eta_k$$

$\frac{dV}{dt} = H(V) = -(I - \gamma P_\pi)V + r_\pi$

- α_k can be interpreted as a time difference allowing to define a time $t_k = \sum_{t' \leq t} \alpha_k$.
- $\theta(t)$ is piecewise affine and defined through its derivative at time $t \in (t_k, t_{k+1})$.
- This piecewise function remains close to any solution of the ODE starting from θ_k for an arbitrary amount of time provided k is large enough.

- More general proofs based on martingale.

Asynchronous Update

$$\text{From} \quad \theta_{k+1} = \theta_k + \alpha_k h_k(\theta_k) \quad \text{with} \quad h_k(\theta) = H(\theta) + \epsilon_k + \eta_k$$
$$\text{to} \quad \forall i, \theta_{k+1}(i) = \theta_k(i) + \alpha_k(i) h_k(\theta_k)(i)$$

Asynchronous Update

- Componentwise action on θ.
- Not necessarily the same stepsize $\alpha_k(i)$ for all components.
- $\alpha_k(i) = 0$ is permitted!
- Previous results hold provided for every component i, $\sum_k \alpha_k(i) \to \infty$ and $\sum_k \alpha_k^2(i) < \infty$,

- Exact setting of TD approximation!

Outline

Planning with Temporal Differencies

A State Value Function Attempt

- $V_\star$ is the fixed point of $\mathcal{T}^\star$.
- Approximate it as the zero of $\mathcal{T}^\star - \mathrm{Id}$.
- By construction

$$\mathcal{T}^\star v(S_t) = \max_a \mathbb{E}[R_{T+1} + \gamma v(S_{t+1}) | S_t, a]$$

- Not an expectation!

A State-Action Value Function Attempt

- $q_\star$ is the fixed point of $\mathcal{T}^\star$.
- Approximate it as the zero of $\mathcal{T}^\star - \mathrm{Id}$.
- By construction

$$\mathcal{T}^\star q(S_t, A_t) = \mathbb{E}\left[R_{t+1} + \gamma \max_a q(S_{t+1}, a) \middle| S_t, A_t\right]$$

- An expectation!

Q Learning

Discounted: Planning by Q-Learning

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, Q(s,a), N(s,a) = 0$, n=0, $t' = 0$
repeat
- $t \leftarrow 0$
- Pick initial state S_0 following μ_0
- **repeat**
  - $N(S_t) \leftarrow N(S_t) + 1$
  - Pick action A_t according to $\pi(\cdot|S_t)$
  - $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(N(S_t, A_t))\left(R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)\right)$
  - $t \leftarrow t + 1$
  - $t' \leftarrow t' + 1$
- **until** *episod ends at time* T' *or* $t' = T$

until $t' = T$
output: Deterministic policy $\tilde{\pi}(s) = \text{argmax}_a Q(s,a)$

Planning with Q Learning

$$Q(S_t, A_t) = Q(S_t, A_t) + \alpha(N(S_t, A_t)) \left(\underbrace{R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)}_{\delta_t} \right)$$

Q-Learning

- Update is independent of the policy Π.
- Convergence of the Q-value function provided the policy is such that $N(s, a)$ tends to ∞ for any state and any action.
- Implies a convergence of the policy.
- Relies on temporal difference.

- Most classical (tabular) planning algorithm!

Outline

Planning with Policy Improvement

from
$$Q(S_t, A_t) = Q(S_t, A_t) + \alpha(N(S_t, A_t)) \left(\underbrace{R_{t+1} + \gamma \max_a Q(S_{t+1}, a) - Q(S_t, A_t)}_{\delta_t} \right)$$

to
$$Q(S_t, A_t) = Q(S_t, A_t) + \alpha(N(S_t, A_t)) \left(\underbrace{R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)}_{\delta_t} \right)$$

$$\Pi(S_t) = \operatorname*{argmax}_a Q(S_t, a) (\text{plus exploration})$$

Policy Improvement

- More emphasis on the policy with a link between the policy used to play and the optimized policy.
- Almost equivalent to use the current policy in the Q-Learning algorithm.

Discounted

SARSA

Discounted: Planning by SARSA

input: MDP environment, initial state distribution μ_0, policy Π and discount factor γ
parameter: Number of step T
init: $\forall s, a, Q(s,a), N(s,a) = 0$, n=0, $t' = 0$
repeat
- $t \leftarrow 0$ Pick initial state S_0 following μ_0
- **repeat**
  - $N(S_t) \leftarrow N(S_t) + 1$
  - Pick action A_t according to $\pi(\cdot|S_t)$
  - $Q(S_{t-1}, A_{t-1}) \leftarrow Q(S_{t-1}, A_{t-1}) + \alpha(N(S_{t-1}, A_{t-1}))\,(R_t + \gamma Q(S_t, A_t) - Q(S_{t-1}, A_{t-1}))$
  - $\Pi(S_{t-1}) = \operatorname{argmax}_a Q(S_{t-1}, a)$ (plus exploration)
  - $t \leftarrow t + 1$
  - $t' \leftarrow t' + 1$
- **until** *episod ends at time T' or $t' = T$*

until $t' = T$
output: Deterministic policy $\tilde{\pi}(s) = \operatorname{argmax}_a Q(s,a)$

- Does this work?

SARSA and exploration

$$\Pi(S_t) = \operatorname*{argmax}_a Q(S_t, a)(\text{plus exploration})$$

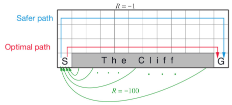
SARSA and Exploration

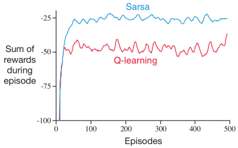
- No hope of convergence if we do not explore all possible actions (and states).
- Impossible if the policy used is deterministic.
- Exploration is required!
- Most classical choice: ϵ-greedy policy with a decaying ϵ.

- Convergence proof is harder than for Q-Learning.
- Relies on the similarity in the limit (when ϵ goes to 0) with the Q-Learning algorithm.

Outline

Q-Learning vs SARSA





How different are they?

- In Q-learning, the exploratory policy used is decoupled from the optimized policy.
- This exploratory policy may yield low rewards on average.
- In SARSA, the two policies are linked with the hope on having higher rewards during the learning step.
- Subtle different behavior even if we modify the exploratory policy in Q-Learning.

Exploration vs Exploitation

Exploration vs Exploitation

- Exploration: explore new policies to be able to discover the best ones.
- Exploitation: use good policies to obtain a good return.
- Exploration is a requirement.

- No tradeoff if we optimize only the final result!
- Tradeoff between the two if we consider that the returns during training matters!
- Q-learning use the first approach and SARSA try to tackle the second.
- Tradeoff if we study a regret:

$$\sum_t \left[\mathbb{E}_{\Pi_\star}[R_t] - \mathbb{E}_{\Pi_t}[R_t] \right]$$

  which forces us to be good as fast as possible.
- No natural definition in the discounted setting.

Outline

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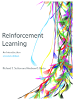

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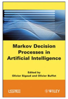

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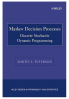

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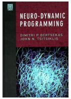

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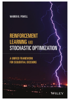

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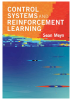

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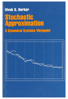

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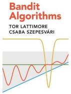

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