

Reinforcement Learning

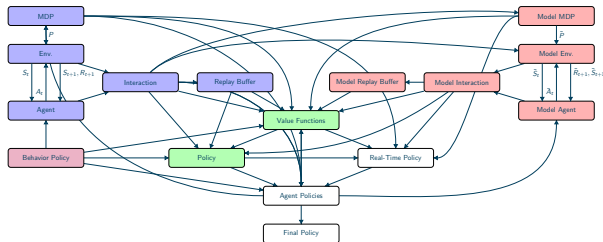
Reinforcement Learning: Advanced Techniques in the Tabular Setting

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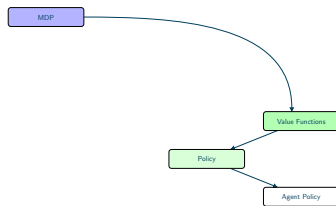


RL: What Are We Going To See?



Outline

- Operations Research and MDP.
- Reinforcement learning and interactions.
- More tabular reinforcement learning.
- Reinforcement and approximation of value functions.
- Actor/Critic: a Policy Point of View



How to find the best policy knowing the MDP?

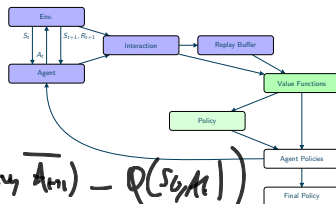
- Is there an optimal policy?
- How to estimate it numerically?
- Finite states/actions space assumption (tabular setting).
- Focus on iterative methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Focus on the discounted setting.

$$Q \leftarrow Q + \alpha (\tau Q - Q)$$

$$Q(s_t, A_t) \leftarrow Q(s_t, A_t)$$

$$+ \alpha \left(R_{t+1} + \gamma Q(s_{t+1}, \bar{A}_{t+1}) - Q(s_t, A_t) \right)$$

δ_t



$$\tau Q$$

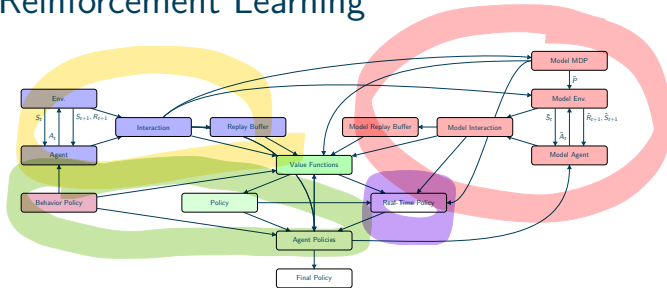
$$E[R_{t+1} + \gamma Q(s_{t+1}, A_{t+1})]$$

$$\tau^* = E[R_{t+1} + \gamma \max_a Q(s_t, a)]$$

How to find the best policy not knowing the MDP?

- How to interact with the environment to learn a good policy?
- Can we use a Monte Carlo strategy outside the episodic setting?
- How to update value functions after each interaction?
- Focus on stochastic methods using tabular value functions (Q learning, SARSA...)
- Policy deduced by a statewise optimization of the value function over the actions.

More Tabular Reinforcement Learning



Can We Do Better?

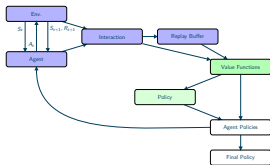
- Is there a gain to wait more than one step before updating?
- Can we interact with a different policy than the one we are estimating?
- Can we use an estimated model to plan?
- Can we plan in real-time instead of having to do it beforehand?
- Finite states/actions space setting (tabular setting).

Outline

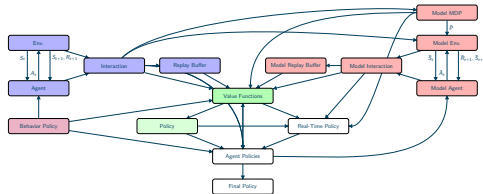
- 1 n -step Algorithms
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Advanced Tabular Reinforcement Learning

From



to

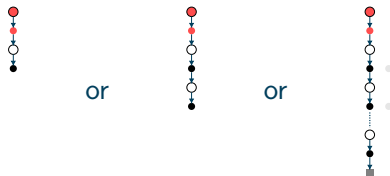


- Core idea: Approximate Bellman Operators with Stochastic Approximation. . .

Advanced Ideas?

- Between MC and TD?
- Off-policy vs on-policy?
- Exploration vs Exploitation?
- Model? Replay?
- Real-Time Planning?

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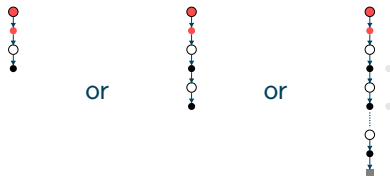
How many steps before backup?

- One step: TD.
- As many steps as required to end the episode: MC.
- n -steps: n -steps TD.

$$(\mathcal{T}^\Pi)^n v(s) = \mathbb{E}_\Pi \left[\underbrace{R_{t+1} + \gamma R_{t+2} + \gamma^{n-1} R_{t+n} + \gamma^n v(S_{t+n})}_{G_{t:t+n}} \middle| S_t = s \right]$$

- Family of stochastic approximation algorithms:

$$V(S_t) \leftarrow V(S_t) + \alpha(N(S_t))(G_{t:t+n} - V(S_t))$$



$$V(S_t) \leftarrow V(S_t) + \alpha(N(S_t)) (G_{t:t+n} - V(S_t))$$

n -steps TD

- Convergence for prediction.
 - Need to be combined with Policy Improvement for planning: n -steps SARSA.
 - n -steps Q -learning could be an extension of API... but this means following the optimized policy Π ... i.e. SARSA!
-
- Best convergence often for intermediate n .
 - No proof beside TD for $n > 1$!

Discounted: Prediction by n -steps TD

input: MDP environment, initial state distribution μ_0 , policy Π and discount factor γ

parameter: Number of step T

init: $\forall s, a, Q(s, a), N(s, a) = 0, n=0, t' = 0$

repeat

$t \leftarrow 0$

 Pick initial state S_0 following μ_0

repeat

$N(S_t) \leftarrow N(S_t) + 1$

 Pick action A_t according to $\pi(\cdot|S_t)$

$Q(S_{t-n}, A_{t-n}) \leftarrow Q(S_{t-n}, A_{t-n}) + \alpha(N(S_t, A_t)) (G_{t-n:t} - Q(S_t, A_t))$

$t \leftarrow t + 1$

$t' \leftarrow t' + 1$

until *episod ends at time T' or $t' = T$*

until $t' = T$

output: State-Action value function Q



Expected SARSA

- The policy Π is known so that we can use it in a formula:

$$R_t + \gamma Q(S_t, A_t) \longrightarrow R_t + \gamma \sum_a \pi(a|S_t) Q(S_t, a)$$

- Make the update independent of the action chosen (and thus of the policy used to play).
- Reduce the variance for a computational cost.
- Amount to use the current estimate for $V(S_t)$...

Discounted: Prediction by Expected SARSA

input: MDP environment, initial state distribution μ_0 , policy Π and discount factor γ

parameter: Number of step T

init: $\forall s, a, Q(s, a), N(s, a) = 0, n=0, t' = 0$

repeat

$t \leftarrow 0$

Pick initial state S_0 following μ_0

repeat

$N(S_t) \leftarrow N(S_t) + 1$

Pick action A_t according to $\pi(\cdot|S_t)$

$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha(N(S_t, A_t)) (R_{t+1} + \gamma \sum_a \pi(a|S_t) Q(S_{t+1}, a) - Q(S_t, A_t))$

$t \leftarrow t + 1$

$t' \leftarrow t' + 1$

until *episod ends at time T' or $t' = T$*

until $t = T$

output: State-Action value function Q



n -steps Tree Backup

- At each time step, use the expected SARSA average over the action while replacing the Q value for the picked action by a deeper estimate.

- 1-step return (Expected Sarsa)

$$G_{t:t+1} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1})Q(S_{t+1}, a)$$

- 2-step return:

$$\begin{aligned} G_{t:t+2} &= R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1})Q_{t+1}(S_{t+1}, a) \\ &\quad + \gamma \pi(A_{t+1}|S_{t+1}) \left(R_{t+2} + \gamma \sum_a \pi(a|S_{t+2})Q(S_{t+2}, a) \right) \\ &= R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1})Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1})G_{t+1:t+2} \end{aligned}$$

- 1-step return (Expected Sarsa)

$$G_{t:t+1} = R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a)$$

- 2-step return:

$$\begin{aligned} G_{t:t+2} &= R_{t+1} + \gamma \sum_{a \neq A_{t+1}} \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) G_{t+1:t+2} \\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) + \gamma \pi(A_{t+1}|S_{t+1}) (G_{t+1:t+2} - Q(S_{t+1}, A_{t+1})) \end{aligned}$$

- Recursive definition of n -step return:

$$\begin{aligned} G_{t:t+n} &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) \\ &\quad + \gamma \pi(A_{t+1}|S_{t+1}) (G_{t+1:t+n} - Q(S_{t+1}, A_{t+1})) \end{aligned}$$

- TD update

$$Q(S_{t-n}, A_{t-n}) = Q(S_{t-n}, A_{t-n}) + \alpha(N(S_{t-n}, Q_{t-n})) (G_{t-n:t} - Q(S_{t-n}, A_{t-n}))$$

$$\mathbb{E}[G_{t-n:t}] Q = (\gamma^n)^n Q$$

Between



and



Sampling or Averaging

- Unifying algorithm!
- Recursive definition of n -step return:

$$\begin{aligned} G_{t:t+n} = & R_{t+1} + \sigma G_{t+1:t+n} \\ & + (1 - \sigma) \left(\gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) \right. \\ & \left. + \gamma \pi(A_{t+1}|S_{t+1}) (G_{t+1:t+n} - Q(S_{t+1}, A_{t+1})) \right) \end{aligned}$$

Averaged n -steps return?

- n -step return:

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

- Averaged n -step return: (compound update)

$$G_t^\omega = \sum_{n=1}^{\infty} \omega_n G_{t:t+n} \quad \text{with} \quad \sum_{n=1}^{\infty} \omega_n = 1$$

- TD(λ): specific averaging

$$G_t^\lambda = (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} G_{t:t+n}$$

$$= (1 - \lambda) \sum_{n=1}^{T-t} \lambda^{n-1} G_{t:t+n} + \lambda^{T-t} G_t \quad (\text{Episodic})$$

$$\omega_n = (1-\lambda)\lambda^{n-1}$$
$$\sum \omega_n = 1$$

interpolating between TD (a.k.a TD(0)) and MC for $\lambda = 1$.

- Can be mixed with tree backup strategies (TB(λ))

True λ -return

- Require to wait until the end of an episode before we can update.
- Unusable in a non episodic setting!

Truncated λ -return

- Truncated λ -return:

$$G_t^\lambda = (1 - \lambda) \sum_{n=1}^{H-t} \lambda^{n-1} G_{t:t+n} + \lambda^{H-t} G_{t:H}$$

- The virtual horizon H may vary during the algorithm.

Temporality

- n -step return

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} + \gamma^n V(S_{t+n})$$

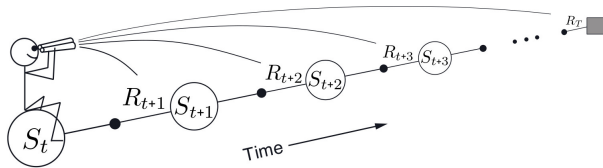
depends on a current estimate V (or Q)!

- In G_λ should we use
 - an estimate available at time t ?
 - an estimate available at time $t + n$?
 - an estimate available at time H ?
- Off-Line vs On-Line!
 - Off-line: keep V constant during the episodes.
 - On-line: Used updated V when available.
 - True on-line (Sutton and Barto): restart algorithm with a growing horizon.

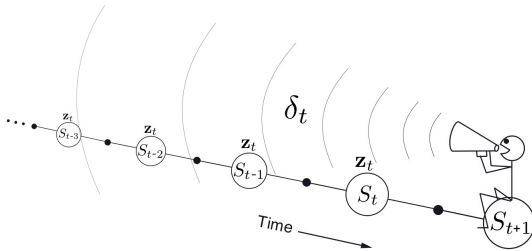
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Forward and Backward Point of View

Eligibility Traces



From a forward view



To a backward one:

Returns and Temporal Differences

- n -step returns:

$$\begin{aligned} G_{t:t+n} - Q(S_t, A_t) &= R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n} \\ &\quad + \gamma^n Q(S_{t+n}, A_{t+n}) - Q(S_t, A_t) \\ &= \sum_{l=1}^n \gamma^{l-1} (R_{t+l} + \gamma Q(S_{t+l}, A_{t+l}) - Q(S_{t+l-1}, A_{t+l-1})) \\ &= \sum_{l=0}^{n-1} \gamma^{l-1} \delta_{t+l} \end{aligned}$$

- λ return:

$$\begin{aligned} G_t^\lambda - Q(S_t, A_t) &= (1 - \lambda) \sum_n \lambda^n (G_{t:t+n} - Q(S_t, A_t)) \\ &= \sum_{n=0} \lambda^n \gamma^n \delta_{t+n} \end{aligned}$$

Forward View

- Updates:

$$Q_t(s, a) = Q_{t-1}(s, a) + \mathbf{1}_{(s,a)=(S_t,A_t)} \alpha_t(s, a) \left(\sum_{t'' \geq t} \lambda^{t''-t} \gamma^{t''-t} \delta_{t''} \right)$$

- Cumulative updates:

$$Q_t(s, a) = Q_0(s, a) + \sum_{t' \leq t} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s, a) \left(\sum_{t'' \geq t'} \lambda^{t''-t'} \gamma^{t''-t'} \delta_{t''} \right)$$

- Limit:

$$Q_\infty(s, a) = Q_0(s, a) + \sum_{t'} \mathbf{1}_{(s,a)=(S_{t'},A_{t'})} \alpha_{t'}(s, a) \left(\sum_{t'' \geq t'} \lambda^{t''-t'} \gamma^{t''-t'} \delta_{t''} \right)$$

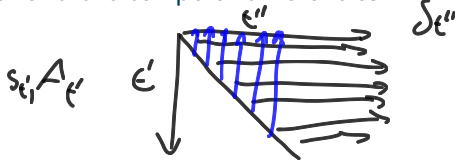
- Focus on the update place.

Limit(s)

- Limit:

$$\begin{aligned}
 Q_{\infty}(s, a) &= \underbrace{Q_0(s, a) + \sum_{t'} \mathbf{1}_{(s,a)=(S_{t'}, A_{t'})} \alpha_{t'}(s, a)}_{\text{Forward View}} \left(\sum_{t'' \geq t'} \lambda^{t''-t'} \gamma^{t''-t'} \delta_{t''} \right) \\
 &= \underbrace{Q_0(s, a)}_{\text{Backward View}} + \sum_{t''} \delta_{t''} \underbrace{\sum_{t' \leq t''} \mathbf{1}_{(s,a)=(S_{t'}, A_{t'})} \alpha_{t'}(s, a) \lambda^{t''-t'} \gamma^{t''-t'}}_{\text{Eligibility Trace}}
 \end{aligned}$$

- Focus on the update place or and the temporal differences...



Backward View

- Same limit with cumulative updates over temporal differences

$$Q_t(s, a) = Q_0(s, a) + \sum_{t'' \leq t} \delta_{t''} \sum_{t' \leq t''} \mathbf{1}_{(s,a)=(S_{t'}, A_{t'})} \alpha_{t'}(s, a) \lambda^{t''-t'} \gamma^{t''-t'}$$

- Updates

$$Q_t(s, a) = Q_{t-1}(s, a) + \underbrace{\delta_t \sum_{t' \leq t} \mathbf{1}_{(s,a)=(S_{t'}, A_{t'})} \alpha_{t'}(s, a) \lambda^{t-t'} \gamma^{t-t'}}_{z_t(s, a)}$$

- Pseudo Eligibility trace:

$$\begin{aligned} z_t(s, a) &= \sum_{t' \leq t} \mathbf{1}_{(s,a)=(S_{t'}, A_{t'})} \alpha_{t'}(s, a) \lambda^{t-t'} \gamma^{t-t'} \\ &= \lambda \gamma z_{t-1}(s, a) + \alpha_t(s, a) \mathbf{1}_{(s,a)=(S_t, A_t)} \end{aligned}$$

- Proof of convergence toward the same target.

$$Q_t(s, a) = Q_{t-1}(s, a) + \alpha_t \delta_t z_t(s, a)$$

Eligibility Trace

- Focus on temporal differences with simultaneous update on all states.
- TD(λ) eligibility trace: $z_t(s, a) = \lambda \gamma z_{t-1}(s, a) + \mathbf{1}_{(s,a)=(S_t,A_t)}$
- Strictly equivalent to the previous scheme for constant stepsize
- Other eligibility trace:

- Replacing trace:

$$z_t(s, a) = \begin{cases} 1 & \text{if } (s, a) = (S_t, A_t) \\ \lambda \gamma z_{t-1}(s, a) & \text{otherwise} \end{cases}$$

- Time dependent trace:

$$z_t(s, a) = c_t \gamma z_{t-1}(s, a) + \mathbf{1}_{(s,a)=(S_t,A_t)}$$

where c_t is defined *in a appropriate way* to ensure the convergence of the algorithm.

- Need to store (and update) this information. . .

$\delta_t?$

Temporal Differences

- Basic temporal differences:

$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$

- Expected temporal differences:

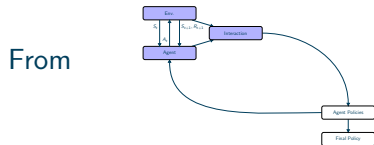
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V(S_{t+1}) - Q(S_t, A_t) \\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\end{aligned}$$

- Average of both:

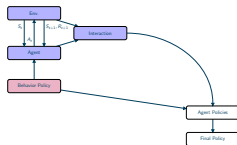
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma \sigma Q(S_{t+1}, A_{t+1}) + \gamma(1 - \sigma)V(S_{t+1}) - Q(S_t, A_t) \\ &= R_{t+1} + \gamma V(S_{t+1}) + \gamma \sigma (Q(S_{t+1}, A_{t+1}) - V(S_{t+1})) - Q(S_t, A_t)\end{aligned}$$

- Only expected temporal average is independent of the next action.
- No generic proof of convergence...

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to



On-Policy vs Off-Policy

- On-Policy: the policy b used to interact is the same than the policy Π evaluated or optimized.
 - Off-Policy: the policy b used to interact may be different from the policy Π evaluated or optimized.
-
- Off-Policy allows in particular to (re)use interactions from previous experiments.
 - Q-learning was possible in off-policy setting.

Importance Sampling

Off-policy vs on-policy



$$p(S_t, A_t, R_{t+1}, S_{t+1}, A_{t+1} | S_t) = \pi(A_t | S_t) p(R_{t+1}, S_{t+1} | S_t, A_t) \pi(A_{t+1} | S_{t+1})$$

$$\rho_{t:t'} = \frac{\mathbb{P}_{\pi}(S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'} | S_t)}{\mathbb{P}_b(S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'} | S_t)} = \frac{\pi(A_t | S_t) \dots \pi(A_{t'} | S_{t'})}{b(A_t | S_t) \dots b(A_{t'} | S_{t'})}$$

Importance Sampling

- For any law p and q , and any function g

$$\mathbb{E}_p[g(x)] = \mathbb{E}_q \left[\frac{p(x)}{q(x)} g(x) \right]$$

provided $q(x) = 0$ implies $p(x) = 0$.

- $\mathbb{V}_{ar}_q \left[\frac{p(x)}{q(x)} g(x) \right]$ may be large with respect to $\mathbb{V}_{ar}_p [g(x)]$ if the ratio $p(x)/q(x)$ is large...

$$\begin{aligned} \mathbb{E}_p[g(x)] &= \sum p(x) g(x) \\ &= \sum q(x) \frac{p(x)}{q(x)} g(x) \\ &= \mathbb{E}_q \left[\frac{p(x)}{q(x)} g(x) \right] \end{aligned}$$

Importance Sampling for Trajectories

- For any trajectory $\tau_{t:t'} = S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'} (, R_{t'+1}, S_{t'+1}), ,$

$$\frac{\mathbb{P}_{\pi}(S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'} (, R_{t'+1}, S_{t'+1}) | S_t)}{\mathbb{P}_b(S_t, A_t, R_{t+1}, S_{t+1}, \dots, R_{t'}, S_{t'}, A_{t'} (, R_{t'+1}, S_{t'+1}) | S_t)} = \frac{\pi(A_t | S_t) \dots \pi(A_{t'} | S_{t'})}{b(A_t | S_t) \dots b(A_{t'} | S_{t'})}$$

$$\mathbb{E}_{\Pi}[g(\tau_{t:t'})|S_t = s] = \mathbb{E}_b[\rho_{t:t'}g(\tau_{t:t'})|S_t = s] \quad \text{with} \quad \rho_{t:t'} = \frac{\pi(A_t|S_t) \dots \pi(A_{t'}|S_{t'})}{b(A_t|S_t) \dots b(A_{t'}|S_{t'})}$$

From b to Π

- Returns:

$$\begin{aligned}\mathbb{E}_{\pi}[G_{t:t'}|S_t = s] &= \mathbb{E}_{\pi}\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1}R_{t''} + \gamma^{t'-t}V(S_{t'}) \middle| S_t = s\right] \\ &= \mathbb{E}_b\left[\rho_{t:(t'-1)} \left(\sum_{t''=t+1}^{t'} \gamma^{t''-t-1}R_{t''} + \gamma^{t'-t}V(S_{t'})\right) \middle| S_t = s\right] \\ &= \mathbb{E}_b\left[\sum_{t''=t+1}^{t'} \rho_{t:(t''-1)}\gamma^{t''-t-1}R_{t''} + \rho_{t:(t'-1)}\gamma^{t'-t}V(S_{t'}) \middle| S_t = s\right]\end{aligned}$$

$$\mathbb{E}_{\Pi}[g(\tau_{t:t'})|S_t, A_t] = \mathbb{E}_b[\rho_{(t+1):t'} g(\tau_{t:t'})|S_t, A_t] \quad \text{with} \quad \rho_{t:t'} = \frac{\pi(A_t|S_t) \dots \pi(A_{t'}|S_{t'})}{b(A_t|S_t) \dots b(A_{t'}|S_{t'})}$$

From b to Π

- Returns:

$$\begin{aligned} \mathbb{E}_{\pi}[G_{t:t'}|S_t, A_t] &= \mathbb{E}_{\pi}\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} Q(S_{t'}, A_{t'}) \middle| S_t, A_t\right] \\ &= \mathbb{E}_b\left[\rho_{(t+1):(t'-1)} \left(\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} R_{t''} + \gamma^{t'-t} Q(S_{t'}, A_{t'})\right) \middle| S_t, A_t\right] \\ &= \mathbb{E}_b\left[\sum_{t''=t+1}^{t'} \gamma^{t''-t-1} \rho_{(t+1):(t''-1)} R_{t''} + \rho_{(t+1):t'} \gamma^{t'-t} Q(S_{t'}, A_{t'}) \middle| S_t, A_t\right] \end{aligned}$$

- No correction if $t' = t + 1$

λ -return

- Recursive definition of the λ -return:

$$G_t^\lambda | S_t = R_{t+1} + \gamma \left((1 - \lambda) V(S_{t+1}) + \lambda G_{t+1}^\lambda \right)$$

$$G_t^\lambda | S_t, A_t = R_{t+1} + \gamma \left((1 - \lambda) \left(\sigma Q(S_{t+1}, A_{t+1}) + (1 - \sigma) \left(\sum_a \pi(a | S_{t+1}) Q(S_{t+1}, a) + \pi(A_{t+1} | S_{t+1}) \left(G_{t+1}^\lambda - Q(S_{t+1}, A_{t+1}) \right) \right) \right) + \lambda G_{t+1}^\lambda \right)$$

- Off-line correction

$$G_t^\lambda | S_t = \rho_{t:t} \left(R_{t+1} + \gamma \left((1 - \lambda) V(S_{t+1}) + \lambda G_{t+1}^\lambda \right) \right)$$

$$G_t^\lambda | S_t, A_t = R_{t+1} + \gamma \left((1 - \lambda) \left(\sigma Q(S_{t+1}, A'_{t+1}) + (1 - \sigma) \left(\sum_a \pi(a | S_{t+1}) Q(S_{t+1}, a) + \pi(A_{t+1} | S_{t+1}) \left(G_{t+1}^\lambda - Q(S_{t+1}, A_{t+1}) \right) \right) \right) + \lambda \rho_{t+1:t+1} G_{t+1}^\lambda \right)$$

where A'_{t+1} is drawn following π (or multiply by $\rho_{t+1:t+1}$ to use A_{t+1}).

$$\delta_t?$$

Temporal Differences

- Basic temporal differences:

$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A'_{t+1}) - Q(S_t, A_t)$$

with A'_{t+1} drawn using π .

- Expected temporal differences:

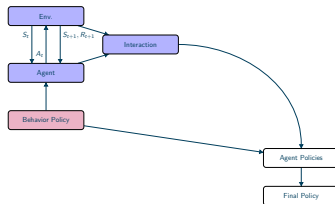
$$\begin{aligned}\delta_t &= R_{t+1} + \gamma V(S_{t+1}) - Q(S_t, A_t) \\ &= R_{t+1} + \gamma \sum_a \pi(a|S_{t+1}) Q(S_{t+1}, a) - Q(S_t, A_t)\end{aligned}$$

without any correction.

- Average of both:

$$\begin{aligned}\delta_t &= R_{t+1} + \gamma \sigma Q(S_{t+1}, A_{t+1}) + \gamma(1 - \sigma)V(S_{t+1}) - Q(S_t, A_t) \\ &= R_{t+1} + \gamma V(S_{t+1}) + \gamma \sigma (Q(S_{t+1}, A'_{t+1}) - V(S_{t+1})) - Q(S_t, A_t)\end{aligned}$$

with A'_{t+1} drawn using π .



Off-Policy Correction

- Replace any estimate of the gain by an importance-sampling corrected one.
- Works well for prediction.
- Can be combined with policy improvement (a la SARSA) but less (no?) theoretical guarantees.

$$\tilde{\mathcal{T}}Q(s, a) = Q(s, a) + \mathbb{E}_b \left[\sum_{t \geq 0} \gamma^t \left(\prod_{t'=1}^t c_{t'} \right) \delta_t \middle| S_0 = s, A_0 = a \right]$$

$$c_t = c(A_t, S_t, A_{t-1}, S_{t-1}, \dots, A_0, S_0)$$

$$\mathbb{E}_b[\delta_t | S_t, A_t] = \mathbb{E}[R_{t+1} + \gamma \mathbb{E}_\pi[Q(S_{t+1}, \cdot)] - Q(S_t, A_t) | S_t, A_t]$$

Generic Off-Policy Algorithm

- Generic off-line algorithm including
 - Importance sampling: $c_t = \rho_{t:t} = \pi(A_t | S_t) / b(A_t | S_t)$
 - TB(λ): $c_t = \lambda \pi(A_t | S_t)$
 - Retrace(λ): $c_t = \lambda \min(1, \pi(A_t | S_t) / b(A_t | S_t))$
- **Prop:** Q_π is a fixed point as $\mathbb{E}_b[\delta_t | S_t, A_t] = \mathbb{E}[\mathcal{T}^\pi Q(S_t, A_t) - Q(S_t, A_t) | S_t, A_t]$.
- **Prop:** $\tilde{\mathcal{T}}$ is a contraction provided $c_t \leq \rho_t = \pi(A_t | S_t) / b(A_t | S_t)$.
- Convergence for Importance sampling, TB(λ) and Retrace(λ) for any b .
- Partial results for policy improvement under more assumptions.

- For $Q(\lambda)$, $c_t = \lambda$, convergence if $\|\pi(\cdot | s) - b(\cdot | s)\|_1 \leq \epsilon$ and $\lambda \leq (1 - \gamma) / (\gamma \epsilon)$.

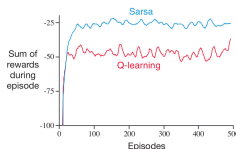
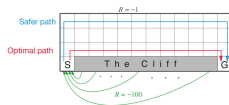
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Q-Learning vs SARSA

Bandits

$$Q^k = \max_{\pi} Q^{\pi}(s, a)$$

$$\hookrightarrow \pi^*$$



$$\pi_1 \dots \pi_T$$

$$E_{\pi_T} (Q_{\pi}(s; a))$$

$$\sum_t E_{\pi_t} (Q_{\pi}(s; a))$$

How different are they?

- In Q-learning, the exploratory policy used is decoupled from the optimized policy.
- This exploratory policy may yield low rewards on average.
- In SARSA, the two policies are linked with the hope on having higher rewards during the learning step.
- Subtle different behavior even if we modify the exploratory policy in Q-Learning.

Exploration vs Exploitation

- Exploration: explore new policies to be able to discover the best ones.
 - Exploitation: use good policies to obtain a good return.
 - Exploration is a requirement.
-
- No tradeoff if we optimize only the final result!
 - Tradeoff between the two if we consider that the returns during training matters!
 - Q-learning use the first approach and SARSA try to tackle the second.
 - Tradeoff if we study a regret:
$$\sum_t \mathbb{E}_{\Pi_*}[R_t] - \mathbb{E}_{\Pi_t}[R_t]$$
which forces us to be good as fast as possible.
 - No natural definition in the discounted setting.

$$\mathcal{S} = \{0\} \quad \text{and} \quad A = \{1, \dots, k\} \quad \text{and} \quad r(s, a) = r_a$$

Bandits

- Very simple toy model where there is only one state!
- Optimal policy: pick $a_* \in \operatorname{argmax} r_a$.
- Q estimation: estimate r_a by playing action a .
- Strategy:
 - Every arm has to be played until we are sure they are bad.
 - Best arm should be played as often as possible to maximize the rewards during the learning phase.
- Simple enough setting to obtain result on the regret

$$r_T = \sum_{t \leq T} (r_{a_*} - R_t)$$

- We will use $\Delta_a = r_{a_*} - r_a$ and assume that $R|a$ is 1-subgaussian.

Explore Then Commit (Random Exploration)

- Play the arm successively during ~~k~~ m steps and then play the optimal one during $T - \cancel{k}m$ steps.

- **Prop:**

$$r_T \leq \min(m, T/K) \sum_{a=1}^k \Delta(a) + \max(T - mK, 0) \sum_{a=1}^k \Delta(a) \exp(-m\Delta(a)^2/4)$$

Furthermore,

$$\mathbb{P}(a_T = a_*) \geq 1 - \sum_{a \neq a_*} \exp(-m\Delta(a)^2/4)$$

- $R_T \leq O(\log T)$ for $m \propto \log T$,
- but $R_T = O(T)$ for any fixed m .

ϵ -greedy Strategy

- Estimate $Q(a) = r_a$ by MC:

$$Q_t(a) = \frac{\sum_{t'=1}^{t-1} \mathbf{1}_{A_{t'}=a} R_{t'}}{\sum_{i=1}^{t-1} \mathbf{1}_{A_{t'}=a}}$$

- Pick arm a at time t using

$$\pi(a) = \begin{cases} \epsilon_t/k + (1 - \epsilon) & \text{if } a = \operatorname{argmax}_{a'} Q_t(a') \text{ (only the smallest if necessary)} \\ \epsilon_t/k & \text{otherwise} \end{cases}$$

- **Prop:**

$$r_T \geq \sum_{t=1}^T \frac{\epsilon_t}{k} \sum_{a=1}^k \Delta(a)$$

ϵ -greedy Strategy

- Prop:**

$$\mathbb{P}(A_T = a_*) \geq 1 - \epsilon_T - \sum_t \exp(-\Sigma_T/(6k)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4k)}$$

with $\Sigma_T = \sum_{s=1}^T \epsilon_s$.

Furthermore,

$$\mathbb{P}(a_* = \operatorname{argmax}_a Q_{T,a}) \geq 1 - \sum_t \exp(-\Sigma_T/(6k)) - \sum_{a \neq a_*} \frac{4}{\Delta(a)^2} e^{-\Delta(a)^2 \Sigma_T/(4k)}$$

If $\epsilon_t = c/t$,

$$r_T \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T) + 1}{k} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

as soon as $c/(6k) > 1$ and $c \min_{a \neq a_*} \Delta(a)/4k < 1$.

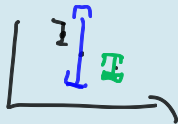
If $\epsilon_t = c \log(t)/t$ then

$$r_T \leq \sum_{a \neq a_*} \left(\Delta(a) \left(c \frac{\log(T)(\log(T) + 1)}{k} + C \right) + \frac{4}{\Delta(a)} C' \right)$$

Upper Confidence Bound

- Use an optimistic strategy to pick the best arm

$$A_t = \operatorname{argmax}_a Q_t(a) + \sqrt{\frac{c \log t}{N_t(a)}}$$



- **Prop:**

$$r_n(t) \leq C_c \sum_a \Delta(a) + \sum_a \frac{4c \ln t}{\Delta(a)}.$$

with $C_c < +\infty$ as soon as $c > 3/2$

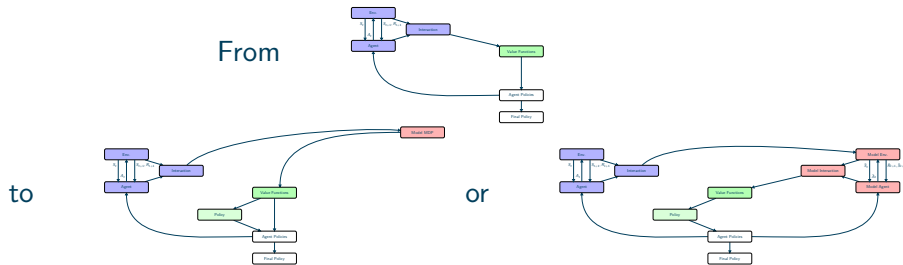
Furthermore

$$\mathbb{P}(A_t = a_*) \geq 1 - 2kt^{-2c+2}$$

as soon as $t \geq \max_a \frac{4c \ln t}{\Delta(a)^2}$.

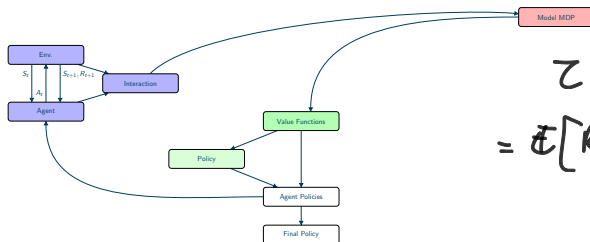
- Optimal regret!
- Hard to extend to RL setting but shows that ϵ -greedy may not be optimal.

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Model Based Approach

- Use the interactions to learn a model. . .
- that can be used to learn a good policy.
- This model can be:
 - a MDP,
 - a simulator.
- Often easier to obtain a simulator.

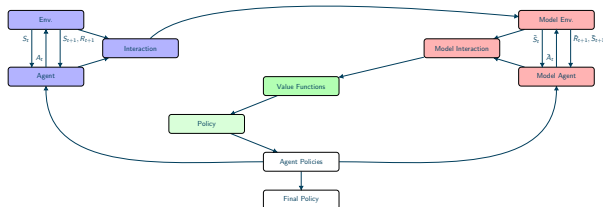


$$TQ - Q$$

$$= E[R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)]$$

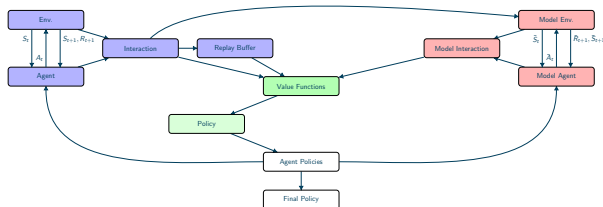
Estimated MDP: back to OR

- MDP can be estimated from trajectories.
- Simple (but maybe slow) even in an off-line setting.
- Once we have an estimated MDP, prediction and planning can be done using OR.
- Implicitly done by TD(0) when doing several passes.
- Model should be checked/improved as much as possible when new trajectories arrive.



Estimated Simulator: back to RL

- Simulator can be estimated from trajectories.
- Simple (but maybe slow) even in an off-line setting.
- Once we have an estimated simulator, prediction and planning can be done using RL.
- Model should be checked/improved as much as possible when new trajectories arrive.



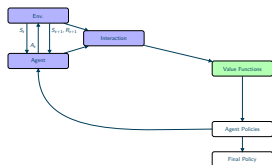
Dyna

- Combine true interactions with simulated ones.
- Simultaneous acting, model learning, OR learning and RL learning.
- Search for a tradeoff between the (slow) learning RL algorithm and the (wrong) model OR algorithm.
- Need to deal with schedule!

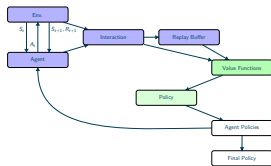
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Replay Buffer and Prioritized Sweeping

From

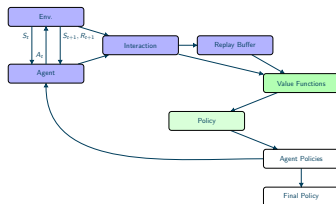


to



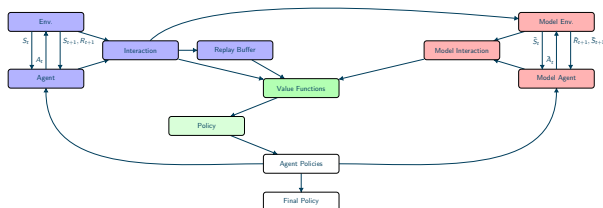
Replay Buffer and Prioritized Sweeping

- Can we reuse previous interactions?
- In which order?



Replay Buffer

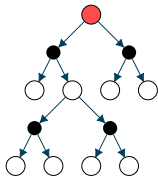
- Store previous interactions (trajectories) in a first-in first-out buffer.
- Draw a subsequence from those interactions (trajectories) and use it in a RL algorithm:
 - On-line: if the trajectory comes from the same policy.
 - Off-line: if the trajectory comes from a different policy.
- Similar to a simulator but no arbitrary choice of state or action.
- Often use with on-line algorithm if the policy has only mildly evolved. . .



Prioritized Sweeping

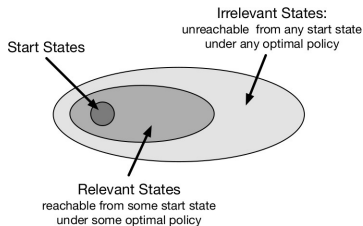
- Plain Replay Buffer: subsequence drawn uniformly.
- Prioritized Sweeping: subsequence drawn favoring states with large temporal differences.
- Can be combined with a model approach.

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Real-Time Planning

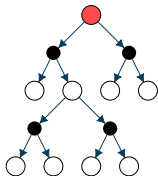
- Can we optimize the policy at the current state?
 - Do we need to optimize it everywhere?
 - What is required?
-
- Planning at decision time...



- Warmup in Dynamic Programming...

RT DP

- Use trajectories to sample the states to update.
 - Convergence holds with exploratory policy.
 - Optimal policy does not require to specify the action in irrelevant states.
 - Convergence holds even without full exploration in some specific cases!
-
- In practice, seems to be computationally efficient.



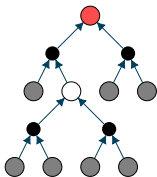
Planning At Decision Time

- Can we find a good action A_t at S_t ... without having it precomputed?
- Policy Improvement

$$A_t = \operatorname{argmax} Q_t(S_t, \cdot)$$

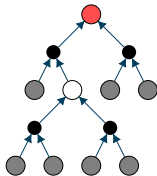
can be seen as a first step.

- How to go deeper?
- **A model or a simulator will be required!**



Heuristic Search

- Requires the knowledge of the MDP and of a heuristic based value function V .
- Strategy:
 - Build a limited depth tree by stopping after a few steps and at some specific states.
 - Backup the heuristic based value function using Dynamic Programming (Optimal Bellman operator).
 - Pick the action having the high value.
- The deeper the better. . . but the more expensive due to branching!
- Requires a *suitable* heuristic. . .



Rollout Policy

- Use a MC estimate with a default policy instead of a heuristic.
 - Backup those estimates using Dynamic Programming.
 - Simulation can even start after the first action (as in Policy Improvement).
-
- The values are (most of the time) discarded for the next state.

- Simultaneous tree growing, rollout and backup by DP.
- Repeat 4 steps:
 - Selection of a sequence of actions according to the current values with a tree policy.
 - Expansion of the tree at the last node without values.
 - Simulation with a rollout policy to estimate the values at this node.
 - Backup of the value by relaxed Dynamic Programming.
- MCTS focuses on promising paths using a UCB approach.



Monte Carlo Tree Search

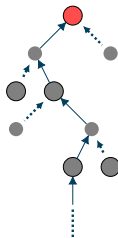
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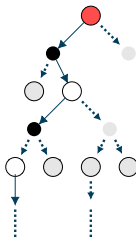
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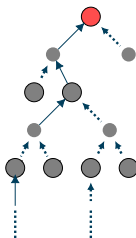
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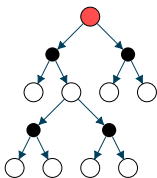
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Model Predictive Control

- Open loop optimization:

$$\max_{a_t, a_{t+1}, \dots, a_{t+h}} \mathbb{E} \left[\sum_{t'=t}^{t+h} R_{t'} \right]$$

using a predictive model (simulator).

- Do not take into account state uncertainties in the control choice. . .
 - But much simpler optimization. . .
 - and equivalence for a linear Gaussian model.
-
- Extensively used for short-term planning in Control.

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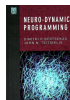
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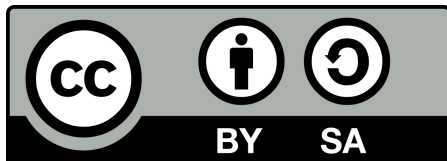
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