Reinforcement Learning
Reinforcement Learning: Policy Approach

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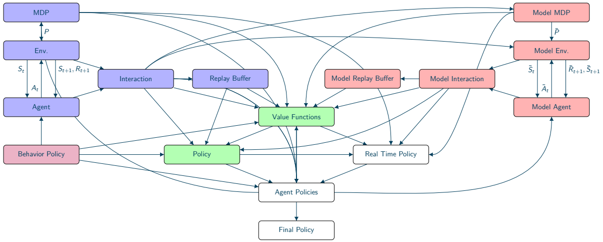
M2DS - Reinforcement Learning – Fall 2023

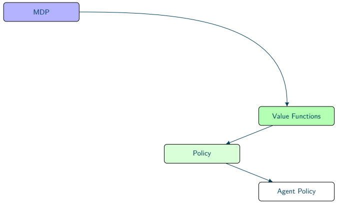
RL: What Are We Going To See?



Outline

- Operations Research and MDP.
- Reinforcement learning and interactions.
- More tabular reinforcement learning.
- Reinforcement and approximation of value functions.
- Actor/Critic: a Policy Point of View

Operations Research and MDP

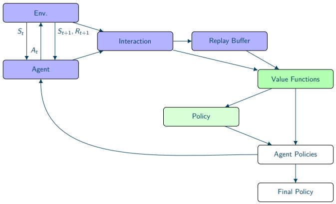


How to find the best policy knowing the MDP?

- Is there an optimal policy?
- How to estimate it numerically?

- Finite states/actions space assumption (tabular setting).
- Focus on interative methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Focus on the discounted setting.

Reinforcement Learning and Interactions

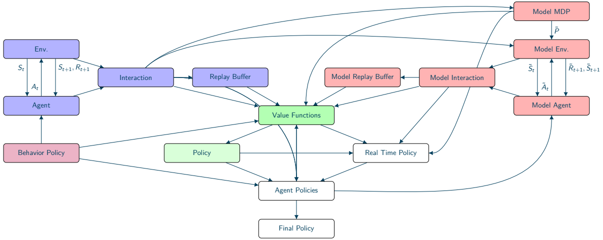


How to find the best policy not knowing the MDP?

- How to interact with the environment to learn a good policy?
- Can we use a Monte Carlo strategy outside the episodic setting?
- How to update value functions after each interaction?

- Focus on stochastic methods using tabular value functions (Q learning, SARSA...)
- Policy deduced by a statewise optimization of the value function over the actions.

More Tabular Reinforcement Learning

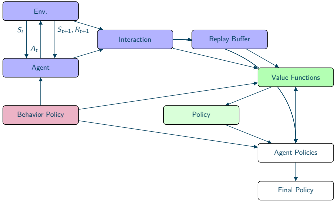


Can We Do Better?

- Is there a gain to wait more than one step before updating?
- Can we interact with a different policy than the one we are estimating?
- Can we use an estimated model to plan?
- Can we plan in real time instead of having to do it beforehand?

- Finite states/actions space setting (tabular setting).

Reinforcement and Approximation of Value Functions

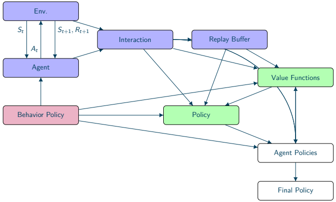


How to Deal with a Large/Infinite states/action space?

- How to approximate value functions?
- How to estimate good approximation of value functions?

- Finite action space setting.
- Stochastic algorithm (Deep Q Learning...).
- Policy deduced by a statewise optimization of the value function over the actions.

Actor/Critic: a Policy Point of View



Could We Directly Parameterized the Policy?

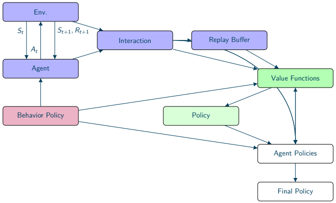
- How to parameterize a policy?
- How to optimize this policy?
- Can we combine parametric policy and approximated value function?

- State Of The Art Algorithms (DPG,PPO, SAC. . .)

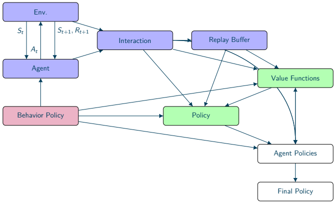
Outline

Policy Point of View

From



to



Policy Point of View

- Optimize policy directely instead of deriving it from a value function.
- Avoid the argmax operator.
- Most natural POV?

- Pontryagin vs Hamilton-Jacobi(-Bellman) in control!

Outline

Policy and Goal

$$J_\mu(\pi) = \sum_s \mu(s) v_\pi(s)$$

Goal: average expected return over the states

- Target used to define the linear programming formulation of an optimal policy in the tabular setting.
- μ can be the initial distribution of the states (independent of π)...
- but may also depends on π (for instance the associated stationary measure)
- Other choices will appear.

- Goal: optimize $J_\mu(\pi)$ in π!

Parametric Policy

$$\pi_\theta(a|s) = \begin{cases} \frac{e^{h_\theta(a,s)}}{\sum_{a'} e^{h_\theta(a,s')}} & \text{(softmax)} \\ P_{h_\theta(s)}(a) & \text{(parametric conditional model)} \\ \mathbf{1}_{a=h_\theta(s)} & \text{(deterministic)} \end{cases}$$

Parametric Policy

- Restriction of the set of policy to a parametrized one.
- Most classical parametrizations:
  - Soft-max with a preference function $h_\theta(a,s)$,
  - Parametric conditional model with parameter $h_\theta(s)$
- To be useful need to be able to sample the distribution.
- h_θ: from linear model to deep learning...
- Most of our result will assume that $\pi_\theta(a|s)$ is differentiable with respect to θ.

- Deterministic policies will be considered with a different analysis.

Episodic Setting: Gradient of Expected Returns

$$v_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta}[G_0 | S_0 = s]$$

$$\nabla_\theta v_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right) G_0 \middle| S_0 = s\right]$$

Expected Returns

- Rely on $v_{\pi_\theta}(s) = \sum_\tau \mathbb{P}_{\pi_\theta}(\tau | S_0 = s)\, G_0(\tau)$ and

$$\begin{aligned}
\nabla \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) &= \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \nabla \log \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \\
&= \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \sum_t (\nabla \log \pi_\theta(A_t|S_t) + \nabla p(R_{t+1}, S_{t+1}|S_t, A_t)) \\
&= \mathbb{P}_{\pi_\theta}(\tau|S_0 = s) \sum_t \nabla \log \pi_\theta(A_t|S_t)
\end{aligned}$$

- In an episodic setting, any trajectory τ ends at a finite time T_τ.

Episodic Setting: Policy Gradient Theorem

$$J_{\mu_0}(\pi_\theta) = \sum_s \mathbb{P}(S_0 = s)\, v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right) G_0\right]$$

Policy Gradient Theorem

- Natural μ: initial state distribution.
- Gradient is an expectation: MC type algorithm...

- Can be interpreted as the gradient of a the maximum likelihood of the actions weighted by the return.
- Favors good actions over bad ones.

Baseline and Variance Reduction

$$J_{\mu_0}(\pi_\theta) = \sum_s \mathbb{P}(S_0 = s)\, v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)\right]$$

Variance Reduction and Baseline

- The previous formulae are valid if one replace G_0 by any function of τ.
- For any constant b, this leads to
$$\nabla \mathbb{E}_{\pi_\theta}[b] = 0 = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right) b\right]$$
- Optimal value for
$b = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t)\right)^2 G_0\right] / \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau-1} \nabla \log \pi_\theta(A_t|S_t)\right)^2\right]$
- Most used value $b = \mathbb{E}_{\pi_\theta}[G_0]$.

Gradient(s) of Expected Return

$$v_{\pi_\theta}(s) = \mathbb{E}_{\pi_\theta}\left[\sum \gamma^t R_t \middle| S_0 = s\right]$$

$$\nabla v_{\pi_\theta}(s) = \sum_t \gamma^t \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t'=0}^{t-1} \nabla \log \pi_\theta(A_{t'}|S_{t'})\right) R_t \middle| S_0 = s\right]$$

$$= \sum_{t'} \mathbb{E}_{\pi_\theta}\left[\nabla \log \pi_\theta(A_{t'}|S_{t'}) \left(\sum_{t \geq t'} \gamma^t R_t\right) |S_0 = s\right]$$

$$= \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}[\nabla \log \pi_\theta(A_{t'}|S_{t'}) q_{\pi_\theta}(S_{t'}, A_{t'})|S_0 = s]$$

$$= \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}\left[\nabla \log \pi_\theta(A_{t'}|S_{t'}) \underbrace{\left(q_{\pi_\theta}(S_{t'}, A_{t'}) - v_{\pi_\theta}(S_{t'})\right)}_{a_{\pi_\theta}(S_{t'}, A_{t'})} |S_0 = s\right]$$

Expected Returns

- Several formulas of stochastic gradients!

More Gradient(s)

$$
\begin{aligned}
\nabla v_{\pi_\theta}(s) &= \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}[\nabla \log \pi_\theta(A_{t'}|S_{t'}) q_{\pi_\theta}(S_{t'}, A_{t'})|S_0 = s] \\
&= \sum_{t'} \gamma^{t'} \mathbb{E}_{\pi_\theta}[\nabla \log \pi_\theta(A_{t'}|S_{t'}) a_{\pi_\theta}(S_{t'}, A_{t'})|S_0 = s] \\
&= \sum_{s} \left(\sum_{t} \gamma^{t} \mathbb{P}_{\pi_\theta}(S_t = s|S_0 = s) \right) \left(\sum_{a} \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) q_{\pi_\theta}(s, a) \right) \\
&= \sum_{s} \left(\sum_{t} \gamma^{t} \mathbb{P}_{\pi_\theta}(S_t = s|S_0 = s) \right) \left(\sum_{a} \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) a_{\pi_\theta}(s, a) \right)
\end{aligned}
$$

Focus on states

- Even more stochastic gradients!

Policy Gradient(s)

$$J_(\mu_0)(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s)$$

$$\begin{aligned}
\nabla J_{\mu_0}(\pi_\theta) &= \sum_s \left(\sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) q_{\pi_\theta}(s,a) \right) \\
&= \sum_s \left(\sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (q_{\pi_\theta}(s,a) - v_{\pi_\theta}(s,a)) \right)
\end{aligned}$$

Discounted Setting

- Average (discounted) return from the beginning.
- Focus on early steps in discounted setting...

Policy Improvement Lemma

$$J_{\mu_0}(\pi') - J_{\mu_0}(\pi) = \sum_t \gamma^t \mathbb{P}_{\pi'}(S_t = s) \left(\sum_a (\pi'(a|s) - \pi(a|s))\, q_\pi(s,a) \right)$$
$$= \sum_t \gamma^t \mathbb{P}_{\pi'}(S_t = s) \left(\sum_a (\pi'(a|s) - \pi(a|s))\, a_\pi(s,a) \right)$$

Proof

- By construction, if S_t is a trajectory using policy π':

$$v_{\pi'}(S_t) - v_\pi(S_t) = \sum_a (\pi'(a|S_t) - \pi(a|S_t))\, q_\pi(S_t,a) + \sum_a \pi'(a|s_t)\,(q_{\pi'}(S_t,a) - q_\pi(S_t,a))$$
$$= \sum_a (\pi'(a|s_t) - \pi(a|S_t))\, v_\pi(S_t,a) + \mathbb{E}_{\pi'}[Vv_{\pi'}(S_{t+1}) - v_{\pi'}(S_{t+1})|S_t]$$

- Discounted setting shortcut

$$v_{\pi'} - v_\pi = r_{\pi'} + \gamma P^{\pi'} v_{\pi'} - r_\pi - \gamma P^\pi v_\pi = r_{\pi'} - r_\pi + \gamma \left(P^{\pi'} - P^\pi \right) v_\pi + \gamma P^{\pi'} (v_{\pi'} - v_\pi)$$
$$v_{\pi'} - v_\pi = (I - \gamma P^{\pi'})^{-1} \left(r_{\pi'} - r_\pi + \gamma \left(P^{\pi'} - P^\pi \right) v_\pi \right)$$

Approximate Policy Improvement Lemma

$$\left| J_{\mu_0}(\pi') - J_{\mu_0}(\pi) - \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left(\sum_a (\pi'(a|s) - \pi(a|s))\, a_\pi(s,a) \right) \right|$$

$$= \left| \sum_t \gamma^t \left(\mathbb{P}_{\pi'}(S_t = s) - \mathbb{P}_\pi(S_t = s) \right) \left(\sum_a (\pi'(a|s) - \pi(a|s))\, a_\pi(s,a) \right) \right|$$

$$\leq \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1^2 \max_{s,a} |a_\pi(s,a)|$$

Approximate Policy Improvement Lemma

- If $\max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1 \leq \epsilon$

$$\mathbb{P}_{\pi'}(S_t = s) = (1-\epsilon)^t \mathbb{P}_\pi(S_t = s) + (1 - (1-\epsilon)^t) \mathbb{P}_{\text{mistake}}(S_t = s)$$

$$\rightarrow |\mathbb{P}_{\pi'}(S_t = s) - \mathbb{P}_\pi(S_t = s)| \leq 2(1 - (1-\epsilon)^t) \leq 2\epsilon t$$

- $\sum_t 2\gamma^t t = \frac{2\gamma}{(1-\gamma)^2}$

Discounted

Approximate Policy Improvement Lemma

$$\left| J_{\mu_0}(\pi') - J_{\mu_0}(\pi) - \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left(\sum_a (\pi'(a|s) - \pi(a|s)) \, a_\pi(s,a) \right) \right|$$
$$\leq \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1^2 \max_{s,a} |a_\pi(s,a)|$$

Approximate Policy Improvement Lemma and Policy Gradient Theorem

- Let $\pi' = \pi_{\theta+h}$ and π_θ
  - $\pi_{\theta+h}(a|s) - \pi_\theta(a|s) = \pi_\theta(a|s)\langle \nabla \log \pi_\theta(a|s), h\rangle + O(\|h\|^2)$
  - $\|\pi_{\theta+h}(\cdot|s) - \pi_\theta(\cdot|s)\|_1 \leq \|h\| \max_a \|\nabla \log \pi_\theta(a|s)\| + O(\|h\|^2)$
- Implies Policy Gradient Theorem:

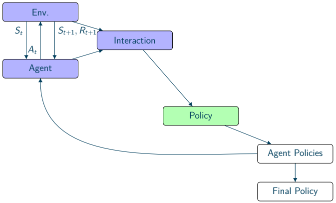
$J_{\mu_0}(\pi_{\theta+h})$

$$= J_{\mu_0}(\pi_\theta) + \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \left(\sum_a \pi_\theta(a|s) \langle \nabla \log \pi_\theta(s,a), h\rangle a_\pi(s,a) \right) + O(\|h\|^2)$$

Discounted

Outline

Monte Carlo Approach



$$G_t = \sum_{t' \geq t} R_{t+1}$$

$$Q_{t,\pi_\theta}(s, a) = \mathbb{E}[G_t | S_t = s, A_t = a]$$

Monte Carlo

- Replace every return by an empirical estimate along episodes.
- Need to wait until the end of the episods.

Episodic

REINFORCE: Monte Carlo Based Policy Gradient

$$J_{\mu_0}(\pi_\theta) = \sum_s \mathbb{P}(S_0 = s)\, v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right) G_0\right]$$

$$= \sum_s \left(\sum_t \mathbb{P}_{\pi_\theta}(S_t = s)\right)\left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) q_{\pi_\theta}(s, a)\right)$$

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right) G_0 \quad \text{or} \quad \widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \nabla \log \pi_\theta(A_t|S_t) G_t$$

REINFORCE

- Plain MC (SGD) algorithm.
- Need to wait until the end of the episods.
- Convergence guarantees (even in off-line setting with importance sampling).

Episodic

REINFORCE with Baseline

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)\right]$$

$$= \sum_s \left(\sum_t \mathbb{P}_{\pi_\theta}(S_t = s)\right)\left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s)\left(q_{\pi_\theta}(s,a) - b(s)\right)\right)$$

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)$$

or $\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \nabla \log \pi_\theta(A_t|S_t)\left(G_t - b(S_t)\right)$

REINFORCE with baseline

- Several choices for b...
- and for $b(s)$ which can be any function (a crude estimate of $V_{t,\pi}(s)$ for instance)!
- Convergence guarantees (even in off-line setting with importance sampling).

Discounted REINFORCE?

$$\nabla J_{\mu_0}(\pi_\theta) = \mathbb{E}_{\pi_\theta}\left[\left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)\right]$$

$$= \sum_s \left(\sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s)\right)\left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s)\left(q_{\pi_\theta}(s,a) - b(s)\right)\right)$$

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \left(\sum_{t=0}^{T_\tau - 1} \nabla \log \pi_\theta(A_t|S_t)\right)(G_0 - b)$$

or $$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \nabla \log \pi_\theta(A_t|S_t)\left(G_t - b(S_t)\right)$$

Discounted REINFORCE

- Can be defined. . .
- but still requires an episodic setting for the discounted return G_t to be computed.

Discounted Measure?

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \nabla \log \pi_\theta(A_t|S_t)\,(G_t - b(S_t))$$

$$\longrightarrow \widehat{\nabla} J_{\mu_{\pi_\theta}}(\pi_\theta) = \frac{1}{1-\gamma} \nabla \log \pi_\theta(A_t|S_t)\,(G_t - b(S_t))?$$

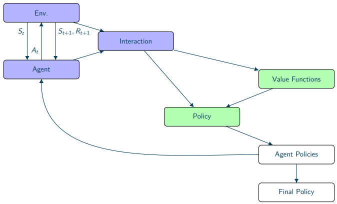
Discounted Measure?

- Much less weights for later states!
- Probability independent of t if the initial distribution is the stationary distribution μ_{π_θ} corresponding to π_θ (it it exists).
- Approximately true after a burning stage if we reach stationarity...
- Better handled by the average return!

- More on this later...

Outline

Actor/Critic



Actor/Critic

- Actor: Parametric policy π_θ used.
- Critic: Q-value function $Q_{\boldsymbol{w}}(\cdot, \cdot)$ approximating Q_{π_θ}.
- Critic follows the Actor, which is optimized using the Critic.

- In Value Approximation, the Actor follows the Critic (through the argmax operator).
- In on-line methods, the Actor is used to interact with the environment.

Actor/Critic

$$J_{(}\mu_0)(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_0}(\pi_\theta) = \sum_s \left(\sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \right) \left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (q_{\pi_\theta}(s,a) - v_{\pi_\theta}(s,a)) \right)$$

$$\widehat{\nabla} J_{\mu_0}(\pi_\theta) = \sum_t \gamma^t \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left(q_{\pi_\theta}(S_t, A_t) - \sum_a \pi(a|S_t) q_{\pi_\theta}(S_t, A_t) \right)$$

$$\simeq \sum_t \gamma^t \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left(Q_{\mathbf{w}}(S_t, A_t) - \sum_a \pi(a|S_t) Q_{\mathbf{w}}(S_t, A_t) \right)$$

Actor/Critic

- Critic update: Stochastic Policy Gradient with plugin.
- Actor update: any Q-value methods estimating q_{π_θ}.
- Requires a two-scales algorithm so that $Q_{\mathbf{w}}$ is always a good estimate of q_{π_θ}.

- Is this a real algorithm in a non-episodic setting?

Actor/Critic

$$J_{\mu_{\pi_\theta}}(\pi_\theta) = \sum_s \mu_{\pi_\theta}(s) v_{\pi_\theta}(s)$$

$$\nabla J_{\mu_{\pi_\theta}}(\pi_\theta) = \sum_s \frac{1}{1-\gamma} \mathbb{P}_{\pi_\theta}(S_t = s) \left(\sum_a \pi_\theta(a|s) \nabla \log \pi_\theta(a|s) (q_{\pi_\theta}(s,a) - v_{\pi_\theta}(s,a)) \right)$$

$$\widehat{\nabla} J_{\mu_{\pi_\theta}}(\pi_\theta) \simeq \frac{1}{1-\gamma} \pi_\theta(A_t|S_t) \nabla \log \pi_\theta(A_t|S_t) \left(Q_{\boldsymbol{w}}(S_t, A_t) - \sum_a \pi(a|S_t) Q_{\boldsymbol{w}}(S_t, A_t) \right)$$

Actor/Critic

- Critic update: Stochastic Policy Gradient with plugin.
- Actor update: any Q-value methods estimating q_{π_θ}.
- Requires a two-scales algorithm so that $Q_{\boldsymbol{w}}$ is always a good estimate of q_{π_θ}.

- Require the existence of a stationary measure... and that this stationary measure is reached *quickly*.
- Much harder to do off-policy algorithm as the stationary measure is not known!

Critic in Actor/Critic

$$Q_{\boldsymbol{w}} \simeq q_{\pi_\theta}$$

Critic

- On-line TD learning with interaction following π_θ.
- Off-Policy TD learning is possible if the policy used for any action is stored.
- Approximate off-policy TD learning is possible using a replay buffer providing π_θ is changing slowly.

- May lead to 3 scales algorithm (Actor/Critic Target/Critic)
- As mentionned in the previous slide, much harder to do off-line update for the actor.

Off-Line Actor

$$J'_\mu(\pi) = \sum_s \mu(s) v_\pi(s)$$

Off-Line Actor

- Idea proposed in 2012.
- Key lemma in the paper
$$\nabla J'_\mu(\pi_\theta) \simeq \sum_s \mu(s) \sum_a \pi_\theta(a|s) \nabla \pi_\theta(a|s) q_{\pi_\theta}(s,a)$$
turns out to be wrong!
- Still used as a heuristic justification of many algorithms!
- Explicit formula for $\nabla J'_\mu(\pi_\theta)$ can be obtained but much harder to use...

Outline

PPO: Minorize-Majorization Algorithm

$$J_{\mu_0}(\pi') \geq J_{\mu_0}(\pi) + \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left(\sum_a (\pi'(s|a) - \pi(s|a))\, a_\pi(s,a) \right) - \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi'(\cdot|s) - \pi(\cdot|s)\|_1^2 \max_{s,a} |a_\pi(s,a)|$$

Ideal Minorize-Majorization Algorithm

- At step k, find θ_{k+1} maximizing

$$J_{\mu_0}(\pi_\theta|\pi_{\theta_k}) = \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a))\, a_{\pi_{\theta_k}}(s,a) \right) - \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)\|_1^2 \max_{s,a} |a_{\pi_{\theta_k}}(s,a)|$$

- By construction, $J_{\mu_0}(\pi_{\theta_{k+1}}) \geq J_{\mu_0}(\pi_{\theta_k})$

- Sample efficient algorithm as the same trajectory can be (re)used in the optimization.

Discounted

PPO: Optimization

$$J_{\mu_0}(\pi_\theta) \geq J_{\mu_0}(\pi_{\theta_k}) + \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a))\, a_{\pi_{\theta_k}}(s,a) \right)$$
$$- \frac{2\gamma}{(1-\gamma)^2} \max_s \|\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)\|_1^2 \max_{s,a} |a_{\pi_{\theta_k}}(s,a)|$$

Optimization

- Gradient descent is possible.
- Gradient of the first term can be approximated using a critic by
$$\sum_s \sum_t \gamma^t \mathbb{P}_\pi(S_t = s) \left(\sum_a \pi_\theta \nabla \pi_\theta(s|a) A_{\pi_{\theta_k}}(s,a) \right)$$
- Gradient of the second term more involved.

- Simpler (TRPO like) strategy: optimize
$$\sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a))\, a_{\pi_{\theta_k}}(s,a) \right)$$
under $\max_s \|\pi_\theta(\cdot|s) - \pi_{\theta_k}(\cdot|s)\|_1^2 \leq \epsilon$ and reduce ϵ there is no gain.

Discounted

PPO: KL Relaxation

$$J_{\mu_0}(\pi_\theta) \geq J_{\mu_0}(\pi_{\theta_k}) + \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a))\, a_{\pi_{\theta_k}}(s, a) \right)$$
$$- \frac{2\gamma R_{\max}}{(1-\gamma)^2} \max_s \mathrm{KL}(\pi_{\theta_k}(\cdot|s), \pi_\theta(\cdot|s))$$

TRPO/PPO Optimization

- Replace the ℓ_1 norm by a KL divergence.
- In practice, replace the max by an average and replace $\frac{2\gamma R_{\max}}{(1-\gamma)^3}$ by parameter β and replace the a_{π_k} by an estimate A_{π_k}.
- PPO: Gradient descent of the relaxed goal.
- TRPO: Constrained optimization.

- Adaptive scheme to set β.
- Can be used with continuous action.

Discounted

PPO: Clipped Objective

$$\sum_s \sum_t \gamma^t \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a \pi_{\theta_k}(s|a) \min\left(\frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)} a_{\pi_{\theta_k}}(s,a), \text{clip}(1-\epsilon, \frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)}, 1+\epsilon) a_{\pi_{\theta_k}}(s,a) \right) \right)$$

Clipped Objective

- Insight by (re)substracting $\sum_a \pi_{\theta_k}(s|a) a_{\theta_k}(s,a) = 0$:

$$\sum_a \min\left(\left(\pi_\theta(s|a) - \pi_{\theta_k}(s,a)\right) a_{\pi_{\theta_k}}(s,a), \text{clip}(-\epsilon, \pi_\theta(s|a) - \pi_{\theta_k}(s,a), \epsilon) a_{\pi_{\theta_k}}(s,a) \right)$$

$$= \sum_a \text{clip}(-\epsilon \pi_{\theta_k}(s,a), \pi_\theta(s|a) - \pi_{\theta_k}(s,a), \epsilon \pi_{\theta_k}(s,a)) a_{\pi_{\theta_k}}(s,a)$$

$$- \max\left(0, -(\pi_\theta(s|a) - \pi_{\theta_k}(s,a)) a_{\pi_{\theta_k}}(s,a) - \epsilon \pi_{\theta_k}(s,a) |a_{\pi_{\theta_k}}(s,a)| \right)$$

- First term amount to replace π_θ by a policy

$$\tilde{\pi}_\theta(a|s) = \text{clip}(\pi_{\theta_k}(a|s)(1-\epsilon), \pi_\theta(a|s), \pi_{\theta_k}(a|s)(1+\epsilon)) + \eta_s \pi_{\theta_k}(a|s)$$

  where η is so that $\tilde{\pi}$ is a probability for all s and $\|\tilde{\pi}_\theta(\cdot,s) - \pi_{\theta_k}(\cdot,s)\|_1 \leq \epsilon$

- Second term: hinge loss type penalization of policy π_θ penalizing *bad* actions.

- Very efficient for discrete actions.

Discounted

PPO: Stationary Objective

$$\sum_{s,t} \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a (\pi_\theta(s|a) - \pi_{\theta_k}(s|a))\, a_{\pi_{\theta_k}}(s,a) \right) - \beta \max_s \mathrm{KL}(\pi_{\theta_k}(\cdot|s), \pi_\theta(\cdot|s))$$

$$\sum_{s,t} \mathbb{P}_{\pi_{\theta_k}}(S_t = s) \left(\sum_a \pi_{\theta_k}(s|a) \min\left(\frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)} a_{\pi_{\theta_k}}(s,a), \mathrm{clip}(1-\epsilon, \frac{\pi_\theta(s|a)}{\pi_{\theta_k}(s,a)}, 1+\epsilon) a_{\pi_{\theta_k}}(s,a) \right) \right)$$

Stationary Objective

- Amount to replace $J_{\mu_0}(\pi)$ by $J_{\mu_\pi}(\pi)$
- Most common implementation of PPO...
- But no way to justify it mathematically!
- May explain the (possible) instabilities of PPO.

Discounted

DPG: Deterministic Policy Gradient

$$J_{\mu_0}(\pi_\theta) = \sum_s \mu_0(s) v_{\pi_\theta}(s) \quad \text{with deterministic policy } \pi_\theta(a|s) = \mathbf{1}_{a=h_\theta(s)}$$

$$\nabla J_{\mu_0}(\pi_\theta) = \sum_s \sum_t \gamma^t \mathbb{P}_{\pi_\theta}(S_t = s) \nabla_a q(S_t, h_\theta(S_t)) \nabla h_\theta(S_t)$$

Deterministic Policy Gradient

- Deterministic policy replaced by a randomized one centered on $h_{\theta(s)}$ in the interactions!.
- Critic trained with a TD variant of DQN.
- Same formula by using a policy $\pi_\theta = \mathrm{N}(h_\theta(s), \sigma^2 \mathrm{Id})$ and letting σ goes to 0.
- Off-Policy as claimed?
- Yes for the actor but no theoretical justification for the critic!
- In practice, the buffer contains only samples using a policy close to the current one. . .

Discounted

SAC: A New Goal

$$R_t \to R_t + \lambda \mathcal{H}(\pi(S_t))$$

A Modified Reward

- Modification of the reward to favor high entropy policy:
$$R_t \to R_t + \lambda \mathcal{H}(\pi(S_t))$$
- Goal:
$$J(\pi) = \mathbb{E}_\pi\left[\sum_t \gamma^t \left(R_t + \lambda \mathcal{H}(\pi(S_t))\right)\right]$$
- Soft value function implicitly defined as the fixed point of
$$\mathcal{T}^\pi q_\pi(s,a) = r_\pi(s,a) + \gamma \sum_{s'} p(s'|s,a) v_\pi(s')$$
$$\text{where} \quad v_\pi(s,a) = \sum_a \pi(a|s)\left(q_\pi(s,a) - \log \pi(a|s)\right)$$

Discounted

SAC: Policy Improvement and Optimal Policy

$$R_t \to R_t + \lambda \mathcal{H}(\pi(S_t))$$

A Modified Policy Improvement Lemma

- Policy improvement rule:
$$\pi^+(\cdot|s) = \operatorname*{argmax}_{\pi(\cdot|s)} \sum_a \pi(a|s) \left(q(s,a) - \lambda \log(\pi(a|s))\right)$$
$$\pi^+(a|s) \propto \exp(-\frac{1}{\lambda} q(s,a))$$
implies $G_{\pi^+}(s,a) \geq G_\pi(s,a)$.
- At convergence, $J(\pi^\star)$ is optimal!
- Convergence in the finite setting.

Discounted

SAC: Parametrization

$$\pi \sim \pi_\theta \quad \text{and} \quad q(s,a) \sim Q_{\boldsymbol{w}}$$

SAC Choices

- Fitted TD learning for Q:
$$\boldsymbol{w} \simeq \operatorname{argmin} \sum_{(S,A,R,S') \in \mathcal{B}} \left(R + \mathbb{E}_{\pi_\theta}\left[\gamma Q_{\overline{\boldsymbol{w}}}(S',a) - \lambda \log \pi_\theta(a|S')\right] - Q_{\boldsymbol{w}}(S,A)\right)^2$$
where the trajectory pieces are samples from a replay buffer and $\overline{\boldsymbol{w}}$ is a slowdown version of $\boldsymbol{w}$ (two-scales algorithm).
- Online version rather than batch...
- Fitted KL for π:
$$\theta \simeq \operatorname{argmin} \sum_{(S,A,R,S') \in \mathcal{B}} \mathrm{KL}(\pi_\theta(\cdot|S) | \exp -\lambda Q_{[\overline{\boldsymbol{w}}]}(S, \dot{)} / Z_{\overline{\boldsymbol{w}}}(S))$$
$$\simeq \sum_{(S,A,R,S') \in \mathcal{B}} \mathbb{E}_{\pi_\theta}\left[\frac{1}{\lambda} \log \pi_\theta(a|S) - Q_\theta(a|s)\right]$$

Discounted

Outline

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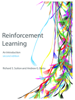

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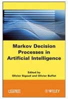

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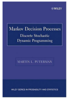

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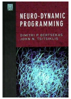

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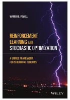

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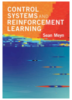

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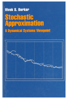

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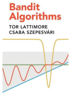

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