Reinforcement Learning
Operations Research: Prediction and Planning

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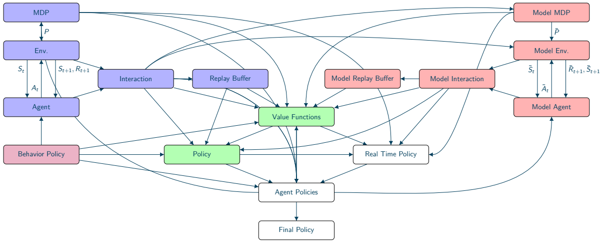
M2 DS - Fall 2022

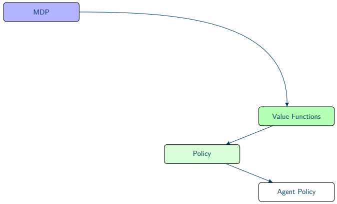
RL: What Are We Going To See?



Outline

- Operations Research and MDP.
- Reinforcement learning and interactions.
- More tabular reinforcement learning.
- Reinforcement and approximation of value functions.
- Actor/Critic: a Policy Point of View

Operations Research and MDP



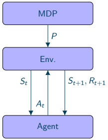
How to find the best policy knowing the MDP?

- Is there an optimal policy?
- How to estimate it numerically?

- Finite states/actions space assumption (tabular setting).
- Focus on interative methods using value functions (dynamic programming).
- Policy deduced by a statewise optimization of the value function over the actions.
- Most results for the discounted setting.

Outline

Markov Decision Process / Operations Research



MDP / OR

- Known MDP model
- Focus on the finite horizon setting

$$G_t^T = \sum_{t'=t+1}^{T} R_{t'}$$

and the discounted setting:

$$G_t^\gamma = \sum_{t'=t+1}^{\infty} \gamma^{t'-(t+1)} R_{t'}$$

- We will later consider the other settings.

Policy

Policy

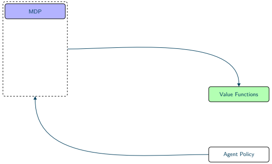
- Finite horizon : emphasis on Markovian policies

$$\Pi_t(A_t = a_t) = \pi_t(A_t = a_t | S_t = s_t) = \pi_t(a_t | s_t)$$

- Discounted return: emphasis on stationary Markovian policies

$$\Pi_t(A_t = a_t) = \pi(A_t = a_t | S_t = s_t) = \pi(a_t | s_t)$$

Prediction



Prediction

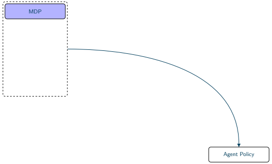
- How to efficently evaluate the quality of a policy

$$v_{t,\Pi}(s) = \mathbb{E}_{\Pi}\left[\sum_{t'=t+1}^{T} \gamma^{t'-(t+1)} R_{t'} \middle| S_t = s \right]$$

when we can ensure that the sum is finite?

- $v_{t,\Pi}$ independent of t in the discounted setting if the policy is stationary.

Planning



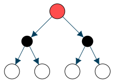
Policy

- How to find a policy π such that
$$\sum_{s,t} \mu(s,t) v_{t,\Pi}(s)$$
is as large as possible?
- Emphasis on $\mu(s,t) = 0$ if $t \neq 0$ and $\mu(s,0) = \mathbb{P}_0(S_0 = s_0)$.

Outline

Bellman Equation

$$v_{t,\Pi}(s) = \sum_a \pi_t(a|s) \sum_{s',r} p(s', r|s, a) \left(r + \gamma v_{t+1,\Pi}(s')\right)$$
$$= \sum_a \pi_t(a|s) r(s, a) + \gamma \sum_{s'} \sum_a p(s'|s, a) \pi_t(a|s) v_{t+1,\Pi}(s')$$



Bellman Equation

- Link between $v_{t,\Pi}$ and $v_{t+1,\Pi}$.
- Straightforward consequence of

$$G_t = \sum_{t'=t+1}^{T} \gamma^{t'-(t+1)} R_{t'} = R_{t+1} + \gamma \sum_{t'=t+2}^{T} \gamma^{t'-(t+2)} R_{t'} = R_{t+1} + \gamma G_{t+1}$$

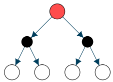
and thus

$$\mathbb{E}[G_t|S_t = s] = \mathbb{E}[R_{t+1}|S_t = s] + \gamma \mathbb{E}[\mathbb{E}[G_{t+1}|S_{t+1}]|S_t = s]$$

Bellman Operator

$$\mathcal{T}^{\pi_t} : \mathbb{R}^{|\mathcal{S}|} \to \mathbb{R}^{|\mathcal{S}|}$$

$$\mathcal{T}^{\pi_t} v(s) = \underbrace{\sum_a \pi_t(a|s) r(s,a)}_{r_{\pi_t}(s)} + \gamma \sum_{s'} \underbrace{p(s'|s,a) \sum_a \pi_t(a|s)}_{P^{\pi_t}(s,s')} v(s')$$



Bellman Operator

- Affine operator from the space of state value functions to the space of state value functions.
- By construction,

$$v_{t,\Pi} = \mathcal{T}^{\pi_t} v_{t+1,\Pi}$$

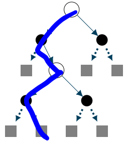
- r_{π_t} is the vector of average immediate rewards using policy π_t while P^{π_t} is the one step state transition matrix using policy π_t.

Outline

Finite Horizon: Naive Approach

$$v_{t,\Pi}^{T}(s) = \sum_{a_t, r_{t+1}, s_{t+1}, \cdots, r_T} \left(\sum_{t'=t+1}^{T} r_{t'} \right) \mathbb{P}_{\Pi}(A_t = a_t \ldots, R_T = r_T | S_t = s)$$

$$= \sum_{a_t, r_{t+1}, s_{t+1}, \cdots, r_T} \left(\sum_{t'=t+1}^{T} r_{t'} \right) \pi_t(a_t|s) \times \cdots \times p(s_T, r_T | s_{T-1}, a_{T-1})$$



Finite Horizon: Naive Approach

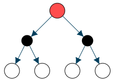
- Exhaustive exploration of the trajectories.
- Complexity of order $(|\mathcal{A}| \times |\mathcal{S}| \times |\mathcal{R}|)^{T-t}$ for the value function at time t.

- Complexity can be reduced to $(|\mathcal{A}| \times |\mathcal{S}|)^{T-t}$ by noticing that

$$v_{t,\Pi}^{T}(s) = \sum_{a_t, s_{t+1}, \cdots, s_{t-1}, a_{t-1}} \left(\sum_{t'=t+1}^{T} r(s_t, a_t) \right) \pi_t(a_t|s) \times \cdots \times p(s_T | s_{T-1}, a_{T-1})$$

Finite Horizon: Recursive Prediction

$$v^T_{T,\Pi} = 0$$
$$v^T_{t-1,\Pi} = \mathcal{T}^{\pi_{t-1}} v^T_{t,\Pi}$$



Finite Horizon: Recursive Prediction

- After time T, the finite horizon return $G_t^T = 0$ hence $v^T_{T,\Pi} = 0$ whatever the policy.
- The Bellman equation yields second equation.
- Equivalent rewriting
$$v^T_{t-1,\Pi}(s) = r_{\pi_{t-1}}(s) + \sum_{s'} P_{\pi_{t-1}}(s, s') v^T_t$$
- Complexity of order only $T \times |\mathcal{S}|^2(|\mathcal{A}| + |\mathcal{S}|)$ to compute all the value functions.

Finite Horizon: Value Iteration

Finite Horizon: Prediction by Value Iteration

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$ and policy Π
parameter: Horizon T
init: $v_T^T(s)=0\,\forall\, s\in\mathcal{S},\ t=T$
repeat
  $t\leftarrow t-1$
  for $\forall\, s\in\mathcal{S}$ **do**

$$v_t^T(s)\leftarrow\sum_{a\in\mathcal{A}}\pi_t(a|s)\left(r(s,a)+\gamma\sum_{s'\in\mathcal{S}}p(s'|s,a)v_{t+1}^T(s')\right)$$

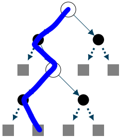
  end
until $t=0$
output: Value functions v_t^T

$$v_t^T = \mathcal{T}^{\pi_t} V_{t+1}^T$$

- Most classical formulation

Discounted: Naive Approach

$$v^{\gamma}_{t,\Pi}(s) = \sum_{t'=t+1}^{\infty} \gamma^{t'-(t+1)} \mathbb{E}_{\Pi}[R_{t'} | S_t = s] \simeq \sum_{t'=t+1}^{T} \gamma^{t} \mathbb{E}_{\Pi}[R_{t'} | S_t = s] = v^{\gamma,T}_{t,\Pi}(s)$$



$$v^{\gamma,T}_{t,\Pi}(s) = \sum_{a_t, s_{t+1}, \cdots, s_{t-1}, a_{t-1}} \left(\sum_{t'=t+1}^{T} \gamma^{t'-(t+1)} r(s_t, a_t) \right) \pi_t(a_t | s) \times \cdots \times p(s_T | s_{t-1}, a_{t-1})$$

Naive approach

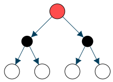
- Exhaustive exploration of truncated trajectories.
- Back to the finite horizon setting...
- **Prop:** Control on the error as $\left| v^{\gamma}_{\Pi} - v^{\gamma,T}_{t,\Pi} \right|_{\infty} \leq \frac{\gamma^{T+1-t}}{1-\gamma} \max_{r \in \mathcal{R}} |r|$
- Relation between the error $\epsilon \simeq \gamma^{T-t}$ and the numerical complexity $C = (|\mathcal{A}| \times |\mathcal{S}|)^{T-t}$ of order $C \simeq \epsilon^{-1}$.

Discounted: Recursive Prediction with Naive Initialization

$G_t = \sum_{t' \geq t} \gamma^{t'-t} R_{t'+1} = \sum_{t'=t}^{T} \gamma^{t'-t} R_{t'+1} + \sum_{t' \geq T} \gamma^{t'-t} R_{t'}$ $\quad |v_t^\gamma| \leq \frac{1}{1-\gamma} \max |R|$

$$v_{T,\Pi}^{\gamma} \simeq v_{T,\Pi}^{\gamma,T'} = \tilde{v}_{T,\Pi}$$

$$v_{t-1,\Pi}^{\gamma} = \mathcal{T}^{\pi_{t-1}} v_{t,\Pi}^{\gamma} \simeq \tilde{v}_{t-1,\Pi} = \mathcal{T}^{\pi_{t-1}} \tilde{v}_{t,\Pi}$$



Recursive Prediction

- Requires an initialization at time T with a horizon T'.
- The Bellman equation yields the second equation.
- Complexity of order only $T \times |\mathcal{S}|^2(|\mathcal{A}| + |\mathcal{S}|)$ to compute all the value functions after the initialization of cost $(|\mathcal{A}| \times |\mathcal{S}|)^{T'-T}$.
- **Prop:** If the approximation error between $v_{T,\Pi}^{\gamma}$ and $v_{T,\Pi}^{\gamma,T'}$ is bounded by ϵ then

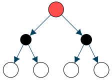
$$\|v_{t,\Pi}^{\gamma} - \tilde{v}_{t,\Pi}\|_\infty \leq \gamma^{T-t}\epsilon, \quad \forall t \leq T$$

Discounted and stationary: Bellman Equation

$\Pi = (\pi, \pi, \cdots, \pi \cdots)$

$$v_\Pi = \mathcal{T}^\pi v_\Pi$$

$$v_\Pi(s) = \sum_a \pi(a|s) r(s,a) + \gamma \sum_{s'} \sum_a p(s'|s,a) \pi(a|s) v_\Pi(s')$$



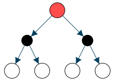
Bellman Equation

- Time independent value function v_Π.
- **Prop:** Unique solution of the linear equation $v_\Pi = \mathcal{T}^\pi v_\Pi$
- Complexity of order $(|A| + |S|) \times |S|^2$ to obtain the solution.

Discounted and stationary: Recursive Implementation

$$v_\Pi = \mathcal{T}^\pi v_\Pi$$

$$v_{k+1} = \mathcal{T}^\pi v_k \quad \text{with arbitrary } v_0$$



Bellman Iteration

- **Prop:** Unique fixed point of the Bellman operator $v \mapsto \mathcal{T}^\pi v$.
- **Prop:** The iterates $v_{k+1} = \mathcal{T}^\pi v_k$ converges toward v_Π and
$$\|v_k - v_\Pi\|_\infty \leq \gamma^k \|v_0 - v_\Pi\|_\infty$$
- Complexity of order $(k + |A|)|S|^2$ to obtain the kth iterate.
- Exponential decay of the error with respect to the complexity.

Bellman Operator and Contraction

$$\|\mathcal{T}^\pi v - \mathcal{T}^\pi v'\|_\infty \leq \gamma \|v - v'\|_\infty$$

Proof

- By definition

$$\|\mathcal{T}^\pi v - \mathcal{T}^\pi v'\|_\infty = \gamma \|P^\pi (v - v')\|_\infty$$

- It suffices then to notice that P^π is a transition matrix, so that

$$\sum_j P^\pi_{i,j} = 1$$

and thus $|\sum_j P^\pi_{i,j} z_j| \leq \max |z_j|$

$\|v - v'\|_\infty = \|\mathcal{T}^\pi v - \mathcal{T}^\pi v'\|_\infty \leq \gamma \|v - v'\|_\infty$

$v' = \mathcal{T}^\pi v'$

$v = \mathcal{T}^\pi v$

Consequences

- Unicity of the solution of $\mathcal{T}^\pi v = v$.
- Linear decay γ^k of the error with the iterates.

$v_{k+1} = \mathcal{T}^\pi v_k$

$$\mathcal{T}^{\pi} v(s) = r_{\pi}(s) + \gamma \mathbb{E}_{\pi}[v(s')]$$

$$+ \gamma \sum_{a} \pi(a|s) \sum_{s'} p(s'|s,a) \, v(s')$$

$$+ \gamma \sum_{s'} \underbrace{\sum_{a} \pi(a|s) p(s'|s,a)}_{P^{\pi}_{ss'}} \, v(s')$$

$$\left| \sum_{s'} P^{\pi}_{(s,s')} (v(s') - v'(s')) \right|$$

$$\leq \sum P^{\pi}_{(s,s')} \left| v(s') - v'(s') \right|$$

$$\leq \| v - v' \|_{\infty}$$

$$\mathcal{T}^{\pi} v = r_{\pi} + \gamma P^{\pi} v$$

$$\mathcal{T}^{\pi} v - \mathcal{T}^{\pi} v' = (r_{\pi} + \gamma P^{\pi} v) - (r_{\pi} + \gamma P^{\pi} v')$$

$$= \gamma P^{\pi} (v - v') \Rightarrow \| \mathcal{T}^{\pi} v - \mathcal{T}^{\pi} v' \|_{\infty} \leq \gamma \| P^{\pi} (v - v') \|_{\infty}$$

$$\leq \gamma \| v - v' \|_{\infty}$$

Bellman Operator and Bellman Equation Solution

$$v_\Pi = \left(\sum_{k=0}^{\infty} \gamma^k \left(P^\pi\right)^k\right) r_\pi$$

$(I - A)^{-1} = \sum_{l \geq 0} A^l$ if $|||A||| < 1$

A Closed Formula for the State Value Function

- $v_\Pi = \mathcal{T}^\pi v_\Pi \Leftrightarrow \left(I - \gamma P^\pi\right) v_\Pi = r_\pi$
- As P^π is a transition matrix, its eigenvalues are smaller than 1 and thus $\left(I - \gamma P^\pi\right)$ is invertible of inverse

$$\left(I - \gamma P^\pi\right)^{-1} = \sum_{k=0}^{\infty} \gamma^k \left(P^\pi\right)^k$$

- Could have been obtained without the Bellman equation as the $\left(\left(P^\pi\right)^k\right)_{s,s'}$ is, by construction, the probability of being at state s' at time k starting from s at time 0 and following Π.

Discounted and stationary: Value Iteration

Discounted: Prediction by Value Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, discount factor γ, and stationary policy π
init: $\tilde{v}(s) \forall s \in \mathcal{S}$
repeat
  $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$
  for $s \in \mathcal{S}$ **do**

$$\tilde{v}(s) \leftarrow \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a) \tilde{v}_{\text{prev}}(s') \right)$$

  end
output: Value function $\tilde{v}$

- When to stop?

Discounted and stationary: Value Iteration

Discounted: Prediction by Value Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, discount factor γ, and stationary policy π
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s) \, \forall \, s \in \mathcal{S}$
repeat
  $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$
  $\Delta \leftarrow 0$
  for $s \in \mathcal{S}$ **do**

$$\tilde{v}(s) \leftarrow \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) \tilde{v}_{\text{prev}}(s') \right)$$

    $\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\text{prev}}(s)|\right)$
  end
until $\Delta < \delta$
output: Value function $\tilde{v}$

$$\Delta = \|v_e - v_{e-1}\|_\infty$$

- **Prop:** when the algorithms stops

$$\|\tilde{v} - v_\Pi\|_\infty \leq \frac{2\delta}{1-\gamma}$$

Discounted and stationary: Value Iteration

Discounted: Prediction by Value Iteration - Gauss-Seidel Version

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, discount factor γ, and stationary policy π
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s) \, \forall \, s \in \mathcal{S}$
repeat
  $\Delta \leftarrow 0$
  for $s \in \mathcal{S}$ **do**
    $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}(s)$

$$\tilde{v}(s) \leftarrow \sum_{a \in \mathcal{A}} \pi(a|s) \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) \tilde{v}(s') \right)$$

    $\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\text{prev}}|\right)$
  end
until $\Delta < \delta$
output: Value function $\tilde{v}$

- Gauss-Seidel variation mostly used in practice.
- No need to store the previous value function.

Outline

Optimal Policy

Optimal Policy

- An optimal policy $\Pi_\star$ should be better than any other policies:

$$\forall s, \forall t, v_{t,\Pi_\star}(s) = \sup_{\pi} v_{t,\Pi}(s)$$

Several Questions

- Do this policy exists?
- Is it unique?
- How to characterize it?
- How to obtain it?

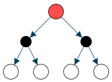
- Even the sup above could be an issue if it is not attained!

Finite Horizon and Optimal Policy

Explicit Recursive Solution

- After horizon T, any policy leads to a 0 return.
- At time $T-1$,
  - the total return G_T is the immediate return at time T and thus
  $$v_{T,\Pi^\star}(s) = \sup_{\pi(a|s)} \sum_a \pi(a|s) r(a,s) = \sup_a r(a,s)$$
  - the optimal policy $\pi^\star_{T-1}$ exists and is determistic.
- By recursion,
  - the total return at time $t-1$ is the immediate return at time t plus the total return at time $t-1$ and thus
  $$v_{t-1,\Pi^\star}(s) = \sup_{\pi(a|s)} \sum_a \pi(a|s) \left(r(a,s) + \sum_{s'} p(s'|s,a) v_{t,\Pi^\star}(s') \right)$$
  $$= \sup_a \left(r(a,s) + \sum_{s'} p(s'|s,a) v_{t,\Pi^\star}(s') \right)$$
  - the optimal policy $\pi^\star_{t-1}$ exists and is determistic.

Discounted Setting and Optimal Stationary Policy



Heuristic

- Optimal policy: $v_{\Pi^\star}(s) = \sup_\pi v_\Pi(s)$
- Stationary solution:

$$
\begin{aligned}
v_{\Pi^\star}(s) &= \sup_{\pi} \left(\mathcal{T}^{\pi} v_{\Pi^\star}\right)(s) \\
&= \sup_{\pi_t(\cdots|s)} \sum_a \pi(a|s) \left(r(a,s) + \sum_{s'} p(s'|s,a) v_{\Pi^\star}(s') \right) \\
&= \sup_a \left(r(a,s) + \sum_{s'} p(s'|s,a) v_{\Pi^\star}(s') \right)
\end{aligned}
$$

- Optimal deterministic policy: $\pi^\star(s) \in \operatorname{argmax} \left(r(a,s) + \sum_{s'} p(s'|s,a) v_{\Pi^\star}(s') \right)$.

- Is everything well defined? Yes but one has to be more cautious!

Optimal Value Function and Bellman Operator

Optimal Value Function

- Optimal value function: $v_\star(s) = \sup_\Pi v_\Pi(s)$
- Defined state by state so that it is not necessarily attained by a single $\Pi^\star$

Optimal Bellman operator

- Similar to the Bellman operator but do not depend on a policy:

$$\mathcal{T}^\star v(s) = \sup_a \left(r(a,s) + \sum_{s'} p(s'|s,a) v(s') \right)$$

Link between the two

- $v \geq \mathcal{T}^\star v$ implies $v \geq v_\star$.
- $v \leq \mathcal{T}^\star v$ implies $v \leq v_\star$.

$\Rightarrow \quad v = v_\pi \quad \Rightarrow \quad v_\pi \leq \mathcal{T} v_\pi$

$v \geq T^* v \quad \Rightarrow \quad v \geq v^*$

$v^*(s) = \sup_{\pi} v_\pi(s)$

$\pi = (\pi_0, \dots, \pi_t, \dots) \Rightarrow v_\pi \quad \rightsquigarrow \quad v_\pi \leq v$

$\hookrightarrow v_{0\pi} = T^{\pi_0} v_{1\pi} \Rightarrow v_{0\pi} = T^{\pi_0}(T^{\pi_1} \dots (T^{\pi_t} v_{t+1,\pi}))$

$v \Rightarrow v \geq T^* v \geq T^{\pi_t} v$

$v \geq T^0(T^1 \dots T^{t} v))$ $\quad \swarrow \gamma^t \|v - v_\pi\|_\infty$

$v = T^0(T^1 \dots T^t v_\pi) + \left| T^0(T^1 \dots (T^t v_\pi) - T \dots T v) \right|$

Optimal Value Function and Bellman Operator

Bellman Operator and Fixed Point

- **Prop:** $\mathcal{T}^\star$ is a γ-contraction for the sup-norm and thus it exists a unique $v_{\star\star}$ such that $v_{\star\star} = \mathcal{T}^\star v_{\star\star}$.

Fixed Point and Optimal Value Function

- **Prop:** : $v_\star = v_{\star\star}$ and is thus the unique fixed point of $\mathcal{T}^\star$.
- **Proof:** $v_{\star\star} = \mathcal{T}^\star v_{\star\star}$ and thus $v_{\star\star} = v_\star$ according the link between the optimal value function and the Bellman operator.

- Does this mean something about policies?

Optimal Policy and Bellman Operator

Bellman Operator and Policy

- **Prop:** For any v, any policy π_v satisfying

$$\pi_v(s) \in \underset{a}{\operatorname{argmax}} \left(r(a,s) + \sum_{s'} p(s'|s,a) v(s') \right)$$

is such that $\mathcal{T}^\star v(s) = \sup_\pi \mathcal{T}^\pi v(s) = \mathcal{T}^{\pi_v} v(s)$

Bellman Operator and Optimal Policy

- **Prop:** Any stationary policy $\pi_\star$ satisfying

$$\pi_\star(s) \in \underset{a}{\operatorname{argmax}} \left(r(a,s) + \sum_{s'} p(s'|s,a) v^\star(s') \right)$$

is optimal.

- **Proof:** Indeed by construction, $\mathcal{T}^\star v_\star = \mathcal{T}^{\pi_\star} v_\star$ and thus, as $\mathcal{T}^\star v_\star = v_\star$, $v_{\pi_\star} = v_\star$.

Optimal Policy and Bellman Operator

Summary

- It exists a unique $v_\star$ such that $\mathcal{T}^\star v_\star = v_\star$
- $\forall s, v_\star(s) = \sup_\pi v_\pi(s)$
- Any policy $\pi_\star$ satisfying:

$$\forall s, \pi_\star(s) \in \operatorname*{argmax}_a \left(r(a,s) + \sum_{s'} p(s'|s,a) v^\star(s') \right)$$

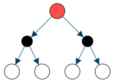
is optimal as $\forall s, v_{\pi_\star}(s) = v_\star(s) = \sup_\pi v_\pi(s)$

- Existence result but not (yet) a constructive algorithm!

Outline

Linear System and Linear Programming

$$v_\pi = \mathcal{T}^\pi v_\pi \qquad v_\star = \mathcal{T}^\star v_\star$$



Explicit Resolution of the Equations?

- Prediction:
  - Simple linear system for v_π.
  - Already mentionned before...
  - Complexity of order $(|A| + |S|)|S|^2$.
- Planning:
  - More complex linear programming system for $v_\star$ due to the max operator.
  - Optimal policy easily deduced from $v_\star$.
  - Complexity of order $(|A||S|)^3$.

Linear Programming

From $\forall s, v(s) = \sup_a r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s')$

to $\min_v \sum_s \mu(s) v(s)$

such that $\forall (s,a), v(s) \geq r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s')$

Different formulations but same solution

- Using $v \geq \mathcal{T}^\star v \Leftrightarrow v \geq v_\star$, the condition implies $v \geq v_\star$
- Now for any μ satisfying $\mu(s) > 0$, $\sum_s \mu(s) v(s) \geq \sum_s \mu(s) v_\star(s)$ as soon as the condition is satisfied, hence $v_\star$ is a solution.
- If for any state $v(s) > v_\star(s)$ then $\sum_s \mu(s) v(s) > \sum_s \mu(s) v_\star(s)$ and thus $v_\star$ is the unique minimizer.

Primal Problem

$$\text{Primal:} \quad \min_{v} \sum_{s} \mu(s) v(s)$$

$$\text{such that } \forall (s,a), v(s) \geq r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s')$$

Some properties

- Can be solved with a linear programming solver.
- Unicity of solution (and thus independence with respect to μ) can be proved without using $v_\star$.
  - **Proof:** let v_1 a solution for μ_1 and v_2 a solution for μ_2 then $\min(v_1, v_2)$ satifies the constraints. Furthermore if exists $v_2(s) < v_1(s)$ then $\min(v_1, v_2)$ is a strictly better solution for μ_2 which is impossible.

Dual Problem

Primal: $$\min_v \sum_s \mu(s) v(s)$$

$$\text{such that } \forall (s,a), v(s) \geq r(s,a) + \gamma \sum_{s'} p(s'|s,a) v(s')$$

Dual: $$\max_{\lambda(s,a)\geq 0} \sum_{s,a} \lambda(s,a) r(s,a)$$

$$\text{such that } \forall s, \sum_a \lambda(s,a) = \mu(s) + \gamma \sum_{s',a} p(s|s',a)\lambda(s',a)$$

Derivation

- Usual derivation through the Lagrangian:

$$\mathcal{L}(v,\lambda) = \sum_s \mu(s)v(s) + \sum_{s,a} \lambda(s,a) \left(r(s,a) + \gamma \sum_{s',a} p(s|s',a) v(s') - v(s) \right)$$

- Strong duality as Slater condition holds when $\gamma < 1$ with $v = \frac{1+\gamma}{1-\gamma} \max_{s,a} r(s,a)$.

Dual and Interpretation

$v_\pi = \sum \gamma^{t-1} P_\pi^t r_\pi$

$$\text{Dual:} \quad \max_{\lambda(s,a)\geq 0} \sum_{s,a} \lambda(s,a) r(s,a)$$

$$\text{such that } \forall s, \sum_a \lambda(s,a) = \mu(s) + \gamma \sum_{s',a} p(s|s',a)\lambda(s',a)$$

$$\text{Interpretation}: \max_\pi \sum_{k=0}^{\infty} \gamma^k \sum_{s,a} \mathbb{P}(S_t = s, A_t = a | S_0 \sim \mu, \pi)\, r(s,a)$$

Interpretation in terms of policy

- For any feasible λ, define $u(s) = \sum_a \lambda(s,a)$ and the policy $\pi(a|s) = \lambda(s,a)/u(s)$.
- **Prop:** $u = (\mathrm{Id} - \gamma P^\pi)\mu = \sum_{k=0}^\infty \gamma^k (P^\pi)^k \mu$.
- **Prop:** $\lambda(s,a) = \pi(a|s)u(s) = \sum_{k=0}^\infty \gamma^k \mathbb{P}(S_t = a, A_t = a | S_0 \sim \mu, \pi)$
- Conversely for any π they is a feasible λ.
- Any optimal $\lambda_\star$ (and thus policy) satisfies $\lambda_\star(s,a) = 0$ if $v_\star(s) > r(s,a) + \gamma \sum_{s'} p(s'|s,a) v_\star(s')$ (optimal policy support)

Outline

Finite Horizon: Planning by Value Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$
parameter: Horizon T
init: $v_T^T(s) = 0 \, \forall s \in \mathcal{S}, \; t = T$
repeat

  $t \leftarrow t - 1$

  for $s \in \mathcal{S}$ **do**

$$v_t^T(s) \leftarrow \max_{a \in \mathcal{A}} \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) v_{t+1}^T(s') \right)$$

  end

until $t = 0$

output: Deterministic policy $\pi_t(s) \in \operatorname{argmax}_{a \in \mathcal{A}} \left(r(s,a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s,a) v_{t+1}^T(s') \right)$

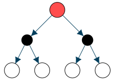
$v_t^T = \mathcal{T}^* v_{t+1}^T$

- Algorithm used to prove the existence of an optimal policy.
- No necessarily unique as argmax may not be unique.

Optimal Value Function, Fixed Point and Contraction

$$v_\star = \mathcal{T}^\star v_\star \quad \text{and} \quad \|\mathcal{T}^\star v - \mathcal{T}^\star v'\|_\infty \leq \gamma \|v - v'\|_\infty$$

$$\implies v_{k+1} = \mathcal{T}^\star v_k \to v_\star$$



Bellman Operator

- Properties of Optimal Bellman Operator:
  - $v_\star$ is a fixed point of $\mathcal{T}^\star$.
  - $\mathcal{T}^\star$ is a γ-contraction for the $\|\cdot\|_\infty$ norm.
- Classical fixed point theorem setting.
- Practical algorithm to approximate $v_\star$.

Value Iteration Algorithm

Discounted: Value Iteration Planning

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$, and discount factor γ
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s)\,\forall\, s\in\mathcal{S}$
repeat
  $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$
  $\Delta \leftarrow 0$
  for $s\in\mathcal{S}$ **do**
    $\tilde{v}(s) \leftarrow \max_{a\in\mathcal{A}} r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}_{\text{prev}}(s')$
    $\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\text{prev}}(s)|\right)$
  end
until $\Delta < \delta$
output: Value function $\tilde{v}$

- Same convergence criterion (and similar proof) than in the planning case.
- Which policy?

Value Iteration Algorithm

Discounted: Value Iteration Planning

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$, and discount factor γ
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s)\,\forall\, s\in\mathcal{S}$
repeat
    $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$
    $\Delta \leftarrow 0$
    for $s\in\mathcal{S}$ **do**
        $\tilde{v}(s) \leftarrow \max_{a\in\mathcal{A}} r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}_{\text{prev}}(s')$
        $\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\text{prev}}(s)|\right)$
    end
until $\Delta < \delta$
output: Deterministic policy $\tilde{\pi}(s) \in \underset{a}{\operatorname{argmax}}\, r(s,a) + \gamma \sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}(s')$

$\tilde{v} \to v_*$

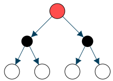
$? \; v_{\tilde{\pi}} \to v_* ?$

- Natural idea: define a policy using the argmax of the existence proof.
- Do we have a convergence guarantee on the resulting policy?

Value and argmax Policy

$$\tilde{\pi}(s) \in \operatorname*{argmax}_a r(s,a) + \gamma \sum_{s'} p(s'|s,a)\tilde{v}(s')$$

$$\implies \|v_{\tilde{\pi}} - v_\star\|_\infty \leq \frac{2\gamma}{1-\gamma}\|\tilde{v} - v_\star\|_\infty$$



Value and argmax Policy

- Bound on the loss of the final policy!
- Rely on the fact that, by construction, $\mathcal{T}^{\tilde{\pi}}\tilde{v} = \mathcal{T}^\star\tilde{v}$
- **Proof:**

$$\begin{aligned}
\|v_{\tilde{\pi}} - v_\star\|_\infty &= \|\mathcal{T}^{\tilde{\pi}} v_{\tilde{\pi}} - \mathcal{T}^{\tilde{\pi}}\tilde{v} + \mathcal{T}^\star\tilde{v} - \mathcal{T}^\star v_\star\|_\infty \\
&\leq \|\mathcal{T}^{\tilde{\pi}} v_{\tilde{\pi}} - \mathcal{T}^{\tilde{\pi}}\tilde{v}\|_\infty + \|\mathcal{T}^\star\tilde{v} - \mathcal{T}^\star v_\star\|_\infty \\
&\leq \gamma\|v_{\tilde{\pi}} - \tilde{v}\|_\infty + \gamma\|\tilde{v} - v_\star\|_\infty \\
&\leq \gamma\|v_{\tilde{\pi}} - v_\star\|_\infty + 2\gamma\|\tilde{v} - v_\star\|_\infty
\end{aligned}$$

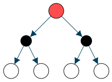
Value Iteration Algorithm

Discounted: Value Iteration Planning

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, and discount factor γ
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{v}(s) \, \forall \, s \in \mathcal{S}$
repeat
  $\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$
  $\Delta \leftarrow 0$
  for $s \in \mathcal{S}$ **do**
    $\tilde{v}(s) \leftarrow \max_{a \in \mathcal{A}} r(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a) \tilde{v}_{\text{prev}}(s')$
    $\Delta \leftarrow \max\left(\Delta, |\tilde{v}(s) - \tilde{v}_{\text{prev}}(s)|\right)$
  end
until $\Delta < \delta$
output: Deterministic policy $\tilde{\pi}(s) \in \operatorname*{argmax}_a r(s, a) + \gamma \sum_{s' \in \mathcal{S}} p(s'|s, a) \tilde{v}(s')$

- **Prop:** $\|v_{\tilde{\pi}} - v_\star\|_\infty \leq \dfrac{4\gamma\delta}{1 - \gamma}$

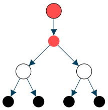
From State Value to State-Action Value Functions



$$v_\pi(s) = \mathbb{E}_\pi\left[\sum_k \gamma^k R_t | S_0 = s\right]$$

$$\mathcal{T}^\pi v(s) = \sum_a \pi(a|s)\left(r(s,a) + \sum_{s'} p(s'|s,a) v(s')\right)$$

$$\mathcal{T}^\star v(s) = \max_a r(s,a) + \sum_{s'} p(s'|s,a) v(s')$$



$$q_\pi(s,a) = \mathbb{E}_\pi\left[\sum_k \gamma^k R_t | S_0 = s, A_0 = a\right]$$

$$\mathcal{T}^\pi q(s,a) = r(s,a) + \sum_{s'} p(s'|s,a) \sum_a \pi(a|s') q(s',a)$$

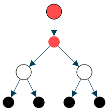
$$\mathcal{T}^\star q(s,a) = r(s,a) + \sum_{s'} p(s'|s,a) \max_a q(s',a)$$

Two equivalent point of view?

- Everything could have been defined using the state-action point of view.
- Knowing v_π is equivalent to knowing q_π as

$$v_\pi(s) = \sum_a \pi(s|a) q_\pi(s,a) \quad \text{and} \quad q_\pi(s,a) = r(s,a) + \sum_{s'} p(s'|s,a) v_\pi(s').$$

State-Action Bellman Operators



$$\mathcal{T}^{\pi} q(s,a) = r(s,a) + \sum_{s'} p(s'|s,a) \sum_{a} \pi(a|s') q(s',a)$$

$$\mathcal{T}^{\star} q(s,a) = r(s,a) + \sum_{s'} p(s'|s,a) \max_{a} q(s',a)$$

Properties

- **Prop:** $\mathcal{T}^{\pi}$ and $\mathcal{T}^{\star}$ are γ contractions for the $\|\cdot\|_{\infty}$ norm.
- **Prop:** q_{π} is the unique solution of $\mathcal{T}^{\pi} q = q$
- **Prop:** $q_{\star}$ defined $q_{\star}(s,a) = \sup_{\pi} q_{\pi}(s,a)$ is the unique solution of $q = \mathcal{T}^{\star} q$ and is attained for any policy $\pi_{\star}$ satisfying $\pi_{\star}(s) \in \operatorname{argmax} q_{\star}(s,a)$.
- **Prop:** Any such policy satisfies: $v_{\pi_{\star}}(s) = q_{\pi_{\star}}(s, \pi_{\star}(s)) = v_{\star}(s)$.

State-Action Value Iteration Algorithm

Discounted: Planning by State-Action Value Iteration

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$, and discount factor γ
parameter: $\delta > 0$ as accuracy termination threshold
init: $\tilde{q}(s,a)\,\forall\,(s,a)\in\mathcal{S}\times\mathcal{A}$
repeat
  $\tilde{q}_{\text{prev}} \leftarrow \tilde{q}$
  $\Delta \leftarrow 0$
  for $s\in\mathcal{S}$ **do**
    for $a\in\mathcal{A}$ **do**

$$\tilde{q}(s,a) \leftarrow \left(r(s,a)+\gamma\sum_{s'\in\mathcal{S}} p(s'|s,a)\max_{a'}\tilde{q}_{\text{prev}}(s',a')\right)$$

$$\Delta \leftarrow \max\left(\Delta, |\tilde{q}(s,a)-\tilde{q}_{\text{prev}}(s,a)|\right)$$

    end
  end
until $\Delta < \delta$
output: Deterministic policy $\tilde{\pi}(s)\in\operatorname*{argmax}_a \tilde{q}(s,a)$

- Same complexity but more storage than with state value function...
- but will be useful later!

Outline

Value Fonction vs Policy Point of View

$$v, q \longrightarrow \Pi \quad \text{or} \quad \Pi \longrightarrow v, q?$$

Planning

- Focus so far on value-fonction point of view!
- Heuristic: find a good approximation of the optimal value function and deduce a good policy.
- Can we work directly on the policy itself?

- For prediction, only the policy point of view makes sense!

Toward Policy Improvement

$$\forall s, \pi_+(s) \in \operatorname*{argmax}_a q_\pi(s,a) \Longrightarrow \forall v_{\pi_+}(s) \geq v_\pi(s)$$

Classical Policy Improvement Lemma

- **Prop:** Given a policy π and its q value-function, one can obtain a better policy with the argmax operator.
- **Prop:** If no improvement is possible, it means that π is already optimal.
- **Proof:** Use $\mathcal{T}^{\pi_+} v_\pi = \mathcal{T}^\star v_\pi \geq \mathcal{T}^\pi v_\pi = v_\pi$ to prove $\left(\mathcal{T}^{\pi_+}\right)^k v_\pi \geq v_\pi$ which implies the result by letting k goes to $+\infty$.

- Leads to a sequential improvement algorith. . .

Policy Improvement Lemma

$$\mathbb{E}[v_{\pi'}(S_0)] - \mathbb{E}[v_\pi(S_0)] = \sum_{k=0}^{\infty} \gamma^k \mathbb{E}_{\pi'}\left[\sum_a \pi'(a|S_t)\left(q_\pi(S_t,a) - v_\pi(S_t)\right)\right]$$
$$= \sum_{k=0}^{\infty} \gamma^k \mathbb{E}_{\pi'}\left[\sum_a \left(\pi'(a|S_t) - \pi(a|S_t)\right) q_\pi(S_t,a)\right]$$

A Generic Improvement Lemma

- No assumptions on π and π'!
- Easy proof.
- Imply the previous lemma as $\max_a Q_\pi(s,a) - v_\pi(s) \geq 0$.
- Show that improvement choices are possible.

- Will prove to be useful later...

Policy Iteration

Discounted: Planning by Policy Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, and discount factor γ
parameter: Initial policy $\tilde{\pi}$
repeat
    Compute $q_{\tilde{\pi}}$.
    for $s \in \mathcal{S}$ **do**
        for $a \in \mathcal{A}$ **do**
            $\tilde{\pi}(s) \leftarrow \text{argmax}\, q_{\tilde{\pi}}(s, a)$
        end
    end
output: Deterministic policy $\tilde{\pi}$.

Some issues

- How to obtain q_{π}?
- When to stop?

Policy Iteration

Discounted: Planning by Policy Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, and discount factor γ
parameter: Initial policy $\tilde{\pi}$
repeat
  stable $\leftarrow 1$
  Compute $q_{\tilde{\pi}}$.
  for $s \in \mathcal{S}$ **do**
    $old - action \leftarrow \tilde{\pi}(s)$
    $\tilde{\pi}(s) \leftarrow \text{argmax}\, q_{\tilde{\pi}}(s, a)$
    if $\tilde{\pi}(s) \neq old - action$ **then**
      stable $\leftarrow 0$
    end
  end
until *stable* $== 1$
output: Deterministic policy $\tilde{\pi}$.

Finite Setting

- Finite set of action-states implies a finite set of policy.
- Convergence of the algorithm in finite time!

Policy Iteration

Convergence Rate

- Crude analysis:
  - Bound after k steps of the algorithm
$$\|v_{\pi_k} - v_\star\|_\infty \leq \gamma \|v_{\pi_{k-1}} - v_\star\|_\infty \leq \gamma^k \|v_{\pi_0} - v_\star\|_\infty$$
$$\|v_{\pi_k} - v_\star\|_\infty \leq \frac{\gamma}{1-\gamma} \|v_{\pi_k} - v_{\pi_{k-1}}\|_\infty$$
  - Not much better than value iteration but much higher complexity as q_{π_k} is obtained by solving the Bellman equation!
- Much faster in practice...
- Clever analysis (Putterman):
  - Under some mild assumptions and provided $\|P^{\pi_k} - P^\star\| \leq K \|v_{\pi_k} - v_\star\|_\infty$ then
$$\|v_{\pi_k} - v_\star\|_\infty \leq \frac{K\gamma}{1-\gamma} \|v_{\pi_{k-1}} - v_\star\|_\infty$$
  - May explain the better convergence in practice!

Outline

Value Iteration: (Relaxed) First Order Method

Value Iteration

- Iteration:

$$v_k = \mathcal{T}^\star v_{k-1}$$
$$= v_{k-1} + \left(\mathcal{T}^\star - \mathrm{Id}\right) v_{k-1}$$

- Relaxation

$$v_k = v_{k-1} - \alpha \left(\mathrm{Id} - \mathcal{T}^\star\right) v_{k-1}$$

  can be proved to converge for any $\alpha < \frac{2}{1+\gamma}$.
- Can be interpreted as a first order method with pseudo-gradient $\left(\mathcal{T}^\star - \mathrm{Id}\right) v_{k-1}$.
- No function corresponding to this gradient!

- Is there a better choice for α than $\alpha = 1$?
- No as the resulting operator is a contraction of constant

$$|1 - \alpha| + \alpha\gamma \geq \gamma$$

Policy Iteration: Newton-Raphson Method

Policy Iteration

- Explicit iteration:

$$\text{Solve } v_{\pi_{k-1}} = \mathcal{T}^{\pi_k} v_{\pi_{k-1}}$$
$$\text{Let } \pi_k \text{ such that } \mathcal{T}^{\pi_k} v_{\pi_{k-1}} = \mathcal{T}^\star v_{\pi_{k-1}}$$

- Implicit iteration on v_{π_k}:

$$\begin{aligned} v_{\pi_k} &= (\mathrm{Id} - \gamma P^{\pi_k})^{-1} r_{\pi_k} \\ &= (\mathrm{Id} - \gamma P^{\pi_k})^{-1} \left(r_{\pi_k} + (\gamma P^{\pi_k} - \mathrm{Id}) v_{\pi_{k-1}} + (\mathrm{Id} - \gamma P^{\pi_k}) v_{\pi_{k-1}} \right) \\ &= v_{\pi_{k-1}} - (\mathrm{Id} - \gamma P^{\pi_k})^{-1} (\mathrm{Id} - \mathcal{T}^{\pi_k}) v_{\pi_{k-1}} \end{aligned}$$

- Can be interpreted as a second order method with pseudo-gradient $(\mathrm{Id} - \mathcal{T}^{\pi_k}) v_{\pi_{k-1}} = (\mathrm{Id} - \mathcal{T}^\star) v_{\pi_{k-1}}$ and pseudo-Hessian $(\mathrm{Id} - \gamma P^{\pi_k})$.

- Not a formal analysis but give a good insight on the better convergence of policy iteration.

Outline

Stability of Value and Policy Iteration

Ideal Value and Policy Iteration?

- Iterative algorithms.
- Convergence proofs assume perfect computation.
- What happens if we make a (small) error at each step?

- Particularly important for Policy Iteration in which one resolves a linear system at each step!

Value Iteration Stability

$$v_k = \mathcal{T}^\star v_{k-1} + \epsilon_{k-1}$$

$$\implies \|v_k - v_\star\|_\infty \leq \gamma^k \|v_0 - v_\star\|_\infty + \frac{\max_{0 \leq k' < k} \|\epsilon_{k'}\|_\infty}{1-\gamma}$$

$$\implies \|v_{\pi_k} - v_\star\|_\infty \leq \frac{2\gamma^{k+1}}{1-\gamma} \|v_0 - v_\star\|_\infty + \frac{2\gamma \max_{0 \leq k' < k} \|\epsilon_{k'}\|_\infty}{(1-\gamma)^2}$$

Stability with respect to the error

- Proof relies on the contraction property of $\mathcal{T}^\star$ (hence similar results for $\mathcal{T}^\pi$).
- Error term $\frac{\max_{0 \leq k' < k} \|\epsilon_{k'}\|_\infty}{1-\gamma}$ can be replaced by $\sum_{k'=0}^{k-1} \gamma^{k-k'} \|\epsilon_{k'}\|_\infty$
- Convergence if $\|\epsilon_k\|_\infty$ tends to 0.
- Remains in a neighborhood of the optimal solution if $\|\epsilon_k\|_\infty$ is bounded.

Policy Iteration

$$v_{k-1} = v_{\pi_{k-1}} + \epsilon_{k-1} \quad \text{and} \quad \mathcal{T}^{\pi_k} v_{k-1} = \mathcal{T}^{\star} v_{k-1}$$

$$\implies \|v_{\pi_k} - v_\star\|_\infty \leq \gamma^k \|v_{\pi_0} - v_\star\|_\infty + \frac{\gamma(2-\gamma)\max_{0\leq k'<k} \|\epsilon_{k'}\|_\infty}{(1-\gamma)^2}$$

Stability with respect to the error

- Quite involved proof but crude results.
- Error term $\frac{\max_{0\leq k'<k} \|\epsilon_{k'}\|_\infty}{1-\gamma}$ can be replaced by $\sum_{k'=0}^{k-1} \gamma^{k-k'} \|\epsilon_{k'}\|_\infty$
- Convergence if $\|\epsilon_k\|_\infty$ tends to 0.
- Remains in a neighborhood of the optimal solution if $\|\epsilon_k\|_\infty$ is bounded.

- Policy Iteration only requires an approximate estimate of $v_{\pi_{k-1}}$, for instance obtained by Bellman iteration...

Outline

Modified Policy Iteration

Modified

Discounted: Planning by ~~Generalized~~ Policy Iteration

input: MDP model $\langle(\mathcal{S}, \mathcal{A}, \mathcal{R}), P\rangle$, and discount factor γ
parameter: Initial q
repeat
  for $s \in \mathcal{S}$ **do**
    $\tilde{\pi}(s) \leftarrow \text{argmax}\, q(s, a)$
  end
  repeat
    $q_{\text{prev}} \leftarrow q$
    for $(s, a) \in \mathcal{S} \times \mathcal{A}$ **do**

$$q(s, a) \leftarrow r(s, a) + \gamma \sum_{s, a'} p(s'|s, a)\tilde{\pi}(a'|s) q_{\text{prev}}(s, a)$$

    end
output: Deterministic policy $\tilde{\pi}$.

- Algorithm driven by q.
- Flexibility in the number of prediction steps after each policy improvement steps.
- Special cases:
  - Large number: Policy Iteration with (small) error.
  - One: Value Iteration!

MPI Analysis

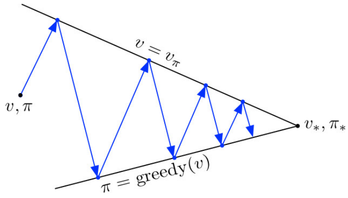
$$\mathcal{T}^{\pi_k} v_k = \mathcal{T}^{\star} v_k \quad \text{and} \quad v_{k+1} = \left(\mathcal{T}^{\pi_k}\right)^{m_k} v_k$$
$$\implies \|v_{k+1} - v_\star\|_\infty \leq \gamma \left(\frac{1 - \gamma^{m_k}}{1 - \gamma} \|P^{\pi_k} - P^\star\| + \gamma^{m_k} \right) \|v_k - v_\star\|_\infty$$

Convergence Results

- Quite technical proof.
- Valid only under the mild assumption $\mathcal{T}^\star v_0 \geq v_0$.
- Very fast decay provided $\|P^{\pi_k} - P^\star\|$ is small.

- No stability with arbitrary errors...

Generalized Policy Iteration



General Policy Iteration

- Two simultaneous interacting processes:
  - One forcing the policy to correspond to the current value function (Policy Improvement)
  - One trying to male the current value function coherent with the current policy (Policy Evaluation)
- Several variations possible on the two processes.

- In GPI, the policy is driven by the value function.
- Typically, stabilizes only if one reaches the optimal value/policy pair.

Source: Sutton and Barto

State Update Order

Discounted: Prediction by Value Iteration - State Update Order

input: MDP model $\langle(\mathcal{S},\mathcal{A},\mathcal{R}),P\rangle$, discount factor γ, and stationary policy π
init: $\tilde{v}(s)\,\forall\, s\in\mathcal{S}$
repeat

$\tilde{v}_{\text{prev}} \leftarrow \tilde{v}$

for $s\in\mathcal{S}'\subset\mathcal{S}$ **do**

$$\tilde{v}(s) \leftarrow \sum_{a\in\mathcal{A}} \pi(a|s)\left(r(s,a)+\gamma\sum_{s'\in\mathcal{S}} p(s'|s,a)\tilde{v}_{\text{prev}}(s')\right)$$

end

output: Value function $\tilde{v}$

Classical strategies

- $\mathcal{S}'=\mathcal{S}$: classical iteration
- $\mathcal{S}'=\{s\}$: Gauss-Seidel
- $\mathcal{S}'=\{s,|\mathcal{T}^\pi\tilde{v}(s)-\tilde{v}(s)|>\epsilon\}$: Prioritized sweeping

- Converges provided all states are visited infinitely often. . .
- Gain in term of storage or focus on most interesting states. . .

Policy Improvement Variation

$$\text{Greedy}: \pi(s) \in \underset{a}{\operatorname{argmax}}\, q(s,a) \iff \pi(|s) \in \underset{\tilde{\pi}}{\operatorname{argmax}} \sum_a \tilde{\pi}(a) q(s,a)$$

$$\text{Restricted}: \pi(\cdot|s) \in \underset{\tilde{\pi}\in\tilde{\Pi}_\epsilon}{\operatorname{argmax}} \sum_a \tilde{\pi}(a) q(s,a)$$

$$\text{Regularized}: \pi(\cdot|s) \in \underset{\tilde{\pi}}{\operatorname{argmax}} \sum_a \tilde{\pi}(a) q(s,a) + \epsilon P(\tilde{\pi})$$

Classical Variations

- ϵ-greedy: Restrict $\tilde{\pi}$ to the set of policy s.t. $\tilde{\pi}(a) \geq \epsilon$
  - Explicit solution: $\pi(a|s) = \epsilon + (1-\epsilon) \operatorname{argmax} q(s,a)$
  - Policy improvement property if ϵ decreases.
- Soft-max: Regularize by $\epsilon H(\tilde{\pi})$ where H is the entropy.
  - Explicit solution: $\pi(a|s) \propto \exp(q(s,a)/\epsilon)$
  - No classical policy improvement...

- Tends to greedy when ϵ goes to 0.
- Will proved to be interesting later...

Outline

Infinite Setting

- No issue with the rewards as only the expectation is used.
- All the theory remains valid if the states are countable, but there is an issue in the algorithms as we need to store/update an infinite number of states.
- The proof of existence of an optimal policy requires the max to be attained, which cannot be ensured in an infinite (even countable setting).

Some results...

- **Thm:** If S is coutable, there exists an ϵ-optimal (stationary) policy for any $\epsilon > 0$.
- **Thm:** If S is a Polish space (completely metrizable topological space),
  - there exists a (P, ϵ)-optimal (stationary policy) for any $\epsilon > 0$.
  - if each A_s is countable, there exists an ϵ-optimal (stationary) policy for any $\epsilon > 0$.
  - if each A_s is finite, there exists an optimal (stationary) policy.
  - if each A_s is a compact metric space, $r(s, a)$ is a bounded u.s.c. function on A_s and $p(B|s, a)$ is continuois in a for each Borel subset B and any s, thre exist an optimal (stationary) policy.

- **Mainly technical difficulties...**

Total Reward

$$v_\Pi(s) = \mathbb{E}_\Pi\left[\sum_{t'=1}^{+\infty} R_{t+1} \middle| S_0 = s\right]$$

$$= \underbrace{\mathbb{E}_\Pi\left[\sum_{t'=1}^{+\infty} \max(0, R_{t+1}) \middle| S_t = s\right]}_{v_{+,\Pi}(s)} - \underbrace{\mathbb{E}_\Pi\left[\sum_{t'=t+1}^{+\infty} \max(0, -R_{t+1}) \middle| S_t = s\right]}_{v_{-,\Pi}(s)}$$

- Total reward not necessarily well defined!
- Need to **assume** this is the case!

Classical Assumptions

- Episodic model: $\forall \Pi, s,\ \mathbb{E}_\Pi\left[\min_{t, \forall t' \geq t, R_{t'}=0} \middle| S_0 = s\right] < +\infty$
- More general assumption: $\forall \Pi, s$ eiher $v_{+,\Pi}(s)$ or $v_\Pi(s)$ is finite.

Bellman Operator and Optimality Equation

$$\sup_{\Pi} v_{\Pi}(s) = v_\star(s) = \underbrace{\max_a r(s,a) + \sum_{s'} p(s'|s,a) v_\star(s')}_{\mathcal{T}^\star(v_\star)(s)}$$

- Similar to the discounted setting as:
  - We can focus on Markovian policy.
  - The optimal value $v_\star$ satisfies the Bellman optimality equation.

But...

- $\mathcal{T}^\star$ is not a contraction and thus there may be several solution of the equation.
- If π is such that $\mathcal{T}^\pi v_\star = \mathcal{T}^\star v_\star$, we need to assume that $\limsup (P^\pi)^n v_\star(s) \leq 0$ to prove that $\Pi = (\pi, \pi, \dots)$ is optimal.
- There may not exists an optimal policy!

- Existence of optimal policies in the finite state-action setting by defining the total reward to the limit of discounted setting when $\gamma \to 1$ and using the finiteness of the policy set...

Positive Bounded and Negative Models

Positive Bounded Models

- $\forall \Pi, s, \;\; v_{+,\Pi}(s) < \infty$
- $\forall s, \exists a, r(s,a) \geq 0$

- Often stronger assumption: $r(s,a) \geq 0$.
- Any discounted model can be put in this framework by adding an absorbing state reached at random at each step with probability $1 - \gamma$.

Negative Models

- $\forall \Pi, s, \;\; v_{+,\Pi}(s) = 0$ and $v_{-,\Pi}(s) < \infty$
- There exists a policy Π such that $\forall s, v_{\Pi}(s) > -\infty$

- Maximization of v_{Π} amounts to the minization of $v_{-,\Pi}$ and the negative reward can be interpreted as the opposite of costs.
- Stochastic Shortest Path within this framework.

Positive Bounded and Negative Models Results

Result	Positive Bounded Models	Negative Models
Optimality equation	$v^\star$ is a minimal solution within $v \leq \mathcal{T}^\star v$	$v^\star$ is a maximal solution within $v \geq \mathcal{T} v$
$\mathcal{T}^\pi v_\star = \mathcal{T}^\star v_\star \Rightarrow \pi$ optimal	Only if $\limsup (P^\pi)^n v_\star(s) = 0$	Always
Existence of optimal stationary policy	S and A finite or existence of optimal policy and $r \geq 0$	A_s finite or A_s compact, r and p *continuous* with respect to a.
Existence of stationary ϵ-optimal policy	If $v^\star$ is bounded	Not always (Always for non stationary policy)
Value Iteration converges	$0 \leq v_0 \leq v_\star$	$0 \geq v_0 \geq v_\star$ and A_s finite or S finite if $v_\star > -\infty$
Policy Iteration converges	Yes	Not always
Modified Policy Iteration converges	$0 \leq v_0 \leq v_\star$ and $v_0 \leq \mathcal{T}^\star v_0$	Not always
Solution by linerar programming	Yes	No

Average Return

$$\bar{v}_\Pi(s) = \lim_{T\to\infty} \frac{1}{T} v_{T,\Pi}(s) = \lim_{T\to\infty} \frac{1}{T} \mathbb{E}_\Pi\left[\sum_{t=1}^{T} R_t \middle| S_0 = s\right]$$

$$\longrightarrow \bar{v}_{+,\Pi}(s) = \limsup_{T\to\infty} \frac{1}{T} v_{T,\Pi}(s)$$

$$\bar{v}_{-,\Pi}(s) = \liminf_{T\to\infty} \frac{1}{T} v_{T,\Pi}(s)$$

Average Return(s)

- Limit $\bar{v}_\Pi$ may not be defined!
- **Prop:** $\bar{v}_\Pi$ is well defined if Π is stationary and $\frac{1}{T}\sum_{t=1}^{T}(P^\pi)^{t-1}$ tends to a stochastic matrix.
- Limits $\bar{v}_{+,\Pi}$ and $\bar{v}_{-,\Pi}$ always defined!

Average Returns and Optimality

$$\overline{v}_{+,\star}(s) = \sup_{\Pi} \overline{v}_{+,\Pi}(s) \quad \text{and} \quad \overline{v}_{-,\star}(s) = \sup_{\Pi} \overline{v}_{-,\Pi}(s)$$

Optimality of $\Pi_\star$

- Average optimal:
$$\overline{v}_{-,\Pi_\star} \geq \overline{v}_{+,\star}(s)$$
- Lim-sup average optimal (best case analysis):
$$\overline{v}_{+,\Pi_\star} \geq \overline{v}_{+,\star}(s)$$
- Lim-inf average optimal (worst case analysis):
$$\overline{v}_{-,\Pi_\star} \geq \overline{v}_{-,\star}(s)$$

- More complex setting!
- Let's start with Prediction...

Prediction for a Stationary Markov Policy

$$\overline{v}_\Pi(s) = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} P_\pi^{t-1} r_\pi = \left(\lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} P_\pi^{t-1} \right) r_\pi = P_\pi^\infty r_\pi$$

Stochastic Matrix P_π^∞

- Measures the average amount of time spend on a state s' starting from state s at $t = 0$ when using policy π.
- Structure linked to the properties of the resulting Markov chain:
  - If aperiodic, $P_\pi^\infty = \lim_T P_\pi^T$ i.e. P_π^∞ is close to the probability of reaching s' from s at any large T.
  - If unichain, then P_π^∞ has identical rows and corresponds to the stationary distribution.
  - If multichhain, then P_π^∞ has a diagonal block structure with rows equal withing each block corresponding to the stationary distribution in each chain.

- Implies that $\overline{v}_\Pi(s) = \overline{v}_\Pi(s')$ in the Markov process is unichain.
- Limit P_π^∞ may be hard to compute...

Average Reward and Relative Value Functions

$$U_\pi(s) = \mathbb{E}_\pi\left[\sum_{t=1}^{\infty}(R_t - \overline{v}_\pi(S_t))\middle| S_0 = s\right] \quad \Leftrightarrow U_\pi = \underbrace{(\mathrm{Id} - P_\pi + P_\pi^\infty)^{-1}(\mathrm{Id} - P_\pi^\infty)}_{H_\pi} r_\pi$$

Link between U_π and $\overline{v}_\pi$

- $(\mathrm{Id} - P_\pi)\overline{v}_\pi = 0$
- $\overline{v}_\pi + (I - P_\pi)U_\pi = r_\pi$

Characterization by a system

- If $(\mathrm{Id} - P_\pi)\overline{v} = 0$ and $\overline{v} + (I - P_\pi)U = r_\pi$ then
  - $\overline{v} = \overline{v}_\pi$,
  - $U = U_\pi + u$ with $(I - P_\pi)u = 0$,
  - If $P_\pi^\infty U = 0$ then $u = 0$.

- Prediction possible by solving this system as we do not need U_π.

Optimality Equations

$$\overline{v}(s) = \max_a \sum_{s'} p(s'|s,a)\overline{v}(s')$$

$$U(s) + \overline{v}(s) = \max_{a \in B_s} r(s,a) + \sum_{s'} p(s'|s,a)U(s) \text{with } B_s = \{a | \sum_{s'} p(s'|s,a)\overline{v}(s') = \overline{v}(s)\}$$

$$\pi_\star(s) \in \operatorname{argmax}_{a \in B_s} r(s,a) + \sum_{s'} p(s'|s,a)U(s)$$

Existence

- If there is a solution $(\overline{v}, U)$ of the system then $\overline{v} = \overline{v}_\star$ and $\pi_\star$ is an optimal policy.
- There may exist other optimal policies not satisfying the argmax property.
- There may not exist solutions to the system.

- Associated relative value iteration and modified policy iteration can be defined.
- Convergence under strong assumptions. . .

Outline

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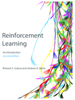

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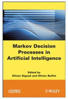

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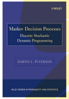

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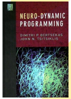

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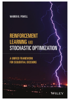

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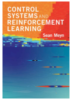

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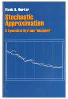

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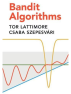

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