The colorful interior of families of convex bodies and its tropical analog

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1 Tropical posynomial systems

A tropical posynomial is a supremum of finitely many affine functions

$$P_{trop}(x) = \max_{a \in A} \left(\langle a, x\rangle + r_a\right)$$

where $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$ is a finite set (the tropical exponents).

Terminology comes from the tropical semifield $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ in which P_{trop} can be thought of as the analog of a posynomial.

For tropical posynomials $Q_1, \dots, Q_n$, let $S_{\mathbb{R}_{\max}}$ be the tropical posynomial system:

$$\forall i \in [n] \quad Q_i(x) = 0\,. \qquad (S_{\mathbb{R}_{\max}})$$

The corresponding feasability problem is NP-complete. These system are the analog of real posynomial systems:

$$\forall i \in [n] \quad \sum_{a \in A_i} r_a x^a = 1\,, \qquad (S_{\mathbb{R}})$$

where $(A_i)_{1 \le i \le n} \subset \mathbb{R}^n$ are finite and the r_a are positive real numbers.

2 From performance evaluation to tropical posynomials

Since 2014, we collaborate with the Parisian Fire Brigade and *Préfecture de Police* to analyze the performance of their joint emergency call center, implementing a bi-level filtering mechanism to prioritize urgent calls.

We have modeled the call center using a generalization of Petri nets [1] in order to take into account synchronization and concurrence phenomena.

Figure 1: Paris' 17-18-112 emergency call center

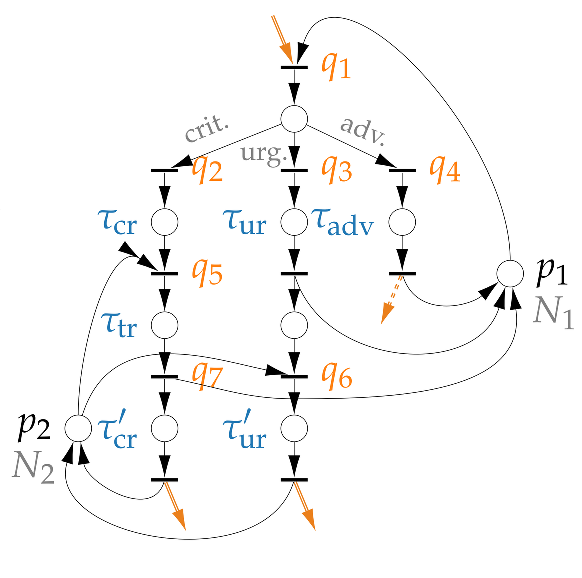

Figure 2: A Petri net model of the call center from [1]

Petri nets provide dynamic equations satisfied by counter variables (*e.g.* counting the number of calls):

$$z_1(t) = N_1 + z_5(t - \tau_{tr}) + \pi_u z_1(t - \tau_{ur}) + \pi_{adv} z_1(t - \tau_{adv})$$
$$z_5(t) = (N_2 + z_5(t - \tau_{tr} - \tau'_{cr}) + z_6(t - \tau'_{ur}) - z_6(t^-)) \wedge \pi_{cr} z_1(t - \tau_{cr})$$
$$z_6(t) = (N_2 + z_5(t - \tau_{tr} - \tau'_{cr}) + z_6(t - \tau'_{ur}) - z_5(t)) \wedge \pi_u z_1(t - \tau_{ur})$$

Computing stationary regimes (*i.e.* $z_i(t) = \rho_i t + u_i$) reduces to a tropical posynomial system over a semifield of germs of affine functions. This builds on a series of work representing several classes of Petri nets by tropical dynamical systems [2, 3, 1].

3 The colorful interior of a family of convex bodies

Let $\mathcal{V} = (V_1, V_2, \dots, V_n)$ be a family of subsets of $\mathbb{R}^n$, each one being assigned a color.

Definition. *We say a vector y is rainbow with respect to $\mathcal{V}$ if $y \in \operatorname{cone}(\bigcup_i V_i)$ and all linear decompositions of y use all the colors:*

$$\forall \mu \in (\mathbb{R}^+)^{\uplus_i V_i} \quad y = \sum_{i=1}^{n} \sum_{v \in V_i} \mu_v v \implies \forall i \in [n] \quad \exists v \in V_i \quad \mu_v > 0\,.$$

The set of rainbow vectors is denoted by $⋒\mathcal{V}$ and called the colorful interior of $\mathcal{V}$.

Proposition. *For all $i \in [n]$, let us denote by $\mathcal{P}_i = \operatorname{cone}(V_i)$, $\widehat{\mathcal{P}}_i = \operatorname{cone}(\bigcup_{j \neq i} V_j)$ and $\mathcal{P} = \operatorname{cone}(\bigcup_j V_j)$. Then*

$$⋒\mathcal{V} = \mathcal{P} \setminus \bigcup_{i \in [n]} \widehat{\mathcal{P}}_i$$

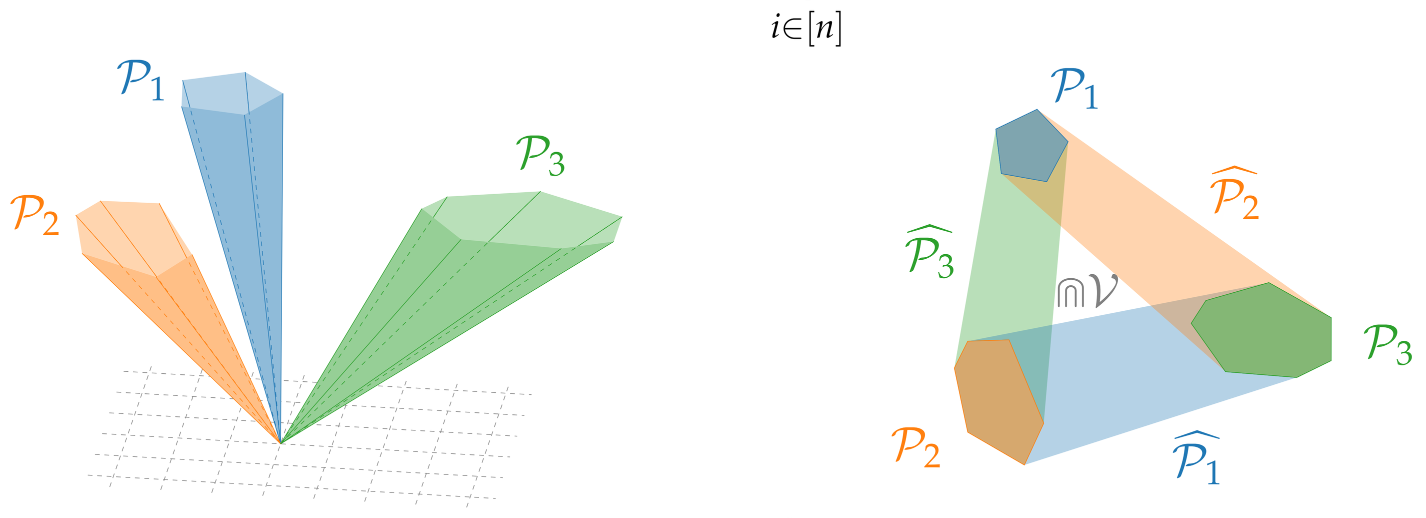

Figure 2: Three subsets V_1, V_2 and V_3 of $\mathbb{R}^3$ and their conic hulls (left) together with a cross-section (right). Here $⋒\mathcal{V}$ is the white triangle.

Definition. *We say the configuration $\mathcal{V}$ is pointed if $\bigcup_i V_i$ is contained in an open halfspace.*

4 Using rainbow vectors to solve tropical and real posynomial systems

Recall the tropical posynomial system we want to solve:

$$\forall i \in [n] \quad \max_{a \in A_i} \left(\langle a, x\rangle + r_a\right) = 0\,. \qquad (S_{\mathbb{R}_{\max}})$$

We fix a vector $y \in \mathbb{R}^n$ and introduce the following linear program:

$$\min_x \langle y, x\rangle \quad \text{s.t.} \quad \forall i \in [n] \quad \max_{a \in A_i} \langle a, x\rangle + r_a \le 0\,, \qquad \text{(LP)}$$

with dual:

$$\sup_{\mu \ge 0} \langle \mu, r\rangle \quad \text{s.t.} \quad -y = \sum_{i=1}^{n} \sum_{a \in A_i} \mu_a a\,. \qquad \text{(LD)}$$

Theorem. *If (LP) is feasible and $-y$ is rainbow with respect to the configuration $(A_1, \dots, A_n)$, then the optimal solutions of (LP) are precisely the solutions of the tropical posynomial system.*

Remark that $(A_1, \dots, A_n)$ being a pointed configuration ensures (LP) is feasible. This theorem carries over to the real case:

Theorem. *Let $S_{\mathbb{R}}$ be the following posynomial system:*

$$\forall i \in [n] \quad \sum_{a \in A_i} r_a x^a = 1\,, \qquad (S_{\mathbb{R}})$$

where $(A_i)_i$ are finite subsets of $\mathbb{R}^n$ and $(r_a)_a$ are positive.
If $(A_1, \dots, A_n)$ is a pointed configuration and has a non-empty colorful interior, then $S_{\mathbb{R}}$ has at least one positive solution (i.e. in $\mathbb{R}^n_{>0}$).

Proof uses the entropic program $\min_X \langle y, X\rangle$ s.t. $\forall i \in [n] \quad \log\left(\sum_{a \in A_i} r_a e^{\langle a, X\rangle}\right) \le 0\,.$

5 Properties of the colorful interior

We assume in this part that $\mathcal{V}$ is a pointed configuration.

Theorem. *$⋒\mathcal{V}$ is either empty or the interior of a simplicial cone.*

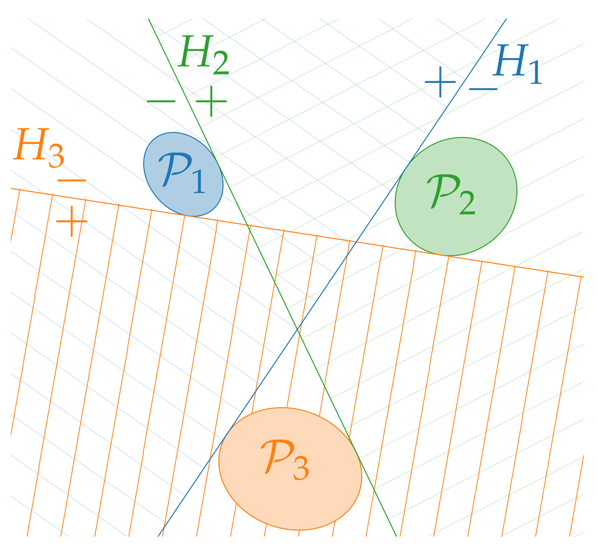

Figure 3: For convex V_1, V_2 and $V_3 \subset \mathbb{R}^3$, we have $⋒\mathcal{V} = H_1^{>0} \cap H_2^{>0} \cap H_3^{>0}$

This simplicial cone is the intersection of n open halfspaces, each one

- being tangent to $n-1$ bodies of $\mathcal{V}$,
- containing the last one.

In [6], Cappell *et al.* have stated when such hyperplanes exist:

Theorem [6]. *Let $\mathcal{A} = (A_1, \dots, A_n)$ be a family of n separated compact convex bodies of $\mathbb{R}^n$, then there are exactly two oriented affine hyperplanes that are tangent and outer to $\mathcal{A}$.*

Recall that if $\mathcal{A} = (A_1, \dots, A_p)$ is a family of convex bodies of $\mathbb{R}^n$, $\mathcal{A}$ is said to be separated if forall $k \le n$ and forall $(x_{i_1}, \dots, x_{i_k}) \in A_{i_1} \times \dots \times A_{i_k}$, $\operatorname{span}(x_{i_1}, \dots, x_{i_k})$ is k-dimensional.

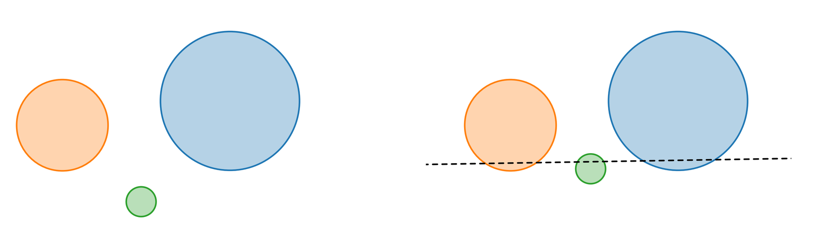

Figure 4: The three bodies are separated on the left figure, but not on the right one

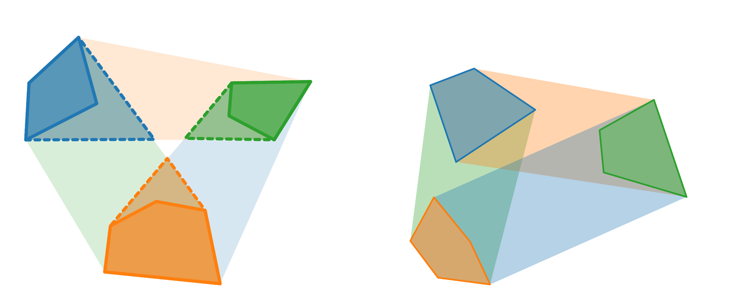

Figure 5: Left: The $(\overline{\mathcal{P}}_i)_i$ bodies visualised. Right: $⋒\mathcal{V} = \emptyset$ despite $(\mathcal{P}_i)_i$ are separated

For all $i \in [n]$, we define $\overline{\mathcal{P}}_i = \bigcap_{j \neq i} \widehat{\mathcal{P}}_j$.

Conjecture. *The following assertions are equivalent:*

(i) $⋒\mathcal{V} \neq \emptyset$

(ii) The family $(\overline{\mathcal{P}}_i)_i$ is separated

We proved $(i) \implies (ii)$ and since for all $i \in [n]$, $\mathcal{P}_i \subset \overline{\mathcal{P}}_i$, the separation of $(\mathcal{P}_i)_i$ is necessary to have $⋒\mathcal{V} \neq \emptyset$ (but not sufficient). We proved $(ii) \implies (i)$ for $n = 3$.

6 From the tropical colorful interior to tropical SVM

The definitions of the colorful interior of n convex bodies of $\mathbb{R}^n$ can be transposed to n tropical convex bodies of $(\mathbb{R}_{\max})^n$.

In [4], Gärtner and Jaggi introduced the notion of tropical support vector machines, *i.e.* tropical hyperplane H separating n data point clouds, each class lying in one unique sector of H.

Theorem. The tropical colorful interior of n tropical convex bodies of $(\mathbb{R}_{\max})^n$ is the locus of all apices of tropical SVM separating these bodies. If non empty, it is a polytrope (*i.e.* it is convex tropically *and* classically).

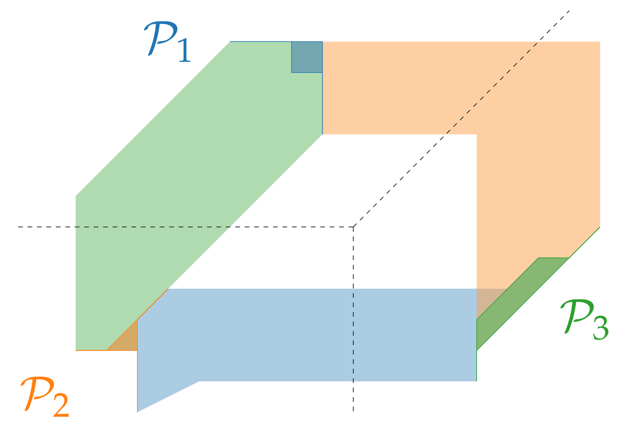

Figure 6: Three tropical simplices of $(\mathbb{R}_{\max})^3$, their colorful interior (in white), and a tropical SVM

Theorem. Deciding wether the tropical colorful interior $⋒\mathcal{V}$ is empty or not (and finding a point inside) can be achieved in strongly polynomial time.

This answers one of the questions raised in [4]. A key element of the proof is the following theorem inspired by a result from Lawrence and Soltan (see [5]) :

Theorem. The tropical colorful interior $⋒\mathcal{V}$ coincides with the intersection of the interior of all multicolor tropical simplices, *i.e.* having one vertex in each color set:

$$⋒\mathcal{V} = \bigcap_{(v_1, \dots, v_n) \in V_1 \times \dots \times V_n} \operatorname{int}(\operatorname{tconv}(v_1, \dots, v_n))\,.$$

7 References

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