Second Order Reflected Backward Stochastic Differential Equations

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Abstract

In this article, we study a class of reflected second order backward stochastic differential equations with a given lower càdlàg obstacle. We prove existence and uniqueness of the solution under a Lipschitz type assumption on the generator, and we investigate some links between our reflected 2BSDEs and non-classical optimal stopping problems. Finally, we show that reflected 2BSDEs provide a super-hedging price for American options in a market with volatility uncertainty.

Key words: Second order backward stochastic differential equation, reflected backward stochastic differential equation, optimal stopping time problem, American option, volatility uncertainty.

AMS 2000 subject classifications: 60H10, 60H30
1 Introduction

Backward stochastic differential equations (BSDEs for short) appeared in Bismuth [4] in the linear case, and then have been widely studied since the seminal paper of Pardoux and Peng [27]. Their range of applications includes notably probabilistic numerical methods for partial differential equations, stochastic control, stochastic differential games, theoretical economics and financial mathematics. On a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})\) generated by an \(\mathbb{R}^d\)-valued Brownian motion \(B\), a solution to a BSDE consists on finding a pair of progressively measurable processes \((Y, Z)\) such that

\[
Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \quad \mathbb{P} - a.s.
\]

where \(f\) (also called the driver) is a progressively measurable function and \(\xi\) is an \(\mathcal{F}_T\)-measurable random variable.

Pardoux and Peng proved existence and uniqueness of the above BSDE provided that the function \(f\) is uniformly Lipschitz in \(y\) and \(z\) and that \(\xi\) and \(f_s(0, 0)\) are square integrable.

Reflected backward stochastic differential equations (RBSDEs for short) were introduced by El Karoui, Kapoudjian, Pardoux, Peng and Quenez in [14], followed among others by El Karoui, Pardoux and Quenez in [16] and Bally, Caballero, Fernandez and El Karoui in [2] to study related obstacle problems for PDE’s and American options pricing. In this case, the solution \(Y\) of the BSDE is constrained to stay above a given obstacle process \(S\). In order to achieve this, a non-decreasing process \(K\) is added to the solution

\[
Y_t = \xi + \int_t^T f_s(Y_s, Z_s)ds - \int_t^T Z_s dB_s + K_T - K_t, \quad t \in [0, T], \quad \mathbb{P} - a.s.
\]

\[
Y_t \geq S_t, \quad t \in [0, T], \quad \mathbb{P} - a.s.
\]

\[
\int_0^T (Y_s - S_s)dK_s = 0, \quad \mathbb{P} - a.s.,
\]

where the last condition, also known as the Skorohod condition means that the process \(K\) is minimal in the sense that it only acts when \(Y\) reaches the obstacle \(S\). This condition is crucial to obtain the wellposedness of the classical RBSDEs.

Following those pioneering works, many authors have tried to relax the assumptions on the driver of the RBSDE and the corresponding obstacle. Hence, Matoussi [25] and Lepeltier, Matoussi and Xu [24] have extended the existence and uniqueness results to generator with arbitrary growth in \(y\). Then, Kobylanski, Lepeltier, Quenez and Torres [21], Lepeltier and Xu [23] and Bayraktar and Yao [3] studied the case of a generator which is quadratic in \(z\). Similarly, Hamadène [18] and Lepeltier and Xu [22] proved existence and uniqueness when the obstacle is no longer continuous.

More recently, motivated by applications in financial mathematics and probabilistic numerical methods for PDEs (see [7], [17], [30]), Cheredito, Soner, Touzi and Victoir [9] introduced the notion of Second order BSDEs (2BSDEs), which are connected to the larger class of fully nonlinear PDEs. Then, Soner, Touzi and Zhang [35] provided a complete theory of
existence and uniqueness for 2BSDEs under uniform Lipschitz conditions similar to those of Pardoux and Peng. Their key idea was to reinforce the condition that the 2BSDE must hold \( P - \text{a.s.} \) for every probability measure \( P \) in a non-dominated class of mutually singular measures (see Section 2 for precise definitions).

Our aim in this paper is to provide a complete theory of existence and uniqueness of Second order RBSDEs (2RBSDEs) under the Lipschitz-type hypotheses of [35] on the driver. We will show that in this context, the definition of a 2RBSDE with a lower obstacle \( S \) is very similar to that of a 2BSDE. We do not need to add another increasing process, unlike in the classical case, and we do not need to impose a condition similar to the Skorohod condition. The only change necessary is in the minimal condition that the increasing process \( K \) of the 2RBSDE must satisfy.

The rest of this paper is organised as follows. In Section 2, we recall briefly some notations, provide the precise definition of 2RBSDEs and show how they are connected to classical RBSDEs. Then, in Section 3, we show a representation formula for the solution of a 2RBSDEs which in turn implies uniqueness. We then provide some links between 2RBSDEs and optimal stopping problems. In Section 4, we give a proof of existence by means of r.c.p.d. techniques, as in [32] for quadratic 2BDEs. Let us mention that this proof required to extend existing results on the theory of \( g \)-martingales of Peng (see [28]) to the reflected case. Since to the best of our knowledge, those results do not exist in the litterature, we prove them in the Appendix in Section A. Finally, we use these new objects in Section 5 to study the pricing problem of American options in a market with volatility uncertainty.

### 2 Preliminaries

Let \( \Omega := \{ \omega \in C([0,1],[\mathbb{R}^d]) : \omega_0 = 0 \} \) be the canonical space equipped with the uniform norm \( \| \omega \|_\infty := \sup_{0 \leq t \leq T} |\omega_t| \), \( B \) the canonical process, \( P_0 \) the Wiener measure, \( \mathcal{F} := \{ \mathcal{F}_t \}_{0 \leq t \leq T} \) the filtration generated by \( B \), and \( \mathcal{F}^+ := \{ \mathcal{F}^+_t \}_{0 \leq t \leq T} \) the right limit of \( \mathcal{F} \). We first recall the notations introduced in [35].

#### 2.1 The Local Martingale Measures

We say a probability measure \( P \) is a local martingale measure if the canonical process \( B \) is a local martingale under \( P \). By Karandikar [20], there exists an \( \mathcal{F} \)-progressively measurable process, denoted as \( \int_0^t B_s dB_s \), which coincides with the Itô’s integral, \( P - \text{a.s.} \) for all local martingale measure \( P \). This allows to provide a pathwise definition of

\[
\langle B \rangle_t := B_t B_t^T - 2 \int_0^t B_s dB_s^T \text{ and } \hat{a}_t := \limsup_{\epsilon \downarrow 0} \frac{1}{\epsilon} (\langle B \rangle_t - \langle B \rangle_{t-\epsilon}),
\]

where \( T \) denotes the transposition and the lim sup is componentwise.

Let \( \mathcal{P}_W \) denote the set of all local martingale measures \( P \) such that

\[
\langle B \rangle_t \text{ is absolutely continuous in } t \text{ and } \hat{a} \text{ takes values in } S_d^{>0}, \ P - \text{a.s.} \quad (2.1)
\]
where $S_d^{>0}$ denotes the space of all $d \times d$ real valued positive definite matrices.

As usual in the theory of 2BSDEs, we will concentrate on the subclass $\mathcal{P}_s \subset \mathcal{P}_W$ consisting of all probability measures

$$
P^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \text{ where } X^\alpha_t := \int_0^t \alpha_s^{1/2} dB_s , \ t \in [0,1], \ \mathbb{P}_0 \text{-a.s.} \ (2.2)$$

for some $\mathbb{F}$-progressively measurable process $\alpha$ taking values in $S_d^{>0}$ with $\int_0^T |\alpha_t| dt < +\infty$, $\mathbb{P}_0$-a.s.

### 2.2 The non-linear Generator

We consider a map $H_t(\omega, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \to \mathbb{R}$, where $D_H \subset \mathbb{R}^d$ is a given subset containing 0.

Define the corresponding conjugate of $H$ w.r.t. $\gamma$ by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} \text{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \right\} \text{ for } a \in S_d^{>0},$$

$$\hat{F}_t(y, z) := F_t(y, z, \hat{a}_t) \text{ and } \hat{F}_t^0 := \hat{F}_t(0, 0).$$

We denote by $D_{F_t(y,z)}$ the domain of $F$ in $a$ for a fixed $(t, \omega, y, z)$.

As in [35] we fix a constant $\kappa \in (1, 2]$ and restrict the probability measures in $\mathcal{P}^\kappa_H \subset \mathcal{P}_S$

**Definition 2.1.** $\mathcal{P}^\kappa_H$ consists of all $\mathbb{P} \in \mathcal{P}_S$ such that

$$a^\mathbb{P} \leq \hat{a} \leq \bar{a}^\mathbb{P}, \ dt \times d\mathbb{P} \text{-as for some } a, \bar{a} \in S_d^{>0}, \text{ and } \mathbb{E}^\mathbb{P} \left[ \left( \int_0^T |\hat{F}_t^0|^\kappa dt \right)^2 \right] < +\infty$$

It is clear that $\mathcal{P}^\kappa_H$ is decreasing in $\kappa$, and $\hat{a}_t \in D_{F_t}$, $dt \times d\mathbb{P}$-as for all $\mathbb{P} \in \mathcal{P}^\kappa_H$.

**Definition 2.2.** We say that a property holds $\mathcal{P}^\kappa_H$-quasi-surely ($\mathcal{P}^\kappa_H$-q.s. for short) if it holds $\mathbb{P}$-a.s. for all $\mathbb{P} \in \mathcal{P}^\kappa_H$.

We now state our main assumptions on the function $F$ which will be our main interest in the sequel

**Assumption 2.1.**  
(i) The domain $D_{F_t(y,z)} = D_{F_t}$ is independent of $(\omega, y, z)$.

(ii) For fixed $(y, z, a)$, $F$ is $\mathbb{F}$-progressively measurable in $D_{F_t}$.

(iii) We have the following uniform Lipschitz-type property in $y$ and $z$

$$\forall (y, y', z, z', t, a, \omega), \quad \left| F_t(\omega, y, z, a) - F_t(\omega, y', z', a) \right| \leq C \left( |y - y'| + |a|^{1/2} |z - z'| \right).$$

(iv) $F$ is uniformly continuous in $\omega$ for the $|| \cdot ||_\infty$ norm.
2.3 The Spaces and Norms

We now recall from \[35\] the spaces and norms which will be needed for the formulation of
the second order BSDEs. Notice that all subsequent notations extend to the case $\kappa = 1$.

For $p \geq 1$, $L^{p,\kappa}_H$ denotes the space of all $\mathcal{F}_T$-measurable scalar r.v. $\xi$ with
\[
\|\xi\|^{p,\kappa}_{L^p_H} := \sup_{P \in \mathcal{P}^\kappa_H} \mathbb{E}^P[|\xi|^p] < +\infty.
\]

$\mathbb{H}^{p,\kappa}_H$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}^d$-valued processes $Z$ with
\[
\|Z\|^{p,\kappa}_{\mathbb{H}^p_H} := \sup_{P \in \mathcal{P}^\kappa_H} \mathbb{E}^P \left[ \left( \int_0^T \tilde{a}_t^{1/2} Z_t^2 dt \right)^{\frac{p}{2}} \right] < +\infty.
\]

$\mathbb{D}^{p,\kappa}_H$ denotes the space of all $\mathbb{F}^+$-progressively measurable $\mathbb{R}$-valued processes $Y$ with $\mathcal{P}^\kappa_H - q.s.$ càdlàg paths, and
\[
\|Y\|^{p,\kappa}_{\mathbb{D}^p_H} := \sup_{P \in \mathcal{P}^\kappa_H} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] < +\infty.
\]

For each $\xi \in L^{1,\kappa}_H$, $P \in \mathcal{P}^\kappa_H$ and $t \in [0, T]$ denote
\[
\mathbb{E}^{H,P}_t[\xi] := \text{ess sup}_{P' \in \mathcal{P}^\kappa_H(t+,P)} \mathbb{E}^{P'}_t[\xi] \quad \text{where} \quad \mathcal{P}^\kappa_H(t+,P) := \left\{ P' \in \mathcal{P}^\kappa_H : P' = P \text{ on } \mathcal{F}^+_t \right\}.
\]

Here $\mathbb{E}^P_t[\xi] := \mathbb{E}^P[\xi|\mathcal{F}_t]$. Then we define for each $p \geq \kappa$,
\[
\mathbb{L}^{p,\kappa}_H := \left\{ \xi \in L^{p,\kappa}_H : \|\xi\|^{p,\kappa}_{L^p_H} < +\infty \right\} \quad \text{where} \quad \|\xi\|^{p,\kappa}_{L^p_H} := \sup_{P \in \mathcal{P}^\kappa_H} \mathbb{E}^P \left[ \text{ess sup}_{0 \leq t \leq T} \left( \mathbb{E}^{H,P}_t[|\xi|^\kappa] \right)^{\frac{p}{\kappa}} \right].
\]

Finally, we denote by $\text{UC}_b(\Omega)$ the collection of all bounded and uniformly continuous maps $\xi : \Omega \to \mathbb{R}$ with respect to the $\|\cdot\|_\infty$-norm, and we let
\[
L^{p,\kappa}_H := \text{the closure of } \text{UC}_b(\Omega) \text{ under the norm } \|\cdot\|^{p,\kappa}_{L^p_H}, \text{ for every } 1 \leq \kappa \leq p.
\]

2.4 Formulation

First, we consider a process $S$ which will play the role of our lower obstacle. We will always
assume that $S$ verifies the following properties

(i) $S$ is $\mathbb{F}$-progressively measurable and càdlàg.

(ii) $S$ is uniformly continuous in $\omega$ in the sense that for all $t$
\[
|S_t(\omega) - S_t(\tilde{\omega})| \leq \rho(\|\omega - \tilde{\omega}\|_t), \quad \forall (\omega, \tilde{\omega}) \in \Omega^2,
\]

for some modulus of continuity $\rho$ and where we define $\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega(s)|$. 

\[5\]
Then, we shall consider the following second order RBSDE (2R BSDE for short) with lower obstacle $S$

$$Y_t = \xi + \int_t^T \hat{F}_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t, \ 0 \leq t \leq T, \ \mathcal{P}_H^\kappa - q.s. \quad (2.3)$$

We follow Soner, Touzi and Zhang [35]. For any $\mathbb{P} \in \mathcal{P}_H^\kappa$, $\mathbb{F}$-stopping time $\tau$, and $\mathcal{F}_\tau$-measurable random variable $\xi \in \mathbb{L}^2(\mathbb{P})$, let $(y^\mathbb{F}, z^\mathbb{F}, k^\mathbb{F}) := (y^\mathbb{F}(\tau, \xi), z^\mathbb{F}(\tau, \xi), k^\mathbb{F}(\tau, \xi))$ denote the unique solution to the following standard RBSDE with obstacle $S$ (existence and uniqueness have been proved under our assumptions by Lepeltier and Xu in [22])

$$\begin{cases}
  y^\mathbb{F}_t &= \xi + \int_t^\tau \hat{F}_s(y^\mathbb{F}_s, z^\mathbb{F}_s) ds - \int_t^\tau z^\mathbb{F}_s dB_s + k^\mathbb{F}_\tau - k^\mathbb{F}_t, \ 0 \leq t \leq \tau, \ \mathbb{P} - a.s. \\
  y^\mathbb{F}_t &\geq S_t, \ \mathbb{P} - a.s. \\
  \int_0^t (y^\mathbb{F}_s - S_s) \, dk^\mathbb{F}_s = 0, \ \mathbb{P} - a.s., \ \forall t \in [0, T].
\end{cases}$$

**Definition 2.3.** For $\xi \in \mathbb{L}^2_H^\kappa$, we say $(Y, Z) \in \mathbb{D}^2_H^\kappa \times \mathbb{H}^2_H^\kappa$ is a solution to the 2RBSDE (2.3) if

- $Y_T = \xi, \ \mathcal{P}_H^\kappa - q.s.$
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$, the process $K^\mathbb{P}$ defined below has non-decreasing paths $\mathbb{P} - a.s.$
  $$K^\mathbb{P}_t := Y_0 - Y_t - \int_0^t \hat{F}_s(Y_s, Z_s) ds + \int_0^t Z_s dB_s, \ 0 \leq t \leq T, \ \mathbb{P} - a.s. \quad (2.4)$$
- We have the following minimum condition
  $$K^\mathbb{P}_t - k^\mathbb{P}_t = \underset{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})}{\text{ess inf}} \mathbb{P}' \mathbb{E}^\mathbb{P}' \left[ K^{\mathbb{P}'}_T - k^{\mathbb{P}'}_T \right], \ 0 \leq t \leq T, \ \mathbb{P} - a.s., \ \forall \mathbb{P} \in \mathcal{P}_H^\kappa. \quad (2.5)$$
- $Y_t \geq S_t, \ \mathcal{P}_H^\kappa - q.s.$

Following [35], in addition to Assumption 2.1, we will always assume

**Assumption 2.2.** (i) $\mathcal{P}_H^\kappa$ is not empty.

(ii) The processes $\hat{F}^0$ and $S$ satisfy the following integrability conditions

$$\phi^2_H := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \text{ess sup}_{0 \leq t \leq T} \left( \mathbb{E}^\mathbb{H}_{t}^\mathbb{P} \left[ \int_0^T |\hat{F}_s^{0, \kappa} ds \right] \right)^\kappa \right] < +\infty \quad (2.6)$$

$$\psi^2_H := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \text{ess sup}_{0 \leq t \leq T} \left( \mathbb{E}^\mathbb{H}_{t}^\mathbb{P} \left[ \left( \sup_{0 \leq s \leq T} (S_s)^+ \right)^\kappa \right] \right)^\kappa \right] < +\infty. \quad (2.7)$$
2.5 Connection with standard RBSDEs

If $H$ is linear in $\gamma$, that is to say

$$H_t(y, z, \gamma) := \frac{1}{2} \text{Tr} \left[ a^0_t \gamma \right] - f_t(y, z),$$

where $a^0 : [0, T] \times \Omega \to S^0_d$ is $\mathbb{F}$-progressively measurable and has uniform upper and lower bounds. As in [35], we no longer need to assume any uniform continuity in $\omega$ in this case. Besides, the domain of $F$ is restricted to $a^0$ and we have

$$\hat{F}_t(y, z) = f_t(y, z).$$

If we further assume that there exists some $P \in \mathcal{P}$ such that $\hat{a}$ and $a^0$ coincide $P$-a.s. and $E_P \left[ \int_0^T |f_t(0, 0)|^2 \, dt \right] < +\infty$, then $\mathcal{P}_H^\kappa = \{P\}$.

Then, unlike with 2BSDEs, it is not immediate from the minimum condition (2.5) that the process $K^P - k^P$ is actually null. However, we know that $K^P - k^P$ is a martingale with finite variation. Since $P$ satisfy the martingale representation property, this martingale is also continuous, and therefore it is null. Thus we have

$$0 = k^P - K^P, \ P - a.s.,$$

and the 2RBSDE is equivalent to a standard RBSDE. In particular, we see that the part of $K^P$ which increases only when $Y_{t-} > S_{t-}$ is null, which means that $K^P$ satisfies the usual Skorohod condition with respect to the obstacle.

3 Uniqueness of the solution and other properties

3.1 Representation and uniqueness of the solution

We have similarly as in Theorem 4.4 of [35]

**Theorem 3.1.** Let Assumptions 2.1 and 2.2 hold. Assume $\xi \in \mathcal{L}^{\kappa}_{H}$ and that $(Y, Z)$ is a solution to 2RBSDE (2.3). Then, for any $P \in \mathcal{P}_H^\kappa$ and $0 \leq t_1 < t_2 \leq T$,

$$Y_{t_1} = \text{ess sup}_{P \in \mathcal{P}_H^\kappa(t_1, t_2)} y^P_{t_1}(t_2, Y_{t_2}), \ P - a.s. \quad (3.1)$$

Consequently, the 2RBSDE (2.3) has at most one solution in $\mathcal{D}_{H}^{\kappa} \times \mathcal{H}_{H}^{\kappa}$.

**Remark 3.1.** Let us now justify the minimum condition (2.5). Assume for the sake of clarity that the generator $\hat{F}$ is equal to 0. By the above Theorem, we know that if there exists a solution to the 2RBSDE (2.3), then the process $Y$ has to satisfy the representation
Therefore, we have a natural candidate for a possible solution of the 2RBSDE. Now, assume that we could construct such a process $Y$ satisfying the representation (3.1) and which has the decomposition (2.3). Then, taking conditional expectations in $Y - y^p$, we end up with exactly the minimum condition (2.5).

**Proof.** The proof follows the lines of the proof of Theorem 4.4 in [35]. First,

$$Y_t = \text{ess sup}_{\mathbb{P}'} y_t^{p'}(T, \xi), \quad t \in [0, T], \quad \mathbb{P} - \text{a.s.,}$$

for all $\mathbb{P} \in \mathcal{P}_H^\kappa$, and thus is unique. Then, since we have that $d\langle Y, B \rangle_t = Z_t d\langle B \rangle_t$, $\mathcal{P}_H^\kappa - \text{q.s.,} Z$ is unique. Finally, the process $K^p$ is uniquely determined. We shall now prove (3.1).

(i) Fix $0 \leq t_1 < t_2 \leq T$ and $\mathbb{P} \in \mathcal{P}_H^\kappa$. For any $\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})$, we have

$$Y_t = Y_{t_2} + \int_{t_1}^{t_2} \delta F_s(Y_s, Z_s) ds - \int_{t_1}^{t_2} Z_s dB_s + K_{t_2}^{p'} - K_{t_1}^{p'}, \quad t_1 \leq t \leq t_2, \quad \mathbb{P}' - \text{a.s.}$$

Now, it is clear that we can always decompose the non-decreasing process $K^p$ into

$$K_{t_2}^{p'} = A_{t_2}^{p'} + B_{t_2}^{p'}, \quad \mathbb{P}' - \text{a.s.,}$$

where $A^{p'}$ and $B^{p'}$ are two non-decreasing processes such that $A^{p'}$ only increases when $Y_{t_2} = 0$ and $B^{p'}$ only increases when $Y_{t_1} > 0$. With that decomposition, we can apply a generalisation of the usual comparison theorem proved by El Karoui et al. (see Theorem 5.2 in [15]), whose proof is postponed to the appendix, under $\mathbb{P}'$ to obtain $Y_{t_1} \geq y_{t_1}^{p'}(t_2, Y_{t_2})$ and $A_{t_1}^{p'} - A_{t_1}^{p'} \leq k_{t_2}^{p'} - k_{t_1}^{p'}$, $\mathbb{P}' - \text{a.s.}$ Since $\mathbb{P}' = \mathbb{P}$ on $F_{t_1}^+$, we get $Y_{t_1} \geq y_{t_1}^{p'}(t_2, Y_{t_2})$, $\mathbb{P} - \text{a.s.}$ and thus

$$Y_{t_1} \geq \text{ess sup}_{\mathbb{P}'} y_{t_1}^{p'}(t_2, Y_{t_2}), \quad \mathbb{P} - \text{a.s.}$$

(ii) We now prove the reverse inequality. Fix $\mathbb{P} \in \mathcal{P}_H^\kappa$. We will show in (iii) below that

$$C_{t_1}^{p_1} := \text{ess sup}_{\mathbb{P}'} \mathbb{E}_{\mathbb{P}'}^{p_1} \left[ \left( K_{t_2}^{p_1} - k_{t_2}^{p_1} - K_{t_1}^{p_1} + k_{t_1}^{p_1} \right)^2 \right] < +\infty, \quad \mathbb{P} - \text{a.s.}$$

For every $\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})$, denote

$$\delta Y := Y - y^{p'}(t_2, Y_{t_2}), \quad \delta Z := Z - z^{p'}(t_2, Y_{t_2})$$

and

$$\delta K^{p'} := K^{p'} - k^{p'}(t_2, Y_{t_2}).$$

By the Lipschitz Assumption (2.1(iii)), there exist two bounded processes $\lambda$ and $\eta$ such that for all $t_1 \leq t \leq T_2$

$$\delta Y_t = \int_t^{t_2} \left( \lambda_s \delta Y_s + \eta_s \delta Z_s \right) ds - \int_t^{t_2} \delta Z_s dB_s + \delta K_{t_2}^{p'} - \delta K_{t_1}^{p'}, \quad \mathbb{P}' - \text{a.s.}$$
Define for $t_1 \leq t \leq t_2$ the following continuous process

$$M_t := \exp \left( \int_{t_1}^{t} \left( \lambda_s - \frac{1}{2} |\eta_s|^2 \right) ds - \int_{t_1}^{t} \eta_s \tilde{a}_s^{-1/2} dB_s \right), \ P' - a.s.$$  

Note that since $\lambda$ and $\eta$ are bounded, we have for all $p \geq 1$

$$\mathbb{E}_{t_1}^{P'} \left[ \sup_{t_1 \leq t \leq t_2} (M_t)^p + \sup_{t_1 \leq t \leq t_2} (M_t^{-1})^p \right] \leq C_p, \ P' - a.s. \quad (3.3)$$

Then, by Itô's formula, we obtain

$$\delta Y_{t_1} = \mathbb{E}_{t_1}^{P'} \left[ \int_{t_1}^{t_2} M_t dK_t^{P'} \right]. \quad (3.4)$$

Let us now prove that the process $K_t^{P'} - k_t^{P'}$ is non-decreasing. By the minimum condition (2.5), it is clear that it is actually a $P'$-submartingale. Let us apply the Doob-Meyer decomposition under $P'$, we get the existence of a $P'$-martingale $N_t^{P'}$ and a non-decreasing process $P_t^{P'}$, both null at 0, such that

$$K_t^{P'} - k_t^{P'} = N_t^{P'} + P_t^{P'}, \ P' - a.s.$$  

Then, since we know that all the probability measures in $\mathcal{P}_H^\kappa$ satisfy the martingale representation property, the martingale $N_t^{P'}$ is continuous. Besides, by the above equation, it also has finite variation. Hence, we have $N_t^{P'} = 0$, and the result follows.

Returning back to (3.4), we can now write

$$\delta Y_{t_1} \leq \mathbb{E}_{t_1}^{P'} \left[ \sup_{t_1 \leq t \leq t_2} (M_t) \left( \delta K_{t_2}^{P'} - \delta K_{t_1}^{P'} \right) \right]$$

$$\leq \left( \mathbb{E}_{t_1}^{P'} \left[ \sup_{t_1 \leq t \leq t_2} (M_t)^3 \right] \right)^{1/3} \left( \mathbb{E}_{t_1}^{P'} \left[ (\delta K_{t_2}^{P'} - \delta K_{t_1}^{P'})^{3/2} \right] \right)^{2/3}$$

$$\leq C \left( \mathbb{E}_{t_1}^{P'} \left[ (\delta K_{t_2}^{P'} - \delta K_{t_1}^{P'})^{1/3} \right] \right)^{1/3}, \ P - a.s. \quad \text{Taking the essential infimum on both sides finishes the proof.}$$

(iii) It remains to show that the estimate for $C_{t_1}^{P}$ holds. But by definition, we clearly have
\[
\mathbb{E}^P \left[ \left( K_{t_2}^P - k_{t_2}^P - K_{t_1}^P + k_{t_1}^P \right)^2 \right] \leq C \left( \|Y\|_{L^2_H}^2 + \|Z\|_{L^2_H}^2 + \phi^2 \right) \\
+ C \sup_{P \in \mathcal{P}_H} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |y_t^P|^2 + \int_0^T |\alpha_t^1 z_t^P|^2 \, ds \right] \\
< +\infty,
\]

since the last term on the right-hand side is finite thanks to the integrability assumed on \( \xi \) and \( \hat{F}_0 \).

Then we can proceed exactly as in the proof of Theorem 4.4 in [35].

\[\square\]

Finally, the following comparison Theorem follows easily from the classical one for RBSDEs (see for instance Theorem 5.2 in [15] and Theorem 3.4 in [22]) and the representation (3.1).

**Theorem 3.2.** Let \((Y, Z)\) and \((Y', Z')\) be the solutions of 2RBSDEs with terminal conditions \(\xi\) and \(\xi'\), lower obstacles \(S\) and \(S'\) and generators \(\hat{F}\) and \(\hat{F}'\) respectively, and let \((y^P, z^P, k^P)\) and \((y'^P, z'^P, k'^P)\) the solutions of the associated RBSDEs. Assume that they both verify our Assumptions 2.1 and 2.2 and that we have

- \(\xi \leq \xi', \mathcal{P}_H - q.s.\)
- \(\hat{F}_t(y_t^P, z_t^P) \leq \hat{F}'_t(y'^P_t, z'^P_t), \mathbb{P} - a.s., \) for all \(P \in \mathcal{P}_H.\)
- \(S_t \leq S'_t, \mathcal{P}_H - q.s.\)

Then \(Y \leq Y', \mathcal{P}_H - q.s.\)

**Remark 3.2.** Note that in our context, in the above comparison Theorem, even if the obstacles \(S\) and \(S'\) are identical, we cannot compare the increasing processes \(K^P\) and \(K'^P\).

This is due to the fact that the processes \(K^P\) do not satisfy the Skorohod condition, since it can be considered, at least formally, to come from the addition of an increasing process due to the fact that we work with second-order BSDEs, and an increasing process due to the reflection constraint. And only the second one is bound to satisfy the Skorohod condition.

### 3.2 Some properties of the solution

Now that we have proved the representation (3.1), we can show, as in the classical framework, that the solution \(Y\) of the 2RBSDE is linked to an optimal stopping problem.
Proposition 3.1. Let \((Y, Z)\) be the solution to the above 2RBSDE (2.3). Then for each \(t \in [0, T]\) and for all \(\mathbb{P} \in \mathcal{P}_H^u\)

\[
Y_t = \text{ess sup}_{\mathbb{P}'} \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^\tau \hat{F}_s(y_s', z_s') ds + S_\tau 1_{\tau < T} + \xi 1_{\tau = T} \right], \quad \mathbb{P} - \text{a.s.} \tag{3.5}
\]

\[
= \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}} \left[ \int_t^\tau \hat{F}_s(Y_s, Z_s) ds + A_\tau^{\mathbb{P}} - A_t^{\mathbb{P}} + S_\tau 1_{\tau < T} + \xi 1_{\tau = T} \right], \quad \mathbb{P} - \text{a.s.} \tag{3.6}
\]

where \(\mathcal{T}_{t,T}\) is the set of all stopping times valued in \([t, T]\) and where \(A_\tau^{\mathbb{P}} := \int_t^\tau 1_{Y_s > S_s} dK_s^{\mathbb{P}}\) is the part of \(K^{\mathbb{P}}\) which only increases when \(Y_s > S_s\).

Remark 3.3. We want to highlight here that unlike with classical RBSDEs, considering an upper obstacle in our context is fundamentally different from considering a lower obstacle. Indeed, having an upper obstacle corresponds, at least formally, to add an increasing process in the definition of a 2BSDE. Since there is already an increasing process in that definition, we still end up with an increasing process. However, in the case of a lower obstacle, we would have to add a decreasing process in the definition, therefore ending up with a finite variation process. This situation thus becomes much more complicated. Furthermore, in that case we conjecture that the above representation of Proposition 3.1 would hold with a sup-inf instead of a sup-sup, indicating that this situation should be closer to stochastic games than to stochastic control. This is an interesting generalization that we leave for future research.

Proof. By Proposition 3.1 in [22], we know that for all \(\mathbb{P} \in \mathcal{P}_H^u\)

\[
b_t^{\mathbb{P}} = \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}} \left[ \int_t^\tau \hat{F}_s(y_s', z_s') ds + S_\tau 1_{\tau < T} + \xi 1_{\tau = T} \right], \quad \mathbb{P} - \text{a.s.}
\]

Then the first equality is a simple consequence of the representation formula (3.1). For the second one, we proceed exactly as in the proof of Proposition 3.1 in [22]. Fix some \(\mathbb{P} \in \mathcal{P}_H^u\) and some \(t \in [0, T]\). Let \(\tau \in \mathcal{T}_{t,T}\). We obtain by taking conditional expectation in (2.3)

\[
Y_t = \mathbb{E}_t^{\mathbb{P}} \left[ Y_\tau + \int_\tau^T \hat{F}_s(Y_s, Z_s) ds + K_\tau - K_t \right]
\]

\[
\geq \mathbb{E}_t^{\mathbb{P}} \left[ \int_\tau^T \hat{F}_s(Y_s, Z_s) ds + S_\tau 1_{\tau < T} + \xi 1_{\tau = T} + A_\tau - A_t \right].
\]

This implies that

\[
Y_t \geq \text{ess sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}} \left[ \int_\tau^T \hat{F}_s(Y_s, Z_s) ds + A_\tau - A_t + S_\tau 1_{\tau < T} + \xi 1_{\tau = T} \right], \quad \mathbb{P} - \text{a.s.}
\]

Fix some \(\varepsilon > 0\) and define the stopping time \(D_t^{\mathbb{P}, \varepsilon} := \inf \{ u \geq t, \ Y_u \leq S_u + \varepsilon, \ \mathbb{P} - \text{a.s.} \} \wedge T\). It is clear by definition that on the set \(D_t^{\mathbb{P}, \varepsilon} < T\), we have \(Y_{D_t^{\mathbb{P}, \varepsilon}} \leq S_{D_t^{\mathbb{P}, \varepsilon}} + \varepsilon\). Similarly,
on the set \( \{ D^{P,\varepsilon}_t = T \} \), we have \( Y_s > S_s + \varepsilon \), for all \( t \leq s \leq T \). Hence, for all \( s \in [t, D^{P,\varepsilon}_t] \), we have \( Y_{s-} > S_{s-} \). This implies that \( K^{P,\varepsilon}_t - K_t = A^{P,\varepsilon}_t - A_t \), and therefore

\[
Y_t \leq E^P_t \left[ \int_t^{D^{P,\varepsilon}_t} \hat{F}_s(Y_s, Z_s) ds + A^{P,\varepsilon}_t - A_t + S_{D^{P,\varepsilon}_t}1_{D^{P,\varepsilon}_t < T} + \xi 1_{D^{P,\varepsilon}_t = T} \right] + \varepsilon,
\]

which ends the proof by arbitrariness of \( \varepsilon \).

Then, if we have more information on the obstacle \( S \), we can give a more explicit representation for the processes \( K^{P} \), just as in the classical case (see Proposition 4.2 in [10]).

**Assumption 3.1.** \( S \) is a semi-martingale of the form

\[
S_t = S_0 + \int_0^t U_s ds + \int_0^t V_s dB_s + C_t, \quad \mathcal{P}_H^\kappa - q.s.
\]

where \( C \) is càdlàg process of integrable variation such that the measure \( dC_t \) is singular with respect to the Lebesgue measure \( dt \) and which admits the following decomposition

\[
C_t = C^+_t - C^-_t,
\]

where \( C^+ \) and \( C^- \) are non-decreasing processes. Besides, \( U \) and \( V \) are respectively \( \mathbb{R} \) and \( \mathbb{R}^d \)-valued \( \mathcal{F}_t \) progressively measurable processes such that

\[
\int_0^T (|U_t|^2 + |V_t|^2) dt + C^+_T + C^-_T \leq +\infty, \quad \mathcal{P}_H^\kappa - q.s.
\]

**Proposition 3.2.** Let Assumptions 2.1, 2.2 and 3.1 hold. Let \((Y, Z)\) be the solution to the 2RBSDE (2.3), then

\[
Z_t = V_t, \quad dt \times \mathcal{P}_H^\kappa - q.s. \text{ on the set } \{ Y_{t-} = S_{t-} \}, \quad (3.7)
\]

and there exists a progressively measurable process \((\alpha_t^P)_{0 \leq t \leq T}\) such that \( 0 \leq \alpha \leq 1 \) and

\[
1_{\{Y_{t-} = S_{t-}\}} dK_t^P = \alpha_t^P 1_{Y_{t-} = S_{t-}} \left( [\hat{F}_t(S_t, V_t) + U_t]^- dt + dC_t^- \right).
\]

**Proof.** First, for all \( P \in \mathcal{P}_H^\kappa \), the following holds \( P - a.s. \)

\[
Y_t - S_t = Y_0 - S_0 - \int_0^t \left( \hat{F}_s(Y_s, Z_s) + U_s \right) ds + \int_0^t (Z_s - V_s) dB_s - K_t^P - C^+_t + C^-_t.
\]

Now if we denote \( L_t \) the local time at 0 of \( Y_t - S_t \), then by Itô-Tanaka formula under each
\((Y_t - S_t)^+ = (Y_0 - S_0)^+ - \int_0^t 1_{Y_s > S_s^-} \left( \hat{F}_s(Y_s, Z_s) + U_s \right) ds + \int_0^t 1_{Y_s > S_s^-} (Z_s - V_s) dB_s\)

\[
- \int_0^t 1_{Y_s > S_s^-} d(K_t^p + C_t^+ - C_t^-) + \frac{1}{2} L_t
+ \sum_{0 \leq s \leq t} (Y_s - S_s)^+ - (Y_s^- - S_s^-)^+ - 1_{Y_s > S_s^-} \Delta(Y_s - S_s)
= (Y_0 - S_0)^+ - \int_0^t 1_{Y_s > S_s^-} \left( \hat{F}_s(Y_s, Z_s) + U_s \right) ds + \int_0^t 1_{Y_s > S_s^-} (Z_s - V_s) dB_s\]

\[
- \int_0^t 1_{Y_s > S_s^-} d(K_t^p + C_t^+ - C_t^-) + \frac{1}{2} L_t
+ \sum_{0 \leq s \leq t} (Y_s - S_s)^+ - (Y_s^- - S_s^-)^+ - 1_{Y_s > S_s^-} \Delta(Y_s - S_s).
\]

However, we have \((Y_t - S_t)^+ = Y_t - S_t\), hence by identification of the martingale part

\[1_{Y_t^- = S_t^-} (Z_t - V_t) dB_t = 0, \ P^\kappa_H - q.s.\]

from which the first statement is clear.

Since the jump part is obviously positive and \(\text{Land} C^+\) are non-decreasing processes, we also have

\[1_{Y_t^- = S_t^-} dK_t^p \leq -1_{Y_t^- = S_t^-} \left( \left( \hat{F}_t(Y_t, Z_t) + U_t \right) dt - dC_t^- \right).\]

The second statement follows then easily. \(\square\)

### 3.3 A priori estimates

We conclude this section by showing some a priori estimates which will be useful in the sequel.

**Theorem 3.3.** Let Assumptions [2.1] and [2.3] hold. Assume \(\xi \in L^2_H \kappa\) and \((Y, Z, K) \in \mathbb{D}^{2,\kappa}_H \times \mathbb{H}^{2,\kappa}_H \times \mathbb{H}^{2,\kappa}_H\) is a solution to the 2RBSDE (2.3). Let \(\{(y^p, z^p, k^p)\}_{P \in P^\kappa_H}\) be the solutions of the corresponding BSDEs (2.4). Then, there exists a constant \(C_\kappa\) depending only on \(\kappa\), \(T\) and the Lipschitz constant of \(\bar{F}\) such that

\[
\|Y\|^{2}_{\mathbb{D}^{2,\kappa}_H} + \|Z\|^{2}_{\mathbb{H}^{2,\kappa}_H} + \sup_{P \in P^\kappa_H} \mathbb{E}^P \left[ (K_T^p)^2 \right] \leq C \left( \|\xi\|^{2}_{L^2_H \kappa} + \phi^{2,\kappa}_H + \psi^{2,\kappa}_H \right),
\]

and

\[
\sup_{P \in P^\kappa_H} \left\{ \|y^p\|^{2}_{\mathbb{D}^{2}(P)} + \|z^p\|^{2}_{\mathbb{H}^{2}(P)} + \|k^p\|^{2}_{l^2(P)} \right\} \leq C \left( \|\xi\|^{2}_{L^2_H \kappa} + \phi^{2,\kappa}_H + \psi^{2,\kappa}_H \right).
\]

**Proof.** By Lemma 2 in [19], we know that there exists a constant \(C_\kappa\) depending only on \(\kappa\), \(T\) and the Lipschitz constant of \(\bar{F}\), such that for all \(\mathbb{P}\)

\[
|y_t^p| \leq C_\kappa \mathbb{E}^P_t \left[ |\xi|^\kappa + \int_t^T |\hat{F}_s|_0^\kappa ds + \sup_{t \leq s \leq T} (S_s^\kappa)^\kappa \right]. \tag{3.8}
\]
Let us note immediately, that in [19], the result is given with an expectation and not a conditional expectation, and more importantly that the process considered are continuous. However, the generalization is easy for the conditional expectation. As far as the jumps are concerned, their proof only uses Itô’s formula for smooth convex functions, for which the jump part can be taken care of easily in the estimates. Then, one can follow exactly their proof to get our result.

This immediately provides the estimate for $y^p$. Now by definition of our norms, we get from (3.8) and the representation formula (3.1) that

$$\|Y\|_{H^{2,\kappa}_H}^2 \leq C_\kappa \left( \|\xi\|_{L^{2,\kappa}_H}^2 + \phi^2_H + \psi^2_H \right). \quad (3.9)$$

Now apply Itô’s formula to $|Y|^2$ under each $\mathbb{P} \in \mathcal{P}^\kappa_H$. We get as usual for every $\epsilon > 0$

$$\mathbb{E}^\mathbb{P} \left[ \int_0^T |\hat{a}_{t/2} Z_t|^2 \, dt \right] \leq C \mathbb{E}^\mathbb{P} \left[ |\xi|^2 + \int_0^T |Y_t| \left( |\hat{F}_t^0| + |Y_t| + |\hat{a}_{t/2} Z_t| \right) \, dt \right]
+ \mathbb{E}^\mathbb{P} \left[ \int_0^T |Y_t| \, dK_t \right]
\leq C \left( |\xi|^2 + \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \left( \int_0^T |\hat{F}_t^0| \, dt \right)^2 \right] \right)
+ \epsilon \mathbb{E}^\mathbb{P} \left[ \int_0^T |\hat{a}_{t/2} Z_t|^2 \, dt + |K_T^p|^2 \right] + \frac{C^2}{\epsilon} \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right]. \quad (3.10)$$

Then by definition of our 2RBSDE, we easily have

$$\mathbb{E}^\mathbb{P} \left[ |K_T^p|^2 \right] \leq C_0 \mathbb{E}^\mathbb{P} \left[ |\xi|^2 + \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |\hat{a}_{t/2} Z_t|^2 \, dt + \left( \int_0^T |\hat{F}_t^0| \, dt \right)^2 \right], \quad (3.11)$$

for some constant $C_0$, independent of $\epsilon$.

Now set $\epsilon := (2(1 + C_0))^{-1}$ and plug (3.11) in (3.10). One then gets

$$\mathbb{E}^\mathbb{P} \left[ \int_0^T \left| \hat{a}_{t/2} Z_t \right|^2 \, dt \right] \leq C \mathbb{E}^\mathbb{P} \left[ |\xi|^2 + \sup_{0 \leq t \leq T} |Y_t|^2 + \left( \int_0^T |\hat{F}_t^0| \, dt \right)^2 \right].$$

From this and the estimate for $Y$, we immediately obtain

$$\|Z\|_{H^{2,\kappa}_H} \leq C \left( \|\xi\|_{L^{2,\kappa}_H}^2 + \phi^2_H + \psi^2_H \right).$$

Then the estimate for $K^p$ comes from (3.11). The estimates for $z^p$ and $k^p$ can be proved similarly. \hfill \Box
Applying Itô’s formula to $Y_t$, we obtain that there exists a constant $C_\kappa$ depending only on $\kappa$, $T$ and the Lipschitz constant of $F$ such that

$$
\|Y^1 - Y^2\|_{D^{2,\kappa}_H} \leq C \|\xi^1 - \xi^2\|_{L^{2,\kappa}_H},
$$

and

$$
\|Z^1 - Z^2\|_{D^{2,\kappa}_H}^2 + \sup_{\mathbb{P} \in \mathcal{P}^\kappa_H} \mathbb{E}_\mathbb{P} \left[ \sup_{0 \leq t \leq T} \left| K^\mathbb{P}_t^{1,1} - K^\mathbb{P}_t^{2,1} \right|^2 \right]
\leq C \|\xi^1 - \xi^2\|_{L^{2,\kappa}_H}^2 \left( \|\xi^1\|_{L^{2,\kappa}_H} + \|\xi^2\|_{L^{2,\kappa}_H} + (\phi^2_H)^{1/2} + (\psi^2_H)^{1/2} \right).
$$

**Proof.** As in the previous Proposition, we can follow the proof of Lemma 3 in [19], to obtain that there exists a constant $C_\kappa$ depending only on $\kappa$, $T$ and the Lipschitz constant of $\tilde{F}$, such that for all $\mathbb{P}$

$$
\|\hat{y}^\mathbb{P}_t - \hat{y}^{\mathbb{P},2}_t\| \leq C_\kappa \mathbb{E}_\mathbb{P} \left[ \|\xi^1 - \xi^2\|^2 \right].
$$

Now by definition of our norms, we get from (3.12) and the representation formula (3.1) that

$$
\|Y^1 - Y^2\|_{D^{2,\kappa}_H}^2 \leq C_\kappa \|\xi^1 - \xi^2\|_{L^{2,\kappa}_H}^2.
$$

Applying Itô’s formula to $|Y^1 - Y^2|^2$, under each $\mathbb{P} \in \mathcal{P}^\kappa_H$, leads to

$$
\mathbb{E}_\mathbb{P} \left[ \int_0^T \left| \frac{1}{2} \hat{a}_t^2 (Z^1_t - Z^2_t) \right|^2 dt \right]
\leq C \mathbb{E}_\mathbb{P} \left[ \|\xi^1 - \xi^2\|^2 \right] + \mathbb{E}_\mathbb{P} \left[ \int_0^T |Y^1_t - Y^2_t| d(K^\mathbb{P}_t^{1,1} - K^\mathbb{P}_t^{2,1}) \right]
\leq C \left( \|\xi^1 - \xi^2\|_{L^{2,\kappa}_H}^2 + \|Y^1 - Y^2\|_{D^{2,\kappa}_H}^2 \right)
\leq C \left( \|\xi^1 - \xi^2\|_{L^{2,\kappa}_H}^2 + \|Y^1 - Y^2\|_{D^{2,\kappa}_H}^2 \right)
+ \frac{1}{2} \mathbb{E}_\mathbb{P} \left[ \int_0^T \left| \frac{1}{2} \hat{a}_t^2 (Z^1_t - Z^2_t) \right|^2 dt \right]
\leq C \|Y^1 - Y^2\|_{D^{2,\kappa}_H}^2 \left( \mathbb{E}_\mathbb{P} \left[ \sum_{i=1}^2 (K^i_T)^2 \right] \right)^{1/2}.
$$

The estimate for $(Z^1 - Z^2)$ is now obvious from the above inequality and the estimates of Proposition 3.3.

Finally the estimate for the difference of the increasing processes is obvious by definition. 

\[\square\]
4 A direct existence argument

In the articles [35], the main tool to prove existence of a solution is the so called regular conditional probability distributions of Stroock and Varadhan [37]. Indeed, it allows to construct a solution to the 2BSDE when the terminal condition belongs to the space $\mathcal{UC}_b(\Omega)$. In this section we will generalize their approach to the reflected case.

4.1 Notations

For the convenience of the reader, we recall below some of the notations introduced in [35].

- For $0 \leq t \leq T$, denote by $\Omega^t := \{ \omega \in C([t,1],\mathbb{R}^d), w(t) = 0 \}$ the shifted canonical space, $B^t$ the shifted canonical process, $\mathbb{P}_0^t$ the shifted Wiener measure and $\mathbb{F}^t$ the filtration generated by $B^t$.

- For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, define the shifted path $\omega^t \in \Omega^t$
  \[ \omega^t_r := \omega_r - \omega_t, \forall r \in [t,T]. \]

- For $0 \leq s \leq t \leq T$ and $\omega \in \Omega^s$, define the concatenation path $\omega \otimes t \bar{\omega} \in \Omega^s$ by
  \[ (\omega \otimes t \bar{\omega})(r) := \omega_r 1_{[s,t]}(r) + (\omega_t + \bar{\omega}_r) 1_{[t,T]}(r), \forall r \in [s,T]. \]

- For $0 \leq s \leq t \leq T$ and a $\mathbb{F}_t$-measurable random variable $\xi$ on $\Omega^s$, for each $\omega \in \Omega^s$, define the shifted $\mathbb{F}_t$-measurable random variable $\xi^t,\omega$ on $\Omega^t$ by
  \[ \xi^t,\omega(\bar{\omega}) := \xi((\omega \otimes t \bar{\omega})), \forall \bar{\omega} \in \Omega^t. \]
  Similarly, for an $\mathbb{F}^s$-progressively measurable process $X$ on $[s,T]$ and $(t,\omega) \in [s,T] \times \Omega^s$, the shifted process $\{X^t,\omega, r \in [t,T]\}$ is $\mathbb{F}^t$-progressively measurable.

- For a $\mathbb{F}$-stopping time $\tau$, we use the same simplification as [35]
  \[ \omega \otimes \tau \bar{\omega} := \omega \otimes \tau(\omega) \bar{\omega}, \xi^{\tau,\omega} := \xi^{\tau(\omega),\omega}, X^{\tau,\omega} := X^{\tau(\omega),\omega}. \]

- For a $\mathbb{F}$-stopping time $\tau$, the r.c.p.d. of $\mathbb{P}$ (noted $\mathbb{P}^{\tau}$) induces naturally a probability measure $\mathbb{P}^{\tau,\omega}$ (that we also call the r.c.p.d. of $\mathbb{P}$) on $\mathcal{F}_T^{\tau(\omega)}$ which in particular satisfies that for every bounded and $\mathcal{F}_T^{\tau}$-measurable random variable $\xi$
  \[ \mathbb{E}^{\mathbb{P}^{\tau}}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}]. \]

- We define similarly as in Section 2 the set $\mathcal{P}^t_S$, by restricting to the shifted canonical space $\Omega^t$, and its subset $\mathcal{P}^t_H$.

- Finally, we define our "shifted" generator
  \[ \hat{\mathbb{F}}^{t,\omega}_s(\bar{\omega}, y, z) := F_s(\omega \otimes t \bar{\omega}, y, z, \hat{a}^t_s(\bar{\omega})), \forall (s, \bar{\omega}) \in [t,T] \times \Omega^t. \]
  Then note that since $F$ is assumed to be uniformly continuous in $\omega$ under the $\mathbb{L}^\infty$ norm, then so is $\hat{\mathbb{F}}^{t,\omega}$.  

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4.2 Existence when $\xi$ is in $\text{UC}_b(\Omega)$

When $\xi$ is in $\text{UC}_b(\Omega)$, we know that there exists a modulus of continuity $\rho$ for $\xi$, $F$ and $S$ in $\omega$. Then, for any $0 \leq t \leq s \leq T$, $(y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$ and $\omega, \omega' \in \Omega$, $\tilde{\omega} \in \Omega^t$,

$$\left| \xi^{t, \omega}(\tilde{\omega}) - \xi^{t, \omega'}(\tilde{\omega}) \right| \leq \rho(\|\omega - \omega'\|_t), \quad \left| \hat{F}^{t, \omega}(\tilde{\omega}, y, z) - \hat{F}^{t, \omega'}(\tilde{\omega}, y, z) \right| \leq \rho(\|\omega - \omega'\|_t)$$

$$\left| S^{t, \omega}(\tilde{\omega}) - S^{t, \omega'}(\tilde{\omega}) \right| \leq \rho(\|\omega - \omega'\|_t).$$

We then define for all $\omega \in \Omega$

$$\Lambda(\omega) := \sup_{0 \leq s \leq t} \Lambda_t(\omega),$$

where

$$\Lambda_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}_{H}^t} \left( \mathbb{E}^{\mathbb{P}} \left[ \left| \xi^{t, \omega} \right|^2 + \int_t^T |\hat{F}_s^{t, \omega}(0, 0)|^2 ds + \left( \sup_{t \leq s \leq T} (S^{t, \omega}_s) \right)^2 \right] \right)^{1/2}.$$

Now since $\hat{F}^{t, \omega}$ is also uniformly continuous in $\omega$, we have

$$\Lambda(\omega) < \infty \text{ for some } \omega \in \Omega \text{ iff it holds for all } \omega \in \Omega. \quad (4.2)$$

Moreover, when $\Lambda$ is finite, it is uniformly continuous in $\omega$ under the $L^\infty$-norm and is therefore $\mathcal{F}_T$-measurable.

Now, by Assumption [2.2], we have

$$\Lambda_t(\omega) < \infty \text{ for all } (t, \omega) \in [0, T] \times \Omega. \quad (4.3)$$

To prove existence, we define the following value process $V_t$ pathwise

$$V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}_{H}^t} \mathcal{Y}_{t, \omega}^{\mathbb{P}}(T), \text{ for all } (t, \omega) \in [0, T] \times \Omega, \quad (4.4)$$

where, for any $(t_1, \omega) \in [0, T] \times \Omega$, $\mathbb{P} \in \mathcal{P}_{H}^{t_1}$, $t_2 \in [t_1, T]$, and any $\mathcal{F}_{t_2}$-measurable $\eta \in L^\infty(\mathbb{P})$, we denote $\mathcal{Y}_{t_1}^{\mathbb{P}, t_1, \omega}(t_2, \eta) := y_{t_1}^{\mathbb{P}, t_1, \omega}$, where $(y_{r}^{\mathbb{P}, t_1, \omega}, z_{r}^{\mathbb{P}, t_1, \omega}, k_{r}^{\mathbb{P}, t_1, \omega})$ is the solution of the following RBSDE with lower obstacle $S^{t_1, \omega}$ on the shifted space $\Omega^{t_1}$ under $\mathbb{P}$

$$y_{s}^{\mathbb{P}, t_1, \omega} = \eta_{t_1, \omega} + \int_{s}^{t_2} F_{r}^{t_1, \omega}(y_{r}^{\mathbb{P}, t_1, \omega}, z_{r}^{\mathbb{P}, t_1, \omega}) dr - \int_{s}^{t_2} z_{r}^{\mathbb{P}, t_1, \omega} dB_{r}^{t_1} + k_{t_2}^{\mathbb{P}, t_1, \omega} - k_{t_1}^{\mathbb{P}, t_1, \omega}, \quad (4.5)$$

$$y_{t}^{\mathbb{P}, t_1, \omega} \geq S_{t}^{t_1, \omega}, \quad \mathbb{P} \text{ a.s.}$$

$$\int_{t_1}^{t_2} (y_{s}^{\mathbb{P}, t_1, \omega} - S_{s}^{t_1, \omega}) dB_{s}^{\mathbb{P}, t_1, \omega} = 0, \quad \mathbb{P} \text{ a.s.} \quad (4.6)$$

In view of the Blumenthal zero-one law, $\mathcal{Y}_{t, \omega}^{\mathbb{P}}(T, \xi)$ is constant for any given $(t, \omega)$ and $\mathbb{P} \in \mathcal{P}_{H}^t$. Moreover, since $\omega_0 = 0$ for all $\omega \in \Omega$, it is clear that, for the $y^{\mathbb{P}}$ defined in (2.3),

$$\mathcal{Y}_{t, \omega}^{\mathbb{P}, 0, \omega}(t, \eta) = y^{\mathbb{P}}(t, \eta) \text{ for all } \omega \in \Omega.$$
Lemma 4.1. Let Assumptions 2.1 and 2.2 hold and consider some $\xi$ in UC$_b(\Omega)$. Then for all $(t, \omega) \in [0, T] \times \Omega$ we have $|V_t(\omega)| \leq C(1 + \Lambda_t(\omega))$. Moreover, for all $(t, \omega, \omega') \in [0, T] \times \Omega^2$, $|V_t(\omega) - V_t(\omega')| \leq C \rho(\|\omega - \omega'||_1)$. Consequently, $V_t$ is $\mathcal{F}_t$-measurable for every $t \in [0, T]$.

**Proof.** (i) For each $(t, \omega) \in [0, T] \times \Omega$ and $\mathbb{P} \in \mathcal{P}_T$, let $\alpha$ be some positive constant which will be fixed later and let $\eta \in (0, 1)$. By Itô's formula we have, since $\widehat{F}$ is uniformly Lipschitz and since by \[10\int_t^T e^{\alpha s} \left(y_{s}^{t,\omega} - S_{s}^{t,\omega}\right) ds = 0\]

\[
e^{\alpha t} \left|y_{t}^{t,\omega}\right|^2 + \int_t^T e^{\alpha s} \left|\left(\widehat{a}_{s}^{t,\omega}\right)^{1/2} z_{s}^{t,\omega}\right|^2 ds \leq e^{\alpha T} \left|\xi_{t,\omega}\right|^2 + 2C \int_t^T e^{\alpha s} \left|y_{s}^{t,\omega}\right| \left|\widehat{F}_{s}^{t,\omega}(0)\right| ds
\]

\[
+ 2C \int_t^T \left|y_{s}^{t,\omega}\right|^2 \left(\left|g_{s}^{t,\omega}\right| + \left|\left(\widehat{a}_{s}^{t,\omega}\right)^{1/2} z_{s}^{t,\omega}\right|\right) ds - 2 \int_t^T e^{\alpha s} y_{s}^{t,\omega} z_{s}^{t,\omega} dB_{s}^t
\]

\[
+ 2 \int_t^T e^{\alpha s} S_{s}^{t,\omega} ds - \alpha \int_t^T e^{\alpha s} \left|y_{s}^{t,\omega}\right|^2 ds
\]

\[
\leq e^{\alpha T} \left|\xi_{t,\omega}\right|^2 + \int_t^T e^{\alpha s} \left|\widehat{F}_{s}^{t,\omega}(0)\right|^2 ds - 2 \int_t^T e^{\alpha s} y_{s}^{t,\omega} z_{s}^{t,\omega} dB_{s}^t + \eta \int_t^T e^{\alpha s} \left|\left(\widehat{a}_{s}^{t,\omega}\right)^{1/2} z_{s}^{t,\omega}\right|^2 ds
\]

\[
+ \left(2C + C^2 + \frac{C^2}{\eta} - \alpha\right) \int_t^T e^{\alpha s} \left|y_{s}^{t,\omega}\right|^2 ds + 2 \sup_{t \leq s \leq T} e^{\alpha s} \left(S_{s}^{t,\omega}(\omega) + (k_{t}^{P,t,\omega} - k_{t}^{P,t,\omega})\right).
\]

Now choose $\alpha$ such that $\nu := \alpha - 2C - C^2 - \frac{C^2}{\eta} \geq 0$. We obtain for all $\epsilon > 0$

\[
e^{\alpha t} \left|y_{t}^{t,\omega}\right|^2 + (1 - \eta) \int_t^T e^{\alpha s} \left|\left(\widehat{a}_{s}^{t,\omega}\right)^{1/2} z_{s}^{t,\omega}\right|^2 ds \leq e^{\alpha T} \left|\xi_{t,\omega}\right|^2 + \int_t^T e^{\alpha s} \left|\widehat{F}_{s}^{t,\omega}(0, 0)\right|^2 ds
\]

\[
+ \frac{1}{\epsilon} \left(\sup_{t \leq s \leq T} e^{\alpha s} \left(S_{s}^{t,\omega}\right)\right)^2
\]

\[
+ \left(e(k_{T}^{P,t,\omega} - k_{t}^{P,t,\omega})\right)^2
\]

\[
- 2 \int_t^T e^{\alpha s} y_{s}^{t,\omega} z_{s}^{t,\omega} dB_{s}^t.
\]

Taking expectation in \[4.7\] yields

\[
\left|y_{t}^{t,\omega}\right|^2 + (1 - \eta) \mathbb{E}^{\mathbb{P}} \int_t^T \left(\widehat{G}_{s}^{t,\omega}\right)^{1/2} z_{s}^{t,\omega} ds \leq C \Lambda_t(\omega)^2 + \epsilon \mathbb{E}^{\mathbb{P}} \left[k_{T}^{P,t,\omega} - k_{t}^{P,t,\omega}\right]^2.
\]

Now by definition, we also have for some constant $C_0$ independent of $\epsilon$

\[
\mathbb{E}^{\mathbb{P}} \left[k_{T}^{P,t,\omega} - k_{t}^{P,t,\omega}\right]^2 \leq C_0 \mathbb{E}^{\mathbb{P}} \left[\xi_{t,\omega}^2 + \int_t^T \left|\widehat{F}_{s}^{t,\omega}(0, 0)\right|^2 ds + \int_t^T \left|y_{s}^{t,\omega}\right|^2 ds\right]
\]

\[
+ \mathbb{E}^{\mathbb{P}} \left[\int_t^T \left|\left(\widehat{a}_{s}^{t,\omega}\right)^{1/2} z_{s}^{t,\omega}\right|^2 ds\right]
\]

\[
\leq C_0 \left(\Lambda_t(\omega) + \mathbb{E}^{\mathbb{P}} \left[\int_t^T \left|y_{s}^{t,\omega}\right|^2 ds + \int_t^T \left|\left(\widehat{a}_{s}^{t,\omega}\right)^{1/2} z_{s}^{t,\omega}\right|^2 ds\right]\right).
\]

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Choosing \( \eta \) small enough and \( \epsilon = \frac{1}{2C_0} \), Gronwall inequality then implies

\[
|y_t|_F^2 \leq C(1 + \Lambda_t(\omega)).
\]

The result then follows from arbitrariness of \( \mathbb{P} \).

(ii) The proof is exactly the same as above, except that one has to use uniform continuity in \( \omega \) of \( \xi^{t,\omega}, \hat{F}^{t,\omega} \) and \( S^{t,\omega} \). Indeed, for each \((t,\omega) \in [0,T] \times \Omega \) and \( \mathbb{P} \in \mathbb{P}_H^t \), let \( \alpha \) be some positive constant which will be fixed later and let \( \eta \in (0,1) \). By Itô’s formula we have, since \( \hat{F} \) is uniformly Lipschitz

\[
e^{\alpha t} \left| y_t - y_t^F \right|^2 + \int_t^T e^{\alpha s} \left( \left( \frac{T}{s} \right) \left( \xi^{t,\omega} - \xi^{t,\omega'} \right) \right)^2 ds \\
+ 2C \int_t^T e^{\alpha s} \left| y_s - y_s^F \right|^2 \left( \left| \xi^{t,\omega} - \xi^{t,\omega'} \right| \right) ds \\
+ 2C \int_t^T e^{\alpha s} \left| y_s - y_s^F \right|^2 \left( \hat{F}^{t,\omega} - \hat{F}^{t,\omega'} \right) ds \\
+ 2 \int_t^T e^{\alpha s} \left( y_s - y_s^F \right) \left( \xi^{t,\omega} - \xi^{t,\omega'} \right) d(k_s - k_s^F) \\
- 2 \int_t^T e^{\alpha s} \left( y_s - y_s^F \right) (z_s - z_s^F) dB_s^t
\]

\[
\leq e^{\alpha T} \left| \xi^{t,\omega} - \xi^{t,\omega'} \right|^2 + 
\int_t^T e^{\alpha s} \left( \hat{F}^{t,\omega} - \hat{F}^{t,\omega'} \right) ds \\
+ \left( 2C + C^2 \frac{C^2}{\eta} \right) \int_t^T e^{\alpha s} \left| y_s - y_s^F \right|^2 ds \\
+ \eta \int_t^T e^{\alpha s} \left( \frac{T}{s} \right) \left( \xi^{t,\omega} - \xi^{t,\omega'} \right) \left( \xi^{t,\omega} - \xi^{t,\omega'} \right) ds \\
- 2 \int_t^T e^{\alpha s} \left( y_s - y_s^F \right) (z_s - z_s^F) dB_s^t \\
+ 2 \int_t^T e^{\alpha s} \left( y_s - y_s^F \right) \left( \xi^{t,\omega} - \xi^{t,\omega'} \right) d(k_s - k_s^F).
\]

By the Skorohod condition [L.6], we also have

\[
\int_t^T e^{\alpha s} \left( y_s^F - y_s^F \right) d(k_s - k_s^F) \leq \int_t^T e^{\alpha s} \left( S^t_s - S^t_s \right) d(k_s - k_s^F).
\]
Now choose \(\alpha\) such that \(\nu := \alpha - 2C - C^2 - \frac{C^2}{\eta} \geq 0\). We obtain for all \(\epsilon > 0\)
\[
e^{\alpha t} \left| y^p_{t},\omega - y^p_t \right|^2 + (1 - \eta) \int_t^T e^{\alpha s} \left| (\xi^p_s)^{1/2} (z^p_s - z^p_t,\omega) \right|^2 \, ds \\
\leq e^{\alpha T} \left| \xi^{t,}\omega - \xi^t \right|^2 + \int_t^T e^{\alpha s} \left| \hat{F}^p_{t,}\omega (y^p_s, z^p_s, z^p_t,\omega) - \hat{F}^p_t (y^p_s, z^p_s, z^p_t,\omega) \right|^2 \, ds \\
+ \frac{1}{\epsilon} \left( \sup_{t \leq s \leq T} e^{\alpha s} (S^t_s - z^t_s)^2 \right)^2 + \epsilon \left( k^p_{t,}\omega - k^p_t + k^p_t - k^p_{t,}\omega \right)^2 \\
- 2 \int_t^T e^{\alpha s} (y^p_s, z^p_s, z^p_t,\omega) dB_s^t.
\] (4.8)

The end of the proof is then similar to the previous step, using the uniform continuity in \(\omega\) of \(\xi, F\) and \(S\). \qed

Then, we show the same dynamic programming principle as Proposition 4.7 in \([36]\).

**Proposition 4.1.** Under Assumptions 2.1 2.2 and for \(\xi \in UC_b(\Omega)\), we have for all \(0 \leq t_1 < t_2 \leq T\) and for all \(\omega \in \Omega\)
\[
V_{t_1}(\omega) = \sup_{P \in \mathcal{P}_H^f} \mathcal{Y}_{t_1,\omega}^p(t_2, V_{t_2}^{t_1,\omega}).
\]

The proof is almost the same as the proof in \([36]\), but we give it for the convenience of the reader.

**Proof.** Without loss of generality, we can assume that \(t_1 = 0\) and \(t_2 = t\). Thus, we have to prove
\[
V_0(\omega) = \sup_{P \in \mathcal{P}_H} \mathcal{Y}_0^p(t, V_t).
\]

Denote \((y^p, z^p, k^p) := (\mathcal{Y}(T, \xi), Z^p(T, \xi), K^p(T, \xi))\)

(i) For any \(P \in \mathcal{P}_H\), we know by Lemma 4.3 in \([36]\), that for \(P - a.e. \omega \in \Omega\), the r.c.p.d. \(P^{t,\omega} \in \mathcal{P}_H^f\). Now thanks to the paper of Xu and Qian \([33]\), we know that the solution of reflected BSDEs with Lipschitz generator can be constructed via Picard iteration. Thus, it means that at each step of the iteration, the solution can be formulated as a conditional expectation under \(P\). By the properties of the r.p.c.d., this entails that
\[
y_t^p(\omega) = \mathcal{Y}_{t,\omega}^p(t, \xi), \text{ for } P - a.e. \omega \in \Omega.
\] (4.9)

Hence, by definition of \(V_t\) and the comparison principle for RBSDEs, we get that \(y_0^p \leq \mathcal{Y}_0^p(t, V_t)\). By arbitrariness of \(P\), this leads to
\[
V_0(\omega) \leq \sup_{P \in \mathcal{P}_H} \mathcal{Y}_0^p(t, V_t).
\]

(ii) For the other inequality, we proceed as in \([36]\). Let \(P \in \mathcal{P}_H\) and \(\epsilon > 0\). By separability of \(\Omega\), there exists a partition \((E_i^j)_{i \geq 1} \subset \mathcal{F}\) such that \(\|\omega - \omega'\|_t \leq \epsilon\) for any \(i\) and any \(\omega, \omega' \in E_i^j\). Now for each \(i\), fix an \(\hat{\omega}_i \in E_i^j\) and let \(P_i^{\epsilon}\) be an \(\epsilon\)-optimizer of \(V_i(\hat{\omega}_i)\).
Now if we define for each \( n \geq 1 \), \( \mathbb{P}^n := \mathbb{P}^{n, \epsilon} \) by

\[
\mathbb{P}^n(E) := \mathbb{E}^\mathbb{P} \left[ \sum_{i=1}^n \mathbb{1}_{E_i} \right] + \mathbb{P}(E \cap \hat{E}_i^n), \quad \text{where } \hat{E}_i^n := \cup_{i>n} E_i^n.
\]

Then, by the proof of Proposition 4.7 in [36], we know that \( \mathbb{P}^n \in \mathcal{P}_H \). Besides, by Lemma 4.1 and its proof, we know that \( V \) and \( \mathcal{Y}^{P,t,\omega} \) are uniformly continuous in \( \omega \) and thus

\[
V_t \leq V_t(\hat{\omega}_i) + C\rho(\epsilon) \leq \mathcal{Y}^{P, t, \hat{\omega}_i}_t(T, \xi) + \epsilon + C\rho(\epsilon) \\
\leq \mathcal{Y}^{P, t, \omega}_t(T, \xi) + \epsilon + C\rho(\epsilon) = \mathcal{Y}^{P^n, t, \omega}_t(T, \xi) + \epsilon + C\rho(\epsilon).
\]

Then, it follows from (4.9) that

\[
V_t \leq y^n_t + \epsilon + C\rho(\epsilon), \quad \mathbb{P}^n - a.s. \text{ on } \bigcup_{i=1}^n E_i^n. \tag{4.10}
\]

Let now \((y^n, z^n, k^n) := (y^{n, \epsilon}, z^{n, \epsilon}, k^{n, \epsilon})\) be the solution of the following RBSDE with lower obstacle \( S \) on \([0, t]\)

\[
y^n_s = \left[ y^n_t + C\rho(\epsilon) \right] \mathbb{1}_{\cup_{i=1}^n E_i} + V_t \mathbb{1}_{\hat{E}_i^n} + \int_s^t \hat{F}_r(y^n_r, z^n_r) dr - \int_s^t z^n_r dB_r + k^n_t - k^n_s, \quad \mathbb{P} - a.s. \tag{4.11}
\]

By the comparison principle for RBSDEs, we know that \( \mathcal{Y}^P_0(t, V_t) \leq y^n_0 \). Then since \( \mathbb{P}^n = \mathbb{P} \) on \( \mathcal{F}_t \), the equality (4.11) also holds \( \mathbb{P} - a.s. \). Using the same arguments and notations as in the proof of Lemma 4.1, we obtain

\[
\left| y^n_0 - y^n_0 \right|^2 \leq C \mathbb{E}^\mathbb{P} \left[ \epsilon^2 + \rho(\epsilon)^2 + \left| V_t - y^n_t \right|^2 \mathbb{1}_{\hat{E}_i^n} \right].
\]

Then, by Lemma 4.1, we have

\[
\mathcal{Y}^P_0(t, V_t) \leq y^n_0 \leq y^n_0 + C \left( \epsilon + \rho(\epsilon) + \left( \mathbb{E}^\mathbb{P} \left[ \Lambda^n_t \mathbb{1}_{\hat{E}_i^n} \right] \right)^{1/2} \right) \\
\leq V_0(\omega) + C \left( \epsilon + \rho(\epsilon) + \left( \mathbb{E}^\mathbb{P} \left[ \Lambda^n_t \mathbb{1}_{\hat{E}_i^n} \right] \right)^{1/2} \right).
\]

Then it suffices to let \( n \) go to \( +\infty \) and \( \epsilon \) to 0. \( \square \)

Define now for all \((t, \omega)\), the \( \mathbb{P}^+ \)-progressively measurable process

\[
V^+_t := \lim_{r \in Q^0(t, T), r \downarrow t} V_r.
\]

We have the following lemma whose proof is postponed to the Appendix.
Lemma 4.2. Under the conditions of the previous Proposition, we have

\[ V_t^+ = \lim_{r \in \mathbb{Q} \cap \langle t, T \rangle, r \downarrow t} V_r, \ P_H - q.s. \]

and thus \( V^+ \) is càdlàg \( P_H - q.s. \).

Proceeding exactly as in Steps 1 et 2 of the proof of Theorem 4.5 in \[36\], we can then prove that \( V^+ \) is a strong reflected \( \hat{F} \)-supermartingale. Then, using the Doob-Meyer decomposition proved in the Appendix in Theorem 4.2 for all \( \mathbb{P} \), we know that there exists a unique \( (\mathbb{P} - a.s.) \) process \( Z_{t} \in H^2(\mathbb{P}) \) and unique non-decreasing càdlàg square integrable processes \( A_{t} \) and \( B_{t} \) such that

\[ V_t^+ = V_0^+ + \int_0^t \hat{F}_s(V_s^+, Z_s^t) ds + \int_0^t Z_s^t dB_s - A_t^\mathbb{P} - B_t^\mathbb{P}, \ \mathbb{P} - a.s., \forall \mathbb{P} \in \mathcal{P}_H. \]

• \( V_t^+ \geq S_t, \ \mathbb{P} - a.s. \ \forall \mathbb{P} \in \mathcal{P}_H. \)

• \( \int_0^T (V_t^+ - S_t) dA_t^\mathbb{P}, \ \mathbb{P} - a.s., \forall \mathbb{P} \in \mathcal{P}_H. \)

• \( A_{t} \) and \( B_{t} \) never act at the same time.

We then define \( K_{t}^\mathbb{P} := A_{t}^\mathbb{P} + B_{t}^\mathbb{P} \). By Karandikar \[20\], since \( V^+ \) is a càdlàg semimartingale, we can define a universal process \( Z \) which aggregates the family \( \{Z_{t}^\mathbb{P}, \mathbb{P} \in \mathcal{P}_H\} \).

We next prove the representation (3.1) for \( V \) and \( V^+ \), and that, as shown in Proposition 4.11 of \[36\], we actually have \( V = V^+, \ P_H - q.s., \) which shows that in the case of a terminal condition in \( UC_b(\Omega) \), the solution of the 2RBSDE is actually \( \mathbb{F} \)-progressively measurable.

**Proposition 4.2.** Assume that \( \xi \in UC_b(\Omega) \) and that Assumptions 2.1 and 2.2 hold. Then we have

\[ V_t = \text{ess sup}_{P' \in \mathcal{P}_H(t, P)} Y_t^{P'}(T, \xi) \text{ and } V_t^+ = \text{ess sup}_{P' \in \mathcal{P}_H(t^+, P)} Y_t^{P'}(T, \xi), \ \mathbb{P} - a.s., \forall \mathbb{P} \in \mathcal{P}_H. \]

Besides, we also have for all \( t \)

\[ V_t = V_t^+, \ P_H - q.s. \]

**Proof.** The proof for the representations is the same as the proof of proposition 4.10 in \[36\], since we also have a stability result for RBSDEs under our assumptions. For the equality between \( V \) and \( V^+ \), we also refer to the proof of Proposition 4.11 in \[36\].

Therefore, in the sequel we will use \( V \) instead of \( V^+ \).

Finally, we have to check that the minimum condition (2.5) holds. Fix \( P \in \mathcal{P}_H^\mathbb{P} \) and \( P' \in \mathcal{P}_H^P(t^+, P) \). By the Lipschitz property of \( F \), we know that there exists bounded processes \( \lambda \) and \( \eta \) such that
\[ V_t - \bar{y}_t = \int_t^T \lambda_s (V_s - \bar{y}_s) ds - \int_t^T \sigma_s^{1/2} (\bar{Z}_s - z_s) (\sigma_s^{1/2} dB_s - \eta_s ds) \]

\[ + K_T^p - K_t^p - k_T^p + k_T^p. \]  

(4.12)

Then, one can define a probability measure \( Q' \) equivalent to \( \mathbb{P}' \) such that

\[ V_t - y_t = e^{-\int_0^T \lambda_u du} \mathbb{E}^{Q'}_t \left[ \int_t^T e^{\int_u^T \lambda_s ds} dB_s \int_0^u \lambda_s ds - \eta_u du \right]. \]

Now define the following c\'adl\'ag non-decreasing processes

\[ K_s^p := \int_0^s e^{\int_u^s \lambda_r dr} dK_u^p, \quad k_s^p := \int_0^s e^{\int_u^s \lambda_r dr} dk_u^p. \]

By the representation (3.1), we deduce that the process

\[ K^p - k^p \]

is a \( Q' \)-submartingale. Using Doob-Meyer decomposition and the fact that all the probability measures we consider satisfy the martingale representation property, we deduce as in Step (ii) of the proof of Theorem 3.1 that this process is actually non-decreasing. Then by definition, this entails that the process \( K^p - k^p \) is also non-decreasing.

Let us denote

\[ P_t^p := K_T^p - k_T^p. \]

Returning to (4.12) and defining a process \( M \) as in Step (ii) of the proof of Theorem 3.1, we obtain that

\[ V_t - \bar{y}_t = \mathbb{E}^{Q'}_t \left[ \int_t^T M_s dP_s^p \right] \geq \mathbb{E}^{Q'}_t \left[ \inf_{t \leq s \leq T} M_s \left( P_T^p - P_t^p \right) \right]. \]

Then, we have

\[ \mathbb{E}^{Q'}_t \left[ \left( \inf_{t \leq s \leq T} M_s \right)^{1/3} \left( P_T^p - P_t^p \right)^{1/3} \right] \]

\[ \leq \left( \mathbb{E}^{Q'}_t \left[ \left( P_T^p - P_t^p \right) \right] \mathbb{E}^{Q'}_t \left[ \left( P_T^p - P_t^p \right)^{1/3} \right] \right)^{1/3} \]

\[ \leq C \left( \sup_{t \leq s \leq T} \mathbb{E}^{Q'}_t \left[ \left( P_T^p - P_t^p \right)^{2} \right] \right)^{1/3} \]

\[ \left( V_t - \bar{y}_t \right)^{1/3}. \]
Arguing as in Step (iii) of the proof of Theorem 3.1, the above inequality shows that we have
\[
\text{ess inf}_{P'} \mathbb{E}^P \left[ P_{T}^{P'} - P_{t}^{P'} \right] = 0,
\]
that is to say that the minimum condition (2.5) is satisfied.

4.3 Main result

We are now in position to state the main result of this section

**Theorem 4.1.** Let \( \xi \in \mathcal{L}^{2,\kappa}_H \). Under Assumptions 2.1 and 2.2, there exists a unique solution \((Y, Z) \in \mathbb{D}^\infty_H \times \mathbb{H}^2_H \) of the 2BSDE (2.3).

**Proof.** The proof follow the lines of the proof of Theorem 4.7 in [35]. In general for a terminal condition \( \xi \in \mathcal{L}^{2,\kappa}_H \), there exists by definition a sequence \((\xi_n)_{n \geq 0} \subset \text{UC}_b(\Omega)\) such that
\[
\lim_{n \to +\infty} \|\xi_n - \xi\|_{\mathcal{L}^{2,\kappa}_H} = 0 \quad \text{and} \quad \sup_{n \geq 0} \|\xi_n\|_{\mathcal{L}^{2,\kappa}_H} < +\infty.
\]

Let \((Y^n, Z^n)\) be the solution to the 2RBSDE (2.3) with terminal condition \(\xi_n\) and
\[
K_t^{P,n} := Y_0^n - Y_t^n - \int_0^t \hat{F}_s(Y^n_s, Z^n_s) \, ds + \int_0^t Z^n_s \, dB_s, \quad P - \text{a.s.}
\]

By the estimates of Proposition 3.4, we have as \(n, m \to +\infty\)
\[
\|Y^n - Y^m\|_{\mathcal{L}^{2,\kappa}_H}^2 + \|Z^n - Z^m\|_{\mathcal{L}^{2,\kappa}_H}^2 + \sup_{P \in \mathcal{P}_n^H} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |K_t^{P,n} - K_t^{P,m}| \right] \leq C_\kappa \|\xi_n - \xi_m\|_{\mathcal{L}^{2,\kappa}_H} 
\]
\[
\to 0.
\]

Extracting a subsequence if necessary, we may assume that
\[
\|Y^n - Y^m\|_{\mathcal{L}^{2,\kappa}_H}^2 + \|Z^n - Z^m\|_{\mathcal{L}^{2,\kappa}_H}^2 + \sup_{P \in \mathcal{P}_n^H} \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |K_t^{P,n} - K_t^{P,m}| \right] \leq \frac{1}{2n}. \quad (4.13)
\]

This implies by Markov inequality that for all \(P\) and all \(m \geq n \geq 0\)
\[
\mathbb{P} \left[ \sup_{0 \leq t \leq T} \left\{ |Y_t^n - Y_t^m|^2 + |K_t^{P,n} - K_t^{P,m}|^2 \right\} + \int_0^T \alpha_t^{1/2} (Z^n_s - Z^m_s)^2 \, dt > n^{-1} \right] \leq C n 2^{-n}. \quad (4.14)
\]

Define
\[
Y := \lim_{n \to +\infty} Y^n, \quad Z := \lim_{n \to +\infty} Z^n, \quad K^P := \lim_{n \to +\infty} K_t^{P,n},
\]
where the \(\lim\) for \(Z\) is taken componentwise. All those processes are clearly \(\mathbb{F}^+\)-progressively measurable.
By (4.14), it follows from Borel-Cantelli Lemma that for all \( P \) we have \( P-a.s. \)
\[
\lim_{n \to +\infty} \left[ \sup_{0 \leq t \leq T} \left\{ |Y_t^n - Y_t|^2 + |K_t^{\alpha,n} - K_t^{\alpha}|^2 \right\} + \int_0^T |\hat{a}_t^{1/2} (Z_s^n - Z_s)|^2 dt \right] = 0.
\]

It follows that \( Y \) is càdlàg, \( P^\kappa \)-q.s., and that \( K^\alpha \) is a càdlàg non-decreasing process, \( P-a.s. \). Furthermore, for all \( P \), sending \( m \) to infinity in (4.13) and applying Fatou’s lemma under \( P \) gives us that \((Y,Z) \in D^2_{H\kappa} \times H^2_{H\kappa} \).

Finally, we can proceed exactly as in the regular case \((\xi \in UC_b(\Omega))\) to show that the minimum condition (2.5) holds.

\[\square\]

## 5 American Options under volatility uncertainty

First let us recall the link between American options and RBSDEs in the classical framework (see [16] for more details). Let \( \mathcal{M} \) be a standard financial complete market (1 risky asset \( S \) and a bond). It is well known that in some constrained cases the pair wealth-portfolio \((X^\alpha, \pi^\alpha)\) satisfies:
\[
X_{t}^\alpha = \xi + \int_t^T b(s, X_s^\alpha, \pi_s^\alpha) ds - \int_t^T \pi_s \sigma_s dW_s
\]
where \( W \) is a Brownian motion under the underlying probability measure \( P \), \( b \) is convex and Lipschitz with respect to \((x, \pi)\). In addition we assume that the process \((b(t,0,0))_{t \leq T}\) is square-integrable and \((\sigma_t)_{t \leq T}\), the volatility matrix of the \( n \) risky assets, is invertible and its inverse \((\sigma_t)^{-1}\) is bounded. The classical case corresponds to \( b(t, x, \pi) = -r_t x - \pi_t \sigma_t \theta_t \), where \( \theta_t \) is the risk premium vector.

When the American option is exercised at a stopping time \( \nu \geq t \) the yield is given by
\[
\tilde{S}_\nu = S_\nu 1_{[\nu < T]} + \xi_T 1_{[\nu = T]}.\]

Let \( t \) be fixed and let \( \nu \geq t \) be the exercising time of the contingent claim. Then, since the market is complete, there exists a unique pair \((X_s^\nu(\nu, \tilde{S}_\nu), \pi_s^\nu(\nu, \tilde{S}_\nu)) = (X_s^\nu, \pi_s^\nu)\) which replicates \( \tilde{S}_\nu \), i.e.,
\[
-dX_s^\nu = b(s, X_s^\nu, \pi_s^\nu) dt - \pi_s^\nu \sigma_s dW_s, \; s \leq \nu; \; X^\nu_\nu = \tilde{S}_\nu.
\]

Therefore the price of the contingent claim is given by:
\[
X_t^\alpha = \text{ess sup}_{\nu \in [t,T]} X^\nu_t(\nu, \tilde{S}_\nu).
\]

Then, the link with RBSDE is given by the following Theorem of [16]
Theorem 5.1. There exist $\pi^P \in H^2(\mathbb{P})$ and a non-decreasing continuous process $k^P$ such that for all $t \in [0, T]$

$$
\begin{align*}
    X_t^P &= \xi + \int_t^T b(s, X_s^P, \pi_s^P)ds - \int_t^T \pi_s^P \sigma_s dW_s + k_T^P - k_t^P \\
    X_t^P &\geq S_t \\
    \int_0^T (X_t^P - S_t)dk_t^P &= 0.
\end{align*}
$$

Furthermore, the stopping time $D_t^P = \inf\{s \geq t, X_s^P = S_s\} \land T$ is optimal after $t$.

Let us now go back to our uncertain volatility framework. The pricing of European contingent claims has already been treated in that context by Avellaneda, Lévy and Paras in [1], Denis and Martini in [11] with capacity theory and more recently by Vorbrink in [34] using the G-expectation framework.

We still consider a financial market with one risky asset $S$, whose dynamics are given by

$$
dS_t = r_t dt + dB_t, \quad \mathcal{P}_H^\kappa - q.s.
$$

and we assume as above that our wealth process has the following dynamic

$$
X_t = \xi + \int_t^T b(s, X_s, \pi_s)ds - \int_t^T \pi_s dB_s, \quad \mathcal{P}_H^\kappa - q.s.
$$

In order to be in our 2RBSDE framework, we have to assume that $b$ satisfies Assumptions 2.1 and 2.2. The main difference is that now $b$ must satisfy stronger integrability conditions and also that it has to be uniformly continuous in $\omega$ (when we assume that $\hat{a}$ in the expression of $b$ is constant). For instance, in the classical case recalled above, it means that $r$ must be uniformly continuous in $\omega$, which is the case if for example it is deterministic. Finally, we must assume that $\xi \in L_2^H(\mathbb{P})$. This is going to be the case for all Lipschitz functions of $S$, if we assume that $r$ is uniformly continuous in $\omega$, which includes Call and Put options. Finally, since $S$ is going to be the obstacle, it has to be uniformly continuous in $\omega$. This is why we consider an asset which is given by a Brownian motion with drift and not a geometric Brownian motion. Indeed, the geometric Brownian motion may not be uniformly continuous in $\omega$.

Remark 5.1. Of course, from a financial point of view, assets driven by a Brownian motion instead of a geometric Brownian motion, especially in a market with volatility uncertainty, have much less interest. Nonetheless, notice that we could get rid of this restriction by doing the same construction as in [35] for 2BSDEs when considering a canonical process which is an exponential Brownian motion under the Wiener measure.

Following the intuitions in the papers mentioned above, it is natural in our now incomplete market to consider as a superhedging price for our contingent claim

$$
X_t = \text{ess sup}_{P'}^P X_t^{P'}, \quad P - a.s., \quad \forall P \in \mathcal{P}_H^\kappa,
$$

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where \( X_t^p \) is the price at time \( t \) of the contingent claim in the complete market mentioned at the beginning, with underlying probability measure \( \mathbb{P} \). Notice immediately that we do not claim that this price is the superreplicating price in our context, in the sense that it would be the smallest one for which there exists a strategy which superreplicates the American option quasi-surely.

The following Theorem is then a simple consequence of the previous one

**Theorem 5.2.** There exist \( \pi \in \mathbb{H}^{2,\mathbb{P}}_H \), a family of non-decreasing càdlàg processes \( K^\pi \) such that for all \( t \in [0,T] \) and for all \( \mathbb{P} \in \mathcal{P}_H^\pi \)

\[
\left\{ \begin{array}{ll}
X_t = \xi + \int_t^T b(s, X_s, \pi_s)ds - \int_t^T \pi_s \sigma_s dW_s + K_T^\pi - Y^\pi_t, \mathbb{P} - a.s. \\
X_t \geq S_t, \mathbb{P} - a.s. \\
K_t^\pi - k_t^\pi = \text{ess inf}_{\mathbb{P} \in \mathcal{P}_H^\pi(t, F)} \mathbb{E}^\pi_t \left[ K_T^\pi - k_T^\pi \right], \mathbb{P} - a.s.
\end{array} \right.
\]

Furthermore, for all \( \epsilon \), the stopping time \( D_t^\epsilon = \inf \{ s \geq t, X_s \leq S_s + \epsilon, \mathbb{P}^\pi_t - q.s. \} \wedge T \) is \( \epsilon \)-optimal after \( t \). Besides, for all \( \mathbb{P} \), if we consider the stopping times \( D_t^{\epsilon, s} = \inf \{ s \geq t, X_s^\mathbb{P} \leq S_s + \epsilon, \mathbb{P} - a.s. \} \wedge T \), which are \( \epsilon \)-optimal for the American options under each \( \mathbb{P} \), then for all \( \mathbb{P} \)

\[
D_t^\epsilon \geq D_t^{\epsilon, s}, \mathbb{P} - a.s. \tag{5.1}
\]

**Proof.** The existence of the processes is a simple consequence of Theorem 4.1 and the fact that \( X \) is the superhedging price of the contingent claim comes from the representation formula (5.1). Then, the \( \epsilon \)-optimality of \( D_t^\epsilon \) and the inequality (5.1) are clear by definition.

\[ \square \]

**Remark 5.2.** The formula (5.1) confirms the natural intuition that the smallest optimal time to exercise the American option when the volatility is uncertain is the supremum, in some sense, of all the optimal stopping times for the classical American options for each volatility scenarii.

**A Appendix**

**A.1 Technical proof**

**Proof.** [Proof of Lemma 4.2] For each \( \mathbb{P} \), let \( (\tilde{Y}^\mathbb{P}, \tilde{Z}^\mathbb{P}) \) be the solution of the BSDE with generator \( \tilde{F} \) and terminal condition \( \xi \) at time \( T \). We define

\[ \tilde{V}^\mathbb{P} := V - \tilde{Y}^\mathbb{P}. \]

Then, \( \tilde{V}^\mathbb{P} \geq 0, \mathbb{P} - a.s. \)

For any \( 0 \leq t_1 < t_2 \leq T \), let \( (y^{t_1, t_2}, z^{t_1, t_2}, k^{t_1, t_2}) := (Y^{t_2, t_1}, Z^{t_2, t_1}, K^{t_2, t_1}) \). Since we have for \( \mathbb{P} - a.e. \omega, \tilde{Y}^{t_1, t_2}(t_2, V_{t_2})(\omega) = Y^{t_1, t_2}(t_2, V_{t_2})(\omega) \), we get from Proposition 4.1

\[ V_{t_1} \geq y^{t_1, t_2}, \mathbb{P} - a.s. \]

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Denote
\[ \tilde{y}^{P}_{t} := y^{P}_{t} - \bar{Y}^{P}_{t}, \quad \tilde{z}^{P}_{t} := \tilde{a}^{-1/2}(z^{P}_{t} - \bar{Z}^{P}_{t}). \]

Then \( \tilde{V}^{P}_{t} \geq \tilde{y}^{P}_{t} \) and \((\tilde{y}^{P}_{t}, \tilde{z}^{P}_{t})\) satisfies the following RBSDE with lower obstacle \( S - \tilde{Y}^{P} \) on \([0, t]\)
\[ \tilde{y}^{P}_{t} = \tilde{V}^{P}_{t} + \int_{t}^{t_2} f^P_t(\tilde{y}^P_s, \tilde{z}^P_s) ds - \int_{t}^{t_2} \tilde{z}^P_s dW^P_s + k^{P}_{t_2} - k^{P}_{t}, \]
where
\[ f^P_t(\omega, y, z) := \tilde{F}_t(\omega, y + \bar{Y}^{P}_{t} - \tilde{a}^{1/2}(\omega)(y + \bar{Z}^{P}_{t} - \bar{Z}^{P}_{t}(\omega))). \]

By the definition given in the Appendix, \( \tilde{V}^{P} \) is a positive weak reflected \( f^P \)-supermartingale under \( P \). Since \( f^P(0, 0) = 0 \), we can apply the downcrossing inequality proved in the Appendix in Theorem A.3 to obtain classically that for \( P \)-a.e. \( \omega \), the limit
\[ \lim_{r \in \mathbb{Q} \cup (t, T], r \downarrow t} \tilde{V}^P_r(\omega) \]
exists for all \( t \).

Finally, since \( \tilde{Y}^{P} \) is continuous, we get the result. \( \square \)

### A.2 Reflected g-expectation

In this section, we extend some of the results of Peng [28] concerning \( g \)-supersolution of BSDEs to the case of RBSDEs. Let us note that the majority of the following proofs follows straightforwardly from the original proofs of Peng, with some minor modifications due to the added reflection. However, we still provide most of them since, to the best of our knowledge, they do not appear anywhere else in the literature.

In the following, we fix a probability measure \( P \)

#### A.2.1 Definitions and first properties

Let us be given the following objects
- A function \( g_s(\omega, y, z) \), \( P \)-progressively measurable for fixed \( y \) and \( z \), uniformly Lipschitz in \( (y, z) \) and such that
  \[ \mathbb{E}^P \left[ \int_{0}^{T} |g_s(0, 0)|^2 ds \right] < +\infty. \]
- A terminal condition \( \xi \) which is \( \mathcal{F}_T \)-measurable and in \( L^2(P) \).
- A càdlàg process \( V \) with \( \mathbb{E}^P \left[ \sup_{0 \leq t \leq T} |V_t|^2 \right] < +\infty. \)
• A càdlàg non-decreasing process $S$ such that $\mathbb{P}\left[\left(\sup_{0 \leq t \leq T} (S_t)^+\right)^2\right] < +\infty$.

We want to study the following problem. Finding $(y, z, k) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$ such that

$$
\begin{cases}
y_t = \xi + \int_t^T g_s(y_s, z_s)ds - \int_t^T z_s dW_s + k_T - k_t + V_T - V_t, & 0 \leq t \leq T, \; \mathbb{P} \text{-a.s.} \\
y_t \geq S_t, & \mathbb{P} \text{-a.s.} \\
\int_0^T (y_s - S_s) \, dk_s = 0, & \mathbb{P} \text{-a.s.}, \forall t \in [0, T].
\end{cases}
$$

We first have a result of existence and uniqueness

**Proposition A.1.** Under the above hypotheses, there exists a unique solution $(y, z, k) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$ to the reflected BSDE (A.2.1).

**Proof.** Consider the following penalized BSDE, whose existence and uniqueness are ensured by the results of Peng [28]

$$
y^n_t = \xi + \int_t^T g_s(y^n_s, z^n_s)ds - \int_t^T z^n_s dW_s + k^n_T - k^n_t + V_T - V_t,
$$

where $k^n_t := n \int_0^t (y^n_s - S_s)^+ \, ds$.

Then, define $\tilde{y}^n_t := y^n_t + V_t$, $\tilde{\xi} := \xi + V_T$, $\tilde{z}^n_t := z^n_t$, $\tilde{k}^n_t := k^n_t$ and $\tilde{g}_t(y, z) := g_t(y - V, z)$. We have

$$
\tilde{y}^n_t = \tilde{\xi} + \int_t^T \tilde{g}_s(\tilde{y}^n_s, \tilde{z}^n_s)ds - \int_t^T \tilde{z}^n_s dW_s + \tilde{k}^n_T - \tilde{k}^n_t.
$$

Then, since we know by Lepeltier and Xu [22], that the above penalization procedure converges to a solution of the corresponding RBSDE, existence and uniqueness are then simple generalization of the classical results in RBSDE theory. \qed

We also have a comparison theorem in this context

**Proposition A.2.** Let $\xi_1$ and $\xi_2 \in L^2(\mathbb{P})$, $V^i$, $i = 1, 2$ be two adapted, càdlàg processes and $g^i_s(\omega, y, z)$ two functions, which all verify the above assumptions. Let $(y^i, z^i, k^i) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$, $i = 1, 2$ be the solutions of the following RBSDEs with lower obstacle $S^i$

$$
y^i_t = \xi^i + \int_t^T g^i_s(y^i_s, z^i_s)ds - \int_t^T z^i_s dW_s + k^i_T - k^i_t + V^i_T - V^i_t, & \mathbb{P} \text{-a.s.}, \; i = 1, 2,
$$

respectively. If

- $\xi_1 \geq \xi_2$, $\mathbb{P}$-a.s.
- $V^1 - V^2$ is non-decreasing, $\mathbb{P}$-a.s.
• \( S^1 \geq S^2, \ \mathbb{P} - a.s. \)

• \( g_s^1(y^1_s, z^1_s) \geq g_s^2(y^1_s, z^1_s), \ dt \times d\mathbb{P} - a.s. \)

then it holds \( \mathbb{P} - a.s. \) that for all \( t \in [0, T] \)

\[
Y_t^1 \geq Y_t^2.
\]

Besides, if \( S^1 = S^2 \), then we also have \( dK^1 \leq dK^2 \).

**Proof.** The first part can be proved exactly as in [14], whereas the second one comes from the fact that the penalization procedure converges in this framework, as seen previously. \( \square \)

**Remark A.1.** If we replace the deterministic time \( T \) by a stopping time \( \tau \), then all the above is still valid.

From now on, we will specialize the discussion to the case where the process \( V \) is actually in \( \mathbb{I}^2(\mathbb{P}) \) and consider the following RBSDE

\[
\begin{align*}
  y_t &= \xi + \int_{t \land \tau}^\tau g_s(y_s, z_s)ds + V_t - V_{t \land \tau} + k_t - k_{t \land \tau} - \int_{t \land \tau}^\tau z_s dW_s, \quad 0 \leq t \leq \tau, \ \mathbb{P} - a.s. \\
  y_t &\geq S_t, \ \mathbb{P} - a.s. \\
  \int_0^\tau (y_s - S_s^-) \, dk_s = 0, \ \mathbb{P} - a.s., \ \forall t \in [0, \tau].
\end{align*}
\]

**Definition A.1.** If \( y \) is a solution of a RBSDE of the form \( (A.2.1) \), then we call \( y \) a reflected \( g \)-supersolution on \( [0, \tau] \). If \( V = 0 \) on \( [0, \tau] \), then we call \( y \) a reflected \( g \)-solution.

We now face a first difference from the case of non-reflected supersolution. Since in our case we have two increasing processes, if a \( g \)-supersolution is given, there can exist several increasing processes \( V \) and \( k \) such that \( (A.2.1) \) is satisfied. Indeed, we have the following proposition

**Proposition A.3.** Given \( y \) a \( g \)-supersolution on \( [0, \tau] \), there is a unique \( z \in \mathbb{H}^2(\mathbb{P}) \) and a unique couple \( (k, V) \in (\mathbb{I}^2(\mathbb{P}))^2 \) (in the sense that the sum \( k + V \) is unique), such that \( (y, z, k, V) \) satisfy \( (A.2.1) \). Besides, there exists a unique quadruple \( (y, z, k', V') \) satisfying \( (A.2.1) \) such that \( k' \) and \( V' \) never act at the same time.

**Proof.** If both \( (y, z, k, V) \) and \( (y, z^1, k^1, V^1) \) satisfy \( (A.2.1) \), then applying Itô’s formula to \( (y_t - y_t)^2 \) gives immediately that \( z = z^1 \) and thus \( k + V = k^1 + V^1, \ \mathbb{P} - a.s. \).

Then, if \( (y, z, k, V) \) satisfying \( (A.2.1) \) is given, then it is easy to construct \( (k', V') \) such that

• \( k' \) only increases when \( y_{t-} = S_{t-} \).

• \( V' \) only increases when \( y_{t-} > S_{t-} \).

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• $V_t' + k_t' = V_t + k_t$, $dt \times d\mathbb{P} - a.s.$

and such a couple is unique. \hfill \Box

**Remark A.2.** We give a counter-example to the general uniqueness in the above Proposition. Let $T = 2$ and consider the following RBSDE

$$
\begin{align*}
  y_t &= -2 + 2 - t + k_2 - k_t - \int_t^2 z_s dW_s, \quad 0 \leq t \leq 2, \ P - a.s. \\
  y_t &\geq -\frac{t^2}{2}, \ P - a.s. \\
  \int_0^2 (y_s - \frac{t^2}{2}) \, dk_s = 0, \ P - a.s., \ \forall t \in [0, 2].
\end{align*}
$$

We then have $z = 0$, $y_t = 1_{0 \leq t \leq 1} (\frac{1}{2} - t) - \frac{t^2}{2} 1_{1 < t \leq 2}$ and $k_t = 1_{t \geq 1} \frac{t^2 - 1}{2}$.

However, we can also take

$$
V_t' = t1_{t \leq 1} + \left( \frac{t^2}{4} + \frac{t}{4} + \frac{1}{2} \right) 1_{1 < t \leq 2} \text{ and } k_t' = 1_{t \geq 1} \left( \frac{t^2}{4} + \frac{3}{4} t - 1 \right).
$$

Following Peng [28], this allows us to define

**Definition A.2.** Let $y$ be a supersolution on $[0, \tau]$ and let $(y, z, k, V)$ be the related unique triple in the sense of the RBSDE (A.2.1), where $k$ and $V$ never act at the same time. Then we call $(z, k, V)$ the decomposition of $y$.

**A.2.2 Monotonic limit theorem**

We now study a limit theorem for reflected $g$-supersolutions, which is very similar to theorems 2.1 and 2.4 of [28].

We consider a sequence of reflected $g$-supersolutions

$$
\begin{align*}
  y^n_t &= \xi^n + \int_t^T g^n_s(y^n_s, z^n_s) \, ds + V^n_T - V^n_t - k^n_T - k^n_t - \int_t^T z^n_s dW_s, \quad 0 \leq t \leq \tau, \ P - a.s. \\
  y^n_t &\geq S_t, \ P - a.s. \\
  \int_0^T (y^n_s - S^n_s) \, dk^n_s = 0, \ P - a.s., \ \forall t \in [0, T],
\end{align*}
$$

where the $V^n$ are in addition supposed to be continuous.

**Theorem A.1.** If we assume that $(y^n_t)$ increasingly converges to $(y_t)$ with

$$
\mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} |y_t|^2 \right] < +\infty,
$$

and that $(k^n_t)$ decreasingly converges to $(k_t)$, then $y$ is a $g$-supersolution, that is to say that there exists $(z, V) \in \mathbb{H}^2(\mathbb{P}) \times \mathbb{L}^2(\mathbb{P})$ such that

$$
\begin{align*}
  y_t &= \xi + \int_t^T g_s(y_s, z_s) \, ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s, \quad 0 \leq t \leq T, \ P - a.s. \\
  y_t &\geq S_t, \ P - a.s. \\
  \int_0^T (y^n_s - S^n_s) \, dk^n_s = 0, \ P - a.s., \ \forall t \in [0, T],
\end{align*}
$$

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Besides, $z$ is the weak (resp. strong) limit of $z^n$ in $\mathbb{H}^2(\mathbb{P})$ (resp. in $\mathbb{H}^p(\mathbb{P})$ for $p < 2$) and $V_t$ is the weak limit of $V^n_t$ in $L^2(\mathbb{P})$.

Before proving the Theorem, we will need the following Lemma

**Lemma A.1.** Under the hypotheses of Theorem A.1, there exists a constant $C > 0$ independent of $n$ such that

$$\mathbb{E}^P \left[ \int_0^T |z^n_s|^2 \, ds + (V^n_T)^2 + (k^n_T)^2 \right] \leq C.$$

**Proof.** We have

$$A^n_T + k^n_T = y^n_T - y^n_T - \int_0^T g_s(y^n_s, z^n_s) \, ds + \int_0^T z^n_s \, dW_s$$

$$\leq C \left( \sup_{0 \leq t \leq T} |y^n_t| + \int_0^T |z^n_s| \, ds + \int_0^T |g_s(0,0)| \, ds + \int_0^T z^n_s \, dW_s \right). \quad (A.1)$$

Besides, we also have for all $n \geq 1$, $y^n_t \leq y^n_T$ and thus $|y^n_t| \leq |y^n_T| + |y_t|$, which in turn implies that

$$\sup_n \mathbb{E}^p \left[ \sup_{0 \leq t \leq T} |y^n_t|^2 \right] \leq C.$$

Reporting this in (A.1) and using BDG inequality, we obtain

$$\mathbb{E}^P \left[ (V^n_T)^2 + (k^n_T)^2 \right] \leq \mathbb{E}^P \left[ (V^n_T + k^n_T)^2 \right]$$

$$\leq C_0 \left( 1 + \mathbb{E}^P \left[ \int_0^T |g_s(0,0)|^2 \, ds + \int_0^T |z^n_s|^2 \, ds \right] \right). \quad (A.2)$$

Then, using Itô’s formula, we obtain classically for all $\epsilon > 0$

$$\mathbb{E}^P \left[ \int_0^T |z^n_s|^2 \, ds \right] \leq \mathbb{E}^P \left[ |y^n_T|^2 + 2 \int_0^T y^n_s g_s(y^n_s, z^n_s) \, ds + 2 \int_0^T y^n_s \, d(V^n_s + k^n_s) \right]$$

$$\leq \mathbb{E}^P \left[ C \left( 1 + \sup_{0 \leq t \leq T} |y^n_t|^2 \right) + \int_0^T \frac{|z^n_s|^2}{2} \, ds + \epsilon \left( |V^n_T|^2 + |k^n_T|^2 \right) \right]. \quad (A.3)$$

Then, from (A.2) and (A.3), we obtain by choosing $\epsilon = \frac{1}{4C_0}$ that

$$\mathbb{E}^P \left[ \int_0^T |z^n_s|^2 \, ds \right] \leq C.$$

Reporting this in (A.1) ends the proof.

**Proof.** [Proof of Theorem A.1] By Lemma A.1 and its proof we first have

$$\mathbb{E}^P \left[ \int_0^T |g_s(y^n_s, z^n_s)|^2 \, ds \right] \leq C \mathbb{E}^P \left[ \int_0^T |g_s(0,0)|^2 + |y^n_s|^2 + |z^n_s|^2 \, ds \right] \leq C.$$
Thus $g_s(y^n_s, z^n_s)$ and $z^n$ are bounded in $\mathbb{H}^2(\mathbb{P})$, and there exists subsequences which converge respectively to some $g_s$ and $z_s$. Therefore, for every stopping time $\tau$, we also have the following weak convergences

$$
\int_0^\tau z^n_s dW_s \rightarrow \int_0^\tau z_s dW_s, \quad \int_0^\tau g_s(y^n_s, z^n_s) ds \rightarrow \int_0^\tau \bar{g}_s ds,
$$

$$
V^n_\tau \rightarrow -y_\tau + y_0 - k_\tau - \int_0^\tau \bar{g}_s ds + \int_0^\tau z_s dW_s.
$$

Then by the section theorem, it is clear that $V$ and $k$ are non-decreasing, and by Lemma 2.2 of [28] we know that $y$, $V$ and $k$ are càdlàg. We now show the strong convergence of $z^n$. Following Peng [28], we apply Itô’s formula between two stopping times $\tau$ and $\sigma$. Since $V^n$ is continuous, we obtain

$$
\mathbb{E}^\mathbb{P}\left[ \int_\sigma^\tau |z^n_s - z_s|^2 ds \right] \leq \mathbb{E}^\mathbb{P}\left[ |y^n_s - y_\tau|^2 + \sum_{\sigma \leq t \leq \tau} (\Delta(V_t + k_t))^2 \right] + 2\mathbb{E}^\mathbb{P}\left[ \int_\sigma^\tau |y^n_s - y_s| |g_s(y^n_s, z^n_s) - \bar{g}_s| ds + \int_\sigma^\tau (y^n_s - y_s) d(V_s + k_s) \right].
$$

Then we can finish exactly as in [28] to obtain the desired convergence. Since $g$ is supposed to be Lipschitz, we actually have

$$
\bar{g}_s = g_s(y_s, z_s), \quad \mathbb{P} - a.s.
$$

Finally, since for each $n$, we have $y^n_t \geq S_t$, we have $y_t \geq S_t$. For the Skorohod condition, we have, since the $k^n$ are decreasing

$$
\mathbb{E}^\mathbb{P}\left[ \int_0^T y^n_t - S_t^+ dt \right] \leq \mathbb{E}^\mathbb{P}\left[ \int_0^T y^n_t - y^n_\tau^+ dt + \int_0^\tau y^n_t - S_t^- dt \right] = \mathbb{E}^\mathbb{P}\left[ \int_0^T y^n_t - y^n_\tau^- dt \right].
$$

Then, we have

$$
\mathbb{E}^\mathbb{P}\left[ \int_0^T y^n_t - y^n_\tau^- dt \right] \leq \left( \mathbb{E}^\mathbb{P}\left[ \sup_{0 \leq t \leq T} |y^n_t - y^n_\tau^-|^2 \right] \right)^{1/2} \left( \mathbb{E}^\mathbb{P}\left[ k_\tau^2 \right] \right)^{1/2} + \infty
$$

Therefore by Lebesgue dominated convergence Theorem, we obtain that

$$
\mathbb{E}^\mathbb{P}\left[ \int_0^T y^n_t - y^n_\tau^- dt \right] \rightarrow 0,
$$

and thus

$$
\mathbb{E}^\mathbb{P}\left[ \int_0^T y^n_t - S_t^- dt \right] \leq 0,
$$

which ends the proof. $\square$
A.2.3 Doob-Meyer decomposition

We now introduce the notion of reflected \(g\)-(super)martingales.

**Definition A.3.** (i) A reflected \(g\)-martingale on \([0,T]\) is a reflected \(g\)-solution on \([0,T]\).

(ii) \((Y_t)\) is a reflected \(g\)-supermartingale in the strong (resp. weak) sense if for all stopping time \(\tau \leq T\) (resp. all \(t \leq T\)), we have \(\mathbb{E}^P[|Y_t|^2] < +\infty\) (resp. \(\mathbb{E}^P[|Y_t|^2] < +\infty\)) and if the reflected \(g\)-solution \((y_s)\) on \([0,\tau]\) (resp. \([0,t]\)) with terminal condition \(Y_\tau\) (resp. \(Y_t\)) verifies \(y_\sigma \leq Y_\sigma\) for every stopping time \(\sigma \leq \tau\) (resp. \(y_s \leq Y_s\) for every \(s \leq t\)).

As in the case without reflection, under mild conditions, a reflected \(g\)-supermartingale in the weak sense corresponds to a reflected \(g\)-supermartingale in the strong sense. Besides, thanks to the comparison Theorem, it is clear that a \(g\)-supersolution on \([0,T]\) is also a \(g\)-supermartingale in the weak and strong sense on \([0,T]\). The following Theorem adresses the converse property, which gives us a non-linear Doob-Meyer decomposition.

**Theorem A.2.** Let \((Y_t)\) be a right-continuous reflected \(g\)-supermartingale on \([0,T]\) in the strong sense with

\[
\mathbb{E}^P \left[\sup_{0 \leq t \leq T} |Y_t|^2\right] < +\infty.
\]

Then \((Y_t)\) is a reflected \(g\)-supersolution on \([0,T]\), that is to say that there exists a unique triple \((z,k,V) \in \mathbb{H}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{L}^2(\mathbb{P})\) such that

\[
\begin{align*}
Y_t &= Y_T + \int_t^T g_s(Y_s,z_s)ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s. \\
Y_t &\geq S_t, \quad \mathbb{P} - a.s. \\
\int_0^T (Y_{s-} - S_{s-}) dk_s &= 0, \quad \mathbb{P} - a.s., \quad \forall t \in [0,T]. \\
V &\text{ and } k \text{ never act at the same time.}
\end{align*}
\]

We follow again [28] and consider the following sequence of RBSDEs

\[
\begin{align*}
y_t^0 &= Y_T + \int_t^T g_s(y_s^0,z_s^0)ds + n \int_t^T (Y_s - y_s^0)ds + k_T^0 - k_t^0 - \int_t^T z_s^0 dW_s, \quad 0 \leq t \leq T \\
y_t^0 &\geq S_t, \quad \mathbb{P} - a.s. \\
\int_0^T (y_{s-}^n - S_{s-}) dk_s^n &= 0, \quad \mathbb{P} - a.s., \quad \forall t \in [0,T].
\end{align*}
\]

We then have

**Lemma A.2.** For all \(n\), we have

\[Y_t \geq y_t^n.\]

**Proof.** The proof is exactly the same as the proof of Lemma 3.4 in [28], so we omit it. \(\square\)

**Proof.** [Proof of Theorem A.2] The uniqueness is due to the uniqueness for reflected \(g\)-supersolutions proved in Proposition A.3. For the existence part, we first notice that since \(Y_t \geq y_t^n\) for all \(n\), by the comparison Theorem for RBSDEs, we have \(y_t^n \leq y_t^{n+1}\) and \(dk_t^n \geq dk_t^{n+1}\). Therefore they converge monotonically to some processes \(y\) and \(k\). Besides,
y is bounded from above by Y. Therefore, all the conditions of Theorem A.1 are satisfied and y is a reflected g-supersolution on [0, T] of the form

\[ y_t = Y_T + \int_t^T g_s(y_s, z_s)ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s, \]

where V_t is the weak limit of V_i^\alpha := n \int_0^t (Y_s - y_s^\alpha)ds.

From Lemma A.1, we have

\[ \mathbb{E}^\mathbb{P}[(V_T^n)^2] = n^2 \mathbb{E}^\mathbb{P} \left( \int_0^T |Y_s - y_s^n|^2 ds \right) \leq C. \]

It then follows that Y_t = y_t, which ends the proof. \(\square\)

### A.2.4 Downcrossing inequality

In this section we prove a downcrossing inequality for reflected g-supersolution in the spirit of the one proved in [8]. We use the same notations as in the classical theory of g-martingales (see [8] and [28] for instance).

**Theorem A.3.** Assume that g(0, 0) = 0. Let (Y_t) be a positive reflected g-supersolution in the weak sense and let 0 = t_0 < t_1 < ... < t_i = T be a subdivision of [0, T]. Let 0 ≤ a < b, then there exists C > 0 such that D_n^b[Y, n], the number of downcrossings of [a, b] by \{Y_t\}, verifies

\[ \mathcal{E}^{-\mu} [D_n^b[Y, n]] \leq \frac{C}{b - a} \mathcal{E}^\mu [Y_0 \wedge b], \]

where \(\mu\) is the Lipschitz constant of g.

**Proof.** Consider

\[
\begin{align*}
  y_t^i &= Y_{t_i} + \int_{t_i}^{t_{i+1}} (\mu |y_s^i| + \mu |z_s^i| + k_s^n - k_s^a) ds + \int_{t_i}^{t_{i+1}} z_s ds, \\
  y_t^i \geq S_t, & \quad \mathbb{P} - a.s. \\
  \int_{t_i}^{t_{i+1}} (y_s^i - S_s^-) ds = 0, & \quad \mathbb{P} - a.s., \forall t \in [0, t_i].
\end{align*}
\]

We define \(a_s^i := -\mu \text{sgn}(z_s^i)\mathbb{1}_{t_{j-1} \leq s \leq t_j}\) and \(a_s := \sum_{i=0}^n a_s^i\). Let \(\mathbb{Q}^a\) be the probability measure defined by

\[ d\mathbb{Q}^a / d\mathbb{P} = \mathcal{E} \left( \int_0^t a_s dW_s \right). \]

We then have easily that \(y_t^i \geq 0\) since \(Y_{t_i} \geq 0\) and

\[ y_t^i = \text{ess sup}_{t_i \leq \tau \leq t_{i+1}} \mathbb{E}^{\mathbb{Q}^a}_{t_i} \left[ e^{-\mu(t_i - \tau)} S_{\tau} 1_{\tau < t_i} + Y_{t_i} e^{-\mu(t_i - \tau)} 1_{\tau = t_i} \right]. \]

Since Y is reflected g-martingale (and thus also a reflected \(g^{-\mu}\)-supersolution where \(g^{-\mu}(y, z) := -\mu(|y| + |z|)\)), we therefore obtain

\[ \text{ess sup}_{t_i \leq \tau \leq t_{i+1}} \mathbb{E}^{\mathbb{Q}^a}_{t_i} \left[ e^{-\mu(t_i - \tau)} S_{\tau} 1_{\tau < t_i} + Y_{t_i} e^{-\mu(t_i - \tau)} 1_{\tau = t_i} \right] \leq Y_{t_{i-1}}. \]
Hence, by choosing $\tau = t_j$ above, we get

$$E_{t_{i-1}}^{Q^a} \left[ Y_{t_i} e^{-\mu(t_i - t_{i-1})} \right] \leq Y_{t_{i-1}},$$

which implies that $(e^{-\mu t_i} Y_{t_i})_{0 \leq i \leq n}$ is a $Q^a$-supermartingale. Then we can finish the proof exactly as in [8].

\[ \square \]

References


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