Prices Expansion in the Wishart model

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Abstract. Using probability change techniques introduced by Drimus for Heston model, we derive a $n^{th}$ order expansion formula of Wishart option price in terms of Black-Scholes price and Black-Scholes Greeks. Numerical results are given for the second order case. Thanks to this new approximation, the smile implied by Wishart model can be better understood. The sensitivity of Delta and Vega to the volatility (respectively Vanna and Volga) indeed appear explicitly in this formula. En route to our formula, we present a number of new - to our knowledge - results on Laplace transforms and moments of the integrated Wishart processes.

Keywords: Stochastic volatility, Wishart model, price approximation, stochastic correlation, probability change

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1 Introduction

Since Black and Scholes introduced their option valuation model, an extensive literature focuses on pointing out its limitations. More precisely, two major features of the equity index options market cannot be captured through Black-Scholes model. First of all, observed market prices for both in-the-money and out-the-money options are higher than Black-Scholes prices with at-the-money volatilities. This effect is known as the volatility smile: the volatility depends both on the option expiry and the option strike. Second, there exists a term structure of implied volatilities. As a matter of fact, a constant volatility parameter does not enable to model correctly this behavior.

In order to model the smile efficiently, stochastic volatility models are a popular approach. They enable to have distinct processes for the stock return and its variance. Thus, they may generate volatility smiles. Moreover, if the variance process embeds a mean reversion term, these models can capture the term-structure in the variance dynamics. Popular stochastic volatility models include Heston [19], SABR and square-root models to name but a few.

On the equity market, basket products have become very common since the appearance of Mountain Range options. These products may involve intricate dependencies on the variance-covariance structure. However, most stochastic volatility models handle one single asset at a time. Correlation then is introduced as an exogenous parameter through Brownian motions. Interest into Wishart model, studied by Da Fonseca et al in [12] is rising rapidly as it proves to be an efficient way to model stochastic covariance behaviors. However, some practitioners may still be reluctant to adopt Wishart processes since they lack confidence into such complex processes. Getting an intuition on parameters seems an impossible task since the only existing call price formula, given in our previous article [16], involves a Fourier integral of a matrix function. In order to better understand Wishart model, we propose to apply the expansion technique.

Through this article, we pursue two goals. First, we would like to reduce the computational effort needed to compute the call price. Second, and this is our primary goal, we would like to obtain a similar result to those obtained independently by Drimus [8] and Miri et al [2] for Heston model. Using expansion techniques, they express the Heston call price as a combination of Black-Scholes price and Greeks. Since such a formulation shows a much clearer result to those obtained independently by Drimus [8] and Miri et al [2] for Heston model. Using expansion techniques, they express the Heston call price as a combination of Black-Scholes price and Greeks. Since such a formulation shows a much clearer result to those obtained independently by Drimus [8] and Miri et al [2] for Heston model.

On the way of obtaining the expansion formula, we carefully study the integrated Wishart process, generalizing earlier results of Dufresne [9] on the square-root process. In partic-
ular, we give explicit formulas for Laplace transforms and moments of any order. To the best of our knowledge, such results are new and original contributions.

After exposing notations that will be used throughout this paper in section 2, we present the Wishart model that we consider in section 3. Then, various results on the integrated Wishart process are derived in section 4. This enables us to derive Wishart expansion formulas in section 5. Numerical results are given for the special case $n = 2$ in the following section and conclusion summarizes our contribution.
2 Notation

- $M_{n,m}(\mathbb{R}), M_{n,m}(\mathbb{C})$: the sets of $n \times m$ real and complex matrix.
- $M_n(\mathbb{R}), M_n(\mathbb{C})$: the sets of real and complex square matrix.
- $S_n(\mathbb{R})$: the sets of real symmetric square matrix.
- $S_n^+(\mathbb{R})$: the sets of real symmetric positive square matrix.
- $S_n^{-}(\mathbb{R})$: the sets of real symmetric negative square matrix.
- $\tilde{S}_n^+(\mathbb{R})$: the sets of real symmetric positive definite square matrix.
- $\tilde{S}_n^{-}(\mathbb{R})$: the sets of real symmetric negative definite square matrix.
- If $A \in S_n^+$, $\sqrt{A}$ is the unique symmetric positive matrix which square equals $A$.
- $A^T$: the transpose of the matrix $A$.
- $\text{det}(A)$: the determinant of the matrix $A$.
- $A^{ij}$ is the element on the $i^{th}$ row and $j^{th}$ column of the matrix $A$.
- $GL_n(\mathbb{R}), GL_n(\mathbb{C})$: the sets of real and complex invertible matrix.
- $\mathcal{N}$ will denote the probability distribution function of the standard Gaussian random variable.

Unless otherwise stated, from now on we will work with a probability space denoted $(\Omega, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, \mathbb{Q})$, where $\mathbb{Q}$ is supposed to be a risk-neutral probability.
3 Model presentation

In this paper we will work with the so-called Wishart stochastic model, a multifactor generalization of the Heston Model. They were first introduced by Bru in [3] and extensively studied in the form that we are interested in by Da Fonseca et al. in [10], [11], [12] and [13] and ourselves in [16]. We stay in the framework of a single-asset model and assume the following dynamics for the price of the asset $S_t$:

$$\frac{dS_t}{S_t} = rdt + \text{Tr} \left( \sqrt{V_t} dZ_t \right)$$

$$dV_t = (\beta Q^2 + MV_t + V_t M)dt + \sqrt{V_t} dW_t Q + Q dW_t^T \sqrt{V_t}, \quad (1)$$

where:

- $r$ is the short rate, supposed constant.
- $Q \in \tilde{S}_n^+ (\mathbb{R})$.
- $M \in \tilde{S}_n^- (\mathbb{R})$.
- $\beta > n - 1$.
- $W_t$ and $Z_t$ are matrix Brownian motions.
- $V_0 \in \tilde{S}_n^+$.

A process which has the same dynamic as the variance matrix above is called a Wishart process, and we will now note such a process $W(M, Q, \beta, V_0)$.

We also assume the following correlation structure, which has been explicitly chosen so that the model could belong to the general class of affine processes (see [6] for a comprehensive study of affine processes) and thus could be analytically tractable:

$$dZ_t = dW_t R + dB_t \sqrt{T_n - R^T R}, \quad (2)$$

where $R = \rho I_n$ is a multiple of the identity matrix with $\rho \leq 0$ and $B_t$ a matrix Brownian motion independent of $W_t$.

Let us note immediately that the choices we made on the parameters of the model are not the most general ones and are for most of them necessary in order to derive the prices expansion in the Wishart model. Nonetheless, they are consistent with several empirical studies. Indeed, since the matrix $M$ plays a similar role as the mean-reversion parameter in the Heston model, Gouriéroux in [17] advocates that it has to be taken in $\tilde{S}_n^- (\mathbb{R})$. Besides, the assumption on the correlation parameter is consistent with earlier studies, e.g. Bakshi et al. [1], which show that the stock and the volatility are negatively correlated.
4 The Integrated Wishart Process

In the rest of this paper, the integrated variance process in the Wishart model will play a crucial role. It is defined by

\[ \Sigma_t = \int_0^t \text{Tr}[V_s] ds. \]  

Unlike its equivalent in the Heston model (that is to say the integrated square-root process) which has been extensively studied by Dufresne in [9], there are almost no references in the literature to the integrated Wishart process. Therefore, as far as we know, the results of this section are new. We will start by deriving its Laplace Transform, before discussing a method to get its moments of any order.

4.1 The Laplace Transform of the Integrated Wishart Process

One way to derive the Laplace transform of this process is to follow one of the methods described by Dufresne in [9], i.e. to compute the price of a zero-coupon bond in a model where the stochastic interest rate \( r_t \) would be the trace of a Wishart process with parameters \( M, Q, \beta \) and \( \rho \). Let us note

\[ L(\gamma, T; M, Q, \beta) = \mathbb{E}[e^{\gamma \Sigma_T}]. \]  

We suppose that the function above is a function of \( (r_t, T) \). Then, using Itô’s lemma and the usual no-arbitrage condition, we easily get the following PDE

\[ \frac{\partial L}{\partial T} = \mathcal{F} L + \gamma L, \]

where \( \mathcal{F} \) is the infinitesimal generator of the Wishart process, which expression is given in the following proposition (see [16] for a proof)

**Proposition 1.** The infinitesimal generator associated with the Wishart process (1) is given by, for \( V \in S_n^+ \):

\[ \mathcal{F} = \text{Tr} \left[ (\beta Q^2 + MV + VM) D + 2V DQ^2 D \right], \]

where

\[ D = \left( \frac{\partial}{\partial V_{ij}} \right)_{1\leq i,j\leq n}. \]

Since it is known that the characteristic function of the Wishart process is exponentially affine (see [16]), we guess that the function \( \mathcal{L} \) will be the exponential of an affine combination of \( V_0 \) and the terms of the Wishart matrix. That is to say

\[ \mathcal{L}(\gamma, T; M, Q, \beta, \rho) = e^{\text{Tr}(A(T)V_0) + b(T)}, \]
where

\[ A : \mathbb{R}^+ \rightarrow M_n (\mathbb{R}) , \quad A(0) = 0 \]
\[ b : \mathbb{R}^+ \rightarrow \mathbb{R} , \quad b(0) = 0. \]

Then replacing (8) in (5) leads to

\[ \text{Tr} \left[ A'(T) V \right] + b'(T) = \text{Tr} \left[ (A(T)M + MA(T) + 2A(T)Q^2 A(T) + \gamma I_n) V \right] + \beta \text{Tr} \left[ Q^2 A(T) \right]. \]

By identifying the coefficients of \( V \), we finally obtain the following system

\[ \text{Tr} \left( A'(T) \right) = \text{Tr} \left[ (A(T)M + MA(T) + 2A(T)Q^2 A(T)) + \gamma \right] \]
\[ b'(T) = \text{Tr} \left[ \beta Q^2 A(T) \right], \]

and using common properties of the Trace operator

\[ A'(T) = A(T)M + MA(T) + 2A(T)Q^2 A(T) + \gamma I_n \]
\[ b'(T) = \text{Tr} \left[ \beta Q^2 A(T) \right]. \]

The first equation above is a matrix Ricatti equation (see [14]) which can be easily solved with a linearization procedure, as presented by Da Fonseca et al. in [13].

We start by doubling the dimension of the problem by putting

\[ A(T) = F_T(\gamma)^{-1} G_T(\gamma), \]

for \( F_T(\gamma) \in GL_n (\mathbb{R}) \), \( G_T(\gamma) \in M_n (\mathbb{R}) \), \( F(0) = I_n \) and \( G(0) = 0 \).

With these new variables the first equation above becomes

\[ -F'(T)A(T) + G'(T) = G_T(\gamma)M + F_T(\gamma)MA(T) + 2G_T(\gamma)Q^2 A(T) + \gamma F_T(\gamma), \]

which in turn leads to:
\[ G'(T) = G_T(\gamma)M + \gamma F_T(\gamma) \]
\[ -F'(T) = F_T(\gamma)M + 2G_T(\gamma)Q^2, \]  
\[(9)\]

which is easily solved by

\[
(G_T(\gamma) \quad F_T(\gamma)) = (0 \quad I_n) \exp \left[ T \begin{pmatrix} M & -2Q^2 \\ \gamma I_n & -M \end{pmatrix} \right].
\]

Then, instead of solving the equation for \( b'(T) \) we use the following remark. From (9) we obtain:

\[ G_T(\gamma) = -\frac{1}{2} \left( F'(T) + F_T(\gamma)M \right) Q^{-2}, \]

and plugging into (4.1) and using the properties of the trace we deduce

\[ b'(T) = -\frac{\beta}{2} \text{Tr} \left( F_T(\gamma)^{-1} F'(T) + M \right). \]

We can easily integrate to get:

\[ b(T) = -\frac{\beta}{2} \left( T \text{Tr}(M) + \text{Tr} (\log F_T(\gamma)) \right). \]

Using properties of the matrix logarithm we get:

\[ \text{Tr} (\log F_T(\gamma)) = \log (\det F_T(\gamma)), \]

and therefore

\[ b(T) = -\frac{\beta}{2} \left( T \text{Tr}(M) + \log (\det F_T(\gamma)) \right). \]

It is important to note that we have not yet verified whether the above calculations were correct for all values of \( \gamma \). Indeed, the matrix \( F_T(\gamma) \) may not always have a real matrix logarithm for example. In order to have a better understanding of the definition problems that may arise, we will now obtain explicit expressions for both \( F_T(\gamma) \) and \( G_T(\gamma) \).

They will be based on the following lemma

**Lemma 1.** Let’s consider the following block-matrix

\[
A = \begin{pmatrix} TM & -2TQ^2 \\ \gamma TI_n & -TM \end{pmatrix}.
\]

Then we have
∀n ∈ N, \( A^{2n} = \begin{pmatrix} * & T^{2n} \left( M^2 - 2\gamma Q^2 \right)^n \\ 0 & \end{pmatrix} \)

\( A^{2n+1} = \begin{pmatrix} * & \gamma T^{2n+1} \left( M^2 - 2\gamma Q^2 \right)^n \\ \gamma TI_n & -TM \end{pmatrix} \),

where the * denote entries which are not relevant here.

**Proof.** We are going to prove the lemma by induction. For \( n = 0 \) the results are trivial. Suppose that the results hold for all values below a certain \( n \geq 1 \). Then we have

\[
A^{2n+1} = A^{2n} A = \begin{pmatrix} * & T^{2n} \left( M^2 - 2\gamma Q^2 \right)^n \\ 0 & \end{pmatrix} \begin{pmatrix} TM & -2TQ^2 \\ \gamma TI_n & -TM \end{pmatrix} = \begin{pmatrix} * & \gamma T^{2n+1} \left( M^2 - 2\gamma Q^2 \right)^n \\ \gamma T^{2n+1} \left( M^2 - 2\gamma Q^2 \right)^n & -T^{2n+1} \left( M^2 - 2\gamma Q^2 \right)^n M \end{pmatrix},
\]

which proves the result for odd powers of \( A \).

The result for even powers of \( A \) can be proven in a similar fashion.

Using the previous lemma, we then easily get that

\[
e^A = \gamma \sum_{n=0}^{\infty} \frac{T^{2n+1} \left( M^2 - 2\gamma Q^2 \right)^n}{(2n+1)!} \sum_{n=0}^{\infty} \frac{T^{2n} \left( M^2 - 2\gamma Q^2 \right)^n}{(2n)!} - \sum_{n=0}^{\infty} \frac{T^{2n+1} \left( M^2 - 2\gamma Q^2 \right)^n}{(2n+1)!} M.
\]

Now suppose that the matrix \( M^2 - 2\gamma Q^2 \) is symmetric definite positive. We can write

\[
G_T(\gamma) = \gamma \text{sh} \left[ T \sqrt{P(\gamma)} \right] P(\gamma)^{-1/2}
\]

\[
F_T(\gamma) = \text{ch} \left[ T \sqrt{P(\gamma)} \right] - \text{sh} \left[ T \sqrt{P(\gamma)} \right] P(\gamma)^{-1/2} M,
\]

where the hyperbolic sine and cosine of a matrix are defined from the matrix exponential as in the real case and where \( P(\gamma) = M^2 - 2\gamma Q^2 \).
Therefore in that case, except maybe \( \log(\det F_T(\gamma)) \) all the quantities that we came across during the computation of the Laplace transform are real. Thus we also need to have \( \det F_T(\gamma) > 0 \). But since it is clear that \( \text{ch} \left[ T \sqrt{P(\gamma)} \right] \) and \( \text{sh} \left[ T \sqrt{P(\gamma)} \right] \) are symmetric definite positive matrix, we have

\[
\det F_T(\gamma) = \det \left( \text{ch} \left[ T \sqrt{P(\gamma)} \right] \right) \det \left( I_n - \text{ch} \left[ T \sqrt{P(\gamma)} \right]^{-1} \text{sh} \left[ T \sqrt{P(\gamma)} \right] \right) P(\gamma)^{-1/2} M .
\]

Since by definition \( \text{ch} \left[ T \sqrt{P(\gamma)} \right] \), \( \text{sh} \left[ T \sqrt{P(\gamma)} \right] \) and any power in \( \mathbb{Z} \) of \( \sqrt{P(\gamma)} \) are symmetric definite positive matrix which commute with each other, it is well-known that their product is a symmetric definite positive matrix. Then using the fact that \( -M \) is also a symmetric definite positive matrix and proposition 12 of [16], it is clear that the determinant above is going to be strictly positive.

Therefore, we have proven that a sufficient condition for the Laplace transform of the integrated Wishart process to be defined is

\[
P(\gamma) > 0.
\] (12)

Using proposition 13 of [16], we can deduce the following more practical sufficient condition

\[
\gamma < \frac{1}{2} \frac{\lambda_{\min}(M^2)}{\lambda_{\max}(Q^2)}.
\] (13)

We now resume this section in the following proposition

**Proposition 2.** Let \( V_t \) be a Wishart process \( W(M, Q, \beta, V_0) \) where the parameters \( M, Q, \beta \) and \( V_0 \) verify the conditions recalled at the beginning of the article. Then we have

\[
\forall \gamma < \frac{1}{2} \frac{\lambda_{\min}(M^2)}{\lambda_{\max}(Q^2)},
\]

\[
\mathcal{L}(\gamma, T; M, Q, \beta) = \left( \frac{e^{-T \text{Tr}(M)}}{\det F_T(\gamma)} \right)^{\beta/2} e^{T \text{Tr} \left[ (F_T(\gamma)^{-1} G_T(\gamma)V_0) \right] ,
\] (14)

where we have the two following definitions for \( F_T(\gamma) \) and \( G_T(\gamma) \)

\[
\begin{pmatrix}
* & * \\
G_T(\gamma) & F_T(\gamma)
\end{pmatrix} = \exp \left[ T \begin{pmatrix}
M & -2Q^2 \\
\gamma I_n & -M
\end{pmatrix} \right],
\]
or
\[ G_T(\gamma) = \gamma \text{sh} \left[ T \sqrt{M^2 - 2\gamma Q^2} \right] (M^2 - 2\gamma Q^2)^{-1/2} \]

\[ F_T(\gamma) = \text{ch} \left[ T \sqrt{M^2 - 2\gamma Q^2} \right] - \text{sh} \left[ T \sqrt{M^2 - 2\gamma Q^2} \right] (M^2 - 2\gamma Q^2)^{-1/2} M. \]

Remarks

- A rapid examination of the corresponding Laplace transform for the Heston model given in [9], highlights the fact that our formula (14) is its direct generalization to the multidimensional case.
- We have kept both formulations for \( F_T(\gamma) \) and \( G_T(\gamma) \) since they are both going to be needed for the derivation of the moments in the next section.

4.2 The Moments of the Integrated Wishart Process

Even though in the case of the Heston model, as pointed out by Dufresne in [9], several methods are available to obtain the moments of the integrated square-root process, among which the most efficient use the dynamic of the process and Itô’s lemma, the simple fact that the Wishart process is matrix-valued makes those type of analysis useless. Indeed, the dynamic of the trace of a Wishart process does not have a form which allows a simple utilization of Itô’s lemma. Therefore, we are left with one last option, that is to say using the Laplace Transform to derive the moments thanks to the well-known formula

\[ \mathbb{E} \left[ \Sigma^k_T \right] = \frac{\partial^k \mathcal{L}}{\partial \gamma^k} (0, T; M, Q, \beta). \]

That being said, we will limit ourself to the calculation of the first two moments in this paper, since the method used can then be readily adapted to higher order moments (with of course a far larger amount of calculations). Before beginning, we will need the following, which recalls well-known results about matrix functions derivatives (see [23] and [18] and the references therein for the proofs)

Lemma 2. Let \( F : \mathbb{R} \to GL_n(\mathbb{R}) \) be a matrix valued function of class \( \mathcal{C}^1 \). Then the following functions are also \( \mathcal{C}^1 \) and we have

\[
\begin{align*}
\frac{d \log (\det F(t))}{dt} & = \text{Tr} \left[ F(t)^{-1} \frac{dF(t)}{dt} \right] \\
\frac{d e^{F(t)}}{dt} & = \int_0^1 e^{(1-u)F(t)} \frac{dF(t)}{dt} e^{uF(t)} du \\
\frac{dF(t)^{-1}}{dt} & = -F(t)^{-1} \frac{dF(t)}{dt} F(t)^{-1}.
\end{align*}
\]

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As a direct corollary of those formulas, we have the following proposition which gives the first two derivatives of \( L(\gamma, T; M, Q, \beta) \) at \( \gamma = 0 \)

**Proposition 3.** We have

\[
\frac{\partial L}{\partial \gamma}(0, T; M, Q, \beta) = \text{Tr} \left[ e^{TM} \frac{dG_T}{d\gamma}(0) V_0 \right] - \frac{\beta}{2} \text{Tr} \left[ e^{TM} \frac{dF_T}{d\gamma}(0) \right] \tag{15}
\]

\[
\frac{\partial^2 L}{\partial \gamma^2}(0, T; M, Q, \beta) = \text{Tr} \left[ \left( e^{TM} \frac{d^2 G_T}{d\gamma^2}(0) - 2e^{TM} \frac{dF_T}{d\gamma}(0)e^{TM} \frac{dG_T}{d\gamma}(0) \right) V_0 \right] \]

\[
- \frac{\beta}{2} \text{Tr} \left[ e^{TM} \frac{d^2 F_T}{d\gamma^2}(0) - e^{TM} \frac{dF_T}{d\gamma}(0)e^{TM} \frac{dF_T}{d\gamma}(0) \right] \]

\[
+ \left( \frac{\partial L}{\partial \gamma}(0, T; M, Q, \beta) \right)^2 \tag{16}
\]

**Proof.** From the formulas (11) we know that

\[ F_T(0) = e^{-TM} \]

\[ G_T(0) = 0. \]

Using lemma 2 the two formulas are then easily shown. \( \Box \)

We still need to compute the first two derivatives at \( \gamma = 0 \) of \( F_T \) and \( G_T \) and some related quantities. This is done by the following lemma which is proven in the appendix

**Lemma 3.** We have

\[ G_T'(0) = \frac{e^{-TM}}{2} \left[ e^{2TM} - I_n \right] M^{-1} \]

\[ F_T'(0) = 2e^{-TM} \int_0^T e^{2uM} \int_0^u e^{-sM} Q^2 e^{-sM} ds du \]

\[ G_T''(0) = 2M^{-2}Q^2M^{-1} \text{sh}(TM) + 2e^{-TM} \int_0^T e^{gM} M^{-1} Q^2 M^{-1} \text{sh}(gM) dg \]

\[ - 2e^{TM} \int_0^T e^{-gM} M^{-2} Q^2 \text{ch}(gM) dg \]

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\[
\text{Tr} \left[ e^{TM} F_T''(0) \right] = \text{Tr} \left[ -2 \int_0^T e^{gM} M^{-3} Q^2 \text{ch}(gM) dgQ^2 
+ 2e^{2TM} \int_0^T e^{-gM} M^{-3} Q^2 \text{ch}(gM) dgQ^2 
- 4 \int_0^T (T - g)e^{gM} M^{-1} Q^2 M^{-1} \text{sh}(gM) dgQ^2 
- 4e^{TM} M^{-2} Q^2 \int_0^T \text{sh} [(T - g)M] M^{-1} Q^2 e^{-gM} dg \right].
\]

Proof. See appendix A \[\square\]

Then, the remaining quantities needed are given in the following lemma

Lemma 4. We have

\[
\text{Tr} \left[ e^{TM} F_T'(0) e^{TM} G_T'(0) V_0 \right] = \text{Tr} \left[ \frac{(e^{2TM} - I_n) V_0}{2} \int_0^T e^{gM} M^{-1} Q^2 M^{-1} e^{-gM} dg 
- e^{2TM} \int_0^T e^{-gM} M^{-1} Q^2 M^{-1} e^{-gM} dg \frac{(e^{2TM} - I_n) V_0}{2} \right]
\]

\[
\text{Tr} \left[ \left( e^{TM} F_T'(0) \right)^2 \right] = - \text{Tr} \left[ e^{2TM} \int_0^T e^{-gM} M^{-2} Q^2 M^{-1} e^{-gM} M^{-1} \text{ch}(gM) dgQ^2 \right] - \int_0^T (T - g)e^{gM} M^{-1} Q^2 M^{-1} e^{-gM} M^{-2} Q^2 M^{-1} e^{-gM} M^{-1} \text{ch}(gM) dgQ^2
- e^{2TM} \int_0^T ge^{gM} M^{-1} Q^2 M^{-1} e^{-gM} M^{-1} e^{-gM} M^{-1} e^{-gM} M^{-1} \text{ch}(gM) dgQ^2
+ \int_0^T \text{sh} [(T - g)M] M^{-2} Q^2 M^{-1} e^{-gM} M^{-1} e^{-gM} M^{-1} e^{-gM} M^{-1} \text{ch}(gM) dgQ^2\right]
\]

Proof. All the calculations are similar to those in the proof of the previous lemma and are left to the courageous readers. \[\square\]

Finally, thanks to all these results, we can now compute the first two moments of the integrated Wishart process.
Proposition 4. We have

$$
E[\Sigma_T] = \text{Tr} \left[ \frac{M^{-1}V_0}{2} (e^{2TM} - I_n) - \frac{\beta T}{2} M^{-1}Q^2 + \frac{\beta M^{-2}Q^2}{4} (e^{2TM} - I_n) \right] \quad (17)
$$

$$
\text{Var}[\Sigma_T] = \text{Tr} \left[ \left( e^{TM} G_T''(0) - 2e^{TM} F_T'(0) e^{TM} G_T'(0) \right) V_0 \right] - \frac{\beta}{2} \text{Tr} \left[ e^{TM} F_T''(0) - \left( e^{TM} F'_T(0) \right)^2 \right] \quad (18)
$$

Remarks

- The formulas above are interesting since actually, they, almost, do not involve any numeric integration. Indeed, the formula for the first moment only requires to be able to compute the exponential of a matrix (see [20] for efficient algorithms). Besides, it is easy to show that

$$
\forall (\alpha, \beta) \in \mathbb{R}^2_+, \forall (A, B) \in M_n(\mathbb{R})^2, \text{ if we note } \exp \left[ T \begin{pmatrix} -\alpha A & B \\ 0 & \beta A \end{pmatrix} \right] = \begin{pmatrix} P_1 & P_2 \\ 0 & P_3 \end{pmatrix},
$$

then

$$
P_1^{-1}P_2 = \int_0^T e^{\alpha gA} B e^{\beta gB} dg.
$$

Therefore almost all the terms in the expressions above can be calculated thanks to this result. Unfortunately, we did not find any simple way to compute the remaining ones where the integrand is multiplied by \( g \).

- The calculations for moments of higher orders can be done the same way, but we think that using approximations, such as finite difference methods for example, may be rather less time-consuming.

5 Wishart Expansions : the General Case

5.1 Deriving the expansion

Since our aim here is to approximate call prices in the Wishart model with Black-Scholes prices and Greeks, we start by recalling the well-known pricing formula, where, following Drimus [8], instead of using the annualized volatility parameter, we will use the total variance. Specifically, we will consider the formula
\[ C^{BS} (S_0, \Sigma; r, K, T) = S_0 N(d_1) - Ke^{-rT} N(d_0), \]

where

\[
d_0 = \frac{1}{\sqrt{V}} \log \left( \frac{S_0}{Ke^{-rT}} \right) - \frac{1}{2} \sqrt{V}
\]

\[
d_1 = d_0 + \sqrt{V}.
\]

Then, using Itô’s lemma applied to \( \log S_t \) in (1) and the chosen correlation structure leads easily to

\[
S_T = S_0 \xi_T \exp \left( rT - \frac{1 - \rho^2}{2} \Sigma_T + \sqrt{1 - \rho^2} \int_0^T \text{Tr} \left[ \sqrt{\Sigma} dB_t \right] \right), \tag{19}
\]

where the process \((\xi_t)_{0 \leq t \leq T}\) is defined as the stochastic exponential of \( \int_0^t \rho \text{Tr} \left[ \sqrt{\Sigma} dW_s \right] \)

\[
\xi_t = e^{\rho \int_0^t \text{Tr} [\sqrt{\Sigma} dW_s] - \frac{\rho^2}{2} \int_0^t \text{Tr} [\Sigma] du}. \tag{20}
\]

Now, since the matrix Brownian motions \( W_t \) and \( B_t \) are independent, if we note by \( (\mathcal{F}_t^W)_{0 \leq t \leq T} \) the natural filtration of the Brownian motion \( W \), we can write the call price in the Wishart model in the following form

\[
C^{WIS} (S_0, K, T, r, V_0, M, Q, \beta, \rho) = e^{-rT} \mathbb{E}^Q \left[ \mathbb{E}^Q \left[ (S_T - K)_+ | \mathcal{F}_T^W \right] \right] = \mathbb{E}^Q \left[ C^{BS} (S_0 \xi_T, (1 - \rho^2) \Sigma_T; r, K, T) \right]. \tag{21}
\]

In other words, we will make use of the fact that, conditional on a realization of the instantaneous variance path, the Wishart option price becomes a Black-Scholes option price with initial spot \( S_0 \xi_T \) and total variance \((1 - \rho^2) \Sigma_T\). In a different context, Romano and Touzi in [25] used a similar representation in their study of market completeness in stochastic volatility models.

Now, it would be interesting if the process \( \xi_t \) happened to be a martingale. According to Novikov’s condition (see [24]), it will be the case if

\[
\mathbb{E}^Q \left[ e^{\frac{\rho^2}{2} \Sigma_T} \right] < +\infty. \tag{22}
\]

Thanks to proposition 2, we know that it is going to be the case if we have

\[
\rho^2 < \frac{\lambda_{min}(M^2)}{\lambda_{max}(Q^2)}. \tag{23}
\]
In practice, this condition is usually satisfied and does not impose a severe limitation on the range of acceptable parameters. Indeed, the mean-reversion parameter \( M \) tends to have coefficients greater than 1 while the volatility of volatility matrix \( Q \) tends to have coefficients smaller than 1. In all of these cases the inequality is very likely to be satisfied since we have \( |\rho| < 1 \).

Then, in order to obtain our approximation, we proceed from relation (21) and use a Taylor expansion around the point \( (S_0, (1 - \rho^2)E^Q[\Sigma_T]) \). Using the fact that \( E^Q[\xi_T] = 1 \), the \( n^{th} \) order approximation will have the form

\[
C_{n}^{WIS} = C^{BS} \left( S_0, (1 - \rho^2)E^Q[\Sigma_T]; r, K, T \right) \\
+ \sum_{k=2}^{n} \sum_{l=0}^{k} \frac{S_0(1 - \rho^2)^{k-l}}{l!(k-l)!} \partial^k C^{BS} \left( \frac{\partial S^T}{\partial \Sigma^{k-l}} \right) \left( S_0, (1 - \rho^2)E^Q[\Sigma_T]; r, K, T \right) \times \\
E^Q \left[ (\xi_T - 1)^l \left( \Sigma_T - E^Q[\Sigma_T] \right)^{k-l} \right].
\]  

In order to obtain closed form expressions for the terms in the Taylor polynomial above, we notice that we require a technique to derive general moments of the form \( E^Q[\xi_T^n \Sigma_T^m] \) for \( n \) and \( m \) non-negative integers. This is going to be the purpose of the next section. We can also remark that, in general, it is not possible to put a bound on the remainder term associated with the Taylor polynomial above because some Greeks of our European call may be unbounded. For example, the Gamma becomes unbounded as the total variance \( V \) goes to zero.

### 5.2 Crossed-moments of \( \xi_T \) and \( \Sigma_T \)

The first step necessary to compute the quantities we are interested in consist on defining a sequence of equivalent probability measures.

#### 5.2.1 A sequence of Probability Measures

We have seen in the previous section that under the correct condition on \( \rho \), the process \( \xi_T \) was a martingale. Therefore, it can be used to define a new probability measure \( Q^1 \), which Radon-Nykodym density is given by

\[
\frac{dQ^1}{dQ} = \xi_T.
\]  

Now, fortunately, equivalent of the celebrated Girsanov formula have been found in [7] as a generalization of results on Bessel and square Bessel processes. We have the following theorem
Theorem 1. Let $\beta \geq n + 1$, let $V_t$ be the unique symmetric positive definite solution of (1), let $\nu > 0$ and let $Q^*$ be the probability measure defined by its Radon-Nikodym density
\[
\frac{dQ^*}{dQ} = e^\nu \int_0^T \text{Tr} [\sqrt{V_s} dW_s] - \frac{\nu^2}{2} \int_0^T \text{Tr} [V_s] ds.
\] (26)

Then the matrix valued stochastic process $W_t^*$ defined by
\[
W_t^* = W_t - \nu \int_0^t \sqrt{V_s} ds,
\]
is a matrix Brownian motion under the probability $Q^*$, and we have
\[
dV_t = (\beta Q^2 + (M + \nu Q) V_t + V_t (M + \nu Q)) dt + \sqrt{V_t} dW^*_t Q + Q (dW^*_t)^T \sqrt{V_t}.
\]

Therefore, we have that under the new probability $Q^1$, the process $V_t$ is still a Wishart process $W(M + \rho Q, Q, \beta, V_0)$. Let us note that this was for obtaining this particular result that we imposed that the correlation matrix $R$ is a multiple of the identity matrix. Otherwise the Wishart structure would not have been conserved by the probability change. Let us now state a straightforward generalization

Lemma 5. If $\rho < 0$ and $\rho^2 < \frac{\lambda_{\min}(M^2)}{\lambda_{\max}(Q^2)}$, it is possible to define a sequence of equivalent probability measures $Q^n$ by
\[
\frac{dQ^n}{dQ^{n-1}} = \xi^{(n-1)}_T
\]
where
\[
\xi_T^{(n-1)} = e^\rho \int_0^T \text{Tr} [\sqrt{V_t} dW_{t-}^{n-1}] - \frac{\rho^2}{2} \int_0^T \text{Tr} [V_t] dt,
\]
where $W_{t-}^n = W_{t-}^{n-1} - \rho \int_0^t \sqrt{V_s} dt$ is a matrix brownian motion under $Q^n$. Besides, $V_t$ is a Wishart process $W(M + n\rho Q, Q, \beta, V_0)$ under $Q^n$.

Proof. The only thing which have to to be verified here is that all the probability measures are well-defined, the other results being simple consequences of theorem 1. We know that a sufficient condition for $Q^n$ to be defined is
\[
\rho^2 < \frac{\lambda_{\min}((M + n\rho Q)^2)}{\lambda_{\max}(Q^2)}.
\] (27)

Using the second part of lemma 1 of [16], we easily get that
\[
\lambda_{\min}((M + n\rho Q)^2) \geq \lambda_{\min}(M^2) + n^2 \rho^2 \lambda_{\min}(Q^2) + n\rho \lambda_{\min}(QM + MQ).
\]
Now using twice the first part of lemma 1 of [16], we have that, since $Q$ is symmetric definite positive
\[\lambda_{\text{min}}(QM + MQ) \leq \frac{1}{n} \text{Tr}[QM + MQ]\]
\[= \frac{2}{n} \text{Tr}[QM]\]
\[\leq \frac{2}{n} \lambda_{\text{max}}(M) \text{Tr}[Q]\]
\[\leq 0,\]

where we also used the fact that \(M\) is symmetric definite negative.

Therefore, since \(\rho < 0\), it is clear that \(\lambda_{\text{min}}((M + n\rho Q)^2) > \lambda_{\text{min}}(M^2)\) and that the condition (27) is a simple consequence of \(\rho^2 < \frac{\lambda_{\text{min}}(M^2)}{\lambda_{\text{max}}(Q^2)}\). Thus all the probabilities of the sequence are well defined.

We will now use those probabilities to compute the crossed-moments of \(\xi_T\) and \(\Sigma_T\).

### 5.2.2 Moments Calculation

We first consider the more simple case where the power of \(\Sigma_T\) in the expectation is 0. We have the following result

**Proposition 5.** If \(\rho < 0\) and \(\rho^2 < \frac{\lambda_{\text{min}}(M^2)}{\lambda_{\text{max}}(Q^2)}\), we have for any strictly non-negative integer \(n\)

\[\mathbb{E}^Q[\xi_T^n] = \mathcal{L}\left(\frac{n(n-1)}{2}\rho^2, T; M + n\rho Q, Q, \beta\right).\] (28)

**Proof.** First of all, notice that we have the following recursive relationship

\[\xi_T^{(n)} = \xi_T^{(n-1)} e^{-\rho^2 \int_0^T \text{Tr}[V]dt},\]

which leads to, for all \(n \geq 1\)

\[\xi_T = \xi_T^{(n)} e^{n\rho^2 \int_0^T \text{Tr}[V]dt},\]
and thus

\[\xi_T^n = \prod_{k=1}^{n} \xi_T \]

\[= \prod_{k=1}^{n} \left[ \xi_T^{(k-1)} e^{(k-1)\rho^2 \int_0^T \text{Tr}[V] dt} \right] \]

\[= \left( \prod_{k=1}^{n} \xi_T^{(k-1)} \right) e^{\frac{n(n-1)}{2} \rho^2 \int_0^T \text{Tr}[V] dt} \]

\[= \frac{dQ^n}{dQ} e^{\frac{n(n-1)}{2} \rho^2 \int_0^T \text{Tr}[V] dt}.\]

Therefore, since under \(Q^n (V_t)_{0 \leq t \leq T}\) is \(W(M + n\rho Q, Q, \beta, V_0)\), we have

\[\mathbb{E}^Q [\xi_T^n] = \mathbb{E}^Q \left[ \frac{dQ^n}{dQ} e^{\frac{n(n-1)}{2} \rho^2 \int_0^T \text{Tr}[V] dt} \right] \]

\[= \mathbb{E}^{Q^n} \left[ e^{\frac{n(n-1)}{2} \rho^2 \int_0^T \text{Tr}[V] dt} \right] \]

\[= \mathcal{L} \left( \frac{n(n-1)}{2} \rho^2, T; M + n\rho Q, Q, \beta \right).\]

Remains to check that these moments are all finite. As usual we have to verify the sufficient condition

\[n(n-1)\rho^2 < \frac{\lambda_{\min} \left( (M + n\rho Q)^2 \right)}{\lambda_{\max} (Q^2)} \]

\[\Leftrightarrow \lambda_{\min} \left( (M + n\rho Q)^2 \right) - n(n-1)\rho^2 \lambda_{\max} (Q^2) > 0.\]

Yet, using the same arguments as in the proof of lemma 5, we have

\[\lambda_{\min} \left( (M + n\rho Q)^2 \right) - n(n-1)\rho^2 \lambda_{\max} (Q^2) \geq \lambda_{\min} (M^2) + n\rho^2 \lambda_{\min} (Q^2) + n\rho \lambda_{\min} (M^2 + Q^2) \]

\[> 0,\]

which ends the proof.
We now get to the following generalization

**Proposition 6.** If \( \rho < 0 \) and \( \rho^2 < \frac{\lambda_{\min}(M^2)}{\lambda_{\max}(Q^2)} \), we have for any strictly non-negative integer \( n \) and \( m \)

\[
\mathbb{E}^Q [\xi^m_T \Sigma^m_T] = \frac{\partial^m \mathcal{L}}{\partial \gamma^m} \left( \frac{n(n-1)}{2} \rho^2, T; M + n \rho Q, Q, \beta \right).
\]  

(29)

**Proof.** As in the proof of the previous proposition, we write

\[
\mathbb{E}^Q [\xi^m_T \Sigma^m_T] = \mathbb{E}^{Q^n} \left[ e^{\frac{n(n-1)}{2} \rho^2 \Sigma_T r \Sigma^m_T} \right].
\]

Before going on we have to check that the moment above is finite. We know from the proof of proposition 5 that

\[
n(n-1)\rho^2 < \frac{\lambda_{\min}((M+n\rho Q)^2)}{\lambda_{\max}(Q^2)}.
\]

Let us then choose \( p \in \left] n(n-1)\rho^2, \frac{\lambda_{\min}((M+n\rho Q)^2)}{\lambda_{\max}(Q^2)} \right] \), we have

\[
e^{\frac{n(n-1)}{2} \rho^2 \Sigma_T r \Sigma^m_T} = e^{\frac{\Sigma^m_T}{m!}} \frac{\Sigma^m_T}{\left( \frac{p}{2} - \frac{n(n-1)}{2} \right)^m} \leq e^{\frac{\Sigma^m_T}{m!}} \frac{m!}{\left( \frac{p}{2} - \frac{n(n-1)}{2} \right)^m},
\]

where we used the fact that \( \forall (m,x) \in \mathbb{N} \times \mathbb{R}^+, e^x \geq \frac{x^m}{m!} \).

Since we have chosen \( p \) so that it verifies the condition for finiteness of the Laplace transform, we have \( \mathbb{E}^{Q^n} \left[ e^{\frac{\Sigma^m_T}{m!}} \right] < +\infty \).

Now since under \( Q^n \) \((V_t)_{0 \leq t \leq T}\) is \( \mathcal{W}(M+n\rho Q,Q,\beta,V_0) \), we know that

\[
\forall \gamma < \frac{1}{2} \frac{\lambda_{\min}((M+n\rho Q)^2)}{\lambda_{\max}(Q^2)}, \quad \mathbb{E}^{Q^n} \left[ e^{\gamma \Sigma_T} \right] = \mathcal{L} (\gamma, T; M + n \rho Q, Q, \beta).
\]

Taking the \( m^{th} \) derivative with respect to \( \gamma \) at \( \gamma = \frac{n(n-1)}{2} \rho^2 \) leads then immediately to the result. \( \square \)
Remark  Given the very complex form of the Laplace function, it is clear that in practice it would be better to approximate the moments above with numerical methods when \( m \) becomes too big. Indeed we were able to obtain almost closed-form formulas for the derivatives of \( \mathcal{L} \) at the point 0 only, and a quick examination of the proof of lemma 3 shows us that it is very unlikely to be able to do the same at any other point.

6 Second-Order Expansion and Numerical Results

6.1 Calculating the Expansion

We now have everything we need to obtain a second order approximation of call option prices in the Wishart model. When \( n = 2 \), the formula (24) reads

\[
C_{2^{nd \text{IS}}}^{W} = C_{BS} + \frac{1}{2} S_0^2 \mathbb{E}^Q \left[ \left( \xi_T - 1 \right)^2 \right] \frac{\partial^2 C_{BS}}{\partial S^2} + \frac{1}{2} \left(1 - \rho^2\right) \mathbb{V} \mathbb{a}r \left[ \Sigma_T \right] \frac{\partial^2 C_{BS}}{\partial \Sigma^2} \\
+ S_0 \left(1 - \rho^2\right) \mathbb{E}^Q \left[ \left( \xi_T - 1 \right) \left( \Sigma_T - \mathbb{E}^Q \left[ \Sigma_T \right] \right) \right] \frac{\partial^2 C_{BS}}{\partial \Sigma \partial S},
\]

where we dropped the common argument \((S_0, (1 - \rho^2)\mathbb{E}^Q[\Sigma_T]; r, K, T)\) for the sake of clarity.

Before commenting on this formula, let us calculate the unknown quantities involved. For the sake of simplicity, if \( V_t \) is a Wishart \( \mathcal{W}(M, Q, \beta, V_0) \), we will note

\[
\mathbb{E}^Q \left[ \Sigma_T \right] := D_1 \left(T, M, Q, \beta, V_0 \right).
\]

Thus we have using propositions 5 and 6

\[
\mathbb{E}^Q \left[ \left( \xi_T - 1 \right)^2 \right] = \mathbb{E}^Q \left[ \xi_T^2 \right] - 1 \\
= \mathcal{L} \left( \rho^2, T; M + 2\rho Q, Q, \beta \right) - 1
\]

and the variance of \( \Sigma_T \) is given in proposition 2.

An important part in our expansions is played by the Black Scholes Greeks. Their role in hedging and risk managing has long been recognized and they are used by practitioners on a daily basis. They are also discussed in many standard textbooks on quantitative finance and in a large number of scientific papers. Among these, we mention [21] on the effects and role of vega and gamma hedging, [15] in which higher order Black Scholes
partial derivatives are introduced and [5] in which it is shown that the values of Greeks of arbitrary order can be interpreted as the prices of certain contingent claims. The Black-Scholes Greeks which appear in our second order approximation are

\[
\Gamma = \frac{\partial^2 C^{BS}}{\partial S^2} = \frac{e^{-\frac{d_1^2}{2}}}{S\sqrt{2\pi}\Sigma}
\]

\[
\text{Volga} = \frac{\partial^2 C^{BS}}{\partial \Sigma^2} = \frac{Se^{-\frac{d_1^2}{2}}}{4\sqrt{2\pi}\Sigma^{3/2}} (d_0d_1 - 1)
\]

\[
\text{Vanna} = \frac{\partial^2 C^{BS}}{\partial S\partial \Sigma} = -\frac{e^{-\frac{d_1^2}{2}}d_0}{2\sqrt{2\pi}\Sigma}
\]

### 6.2 Numerical Results

In order to test our approximation, we have compared the implied volatilities deduced from the true Call prices in the Wishart Model (which can be obtained thanks to Fourier Inversion of the characteristic function of the log price, see [16] for details) with the implied volatilities obtained for our approximations. The tests have been made for two sets of parameters, namely

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_0 )</td>
<td>100</td>
</tr>
<tr>
<td>( T )</td>
<td>1</td>
</tr>
<tr>
<td>( r )</td>
<td>3%</td>
</tr>
<tr>
<td>( \beta )</td>
<td>4</td>
</tr>
<tr>
<td>( \rho )</td>
<td>-50% or 0</td>
</tr>
</tbody>
</table>

Table 1: Parameters used

<table>
<thead>
<tr>
<th>( V_0 )</th>
<th>( Q )</th>
<th>( M )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (40% \ 0) )</td>
<td>( (30% \ 0) )</td>
<td>( (-1 \ 0) )</td>
</tr>
<tr>
<td>( (0 \ 35%) )</td>
<td>( (0 \ 20%) )</td>
<td>( (0 \ -0.8) )</td>
</tr>
</tbody>
</table>

Table 2: Parameters used

We represent below the results we obtained
We remark that for the case $\rho = 0$, the agreement between the true implied volatilities and the approximated ones is almost perfect, and this over our whole strike range of $[10\%, 200\%]$. However, when the correlation is no longer zero, for strike below 50% of the spot price, our second order approximation underestimate the true implied volatility. It is not surprising, since as we move towards the downside strikes, skewness becomes more pronounced and this is an effect which is not captured by a second order approximation, and of course when there is no correlation between the asset and the volatility, the effects of the skewness vanish. Nonetheless, for strikes above 50% of the spot, the approximation remains very good. As an illustration, the mean of the relative error between the true and approximated implied volatilities over the strike range $[50\%, 200\%]$ is only $0.33\%$.

### 6.3 Computation Time

It is also interesting to check whether the calculations with the approximate price are faster than the original ones. When using our parameters, we have obtained the following results

<table>
<thead>
<tr>
<th></th>
<th>Fourier Price</th>
<th>Approximate Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>0.438</td>
<td>0.047</td>
</tr>
</tbody>
</table>

Table 3: Computation Times for the two pricing methods (s)

Thus, our approximation is at least 10 times faster than the original pricing formula, and can therefore be of a great interest when trying to calibrate Wishart models to market date for example.

### 7 Conclusion

Thanks to new results on the integrated variance process of a Wishart model, we have expressed the Wishart call price as Black-Scholes call price plus a weighted sum of Black-Scholes Greeks. Thus, practitioners may get an intuition on how Wishart parameters
impact the volatility smile shape. In particular, one can identify which parameters change
the slope and the curvature of the smile. Besides, having explicitly Black-Scholes Greeks
in the call price may turn out to be useful for hedging and risk managing purposes.

References


Finance, 4(2), 200.


[7] Catherine Donati-Martin, Yan Doumerc, Hiroyuki Matsumoto, and Marc Yor. Some
properties of the Wishart processes and a matrix extension of the Hartman-Watson

[8] Gabriel G. Drimus. Closed form convexity and cross-convexity adjustments for

[9] Daniel Dufresne. The integrated square-root process. Working Paper, University of

stochastic correlation model using the empirical characteristic function. Research


[12] José Da Fonseca, Martino Grasselli, and Claudio Tebaldi. A multifactor volatility

[13] José Da Fonseca, Martino Grasselli, and Claudio Tebaldi. Option pricing when corre-
lations are stochastic: an analytical framework. Research report, ESILV, September
2006.


A Technical Proofs

Proof of lemma 3. To prove the formulas, we will use the derivative of a matrix exponential given in lemma 2 and the fact that

\[
\begin{pmatrix}
* & * \\
G_T(\gamma) & F_T(\gamma)
\end{pmatrix} = \exp\left[T \begin{pmatrix}
M & -2Q^2 \\
\gamma I_n & -M
\end{pmatrix}\right],
\]

Let us note

\[
A(\gamma) = T \begin{pmatrix}
M & -2Q^2 \\
\gamma I_n & -M
\end{pmatrix}.
\]

Since \(A'' = 0\), we have that

\[
(\exp(A))'(0) = \int_0^1 e^{(1-u)A(0)} A'(0) e^{uA(0)} du,
\]

\[
(\exp(A))''(0) = \int_0^1 \int_0^1 e^{(1-v)(1-u)A(0)} (1-u)A'(0) e^{v(1-u)A(0)} A'(0) e^{uA(0)} dvdu + \int_0^1 \int_0^1 e^{(1-u)A(0)} A'(0) e^{(1-v)uA(0)} uA'(0) e^{uA(0)} dvdu,
\]

where we used Fubini’s theorem, which is justified since we integrate continuous functions on compact sets.

Then, since \(A(0)\) is block upper-triangular, it is easy to show (e.g. using techniques developed in [22]) that

\[
\forall x \in \mathbb{R}, e^{xA(0)} = \begin{pmatrix}
e^{xT M} & -2e^{xT M} \int_0^x e^{-gM} Q^2 e^{-gM} dg \\
0 & e^{-xT M}
\end{pmatrix}.
\]

Then using simple matrix multiplication we get that

\[
G'_T(0) = Te^{-TM} \int_0^1 e^{2uTM} du
\]

\[
F'_T(0) = -2e^{-TM} \int_0^1 e^{2uTM} \int_0^{uT} e^{-sM} Q^2 e^{-sM} dsdu
\]
\[ G_T''(0) = -2T^2 e^{-TM} \left[ \int_0^1 \int_0^1 \int_0^1 \int_0^1 (1-u)e^{(a+2u(1-u))TM-sM}Q^2 e^{uTM-sM} ds dv du \right] \\
+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{(2u-uv)TM-sM}Q^2 e^{uvTM-sM} ds dv du \] \\
\[ F_T''(0) = 4T^2 e^{-TM} \left[ \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{2uT-sM}M Q^2 e^{(uT-s-x)M} e^{-xM} dx dudvdu \right] \\
+ \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{(2u-uv)T-sM}M Q^2 e^{(uT-s-x)M} e^{-xM} dx dudvdu \right]. \]

The statement for \( G_T \) then comes from the fact a primitive of \( e^{xM} \) is \( e^{xM} M^{-1} \) and to obtain the formula for \( F_T \), it suffices to make the variable change \( v = uT \). The calculations for the two other terms are quite long but rather straightforward. We will do them for the first integral in \( F_T''(0) \), which we now note \( I_F \), the other cases being similar.

The only things needed for these calculations are Fubini’s theorem, some very simple variable changes, the primitive recalled above, the fact that \( e^{xM} \) commutes with any power of \( M \) and the cyclic property of the trace operator. That being said, the first step is to get rid off the \((1-u)\) term. In order to do that, let’s put \( w = (1-u)v \). We have, thanks to the cyclic property of the trace

\[ \text{Tr} \left[ e^{-MT} I_F \right] = 4T^2 \text{Tr} \left[ \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{(uT+2wT-s-x)M}Q^2 e^{(uT-s-x)M} dx dwdvdu \right] \]

\[ = 4 \text{Tr} \left[ \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{(u+2w-s-x)M}Q^2 e^{(u-s-x)M} dx dwdvdu \right]. \]

Then we put \( g = u-s \) and we use Fubini’s theorem to inverse the order of integration

\[ \text{Tr} \left[ e^{-MT} I_F \right] = 4 \text{Tr} \left[ \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{(g+2w-x)M}Q^2 e^{(g-x)M} dx dwdvdu \right] \]

\[ = 2 \text{Tr} \left[ \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{(g-x)M} \left[ e^{2(T-u)M} e^{2(u-g)M} \right] M^{-1}Q^2 e^{(g-x)M} dx dwdvdu \right]. \]
Then we separate into two integrals, we use Fubini’s theorem to inverse the order of integration and integrate when possible

\[
\text{Tr} \left[ e^{-MT} I_F \right] = 2 \text{Tr} \left[ e^{2TM} \int_0^T \int_{2x-T}^T \int_{\max(g,x)}^{g+T/2} e^{(g-2u-x)M} M^{-1} Q^2 e^{(g-x)M} dudgdx Q^2 \right] 
- 2 \text{Tr} \left[ \int_0^T \int_{2x-T}^T \int_{\max(g,x)}^{g+T/2} e^{(2u-g-x)M} M^{-1} Q^2 e^{(g-x)M} dudgdx Q^2 \right] 
= \text{Tr} \left[ e^{2TM} \int_0^T \int_{2x-T}^T e^{(g-3x)M} M^{-2} Q^2 e^{(g-x)M} dgdQ^2 \right] 
+ e^{2TM} \int_0^T \int_{2x-T}^T e^{(-g-x)M} M^{-2} Q^2 e^{(g-x)M} dgdQ^2 
- e^{TM} M^{-2} Q^2 e^{TM} \int_0^T e^{-xM} M^{-1} Q^2 e^{-xM} dx 
+ e^{TM} M^{-2} Q^2 e^{TM} \int_0^T e^{xM} M^{-1} Q^2 e^{-xM} dx 
+ \int_0^T \int_{2x-T}^T e^{(x-g)M} M^{-2} Q^2 e^{(g-x)M} dgdQ^2 
- e^{TM} M^{-2} Q^2 e^{TM} \int_0^T e^{-xM} M^{-1} Q^2 e^{-xM} dx 
+ \int_0^T \int_x^{2x-T} e^{(g-x)M} M^{-2} Q^2 e^{(g-x)M} dgdQ^2 
+ M^{-2} Q^2 \int_0^T e^{xM} M^{-1} Q^2 e^{-xM} dx \right].
\]

Finally, we put \( y = g - x \), we inverse the order of integration for the remaining double integral, integrate and reunite the different quantities to get
\[
\text{Tr} [e^{-MT} I_F] = \text{Tr} \left[ -4e^{TM} M^{-2} Q^2 \int_0^T \text{sh}((T-g)M)M^{-1}Q^2 e^{-gM} dg \\
+ 2 \int_0^T (T-g)e^{gM} M^{-2} Q^2 \text{ch}(gM)dgQ^2 \\
+ e^{2TM} \int_0^T e^{-gM} M^{-3} Q^2 \text{ch}(gM)dgQ^2 \\
- \int_0^T e^{9M} M^{-3} Q^2 \text{ch}(gM)dgQ^2 \right].
\]

All the other calculations are similar.