A Robust Coarse Space for Systems of PDEs with high heterogeneities: GenEO

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Abstract

Coarse spaces are instrumental in obtaining scalability for domain decomposition methods for partial differential equations (PDEs). However, it is known that most popular choices of coarse spaces perform rather weakly in the presence of heterogeneities in the PDE coefficients, especially for systems of PDEs. Here, we introduce in a variational setting a new coarse space that is robust even when there are such heterogeneities. We achieve this by solving local generalized eigenvalue problems in the overlaps of subdomains that isolate the terms responsible for slow convergence. We have proved a general theoretical result that rigorously establishes the robustness of the new coarse space and we give some numerical examples on two and three dimensional heterogeneous PDEs and systems of PDEs that confirm this.

Problems we solve

Let V^h be a finite element space of functions in Ω based on a mesh $\mathcal{T}^h = \{\tau\}$ of domain Ω.
Given $f \in (V^h)^*$ find $u \in V^h$

$$a(u,\, v) \;=\; \langle f,\, v\rangle \qquad \forall v \in V^h$$
$$\iff \qquad \mathbf{A}\,\mathbf{u} \;=\; \mathbf{f}$$

Assumptions:

1. $\mathbf{A}$ *symmetric positive definite*
2. $\mathbf{A}$ is given as a set of **element stiffness matrices** + connectivity (list of DOF per element)

   and verifies the assembling property:
   $$a(v,\, w) \;=\; \textstyle\sum_{\tau} a_\tau(v_{|\tau},\, w_{|\tau})$$
   where $a_\tau(\cdot,\cdot)$ *symmetric positive semi-definite*
3. The finite element basis $\{\phi_k\}_{k=1}^n$ of V^h verifies a *unisolvence* property on each element τ.
4. Two more technical assumptions on $a(\cdot,\cdot)$ later!

Examples:

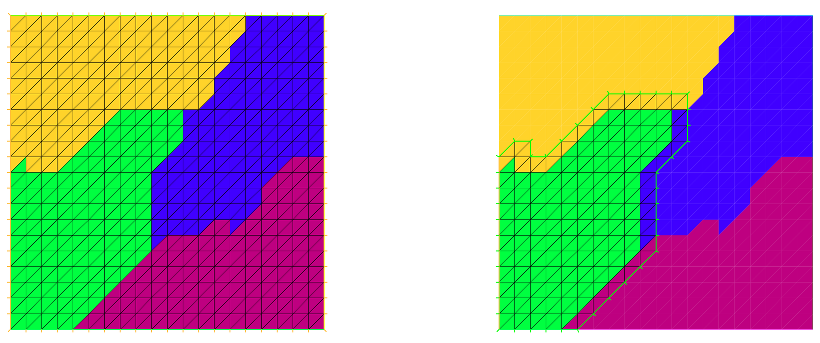
- Darcy $\quad a(u,\, v) = \int_\Omega \boldsymbol{\kappa} \nabla u \cdot \nabla v \, dx$
- Elasticity $\quad a(\boldsymbol{u},\, \boldsymbol{v}) = \int_\Omega \boldsymbol{C}\, \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx$
- Eddy current
  $a(\boldsymbol{u},\, \boldsymbol{v}) = \int_\Omega \boldsymbol{\nu}\, \mathrm{curl}\, \boldsymbol{u} \cdot \mathrm{curl}\, \boldsymbol{v} + \boldsymbol{\sigma}\, \boldsymbol{u} \cdot \boldsymbol{v}\, dx$

with heterogeneities, high contrast in parameters

General Setting: Additive Schwarz

The following is done using only the connectivity information and a graph partitioner such as Metis.

1. Build a non overlapping partition of Ω.
2. Add one layer of elements to each subdomain $j = 1, \ldots, N$ to get a partition into overlapping subdomains Ω_j.



Adding one layer of overlap to the green subdomain.

3. Define the local finite element spaces: $V_j := \mathrm{span}\{\phi_k : \mathrm{supp}(\phi_k) \subset \Omega_j\}$. Then denote by $\mathbf{R}_j^\top : V_j \to V^h$ the natural local/global embedding and by $a_{\Omega_j}(u, v) := \sum_{\tau \subset \Omega_j} a_\tau(u_{|\tau}, v_{|\tau})$ the local bilinear form.
4. Define a coarse space V_H and denote by $\mathbf{R}_H^\top : V_H \to V^h$ the natural coarse/global embedding.

Two level additive Schwarz

$$\mathbf{M}_{AS,2}^{-1} \;:=\; \mathbf{R}_H^\top \mathbf{A}_H^{-1} \mathbf{R}_H + \sum_{j=1}^N \mathbf{R}_j^\top \mathbf{A}_j^{-1} \mathbf{R}_j$$

where $\quad \mathbf{A}_j = \mathbf{R}_j^\top \mathbf{A} \mathbf{R}_j \quad$ and $\quad \mathbf{A}_H = \mathbf{R}_H^\top \mathbf{A} \mathbf{R}_H$.

If we prove the existence of a C_0-stable decomposition (as defined next) for each $v \in V_h$ then the general Schwarz theory tells us that the condition number of the preconditioned operator is bounded by

$$\kappa(\mathbf{M}_{AS,2}^{-1}\mathbf{A}) \le C_0{}^2(k_0 + 1),$$

where each point belongs to at most k_0 subdomains.

Definition (C_0-Stable decomposition)

Given a coarse space $V_H \subset V_h$, local subspaces $\{V_j\}_{1\le j \le N}$ and a constant C_0, a C_0-stable decomposition of $v \in V_h$ is a family of functions,

$$(v_H, v_1, \ldots, v_N) \in V_H \times V_1 \times \ldots \times V_N,$$

which satisfies

$$v = v_H + \sum_{j=1}^N v_j,$$

and

$$a(v_H, v_H) + \sum_{j=1}^N a_{\Omega_j}(v_j, v_j) \le C_0^2 a(v, v).$$

A sufficient condition for this last inequality is: there exists a constant C_1 such that

$$a_{\Omega_j}(v_j, v_j) \le C_1\, a_{\Omega_j}(v_{|\Omega_j}, v_{|\Omega_j}) \text{ for all } j = 1, \ldots, N. \quad (1)$$

Then the decomposition is C_0-stable with

$$C_0^2 = 2 + C_1 k_0 (2k_0 + 1).$$

Objective: define the coarse space in such a way that there exists a decomposition of any $v \in V^h$ which fulfills (1) for a C_1 which is independant of the heterogeneities and the decomposition. Then the bound on the condition number and hence on the convergence rate will also be independant of these quantities leading to a robust method.

In order to do this we need to introduce partition of unity operators which will allow us to define the coarse space and the local components.

Definition ('Discrete' partition of unity)

For any $j = 1, \ldots N$, let

$$dof(\Omega_j) \;:=\; \{k : \mathrm{supp}(\phi_k) \cap \Omega_j \neq \emptyset\}$$

denote the space of all degrees of freedom in Ω_j, and

$$idof(\Omega_j) \;:=\; \{k : \mathrm{supp}(\phi_k) \subset \overline{\Omega}_j\}$$

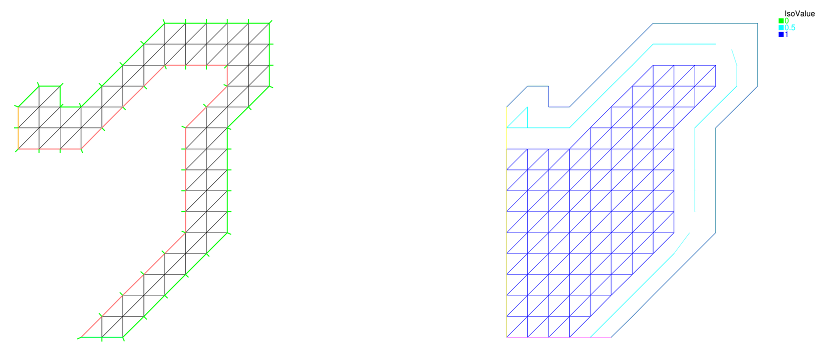
denote the space of internal degrees of freedom in Ω_j.
Notice that: $(V^h)_{|\Omega_j} = \mathrm{span}\{\phi_k\}_{k \in dof(\Omega_j)} \not\subset V^h$.
and $V_j = \mathrm{span}\{\phi_k\}_{k \in idof(\Omega_j)} \subset V^h$.
Then for any $v = \sum_{k=1}^n v_k \phi_k \in V^h$ define the partition of unity operator as:

$$\Xi_j(v) \;:=\; \sum_{k \in idof(\Omega_j)} \frac{1}{\#\{j : k \in idof(\Omega_j)\}} v_k \phi_k \in V_j.$$

It is indeed a partition of unity: $\sum_{j=1}^N \Xi_j v = v$.

Definition (Ω_j°)

Let Ω_j° denote the part of Ω_j that is overlapped (left), then $(\Xi_j v)|_{\Omega_j \setminus \Omega_j^\circ} = v_{\Omega_j \setminus \Omega_j^\circ}$ (right).



Finally define, $a_{\Omega_j^\circ}(v, v) = \sum_{\tau \subset \Omega_j^\circ} a_\tau(v_{|\tau}, v_{|\tau})$.

Theorem: GenEO Coarse Space and convergence result

On each subdomain Ω_j, $j = 1 \ldots N$, find $p_{j,k} \in V_{h|\Omega_j}$ and $\lambda_{j,k} \ge 0$:

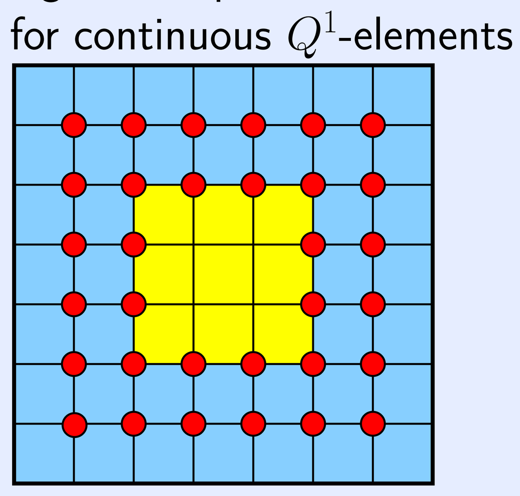
$$a_{\Omega_j}(p_{j,k},\, v) \;=\; \lambda_{j,k}\, a_{\Omega_j^\circ}(\Xi_j p_{j,k},\, \Xi_j v) \qquad \forall v \in V_{h|\Omega_j}$$

$$\Leftrightarrow \mathbf{A}_j \mathbf{p}_{j,k} \;=\; \lambda_{j,k}\, \mathbf{X}_j \mathbf{A}_j^\circ \mathbf{X}_j\, \mathbf{p}_{j,k} \qquad (\mathbf{X}_j \ldots \text{diagonal})$$

Select the first $m_j \;:=\; \min\left\{m : \lambda_{m+1}^j > \frac{\delta_j}{H_j}\right\}$ ($H_j \ldots$ subdomain diameter, δ_j overlap width), eigenvectors per subdomain and define the coarse space as $V_H \;=\; \mathrm{span}\{\Xi_j p_{j,k}\}_{k=1,\ldots,m_j}^{j=1,\ldots,N}$. Then the condition number of the preconditioned operator is bounded by:

$$\kappa(\mathbf{M}_{AS,2}^{-1}\mathbf{A}) \;\le\; (1 + k_0)\left[2 + k_0\,(2k_0 + 1)\,\max_{j=1}^N \left(1 + \frac{H_j}{\delta_j}\right)\right]$$

DOFs that are free in the eigenvalue problem for continuous Q^1-elements



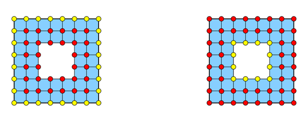
Both matrices typically singular $\implies \lambda_{j,k} \in [0,\, \infty]$

The proof requires two technical assumptions.

Assumption 1: $a_{\Omega_j}(\cdot,\cdot)$ SPD on $\mathrm{span}\{\phi_{k|\Omega_j}\}_{k \in dof(\Omega_j) \setminus idof(\Omega_j)}$

Assumption 2: $a_{\Omega_j^\circ}(\cdot,\cdot)$ SPD on $\mathrm{span}\{\phi_{k|\Omega_j}\}_{k \in idof(\Omega_j) \setminus idof(\Omega_j \setminus \Omega_j^\circ)}$

Assumptions 1 and 2 hold if certain mixed "boundary" value problems are solvable: (red: free dofs, yellow: fixed dofs)



Stable decomposition

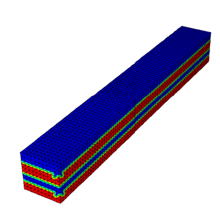
Coarse component: $v_H = \sum_{j=1}^N \Xi_j \Pi_j v_{|\Omega_j} \in V_H$, and **Local components:** $v_j = \Xi_j(v - \Pi_j v) \in V_j$,
where Π_j is the local projector onto $\mathrm{span}\{\Xi_j p_{j,k}\}_{k=1,\ldots,m_j}$.

$$(1) \Leftrightarrow \underbrace{\Xi_j(v - \Pi_j v)}_{v_j}|^2_{a,\Omega_j} \le C_1 |v|^2_{a,\Omega_j}, \Leftrightarrow \Xi_j(v - \Pi_j v)|^2_{a,\Omega_j^\circ} + \underbrace{|\Xi_j(v - \Pi_j v)|^2_{a,\Omega_j \setminus \Omega_j^\circ}}_{=|v - \Pi_j v|^2_{a,\Omega_j\setminus\Omega_j^\circ} \le |v - \Pi_j v|^2_{a,\Omega_j}} \le C_1 |v|^2_{a,\Omega_j}.$$

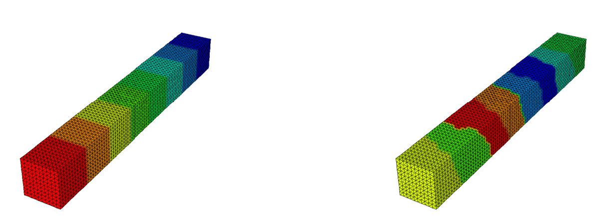
So the only term that we are left to work on is: $\Xi_j(v - \Pi_j v)|^2_{a,\Omega_j^\circ} \overset{\text{HOW?}}{\le} C_1 |v|^2_{a,\Omega_j}$, and the generalized eigenvalue problem bounds just that.

Numerical results

Coefficients



Decompositions



	AS		ZEM			GenEO		
κ_2	it	*cond*	it	*cond*	dim	it	*cond*	dim
1	16	*229*	11	*6.3*	8	11	*8.4*	7
10^2	27	*230*	19	*22*	8	13	*8.4*	14
10^4	29	*230*	23	*210*	8	15	*8.4*	14
10^6	26	*230*	22	*230*	8	11	*8.4*	14

Table: 3D Darcy: number of PCG iterations (it), condition number (*cond) and coarse space dimension (dim) vs. jump in* κ *for* $\kappa_1 = 1$, $\ell = 1$ *added layers,* $L = 8$ *regular subdomains*

		AS		ZEM			GenEO		
L	glob DOF	it	*cond*	it	*cond*	dim	it	*cond*	dim
4	14520	79	$2.4 \cdot 10^3$	54	$2.9 \cdot 10^2$	24	16	*10*	46
8	29040	177	$1.3 \cdot 10^4$	87	$1.0 \cdot 10^3$	48	16	*10*	102
16	58080	378	$1.5 \cdot 10^5$	145	$1.4 \cdot 10^3$	96	16	*10*	214

Table: 3D Elasticity: number of PCG iterations (it), condition number (*cond*), and coarse space dimension (dim) vs. number of regular subdomains, for $\ell = 1$ added layers, $g = 10$, $(E_1, \nu_1) = (2 \cdot 10^{11}, 0.3)$ and $(E_2, \nu_2) = (2 \cdot 10^7, 0.45)$.