Convergence analysis of moment-based approximations in kinetic theory

Teddy Pichard

Abstract

As for Galerkin approximations, the method of moments consists in reducing the number of test functions in a variational approach to a finite number, then performing an approximation of the solution such that the resulting reduced problem is well-posed. It is commonly used in kinetic theory, relying on physical considerations, but few error estimates are available in the literature. This paper provides a functional framework adapted to study the convergence of these methods. Its use is illustrated for the convergence study of a vector of moments toward a kinetic distribution, pointwise, on a homogeneous time-dependent problem, and on a 1D-1V problem.

1 Introduction

Consider the transport of a population represented by its distribution function $f(t,x,v)$ depending on time $t \in (0,T)$, position $x \in \mathbb{R}$ and velocity variable $v \in \mathbb{R}$, and following an equation of the form

$$\partial_t f + v\partial_x f = c(f). \tag{1}$$

The distribution function f solving this PDE carries more information than is required for physical purposes. Based on this observation, the method of moments aims at reducing (1) into a system of equations involving only over the first few moments of f. However, extracting the moments of (1) leads to an undetermined system which requires additional closure relations. Such closures are commonly constructed by solving a problem of moments, i.e. by reconstructing a distribution function f^R from its first moments and then substituting f by f^R in the definition of the terms to close (see e.g. the survey paper [36] and the references therein for details on this construction). The solutions to such problems of moments are not unique and needs to be chosen. This choice is generally motivated more by qualitative properties to be imposed on this approximation rather than by quantitative estimations, which are sparse in the literature.

The estimation of the errors arising from such approximations is studied in two contexts: The hydrodynamic limits (see e.g. [6, 7]) correspond to such approximations based on the moments up to second order. Relying on techniques from PDE analysis, these studies focus on specific physical regimes, typically highly collisional, and provide error estimations with respect to the collision rate. However, the main objective of the method of moments is the numerical treatement of out-of-equilibrium regimes where such an assumption fails. In numerical analysis, Hermite spectral methods recently received a growing interests (see e.g. [38, 9, 10, 11]). Such methods are a linear (or linearized) version of moment-based approximations, and provide error estimates depending on various parameters, including the number of moments used to construct the approximation.

In this context, the present study focuses on error estimates for general nonlinear moment-based approximations with respect to the number of moments. The main objective is to construct a framework adapted to convergence studies of such approximations towards the kinetic model (1). This framework encompasses both the kinetic and the moment systems, providing a natural notion of error. It is presented in the next section. The following three sections focus on the convergence of moment-based approximations towards kinetic distribution functions respectively locally, i.e. without time and space, in a homogeneous case, i.e. with time but without space, and in a 1D-1V case.

2 Moments sets

This section introduces the topological framework used in the next sections. Most of the results are recalled without proofs, which can be found in the indicated literature. The space-time variables (t, x) play a different role in this work compared to the kinetic ones v. This first section focuses on the dependence of the solution with respect to the variable v, and a special attention is given to the notion of solution.

2.1 Dual spaces for the v-dependence

Denote $\mathcal{E}$ the space of entire functions with an infinite radius of convergence

$$\mathcal{E} = \left\{ p \;\mid\; p(v) = \sum_{k=0}^{\infty} a_k v^k \text{ and } |p(v)| < \infty \; \forall v \in \mathbb{R} \right\},$$

and denote $\mathcal{E}'$ the space of linear forms over $\mathcal{E}$. In the following, the solutions $f(t, x) \in \mathcal{E}'$ are considered in a weak sense, i.e. satisfying an equation for all test functions $\phi \in \mathcal{E}$. The notation $g[\phi]$ is used for a linear form $g \in \mathcal{E}'$ applied on $\phi \in \mathcal{E}$.

Remark 1.

- Such solutions $f \in \mathcal{E}'$ are implicitly assumed to have bounded infinite moments. This hypothesis can be restrictive for some applications, and it can be relaxed when the number of considered moments is fixed. But it is necessary when studyinh the convergence in number of moments. In practice, the space $\mathcal{E}'$ encompasses the Maxwellians (Gaussians in v), Dirac distributions and most of the special distribution functions considered in kinetic theory.
- From this definition, one can construct the distribution function in the classical functional sense. By abusing notations, the value of a distribution function in a point v can be retrieved as the limit $\lim_{\epsilon \to 0} f[\delta_v^\epsilon]$ where $\delta_v^\epsilon \in \mathcal{E}$ is a Dirac mollifier, e.g. the Gaussians are in $\mathcal{E}$, when this limit exists.

In order to define a norm (and a Banach space) adapted to our problem, the space $\mathcal{E}'$ is first reduced to a subset.

2.2 Cones of moment sequences

Further definitions are needed:

Definition 2.1. Denote $\mathbf{b}(v) = (v^k)_{k \in \mathbb{N}}$ the sequence of monomials of v (which forms a basis of $\mathcal{E}$).

- A form $f \in \mathcal{E}'$ is represented by a sequence $\mathbf{U} \in \mathbb{R}^{\mathbb{N}}$ if $\mathbf{U}_k = f[\mathbf{b}_k]$ for all $k \in \mathbb{N}$. This notation is also applied componentwise to vectors, sequences, matrices or infinite matrices of polynomials, such that $\mathbf{U} = f[\mathbf{b}] \in \mathbb{R}^{\mathbb{N}}$.
- Conversely, the Riesz functional $L_{\mathbf{U}} \in \mathcal{E}'$ associated with a sequence $\mathbf{U} \in \mathbb{R}^{\mathbb{N}}$ is the linear form:

$$L_{\mathbf{U}} : \begin{cases} \mathcal{E} & \to \mathbb{R}, \\ \sum_{k=0}^{\infty} \alpha_k v^k & \mapsto \sum_{k=0}^{\infty} \alpha_k \mathbf{U}_k. \end{cases}$$

The notation $f \in \mathcal{E}'$, then also $L_{\mathbf{U}} \in \mathcal{E}'$, is also used componentwise to vectors, sequences, matrices or infinite matrices of polynomials, e.g. $f[\mathbf{b}] \in \mathbb{R}^{\mathbb{N}}$.

One observes that

$$L_{\mathbf{U}}(\mathbf{b}) = \mathbf{U}, \qquad L_{f[\cdot]} = f$$

are the identity operators on $\mathbb{R}^{\mathbb{N}}$, resp. $\mathcal{E}'$, and those spaces are in bijection. In the following, the considered solutions evolve only into a subset of $\mathcal{E}'$.

Definition 2.2. The non-negativity in $\mathcal{E}'$ is understood in a weak sense:

- Denote $\mathcal{E}_+$ the set of non-negative entire functions

$$\mathcal{E}_+ = \{p \in \mathcal{E} \mid p(x) \geq 0 \ \forall x \in \mathbb{R}\}.$$

- A form $f \in \mathcal{E}'$ is non-negative, denoted $f \geq 0$, if it preserves the non-negativity, i.e. if it belongs to

$$\mathcal{E}'_+ = \{f \in \mathcal{E}' \mid f[p] \geq 0 \ \forall p \in \mathcal{E}_+\}.$$

- A sequence $\mathbf{U} \in \mathbb{R}^{\mathbb{N}}$ is realizable if $L_{\mathbf{U}} \in \mathcal{E}'_+$ is non-negative. The realizability domain R is

$$\mathsf{R} = \left\{\mathbf{U} \in \mathbb{R}^{\mathbb{N}} \mid L_{\mathbf{U}} \in \mathcal{E}'_+\right\}.$$

By construction, the sets $\mathcal{E}'_+$ and R are convex cones, i.e. they are stable by positive combinations. Such non-negativity requirements were characterized, for entire functions over $\mathbb{R}$, by numerical conditions through the Hamburger-Curto-Fialkow theorem:

Theorem 2.3 ([25, 20])**.** *A sequence $\mathbf{U} \in \mathsf{R}$ is realizable if and only if one of these condition holds:*

- *The Hankel matrices $(\mathbf{U}_{i+j})_{0\leq i,j\leq N}$ are symmetric positive and definite for all $N \in \mathbb{N}$.*
- *The Hankel matrices $(\mathbf{U}_{i+j})_{0\leq i,j\leq N}$ are symmetric non-negative for all $N \in \mathbb{N}$, and there exists $r \in \mathbb{N}$ such that*

$$r + 1 = Rank((\mathbf{U}_{i+j})_{0\leq i,j\leq r}) = Rank((\mathbf{U}_{i+j})_{0\leq i,j\leq M}) \qquad \forall M > r.$$

  In such a case, $\mathbf{U}$ is said to be positively recursively generated.

2.3 Measure representation and determinacy

Another equivalent representation for $\mathcal{E}'_+$ or R is available through the Riesz-Haviland theorem:

Theorem 2.4 ([37, 26, 27])**.** *A sequence $\mathbf{U} \in \mathsf{R}$ is realizable if and only if there exists a positive Borel measure $\mu \in \mathcal{M}(\mathbb{R})^+$ realizing its moments*

$$L_{\mathbf{U}}(p) = \int_{\mathbb{R}} p(v)d\mu(v) \qquad \forall p \in \mathcal{E}. \tag{2}$$

Then, one rewrites equivalently

$$\mathsf{R} = \left\{ \left(\int_{\mathbb{R}} v^i d\mu(v) \right)_{i\in\mathbb{N}} \mid \mu \in \mathcal{M}(\mathbb{R})^+ \right\}.$$

Theorems 2.3 and 2.4 provide a numerical characterization of the existence of a solution $\mu \in \mathcal{M}(\mathbb{R})^+$ to the problem of moments (2). However, this solution is a priori not unique.

Definition 2.5. For a sequence $\mathbf{U} \in \mathsf{R}$, the problem (2) is determinate if its solution $\mu \in \mathcal{M}(\mathbb{R})^+$ is unique.

The determinacy of problems of moments has been studied by various authors and the reader is e.g. referred to the books [2, 41] and references therein:

- For the negative answer, one common example of undeterminate problem of moments is given by the moment sequence of a log-normal distribution $f(v) = v^{-1}\exp(-ln(v)^2)$ (among other distributions). This example led to Krein's sufficient condition of indeterminacy ([31]).
- For the positive answer, Cramér's sufficient condition of determinacy ([18] ; see also Corollaries 4.10 and 4.11 from [39] and Carleman's condition [17]) is recalled:

Proposition 1 ([18, 39])**.** *Consider a realizable sequence* $\mathbf{U} \in \mathsf{R}$ *and suppose that there exists* $c > 0$ *such that*

$$\mathbf{U}_{2n} \leq c^n (2n)! \qquad \forall n \in \mathbb{N}. \tag{3a}$$

Then the problem (2) *is determinate. Furthermore, for all* $\alpha > \sqrt{c}$*, the unique representation* $\mu \in \mathcal{M}(\mathbb{R})^+$ *satisfies*

$$\int_{\mathbb{R}} \exp\left(\frac{|v|}{\alpha}\right) d\mu(v) < \infty. \tag{3b}$$

Proof. The proof of determinacy is given in [39]. Then, there exists $\mu \in \mathcal{M}(\mathbb{R})^+$ such that

$$\frac{\mathbf{U}_{2n}}{(2n)!\alpha^{2n}} = \int_{\mathbb{R}} \frac{1}{(2n)!} \left(\frac{v}{\alpha}\right)^{2n} d\mu(v) \leq \left(\frac{c}{\alpha^2}\right)^n.$$

Assuming $\alpha > \sqrt{c}$ and summing over n provides

$$\int_{\mathbb{R}} \exp\left(\frac{|v|}{\alpha}\right) d\mu(v) \leq \int_{\mathbb{R}} cosh\left(\frac{v}{\alpha}\right) d\mu(v) \leq \exp\left(\frac{c}{\alpha^2}\right) < \infty.$$

□

Denote the set of positive measures satisfying this property by

$$\mathcal{M}(\mathbb{R})_{\exp} = \left\{ \mu \in \mathcal{M}(\mathbb{R})^+ \ \Big| \ \int_{\mathbb{R}} \exp(|v|) d\mu(v) < \infty \right\}.$$

The two specific regimes of interest (see Remark 1), i.e. the sequence of moments of a Dirac measure and of a Maxwellian, both satisfy the relation (3). The distribution functions with such exponential tails have been the source of a vast literature in kinetic theory (see e.g. [13, 12, 4, 35, 22, 24, 5]).

2.4 Products, norms and spaces

For the convergence of approximate moment sequences, an appropriate norm and the completeness of the associated normed space are required.

Notation.

- Denote the spaces

$$\ell^p_{\exp} = \left\{ \mathbf{U} \in \mathbb{R}^{\mathbb{N}} \ \big| \ \|\mathbf{U}\|_{\exp,p} < \infty \right\}, \tag{4a}$$

$$\text{for } 1 \leq p < \infty, \quad \|\mathbf{U}\|_{\exp,p} = \left(\sum_{i=0}^{\infty} \frac{|\mathbf{U}_i|^p}{i!} \right)^{1/p} \qquad \text{and} \quad \|\mathbf{U}\|_{\exp,\infty} = \sup_{i\in\mathbb{N}} \frac{|\mathbf{U}_i|}{i!}. \tag{4b}$$

- Abusing notation, denote the space and norm

$$\mathcal{E}^p_{\exp}{}' = \left\{ L_{\mathbf{U}} \ \big| \ \mathbf{U} \in \ell^p_{\exp} \right\}, \qquad \|L_{\mathbf{U}}\|_{\exp,p} = \|\mathbf{U}\|_{\exp,p}. \tag{5a}$$

- Denote the set of realizable sequences of $\ell^p_{\exp}$ and forms of $\mathcal{E}^p_{\exp}{}'$

$$\mathsf{R}^p_{\exp} := \mathsf{R} \cap \ell^p_{\exp}, \qquad \mathcal{R}^p_{\exp}{}' = \mathcal{E}'_+ \cap \mathcal{E}^p_{\exp}{}'.$$

Since these are weighted ℓ^p spaces, one has:

Property 1.

- *The space* $\ell^2_{\exp}$ *is of Hilbert type with the product* $(\mathbf{U}, \mathbf{W})_{\exp} := \sum_{i=0}^{\infty} \frac{\mathbf{U}_i \mathbf{W}_i}{i!}$,

- *For all $1 < p < \infty$, the space $\ell^p_{\exp}$ is reflexive and its dual is $\ell^q_{\exp}$ with $q^{-1} + p^{-1} = 1$,*
- *For all $1 \leq p \leq \infty$, the space $\ell^p_{\exp}$ is of Banach type with the norm defined in* (4),
- *For $p \leq q$, the space $\ell^p_{\exp} \subset \ell^q_{\exp}$.*
- *The set $\mathsf{R}^p_{\exp}$ is a closed convex cone in $\ell^p_{\exp}$.*

Remark 2.

- Even though $\mathsf{R}^1_{\exp}$ is strictly smaller than R, it still contains all the sequences of moments of Diracs and of Gaussians centered around a bounded $u \in \mathbb{R}$. On the contrary, the cone $\ell^1 \cap \mathsf{R}$ does not contain these sequences, which justifies the present choice of weighted products and norms.
- Since $\ell^1_{\exp} \subset \ell^2_{\exp}$, then the Hilbertian structure of $\ell^2_{\exp}$ can also be used for sequences in $\mathsf{R}^1_{\exp} \subset \ell^1_{\exp}$.
- The sequences in the cone $\mathsf{R}^\infty_{\exp}$ (and therefore all those in $\mathsf{R}^p_{\exp}$ for all $p \geq 1$) satisfy (3a) such that they have a unique representing measure. Furthermore, those in $\mathsf{R}^\infty_{\exp}$ satisfy (3b).

The next sections focus on the convergence of moment-based approximations, i.e. approximations in $\mathsf{R}^1_{\exp}$. This convex cone is a closed subset of the Banach spaces $\ell^1_{\exp}$ and of the Hilbert space $\ell^2_{\exp}$, such that the aforementioned properties can be exploited. This set also offers interpretations of the moment sequences both in the sense of measures or in a dual sense.

3 Pointwise convergence

This section focuses on the convergence of a sequence of approximations $\mathbf{U}^N \in \mathsf{R}^p_{\exp}$ toward a moment sequence $\mathbf{U} \in \mathsf{R}^p_{\exp}$, or equivalently of a sequence of approximations $f^N \in {\mathcal{R}^p_{\exp}}'$ towards $f \in {\mathcal{R}^p_{\exp}}'$, when $N \to \infty$.

3.1 General results

First, the defintions of approximation, restriction and extension operators are fixed.

Definition 3.1.

- For a moment sequence $\mathbf{U} \in \mathsf{R}^p_{\exp}$, resp. a form $f \in {\mathcal{R}^p_{\exp}}'$, an approximation operator a^N, resp. a^N, of order N is an operator

$$\mathsf{a}^N : \begin{cases} \mathsf{R}^p_{\exp} & \to \mathsf{R}^p_{\exp}, \\ \mathbf{U} & \mapsto \mathbf{U}^N = \mathsf{a}^N(\mathbf{U}), \end{cases} \qquad a^N : \begin{cases} {\mathcal{R}^p_{\exp}}' & \to {\mathcal{R}^p_{\exp}}', \\ f & \mapsto f^N = a^N(f), \end{cases}$$

satisfying

$$\mathbf{U}^N_i = \mathbf{U}_i, \qquad f^N[v^i] = f[v^i], \qquad \text{for all} \quad i \leq N. \tag{6}$$

- Define restriction operators by:

$$\mathsf{r}^N : \begin{cases} \mathsf{R}^p_{\exp} & \to \mathbb{R}^{N+1}, \\ \mathbf{U} & \mapsto \mathsf{r}^N(\mathbf{U}) = (\mathbf{U}_i)_{i=0,\dots,N}, \end{cases} \qquad r^N : \begin{cases} {\mathcal{R}^p_{\exp}}' & \to \mathbb{R}^{N+1}, \\ f & \mapsto r^N(f) = (f[v^i])_{i=0,\dots,N}, \end{cases}$$

which provides the vector of the first N moments from a sequence $\mathbf{U} \in \mathsf{R}^p_{\exp}$, resp. from a form $f \in {\mathcal{R}^p_{\exp}}'$.

- Define extension operators $\mathsf{e}^N : \mathbb{R}^N \to \mathsf{R}^p_{\exp}$, resp. $e^N : \mathbb{R}^N \to {\mathcal{R}^p_{\exp}}'$, associated with an approximation a^N, resp. a^N, by

$$\mathsf{a}^N = \mathsf{e}^N \circ \mathsf{r}^N, \qquad a^N = e^N \circ r^N.$$

Remark that, for simplicity, these approximations $\mathsf{a}^N(\mathbf{U})$ are assumed to have values in the same set $\mathsf{R}^p_{\exp}$. Especially, all approximations at any order N are realizable, satisfy (3a) and have therefore a unique representing measure satisfying (3b).

The following proposition provides the convergence of the approximation under an appropriate control on the norm of the approximation.

Proposition 2. *Suppose that* $\|\mathsf{a}^N(\mathbf{U})\|_{p,\exp} \underset{N\to\infty}{\to} \|\mathbf{U}\|_{p,\exp}$. *Then,* $\mathsf{a}^N(\mathbf{U}) \to \mathbf{U} \in \mathsf{R}^p_{\exp}$.

Proof. For $p < \infty$, using the property (6) of the approximation, one has

$$\|\mathbf{U} - \mathsf{a}^N(\mathbf{U})\|^p_{p,\exp} = \sum_{i=N+1}^{\infty} \frac{|\mathbf{U}_i - \mathsf{a}^N(\mathbf{U})_i|^p}{i!} \leq \sum_{i=N+1}^{\infty} \frac{|\mathbf{U}_i|^p + |\mathsf{a}^N(\mathbf{U})_i|^p}{i!}.$$

The first term cancels out in the limit due to the convergence of the series $\|\mathbf{U}\|_{p,\exp}$. The second term also cancels in the limit by combining the convergence $\|\mathsf{a}^N(\mathbf{U})\|_{p,\exp} \underset{N\to\infty}{\to} \|\mathbf{U}\|_{p,\exp}$ and the convergence of the series defined by the sequeces $\mathbf{U}$ and $a^N(\mathbf{U})$.

The case $p = \infty$ is obtained by similar arguments. □

3.2 Convergence of approximations of entropy-based type

One of the most common types of moment-based approximations is given by the solution of a strictly convex minimization problem.

Definition 3.2.
- For $\mathbf{U} \in \mathsf{R}^1_{\exp}$, denote $\mu^R(\mathbf{U}) \in \mathcal{M}(\mathbb{R})_{\exp}$ the unique reconstructed measure having $\mathbf{U}$ for moments. By abuse, the same notation $\mu^R(f)$ is used for $f \in {\mathcal{R}^1_{\exp}}'$.
- A moment entropy is a function $\mathsf{h} : \mathsf{R}^p_{\exp} \to \mathbb{R}$, or equivalently $h : {\mathcal{R}^p_{\exp}}' \to \mathbb{R}$ of the form

$$\mathsf{h}(\mathbf{U}) = \begin{cases} \displaystyle\int_{\mathbb{R}} \eta\left(\frac{d\mu^R}{dv}(\mathbf{U})(v)\right) dv & \text{if} \quad \mu^R(\mathbf{U}) \ll dv, \\ +\infty & \text{otherwise} \end{cases} \tag{7}$$

$$h(f) = \begin{cases} \displaystyle\int_{\mathbb{R}} \eta\left(\frac{d\mu^R}{dv}(f)(v)\right) dv & \text{if} \quad \mu^R(f) \ll dv, \\ +\infty & \text{otherwise,} \end{cases}$$

  where the density $\frac{d\mu}{dv}$ is the Radon-Nikodym derivative of μ against the Lebesgues measure dv, and the kinetic entropy $\eta : \mathbb{R}^+ \to \mathbb{R}$ is a strictly convex.

One observes that the reconstruction function μ^R is linear. Indeed, for two sequences $\mathbf{U}, \mathbf{W} \in \mathsf{R}^1_{\exp}$ and $\alpha, \beta \in \mathbb{R}^+$, then $\alpha\mathbf{U} + \beta\mathbf{W} \in \mathsf{R}^1_{\exp}$ is realized by $\alpha\mu^R(\mathbf{U}) + \beta\mu^R(\mathbf{W})$. Therefore, by combination, the moment entropies are strictly convex functions.

Remark 3. Commonly, the kinetic entropy satisfies $\lim_{g\to 0^+} \eta(g) = +\infty$. In this case, the moment entropies $\mathsf{h}(\mathbf{U})$ are infinite when the underlying measure $\mu^R(\mathbf{U})$ is singular. This allows to reduce the solution set $L^1_{\exp}(\mathbb{R})^+$ to positive functions

$$L^1_{\exp}(\mathbb{R})^+ = \left\{ g \in L^1(\mathbb{R}) \mid g \geq 0 \text{ and } \int_{\mathbb{R}} \exp(|v|) g(v) dv \right\}.$$

when appropriate. The moment sequences are commonly studied in this functional sense, i.e.

$$\mathbf{U} \in \left\{ \left(\int_{\mathbb{R}} v^i g(v) dv \right)_{i\in\mathbb{N}} \mid g \in L^1_{\exp}(\mathbb{R})^+ \right\} \subsetneq \mathsf{R}^1_{\exp},$$

which excludes the moment sequences of singular measures.

The classical construction of the entropy-based approximation (see e.g. [32, 23, 34]) consists in fixing

$$\mathsf{a}^N_{BS}(\mathbf{U}) = \underset{\substack{\mathbf{U}_i=\mathbf{W}_i\\ i=0,\dots,N}}{\operatorname{argmin}}\ \mathsf{h}_{BS}(\mathbf{W}), \qquad \text{with} \quad \eta(g) = \eta_{BS}(g) = g(\log g - 1), \tag{8}$$

using Boltzmann-Shannon entropy η_{BS} ([28]) as kinetic entropy in (7). The optimization problem involving such constraints and such an objective function has been widely studied in the literature (see e.g. [14, 15, 16, 33, 19]). However, it was shown that this problem is ill-posed for some $\mathsf{U} \in \mathsf{R}^1_{\exp}$ in the neighborhood of the Maxwellians moments ([30, 29, 40]). For this purpose, consider approximations of the form

$$\mathsf{a}^N(\mathbf{U}) = \underset{\substack{\mathbf{U}_i=\mathbf{W}_i\\ i=0,\dots,N}}{\operatorname{argmin}}\ \mathsf{h}^N(\mathbf{W}), \tag{9}$$

for some sequences $(\mathsf{h}^N)_{N\in\mathbb{N}}$ of moment entropies such that:

- The entropies h^N have an infinite limit in infinity, i.e.

$$\lim_{\|\mathbf{U}\|_{p,\exp}\to\infty} h^N(\mathbf{U}) = +\infty. \tag{10}$$

- There exists $K \geq 0$, such that for all $N \geq K$, the problem (9) is well-posed.
- The sequence of kinetic entropies η^N converges uniformly to η on a neighborhood of $\mathbf{U}$.

Proposition 3. *For all $p > 1$ and $\mathbf{U} \in \mathsf{R}^p_{\exp}$ such that $\mathsf{h}(\mathbf{U}) < \infty$, the sequence $\mathsf{a}^N(\mathbf{U}) \underset{N\to\infty}{\to} \mathbf{U} \in \ell^p_{\exp}$.*

Proof. Since h^N converges uniformly toward h, then for all $\eta > 0$, there exists $M \in \mathbb{N}$ such that for all $K > M$, then

$$|\mathsf{h}^K(\mathbf{U}) - \mathsf{h}(\mathbf{U})| \leq \eta, \qquad |\mathsf{h}^K(\mathbf{U}^K) - \mathsf{h}(\mathbf{U}^K)| \leq \eta.$$

By construction, $\mathsf{h}^K(\mathbf{U}^K) \leq \mathsf{h}^K(\mathbf{U})$. Then,

$$\mathsf{h}(\mathbf{U}^K) \leq \mathsf{h}^K(\mathbf{U}^K) + \eta \leq \mathsf{h}^K(\mathbf{U}) + \eta \leq \mathsf{h}(\mathbf{U}) + 2\eta.$$

Since (10) holds, then $\mathbf{U}^K$ is uniformly bounded when $K \to \infty$. Then applying Proposition 2 provides the result. ☐

Two such sequences of entropies are considered:

- The φ-divergence technique proposed in [1] consists in choosing η^N of the form

$$\mathsf{h}^N_\varphi(\mathbf{U}) = \int_{\mathbb{R}} \eta^N_\varphi(g(v))dv, \qquad \eta^N_\varphi(g) = g\left(\frac{N^2}{1+N}\left(\frac{g}{g^0}\right)^{1/N} - N\right) + g^0\frac{N}{1+N}, \tag{11a}$$

  where the Maxwellian $g^0 = \mu^R\left(\mathsf{A}^2(\mathbf{U})\right)$ minimizes the Boltzmann-Shannon entropy under the constraints of having the same moments as g up to order 2.

- Since the entropy h_{BS} defined in (8) does not satisfy (10) in $\mathsf{R}^p_{\exp}$, this property can be enforced by adding a Tikhonov regularization ([42]) vanishing when $N \to \infty$, e.g. by choosing

$$\mathsf{h}^N_{reg}(\mathbf{U}) = \mathsf{h}^N_{BS}(\mathbf{U}) + \epsilon^N\|\mathbf{U}\|^p_{p,\exp}, \tag{11b}$$

  with a sequence $(\epsilon^N)_{N\in\mathbb{N}} \in (\mathbb{R}^+_*)^{\mathbb{N}}$ such that $\lim\limits_{N\to\infty} \epsilon^N = 0$. This entropy is ϵ^N-convex and therefore the problem (9) is well-posed. The Tikhonov regularization of the entropy minimization problem was studied in [3] in a different space.

4 Convergence in the homogeneous case

In this part, the following equation is studied using both sequence and form notations:

$$\partial_t \mathbf{U} = \mathsf{c}(\mathbf{U}), \qquad \partial_t f = c(f). \tag{12}$$

4.1 Collision operator

The collision operator is denoted:

$$\mathsf{c} : \begin{cases} \mathsf{R}^p_{\exp} & \to \mathbb{R}^{\mathbb{N}}, \\ \mathbf{U} & \mapsto \mathsf{c}(\mathbf{U}), \end{cases} \qquad c : \begin{cases} {\mathcal{R}^p_{\exp}}' & \to \mathcal{E}', \\ f & \mapsto c(f). \end{cases}$$

The collision operator is assumed to be locally Lipschitz continuous in $\mathsf{R}^p_{\exp}$ such that (12) has a unique strong solution in $C^1([0,T[;\mathsf{R}^p_{\exp})$. This operator needs to satisfy additional properties:

- **Collision invariants**: Define

$$\mathcal{I} = \bigcap_{\mathbf{U}\in\mathsf{R}^p_{\exp}} Ker(L_{\mathsf{c}(\mathbf{U})}), \qquad \mathcal{I} = \bigcap_{g\in{\mathcal{R}^p_{\exp}}'} Ker(c(g)) = c({\mathcal{R}^p_{\exp}}')^{\perp},$$

  using either the notation Ker for the linear form interpretation or the orthogonality notation for the duality interpretation. The collision invariants are a subspace of $\mathcal{E}$, commonly fixed by modelling hypotheses. For 1V problems, $\mathcal{I} = \mathbb{P}_2$ typically represent mass, momentum and energy conservation.

- **Entropy dissipation**: A strictly convex entropy h, resp. h, as defined in (7), is dissipated by the collision operator c, resp. c, if

$$d\mathsf{h}(\mathbf{U})(\mathsf{c}(\mathbf{U})) \leq 0 \quad \forall \mathbf{U} \in \mathsf{R}^p_{\exp}, \qquad dh(f)(c(f)) \leq 0 \quad \forall f \in {\mathcal{R}^p_{\exp}}'.$$

- **Local equilibrium**: Following Boltzmann's H-theorem, the equality case describes the equilibria: Using the sequence notation, assume that $1 \leq p < \infty$ such that the dual of $\ell^p_{\exp}$ is identified with $\ell^q_{\exp}$ with $q^{-1} + p^{-1} = 1$. Denote

$$(\mathsf{I}^p_{\exp})^{\perp} = \left\{ \mathbf{W} \in \ell^q_{\exp} \;\mid\; \sum_{i=0}^{\infty} \mathbf{W}_i \mathbf{b}_i \in \mathcal{I} \right\}.$$

  Then, the collision operator is assumed to satisfy

$$d\mathsf{h}(\mathbf{U})(\mathsf{c}(\mathbf{U})) = 0 \quad \Leftrightarrow \quad d\mathsf{h}(\mathbf{U}) \in (\mathsf{I}^p_{\exp})^{\perp}.$$

  Using the dual notation, the dual of ${\mathcal{E}^p_{\exp}}'$ is also identified with ${\mathcal{E}^q_{\exp}}'$. Denote

$$(I^p_{\exp})^{\perp} = \left\{ g \in {\mathcal{E}^q_{\exp}}' \quad | \quad g[\phi] = 0 \quad \forall \phi \in \mathcal{E}\backslash\mathcal{I} \right\}.$$

  The collision operator is assumed to satisfy

$$dh(f)(c(f)) = 0 \quad \Leftrightarrow \quad dh(f) \in (I^q_{\exp})^{\perp}.$$

  The strict convexity of h and h provides equilibrium distribution of the form

$$\mathbf{M} = (d\mathsf{h})^{-1}(\mathbf{W}) \qquad \text{or} \qquad M = (dh)^{-1}(g), \tag{13}$$

  for some $\mathbf{W} \in (\mathsf{I}^p_{\exp})^{\perp}$ and $\phi \in (I^p_{\exp})^{\perp}$.

Remark 4. One identifies $(\mathsf{I}^p_{\exp})^{\perp} \equiv \mathcal{I} \equiv (I^p_{\exp})^{\perp}$ such that the definitions (13) match the classical definition of a Maxwellian on the density $\frac{d\mu^R}{dv}(\mathbf{M})$, resp. $\frac{d\mu^R}{dv}(M)$, in the classical $L^1_{\exp}(\mathbb{R})^+$ functional sense when $\mathcal{I} = \mathbb{P}_2$ and $\eta = \eta_{BS}$.

Eventually, the solution to the Cauchy problem (12) with an initial condition $f(t=0) = f^0 \in {\mathcal{R}^p_{\exp}}'$, resp. $\mathbf{U}(t=0) = \mathbf{U}^0 \in \mathsf{R}^p_{\exp}$, is a function $f \in C^1([0,T);{\mathcal{R}^p_{\exp}}')$, resp. $\mathbf{U} \in C^1([0,T);\mathsf{R}^p_{\exp})$, satisfying (12) at all times $t \in [0,T)$.

4.2 Relating vectorial and sequencial homogeneous equations

Consider an approximation $f^N \in C^1([0,T);\ \mathcal{R}^p_{\text{exp}}{}')$ of the solution f, or $\mathbf{U}^N \in C^1([0,T);\ \mathsf{R}^p_{\text{exp}})$, of (12), that is given by the equation

$$\begin{cases} \partial_t \mathbf{U}^N & = \mathsf{c}^N(\mathbf{U}^N), \\ \mathbf{U}^N(t=0) & = \mathbf{U}^{0,N}, \end{cases} \qquad \begin{cases} \partial_t f^N & = c^N(f^N), \\ f^N(t=0) & = f^{0,N}, \end{cases} \tag{14}$$

where c^N, resp. c^N, approximates c, resp. c.

To match the approximation framework introduced with (6), these operators are assumed to satisfy the following compatibility conditions associated with an approximation operator a^N, resp. a^N:

Property 2. *The operator c^N, resp. c^N, only depends on the first $N+1$ components of $\mathbf{U}$, resp. $f[v^i]$ for $i \leq N$, i.e.*

$$\partial_{\mathbf{U}_i} \mathsf{c}^N(\mathbf{U}) = 0 \quad \forall i > N, \qquad dc^N(f)(g) = 0 \quad \textit{if} \quad g[v^i] = 0 \quad \forall i > N.$$

Consider the vectorial equation over $\mathbf{V} \in C^1([0,T);\ \mathsf{r}^N(\mathsf{R}))$

$$\begin{cases} \partial_t \mathbf{V} & = \mathbf{c}(\mathbf{V}), \\ \mathbf{V}(t=0) & = \mathbf{V}^0. \end{cases} \tag{15}$$

Proposition 4.

- *Suppose that $\mathbf{V}(t=0) = \mathsf{r}^N(\mathbf{U}^N(t=0))$ and*

$$\mathbf{c}(\mathbf{V}) = \mathsf{r}^N(\mathsf{c}^N(\mathsf{e}^N(\mathbf{V}))). \tag{16}$$

Then the solutions $\mathbf{V}(t) \in \mathsf{r}^N(\mathsf{R})$ of (15) and $\mathbf{U}^N(t) \in \mathsf{R}^p_{\text{exp}}$ of (14) satisfy

$$\mathbf{V}(t) = \mathsf{r}^N(\mathbf{U}^N(t)) \qquad \textit{for all } t \in [0,T).$$

- *Suppose furthermore that $\mathbf{U}^N(t=0) = \mathsf{e}^N(\mathbf{V}(t=0))$ and*

$$\mathsf{c}^N(\mathbf{U}) = d\mathsf{a}^N(\mathbf{U})(\mathsf{c}^N(\mathbf{U})) \qquad \textit{for all } \mathbf{U} \in \mathsf{R}^p_{\text{exp}}. \tag{17}$$

Then the solutions $\mathbf{V}(t) \in \mathsf{r}^N(\mathsf{R})$ of (15) and $\mathbf{U}^N(t) \in \mathsf{R}^p_{\text{exp}}$ of (14) satisfy additionally

$$\mathbf{U}^N(t) = \mathsf{e}^N(\mathbf{V}(t)) \qquad \textit{for all } t \in [0,T).$$

Proof. Under the first conditions (16), one verifies that the equations satisfied by $\mathbf{V}$ and by $\mathsf{r}^N\mathbf{U}^N$ are identical.

Under the second conditions (17), one verifies that

$$\partial_t \mathsf{e}^N(\mathbf{V}) = d\mathsf{e}^N(\mathbf{V})(\partial_t \mathbf{V}) = d\mathsf{e}^N(\mathbf{V})(\mathsf{r}^N(\mathsf{c}^N(\mathsf{e}^N(\mathbf{V})))) = d\mathsf{a}^N(\mathsf{e}^N(\mathbf{V}))(\mathsf{c}^N(\mathsf{e}^N(\mathbf{V})) = \mathsf{c}^N(\mathsf{e}^N(\mathbf{V})).$$

Then, one has

$$\partial_t(\mathsf{e}^N(\mathbf{V}) - \mathbf{U}^N) = \mathsf{c}^N(\mathsf{e}^N(\mathbf{V})) - \mathsf{c}^N(\mathbf{U}^N),$$

which remains zero at all times. □

The second conditions (17) aim to ensure that the solution of (14) evolves within the manifold defined by the range of the approximation operator $\mathsf{a}^N(\mathcal{R}^p_{exp}{}')$. It provides a unique definition of the terms $\mathsf{c}^N(\mathbf{U})_i$ for $i > N$. However, these conditions are too strong to derive the estimates in the next section, and alternative weaker conditions are imposed on $\mathsf{c}^N(\mathbf{U})$.

4.3 Convergence result in 0D

By analogy with numerical analysis, the present convergence results are obtained by combining error estimation and stability. The same terminology is used:

Definition 4.1.

- The consistency error $\varepsilon^N(\mathbf{U})$ is defined, locally in time, by applying the operator (14) to the exact solution $\mathbf{U}$ of (12). This rewrites under the form

$$\varepsilon^N(\mathbf{W}) = \mathsf{c}(\mathbf{W}) - \mathsf{c}^N(\mathbf{W}).$$

- A sequence of operators $(\mathsf{c}^N)_{N\in\mathbb{N}}$ from $\mathsf{R}^p_{\exp}$ to $\mathbb{R}^{\mathbb{N}}$ is $\ell^p_{\exp}$-consistent with the operator c on a set $\mathsf{S} \subset \mathsf{R}^p_{\exp}$ if it converges uniformly on S, i.e. if

$$\lim_{N\to\infty} \sup_{\mathbf{W}\in\mathsf{S}} \|\varepsilon^N(\mathbf{W})\|_{p,\exp} = 0. \tag{18}$$

- A sequence of operators $(\mathsf{c}^N)_{N\in\mathbb{N}}$ from $\mathsf{R}^p_{\exp}$ to $\mathbb{R}^{\mathbb{N}}$ is $\ell^p_{\exp}$-stable on a set $\mathsf{S} \subset \mathsf{R}^p_{\exp}$ if it is Lipschitz continuous over S uniformly with respect to $N \in \mathbb{N}$, i.e. if there exists a constant $0 \leq L < \infty$ such that for all $N \in \mathbb{N}$ and for all $(\mathbf{W}, \mathbf{X}) \in \mathsf{S}^2$, one has

$$\|\mathsf{c}^N(\mathbf{W}) - \mathsf{c}^N(\mathbf{X})\|_{p,\exp} \leq L\|\mathbf{W} - \mathbf{X}\|_{p,\exp}. \tag{19}$$

Under these hypotheses, one obtains the following convergence result:

Proposition 5. *Suppose that the collision operator* c *and the initial condition* $\mathbf{U}^0$ *are such that* (12) *is well-posed. Consider a sequence of operators* $(\mathsf{c}^N)_{N\in\mathbb{N}}$ *that is* $\ell^p_{\exp}$*-consistent with* c *and* $\ell^p_{\exp}$*-stable on a neighborhood* N *of the solution values* $\mathbf{U}([0,T))$*. Then, the sequence of solutions* $(\mathbf{U}^N)_{N\in\mathbb{N}}$ *to* (14) *converge toward the solution* $\mathbf{U}$ *of* (12) *uniformly over* $t \in [0,T]$*, i.e.*

$$\lim_{N\to\infty} \sup_{t\in[0,T)} \|\mathbf{U}^N(t) - \mathbf{U}(t)\|_{p,\exp} = 0.$$

Proof. One computes the error

$$(\mathbf{U} - \mathbf{U}^N)(t) = \int_0^t \varepsilon^N(\mathbf{U}(\tau)) + \left(\mathsf{c}^N(\mathbf{U}(\tau)) - \mathsf{c}^N(\mathbf{U}^N(\tau))\right) d\tau + \left(\mathbf{U}(t=0) - \mathbf{U}^N(t=0)\right).$$

Then, taking the norm provides

$$\|\mathbf{U} - \mathbf{U}^N\|_{p,\exp}(t) \leq \int_0^t \|\varepsilon^N(\mathbf{U}(\tau))\|_{p,\exp} + \|\mathsf{c}^N(\mathbf{U}(\tau)) - \mathsf{c}^N(\mathbf{U}^N(\tau))\|_{p,\exp} d\tau + \|\mathbf{U}(t=0) - \mathbf{U}^N(t=0)\|_{p,\exp}.$$

By stability, one has

$$\|\mathbf{U} - \mathbf{U}^N\|_{p,\exp}(t) \leq \int_0^t \|\varepsilon^N(\mathbf{U}(\tau))\|_{p,\exp} + L\|\mathbf{U} - \mathbf{U}^N\|_{p,\exp}(\tau) d\tau + \|\mathbf{U}(t=0) - \mathbf{U}^N(t=0)\|_{p,\exp}.$$

Then, applying Gronwal's lemma provides

$$\|\mathbf{U} - \mathbf{U}^N\|_{p,\exp}(t) \leq \left(\int_0^t \|\varepsilon^N(\mathbf{U}(\tau))\|_{p,\exp} d\tau + \|\mathbf{U}(t=0) - \mathbf{U}^N(t=0)\|_{p,\exp}\right) \exp(Lt).$$

Convergence follows by consistency and by convergence of the initialization $\mathbf{U}^N(t=0) = \mathsf{a}^N(\mathbf{U}(t=0))$ toward $\mathbf{U}$. ☐

4.4 Sequence of relaxation equations

Consider the relaxation equation (12) with

$$\mathsf{c}(\mathbf{U}) = \frac{\mathbf{M}(\mathbf{U}) - \mathbf{U}}{\tau} \tag{20a}$$

for some relaxation rate $\tau > 0$. The equilibrium point $\mathbf{M}(\mathbf{U})$ is given by the solution of the following problem

$$\mathbf{M}(\mathbf{U}) = \underset{\substack{\mathbf{W} \in \mathsf{R}^p_{\text{exp}} \\ \mathbf{W} - \mathbf{U} \in (\mathsf{I}^p_{\text{exp}})^{\perp}}}{\operatorname{argmin}} \mathsf{h}(\mathbf{W}). \tag{20b}$$

When (20b) is well-posed, the first-order optimality condition yields a solution of the form (13), which justifies the notation $\mathbf{M}$. One observes:

- For all $\mathbf{W} \in \mathbb{R}^{\mathbb{N}}$ such that $\sum_i \mathbf{W}_i v^i \in \mathcal{I}$, the components $\sum_i \mathbf{U}_i(t)\mathbf{W}_i = \sum_i \mathbf{U}^0_i \mathbf{W}_i$ are constant. Therefore, $\mathbf{M}(\mathbf{U}(t)) = \mathbf{M}(\mathbf{U}^0)$, and $\mathsf{c}(\mathbf{U})$ reduces to an affine function.
- The function $\mathbf{U} \mapsto \mathsf{h}(\mathbf{U}) - \mathsf{h}(\mathbf{M}(\mathbf{U}^0))$ is a Lyapunov function for (12). Therefore, $\mathbf{M}(\mathbf{U}^0)$ is asymptotically stable.

Consider the sequence of relaxation equations of the form (12) with a sequence of relaxation operators given by:

$$\mathsf{c}^N(\mathbf{U}) = \frac{\mathbf{M}^N(\mathbf{U}) - \mathbf{U}}{\tau}, \qquad \text{where} \qquad \mathbf{M}^N(\mathbf{U}) = \underset{\substack{\mathbf{W} \in \mathsf{R}^p_{\text{exp}} \\ \mathbf{W} - \mathbf{U} \in (\mathsf{I}^p_{\text{exp}})^{\perp}}}{\operatorname{argmin}} \mathsf{h}^N(\mathbf{W}) \tag{21}$$

where the approximated functions h^N are strictly convex entropies having infinite limits at infinity (10), converging uniformly to h on a neighborhood of the solution values $\mathbf{U}([0,T])$.

Proposition 6. *Suppose that there exists a neighborhood* N *of all solution values* $\mathbf{U}([0,T))$ *on which the sequence of operators* $\mathbf{M}^N$ *defined as the solution of* (21) *converges uniformly toward the operator* $\mathbf{M}$ *defined as the solution of* (20a)*, i.e.*

$$\lim_{N\to\infty} \sup_{\mathbf{W}\in\mathsf{N}} \|\mathbf{M}^N(\mathbf{W}) - \mathbf{M}(\mathbf{W})\|_{p,\text{exp}} = 0.$$

Then, the sequence of relaxation operators $(\mathsf{c}^N)_{N\in\mathbb{N}}$ *defined in* (21) *is* ℓ^p_{exp}*-consistent on* N.

Proof. One simply observes that

$$\|\mathsf{c}^N(\mathbf{W}) - \mathsf{c}(\mathbf{W})\|_{p,\text{exp}} = \frac{1}{\tau}\|\mathbf{M}^N(\mathbf{U}) - \mathbf{M}(\mathbf{W})\|_{p,\text{exp}}.$$

□

Stability can be characterized for relaxation operators:

Proposition 7. *Suppose that the operator* $\mathbf{M}^N$ *defined as the solution of* (21) *and its derivative in* ℓ^p_{exp} *converge uniformly toward the operator* $\mathbf{M}$ *defined as the solution of* (20a) *and its derivative on a neighborhood* N *of all solution values* $\mathbf{U}([0,T))$*. Then, the sequence of relaxation operators* $(\mathsf{c}^N)_{N\in\mathbb{N}}$ *defined in* (21) *is* ℓ^p_{exp}*-stable on* N.

Proof. By construction, one has

$$\|\mathsf{c}^N(\mathbf{U}) - \mathsf{c}^N(\mathbf{W})\|_{p,\text{exp}} \le \frac{1}{\tau}\left(\|\mathbf{U} - \mathbf{W}\|_{p,\text{exp}} + \|\mathbf{M}^N(\mathbf{U}) - \mathbf{M}^N(\mathbf{W})\|_{p,\text{exp}}\right).$$

Under the hypothesis, the sequence of operators $\mathbf{M}^N$ are uniformly Lipschitz continuous. □

Remark 5. In practice, the operator $\mathbf{M}$ is defined as the sequence of moments of a Maxwellian distribution with $\eta = \eta_{BS}$ depending on the first three moments, i.e. $\mathcal{I} = \mathbb{P}_2$. Consider the operator $\mathbf{M}^N$ defined as the sequences of moments of the solutions to (21) using the entropy (11a) or (11b):

- Reproducing the computations from [1] and [3], one verifies that the sequences of operators $\mathbf{M}^N$ defined with such choices of entropies converge uniformly on all bounded subsets of $\mathsf{R}^p_{\exp}$ toward the operator $\mathbf{M}$. Therefore, the corresponding c^N are $\ell^p_{\exp}$-consistent on N.
- Similarly, the operator $\mathbf{M}$ is Lipschitz continuous in $\ell^p_{\exp}$-norm and one also verifies that the operators $\mathbf{M}^N$ with the entropy (11a) or (11b) are also uniformly Lipschitz-continuous on bounded subsets $\mathsf{N} \subset \mathsf{R}^p_{\exp}$, and therefore satisfy the hypothesis of Proposition 7.

In more general cases, sensitivity analysis techniques can be used to demonstrate such results. Typically, extensions of the Danskin theorem (see [21, 8]) can be useful for this aim. Such an analysis requires technical hypotheses and computations, and the examples given by (11) are sufficient for the present purpose.

Corollary 1. *Suppose that $\mathcal{I} = \mathbb{P}_2$ and h^N is given by* (11a) *or* (11b)*. Then the solution $\mathbf{U}^N$ of* (14) *converges uniformly toward the solution $\mathbf{U}$ of* (12) *on* $[0, T)$.

5 Convergence in the 1D-1V case

Lastly, the full 1D-1V kinetic equation (1) is considered.

5.1 Transport operator

The kinetic variable v plays a special role in the considered models, so they are treated separately from the other variables (t, x). Define first the "product by v" operators as

$$P_v : \begin{cases} \mathcal{E} & \to \mathcal{E} \\ \phi & \mapsto \{v \mapsto v\phi(v)\}, \end{cases} \qquad P_v^* : \begin{cases} \mathcal{E}' & \to \mathcal{E}' \\ f & \mapsto \{\phi \mapsto f[P_v(\phi)]\}, \end{cases}$$
$$\mathsf{P}_v : \begin{cases} \mathbb{R}^{\mathbb{N}} & \to \mathbb{R}^{\mathbb{N}} \\ \mathbf{U} & \mapsto (\mathbf{U}_{i+1})_{i\in\mathbb{N}}, \end{cases} \qquad \mathsf{P}_v^* : \begin{cases} \mathbb{R}^{\mathbb{N}} & \to \mathbb{R}^{\mathbb{N}} \\ \mathbf{U} & \mapsto (\mathbf{U}_{i-1})_{i\in\mathbb{N}}, \end{cases}$$

where the adjoint notation is used by abuse. While the solution to the ODE (12) can be understood in a strong sense, solutions to transport equations commonly generate discontinuities and must be understood in a weak sense. For the sake of simplicity, the following definition is chosen as it is adapted to the moment framework and to the convergence study of the next part.

Definition 5.1. A moment solution of (1) is a function $f : \mathbb{R}^+ \times \mathbb{R} \to {\mathcal{R}^p_{\exp}}'$ satisfying, for all $\psi \in C^1_c(\mathbb{R}^+ \times \mathbb{R})$ and all $\phi \in \mathcal{E}$

$$\int_{\mathbb{R}^+\times\mathbb{R}} f(t,x)[\phi]\partial_t\psi(t,x) + P_v^*(f(t,x))[\phi]\partial_x\psi(t,x) + c(f(t,x))[\phi]\psi(t,x)dtdx$$
$$+ \int_{\mathbb{R}} f^0(x)[\phi]\psi(0,x)dx = 0, \tag{22a}$$

for some initial condition f^0. It is equivalently represented as a function $\mathbf{U} : \mathbb{R}^+ \times \mathbb{R} \to \mathsf{R}^p_{\exp}$ satisfying, for all $\psi \in C^1_c(\mathbb{R}^+ \times \mathbb{R})$ and all sequences $\mathbf{W} \in \mathbb{R}^{\mathbb{N}}$ such that the series $|\sum\limits_{i\in\mathbb{N}} \mathbf{W}_i v^i|$ has an infinite convergence radius

$$\int_{\mathbb{R}^+\times\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i \left[\mathbf{U}(t,x)_i \partial_t\psi(t,x) + \mathsf{P}_v(\mathbf{U}(t,x))_i\partial_x\psi(t,x) + \mathsf{c}(\mathbf{U}(t,x))_i\psi(t,x)\right] dtdx$$
$$+ \int_{\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i \left[\mathbf{U}^0(x)_i\psi(0,x)\right] dx = 0, \tag{22b}$$

for some initial condition $\mathbf{U}^0$.

5.2 Relating vectorial and sequencial 1D equations

Formulation (22) naturally extends the definition of weak solutions of hyperbolic systems of (finite number of) balance laws with a flux $\mathbf{F}$ and a source $\mathbf{c}$, i.e. functions $\mathbf{V} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^N$ such that

$$\int_{\mathbb{R}^+\times\mathbb{R}} \mathbf{V}(t,x)_i \partial_t \psi(t,x) + \mathbf{F}(\mathbf{V}(t,x))_i \partial_x \psi(t,x) + \psi(t,x)\mathbf{c}(\mathbf{V}(t,x))_i dxdt + \int_{\mathbb{R}} \mathbf{V}^0(x)\psi(0,x)dx = 0 \quad (23)$$

for all $i = 1,\dots,N$ and $\psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$. As in the homogeneous case, consider an approximation f^N, or $\mathbf{U}^N$, of the solution f of (22), satisfying for all $\phi(v) = \sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{E}$ and all $\psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$

$$\int_{\mathbb{R}^+\times\mathbb{R}} f^N(t,x)[\phi]\partial_t\psi + P_v^*(a^N(f^N(t,x)))[\phi]\partial_x\psi(t,x) + c^N(f^N(t,x))[\phi]\psi(t,x)dtdx \qquad (24a)$$
$$+ \int_{\mathbb{R}} f^{0,N}(x)[\phi]\psi(0,x)dx = 0,$$

$$\int_{\mathbb{R}^+\times\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i \left[\mathbf{U}_i^N(t,x)\partial_t\psi(t,x) + (\mathsf{P}_v(\mathsf{a}^N(\mathbf{U}^N(t,x))))_i\partial_x\psi(t,x) + \mathsf{c}^N(\mathbf{U}^N(t,x))_i\right] dtdx \qquad (24b)$$
$$+ \int_{\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i \left[\mathbf{U}^{0,N}(x)_i\psi(0,x)\right] dx = 0,$$

for some approximations c^N and c^N of the collision operator c and c and for some approximation operators a^N and a^N.

Proposition 8. • *Suppose that $\mathbf{V}(t=0,x) = \mathsf{r}^N(\mathbf{U}^N(t=0,x))$ a.e. in x and that*

$$\mathbf{c}(\mathbf{V}) = \mathsf{r}^N\left(\mathsf{c}^N\left(\mathsf{e}^N(\mathbf{V})\right)\right), \qquad \mathbf{F}(\mathbf{V}) = \mathsf{r}^N\left(\mathsf{P}_v^*\left(\mathsf{e}^N(\mathbf{V})\right)\right).$$

Then, the weak solutions $\mathbf{U}^N(t,x) \in \mathsf{R}^p_{\text{exp}}$ of (24b) and $\mathbf{V}(t,x) \in \mathbb{R}^N$ of (23) satisfy

$$\mathbf{V}(t,x) = \mathsf{r}^N(\mathbf{U}^N(t,x)), \qquad \textit{for almost all } (t,x).$$

• *Suppose furthermore that $\mathbf{U}^N(t=0,x) = \mathsf{e}^N(\mathbf{V}(t=0,x))$ a.e. in x and that*

$$\mathsf{c}^N(\mathbf{U}) - \partial_x\left(\mathsf{P}_v^*\left(\mathsf{a}^N(\mathbf{U})\right)\right) = d\mathsf{a}^N(\mathbf{U})\left(\mathsf{c}^N(\mathbf{U}) - \partial_x\left(\mathsf{P}_v^*\left(\mathsf{a}^N(\mathbf{U})\right)\right)\right).$$

Then, the weak solutions $\mathbf{U}^N(t,x) \in \mathsf{R}^p_{\text{exp}}$ of (24b) and $\mathbf{V}(t,x) \in \mathbb{R}^N$ of (23) satisfy

$$\mathbf{U}^N(t,x) = \mathsf{e}^N(\mathbf{V}(t,x)) \qquad \textit{for almost all } (t,x).$$

The proof of Proposition 4 extends to a weak sense and to generic sequence-valued equations of the form $\frac{d}{dt}\mathbf{U}^N = \mathsf{g}^N(\mathbf{U}^N)$ and vector-valued equations $\frac{d}{dt}\mathbf{V}^N = \mathbf{g}(\mathbf{V})$ with operators $\mathbf{g}$ and g^N satisfying the aforementioned conditions.

This proposition provides a unique definition of the terms $\mathsf{c}^N(\mathbf{U})_i$ for $i > N$, depending on both the first moments $\mathbf{V} = \mathsf{r}^N(\mathbf{U}^N)$ and their space derivative. These conditions are not used in the next subsection, and alternative conditions are imposed on $\mathsf{c}^N(\mathbf{U})$.

5.3 Convergence result in 1D

In this paragraph, we compare the solution of (24) with given approximation operators a^N and c^N to the solution of (22) when $N \to \infty$.

Proposition 9. *Suppose that:*

- *The operator* c *is such that* (22) *has a unique solution* $\mathbf{U}$.
- *The sequence of approximations* $(\mathsf{a}^N)_{N\in\mathbb{N}}$ *converges toward* Id *uniformly on a neighborhood* $\mathsf{N} \supset \{\mathbf{U}(t,x),\ t \in \mathbb{R}^+, x \in \mathbb{R}\}$ *of all solution values in norm* $\ell^1_{\exp}$, *i.e.*

$$\lim_{N\to\infty} \sup_{\mathbf{U}\in\mathsf{N}} \|\mathbf{U} - \mathsf{a}_N(\mathbf{U})\|_{\exp,1} = 0.$$

- *The sequence of approximations* $(\mathsf{c}^N)_{N\in\mathbb{N}}$ *is consistent and stable in norm* $\ell^1_{\exp}$ *over* N, *and such that* (24b) *has a unique solution* $\mathbf{U}^N$ *for all* $N \in \mathbb{N}$.

Then, the sequence $(\mathbf{U}^N)_{N\in\mathbb{N}}$ *converges weakly towards* $\mathbf{U}$ *in norm* $\ell^1_{\exp}$, *i.e.* $\forall \psi \in C_c(\mathbb{R}^+ \times \mathbb{R})$

$$\int_{\mathbb{R}^+\times\mathbb{R}} \|\mathbf{U} - \mathbf{U}^N\|_{\exp,1}(t,x)\psi(t,x)dtdx \to 0.$$

The proof starts with a lemma:

Lemma 5.2. *If* $\mathbf{U} \in \mathsf{R}^1_{\exp}$, *then* $\|\mathsf{P}_v(\mathbf{U})\|_{\exp,1} < \infty$.

Proof. Having $\mathbf{U} \in \mathsf{R}^1_{\exp}$, then there exists $0 < c < 1$ such that (3a) holds. One verifies that

$$\|\mathbf{U}\|_{\exp,1} = \sum_{i=0}^{\infty} \frac{|\mathbf{U}_i|}{i!} \le \int_{\mathbb{R}} \exp(|v|)d\mu(v) \le \int_{\mathbb{R}} \exp\left(\frac{|v|}{\alpha}\right) d\mu(v) < \infty,$$

where μ is the unique representing measure for $\mathbf{U}$. Following Proposition 1, the last inequality holds for all $1 \ge \alpha > \sqrt{c} > 0$. Since $|v| \le \delta \exp(\delta^{-1}|v|) - 1$ for all $\delta > 0$, then

$$\|\mathsf{P}_v(\mathbf{U})\|_{\exp,1} = \sum_{i=0}^{\infty} \frac{|\mathbf{U}_{i+1}|}{i!} \le \int_{\mathbb{R}} |v|\exp(|v|)d\mu(v) \le \int_{\mathbb{R}} \left(\delta\exp\left(\frac{|v|}{\delta}\right) - 1\right) \exp\left(|v|\right) d\mu(v),$$

which is bounded for all δ satisfying $1 \ge \delta(1+\delta)^{-1} \ge \sqrt{c} > 0$. ☐

Proof. of Proposition 9 Subtracting the two equations yields

$$I_1 + I_2 + I_3 = J_1 + J_2 + J_3,$$

with

$$I_1 = \int_{\mathbb{R}^+\times\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i(\mathbf{U} - \mathbf{U}^N)_i \partial_t\psi dtdx, \qquad J_1 = \int_{\mathbb{R}^+\times\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i(\mathbf{U} - \mathsf{a}^N(\mathbf{U}))_{i+1}\partial_x\psi dtdx,$$

$$I_2 = \int_{\mathbb{R}^+\times\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i(\mathsf{a}^N(\mathbf{U}) - \mathsf{a}^N(\mathbf{U}^N))_{i+1}\partial_x\psi dtdx, \qquad J_2 = \int_{\mathbb{R}^+\times\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i(\mathsf{c}(\mathbf{U}) - \mathsf{c}^N(\mathbf{U}))_i\psi dtdx,$$

$$I_3 = \int_{\mathbb{R}^+\times\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i(\mathsf{c}^N(\mathbf{U}) - \mathsf{c}^N(\mathbf{U}^N))_i\psi dtdx, \qquad J_3 = \int_{\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i(\mathbf{U}^0 - \mathbf{U}^{0,N})(x)_i\psi(0,x)dx,$$

One verifies that $J_i \underset{N\to\infty}{\longrightarrow} 0$ for $i = 1, 2, 3$:

- Defining $\mathbf{U}^{0,N}(x) = \mathsf{a}^N(\mathbf{U}^0(x))$, then $\sum\limits_{i=0}^{\infty} \mathbf{W}_i(\mathbf{U}^0 - \mathbf{U}^{0,N})_i(x) \leq \sum\limits_{i=0}^{\infty} |\mathbf{W}_i||\mathbf{U}^0 - \mathbf{U}^{0,N}|_i(x)$ then for all sequences $\mathbf{W}$ such that $\sum\limits_i |\mathbf{W}_i||\mathbf{U}_i| \leq K\|\mathbf{U}\|_{\exp,1}$ then $J_3 \underset{N\to\infty}{\longrightarrow} 0$. Especially, this holds for $\mathbf{W}_i = \delta_{i,j}$ for any $j \in \mathbb{N}$ and for $\mathbf{W}_i = (i!)^{-1}$.
- Similarly, $\sum\limits_{i=0}^{\infty} \mathbf{W}_i(\mathbf{U} - \mathsf{a}^N(\mathbf{U}))_{i+1} \leq \sum\limits_{i=0}^{\infty} |\mathbf{W}_i||\mathbf{U} - \mathsf{a}^N(\mathbf{U})|_{i+1}$ and, using Lemma 5.2, then $J_1 \to 0$ for all sequences $\mathbf{W}$ such that $\sum\limits_i |\mathbf{W}_i||\mathbf{U}_i| \leq K\|\mathbf{U}\|_{\exp,1}$.
- Againg $\sum\limits_{i=0}^{\infty} \mathbf{W}_i(\mathsf{c}(\mathbf{U}) - \mathsf{c}^N(\mathbf{U}))_i \leq \sum\limits_{i=0}^{\infty} |\mathbf{W}_i||\mathsf{c}(\mathbf{U}) - \mathsf{c}^N(\mathbf{U})|_i$ and the assumptions provide $J_2 \to 0$ for all such sequences $\mathbf{W}$.

Then, for all sequences $\mathbf{W}$ such that $\sum\limits_i |\mathbf{W}_i||\mathbf{U}_i| \leq K\|\mathbf{U}\|_{\exp,1}$ and all $\psi \in C_c^1$, one finds

$$0 = \lim_{N\to\infty} \int_{\mathbb{R}^+\times\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i \left((\mathbf{U} - \mathbf{U}^N)_i \partial_t \psi + (\mathsf{a}^N(\mathbf{U}) - \mathsf{a}^N(\mathbf{U}^N))_{i+1} \partial_x \psi + (\mathsf{c}(\mathbf{U}) - \mathsf{c}^N(\mathbf{U}))_i \psi \right) dtdx.$$

For all $i \in \mathbb{N}$, define a smooth vector field X^i that changes sign where $(\mathsf{a}^N(\mathbf{U}) - \mathsf{a}^N(\mathbf{U}^N))_{i+1} \times (\mathbf{U} - \mathbf{U}^N)_i$ changes sign. Define also a function $\phi^i \in C_c^1(\mathbb{R})$ and a constant K^i such that the function $\psi^i = K^i + \exp(\phi^i \circ X^i) \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$ changes sign where $(\mathsf{c}(\mathbf{U}) - \mathsf{c}^N(\mathbf{U}))_i$ does. Substituting such functions ψ^i in the last equation yields

$$\begin{aligned} 0 = \lim_{N\to\infty} \int_{\mathbb{R}^+\times\mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i \Big(&\left(|\mathbf{U} - \mathbf{U}^N|_i + |\mathsf{a}^N(\mathbf{U}) - \mathsf{a}^N(\mathbf{U}^N)|_{i+1}\right) |\phi^{i\prime} \times \exp(\phi^i \circ X^i)(t,x)| \\ &+ |\mathsf{c}(\mathbf{U}) - \mathsf{c}^N(\mathbf{U})|_i |K^i + \exp(\phi^i \circ X^i)(t,x)| \Big) dtdx = 0. \end{aligned}$$

Up to a normalization, and with the variety of such functions ϕ^i and X^i, this provides

$$0 = \lim_{N\to\infty} \int_{\mathbb{R}^+\times\mathbb{R}} \left(\|\mathbf{U} - \mathbf{U}^N\| + \|\mathsf{P}_v(\mathsf{a}^N(\mathbf{U}) - \mathsf{a}^N(\mathbf{U}^N))\| + \|\mathsf{c}(\mathbf{U}) - \mathsf{c}^N(\mathbf{U})\| \right) |\psi(t,x)| dtdx.$$

Using the assumptions and Lemma 5.2 reduces this limit into the desired result. $\square$

References

[1] M. Abdelmalik and H. van Brummelen. Moment closure approximations of the boltzmann equation based on φ-divergences. *J. Stat. Phys.*, 164(1):77–104, 2016.

[2] N. I. Akhiezer. *The classical moment problem.* Edinburgh : Oliver & Boyd, 1965.

[3] G. W. Alldredge, M. Frank, and C. D. Hauck. A regularized entropy-based moment method for kinetic equations. *SIAM J. Appl. Math.*, 79(5):1627–1653, 2019.

[4] R. Alonso, J. Cañizo, I. Gamba, and C. Mouhot. A new approach to the creation and propagation of exponential moments in the boltzmann equation. *Commun. PDE.*, 38(1):155–169, 2012.

[5] R. Alonso, I. Gamba, and M. Tasković. Exponentially-tailed regularity and decay rate to equilibrium for the boltzmann equation. *Nonlinear Analysis*, 262:113920, 2026.

[6] C. Bardos, F. Golse, and D. Levermore. Fluid dynamic limits of kinetic equations i: Formal derivations. *J. Stat. Phys.*, 63:323–344, 1991.

[7] C. Bardos, F. Golse, and D. Levermore. Fluid dynamic limits of kinetic equations ii: Convergence proofs for the boltzmann equation. *Comm. Pure Appl. Math.*, XLVI:667–753, 1993.

[8] D. P. Bertsekas. *Nonlinear programming.* Athena, 1999.

[9] M. Bessemoulin-Chatard and F. Filbet. On the stability of conservative discontinuous galerkin/hermite spectral methods for the vlasov-poisson system. *J. Comput. Phys.*, 451, 2022.

[10] M. Bessemoulin-Chatard and F. Filbet. On the convergence of discontinuous galerkin/hermite spectral methods for the vlasov-poisson system. *SIAM J. Numer. Anal.*, 61, 2023.

[11] A. Blaustein and F. Filbet. On a discrete framework of hypocoercivity for kinetic equations. *Math. Comput.*, 93(345):163–202, 2024.

[12] A. Bobylev and I. Gamba. Boltzmann equations for mixtures of maxwell gases: Exact solutions and power like tails. *J. Stat. Phys.*, 124(2-4), 2006.

[13] A. Bobylev, I. Gamba, and V. Panferov. Moment inequalities and high-energy tails for boltzmann equations with inelastic interactions. *J. Stat. Phys.*, 116(5/6), 2004.

[14] J. Borwein and A. Lewis. Duality relationships for entropy-like minimization problems. *SIAM J. Control Optim.*, 29(2):325–338, 1991.

[15] J. Borwein and A. Lewis. Partially finite convex programming, part I: Quasi relative interiors and duality theory. *Math. Programming*, 57:15–48, 1992.

[16] J. Borwein and A. Lewis. Partially finite convex programming: Part II. *Math. Programming*, 57:49–83, 1992.

[17] T. Carleman. *Les fonctions quasi-analytiques.* Gauthier-Villars, 1926.

[18] H. Cramér. Sur un nouveau théorème-limite de la théorie des probabilités. *Actual Sci Ind.: Colloque consacré à la théorie des probabilités*, 763:5–23, 1938.

[19] I. Csiszár. I-divergence geometry of probability distributions and minimization problems. *Ann. Proba.*, 3:146–158, 1975.

[20] R. Curto and L. Fialkow. Recursiveness, positivity, and truncated moment problem. *Houston j. Math.*, 17(4):603–635, 1991.

[21] J. M. Danskin. *The theory of Max-Min and its application to weapons allocation problems.* Springer, 1967.

[22] L. Desvillettes. Some applications of the method of moments for the homogeneous boltzmann and kac equations. *Arch. Rational Mech. Anal.*, 123(4):387–404, 1993.

[23] W. Dreyer. Maximization of the entropy in non-equilibrium. *J. Phys. A: Math. Gen.*, 20:6505, 1987.

[24] I. Gamba, V. Panferov, and C. Villani. Upper maxwellian bounds for the spatially homogeneous boltzmann equation. *Arch. Rational Mech. Anal.*, 194:253–282, 2009.

[25] H. Hamburger. Über eine erweiterung des Stieltjesschen momentenproblems. Teil 1. *Math. Ann.*, 81:235–319, 1920.

[26] E. K. Haviland. On the momentum problem for distributions in more than one dimension i. *Amer. J. Math.*, 57(3):562–568, 1935.

[27] E. K. Haviland. On the momentum problem for distributions in more than one dimension ii. *Amer. J. Math.*, 58(1):164–168, 1936.

[28] E. T. Jaynes. Information theory and statistical physics. *Phys. Rev.*, 106(4):620–630, 1957.

[29] M. Junk. Domain of definition of levermore's five-moment system. *J. Stat. Phys.*, 93(5/6):1143–1167, 1998.

[30] M. Junk. Maximum entropy for reduced moment problems. *Math. Mod. Meth. Appl. Sci.*, 10(1001–1028):2000, 1998.

[31] M. G. Krein. On an extrapolation problem due to Kolmogorov. *Dokl. Akad. Nauk SSSR ser. matem.*, 45:306–309, 1945.

[32] C. D. Levermore. Moment closure hierarchies for kinetic theories. *J. Stat. Phys.*, 83(5–6):1021–1065, 1996.

[33] L. Mead and N. Papanicolaou. Maximum entropy in the problem of moments. *J. Math. Phys.*, 25:2404, 1984.

[34] I. Müller and T. Ruggeri. *Rational Extended Thermodynamics.* Springer, 1998.

[35] M. Pavić-Čolić and M. Tasković. Propagation of stretched exponential moments for the kac equation and boltzmann equation with maxwell molecules. *Kin. Rel. Mod.*, 11:597–613, 2018.

[36] T. Pichard. Some recent advances on the method of moments in kinetic theory. *ESAIM: ProcS*, 75:86–95, 2023.

[37] M. Riesz. Sur le problème des moments, troisième note. *Ark. Math. Astr. Fys.*, 17(16):1–52, 1923.

[38] C. Schmeiser and A. Zwirchmayr. Convergence of moment methods for linear kinetic equations. *SIAM J. Num. Anal.*, 36:74–88, 1998.

[39] K. Schmüdgen. *The moment problem.* Springer, 2017.

[40] J. Schneider. Entropic approximation in kinetic theory. *ESAIM: M2AN*, 38(3):541–561, 2004.

[41] J. A. Shohat and J. D. Tamarkin. *The problem of moments.* American Mathematical Society, 1943.

[42] A. N. Tikhonov. Solution of incorrectly formulated problems and the regularization method. *Dokl. Akad. Nauk. SSSR*, 151:501–504, 1963.