

Nonequilibrium and multiscale fluids: Models and approximations

Teddy Pichard

Habilitation defense
IP Paris

Defense date: 19/06/26

Jury members:

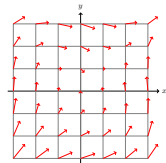
Reviewers: I. Gamba, V. Giovangigli, P. Lafitte

Examiners: M. Bessemoulin-Chatard, F. Bouchut,
N. Crouseilles, H. van Brummelen

General aims

Long term objective: Accurate simulation of complex fluid regimes

Consists of: Model - Reduction - Discretization



$$\rightarrow \partial_t f + \dots \rightarrow \partial_t \mathbf{U} + \dots \rightarrow \mathbf{U}_{i,j}^{n+1} = \dots$$

Fluid models: Different scales:

Microscopic



$$\begin{aligned} \frac{dx_i}{dt} &= \dots \\ \frac{dv_i}{dt} &= \dots \end{aligned}$$

Mesoscopic



$$f(t, x, v)$$

Macroscopic



$$\begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix} (t, x)$$

Context

Starting point: Kinetic models of rarefied gas (academic example)

$$\partial_t f + \operatorname{div}_x(vf) = Q(f)$$

with:

$f(t, x, v)$: distribution function of time t , position x , velocity v

$Q(f)$: change of state (collisions)

Starting point: Kinetic models of rarefied gas (academic example)

$$\partial_t f + \operatorname{div}_x(vf) = Q(f)$$

Major collective properties:

- *Transport phenomena:*

$$f(t, x, v) = f^0(x - vt, v) + \int_0^t Q(f)(\tau, x - v(t - \tau), v) d\tau$$

Starting point: Kinetic models of rarefied gas (academic example)

$$\partial_t f + \operatorname{div}_x(vf) = Q(f)$$

Major collective properties:

- *Transport phenomena:*

$$f(t, x, v) = f^0(x - vt, v) + \int_0^t Q(f)(\tau, x - v(t - \tau), v) d\tau$$

- *H-theorem:*
 - Dissipation phenomena

 - Equilibrium characterization

Starting point: Kinetic models of rarefied gas (academic example)

$$\partial_t f + \operatorname{div}_x(vf) = Q(f)$$

Major collective properties:

- *Transport phenomena:*

$$f(t, x, v) = f^0(x - vt, v) + \int_0^t Q(f)(\tau, x - v(t - \tau), v) d\tau$$

- *H-theorem:*

- Dissipation phenomena: $\exists \eta$ strictly convex

$$\underbrace{\partial_t \left(\int_v \eta(f) \right)}_{\text{entropy } H(f)} + \underbrace{\operatorname{div}_x \left(\int_v v \eta(f) \right)}_{\text{entropy flux } G(f)} = \underbrace{\left(\int_v \eta'(f) Q(f) \right)}_{\text{entropy source } S(f)} \leq 0 \quad \forall f$$

- Equilibrium characterization

Starting point: Kinetic models of rarefied gas (academic example)

$$\partial_t f + \operatorname{div}_x(vf) = Q(f)$$

Major collective properties:

- *Transport phenomena:*

$$f(t, x, v) = f^0(x - vt, v) + \int_0^t Q(f)(\tau, x - v(t - \tau), v) d\tau$$

- *H-theorem:*

- Dissipation phenomena: $\exists \eta$ strictly convex

$$\partial_t \left(\int_v \eta(f) \right) + \operatorname{div}_x \left(\int_v v \eta(f) \right) = \left(\int_v \eta'(f) Q(f) \right) \leq 0 \quad \forall f$$

- Equilibrium characterization:

Definition

Collision invariants: $\mathcal{I} = \left\{ \phi \mid \int_v \phi Q(f) = 0 \quad \forall f \right\} \equiv \operatorname{Span}(1, v, v^2)$

Hypothesis

$\forall f, \exists!$ *minimizer*

$$\forall \phi \in \mathcal{I}, \int_v \phi(f - g) = 0 \quad \operatorname{argmin}_g \int_v \eta(g) =: M(f) \equiv (\eta^*)'(\tilde{\phi}) \quad \text{for some } \tilde{\phi} \in \mathcal{I}$$

Starting point: Kinetic models of rarefied gas (academic example)

$$\partial_t f + \operatorname{div}_x(vf) = Q(f)$$

Major collective properties:

- *Transport phenomena:*

$$f(t, x, v) = f^0(x - vt, v) + \int_0^t Q(f)(\tau, x - v(t - \tau), v) d\tau$$

- *H-theorem:* $\equiv \exists$ Lyapunov function

- Dissipation phenomena: $\exists \eta$ strictly convex

$$\partial_t \left(\int_v \eta(f) \right) + \operatorname{div}_x \left(\int_v v \eta(f) \right) = \left(\int_v \eta'(f) Q(f) \right) \leq 0 \quad \forall f$$

- Equilibrium characterization

$$\int_v \eta'(f) Q(f) = 0 \quad \Leftrightarrow \quad f = M(f)$$

Multiscale and nonequilibrium fluids

Non-dimensional form:

$$\alpha \partial_t f + \beta \operatorname{div}_x(vf) = \frac{1}{\tau} Q(f)$$

Multiscale and nonequilibrium fluids

Non-dimensional form:

$$\alpha \partial_t f + \beta \operatorname{div}_x(vf) = \frac{1}{\tau} Q(f)$$

- **Multiscale:** Phenomena arise at different time-space scales
↪ characterized by $\alpha, \beta, \frac{1}{\tau}$ very different

Multiscale and nonequilibrium fluids

Non-dimensional form:

$$\alpha \partial_t f + \beta \operatorname{div}_x(vf) = \frac{1}{\tau} Q(f)$$

- **Multiscale:** Phenomena arise at different time-space scales
↪ characterized by $\alpha, \beta, \frac{1}{\tau}$ very different
- **Equilibrium:** Collisional phenomena dominate
↪ characterized by $\tau \ll 1$

Multiscale and nonequilibrium fluids

Non-dimensional form:

$$\alpha \partial_t f + \beta \operatorname{div}_x(vf) = \frac{1}{\tau} Q(f)$$

- **Multiscale:** Phenomena arise at different time-space scales
↪ characterized by $\alpha, \beta, \frac{1}{\tau}$ very different
- **Equilibrium:** Collisional phenomena dominate
↪ characterized by $\tau \ll 1$
 - $f \approx M(f)$ is a good approximation
 - $M(f)$ is defined only out of $\mathbf{U}_i = \int_v \phi_i f$ for all $\phi_i \in \mathcal{I}$, s.t.

$$\alpha \partial_t \mathbf{U} + \beta \operatorname{div}_x \mathbf{F}(\mathbf{U}) = 0 \quad (\text{Euler})$$

Multiscale and nonequilibrium fluids

Non-dimensional form:

$$\alpha \partial_t f + \beta \operatorname{div}_x(vf) = \frac{1}{\tau} Q(f)$$

- **Multiscale:** Phenomena arise at different time-space scales
↪ characterized by $\alpha, \beta, \frac{1}{\tau}$ very different
- **Equilibrium:** Collisional phenomena dominate
↪ characterized by $\tau \ll 1$
 - $f \approx M(f)$ is a good approximation
 - $M(f)$ is defined only out of $\mathbf{U}_i = \int_v \phi_i f$ for all $\phi_i \in \mathcal{I}$, s.t.

$$\alpha \partial_t \mathbf{U} + \beta \operatorname{div}_x \mathbf{F}(\mathbf{U}) = 0 \quad (\text{Euler})$$

- **Nonequilibrium:** $f \approx M(f)$ is not a sufficient approximation

Position of the work

Focus on **nonequilibrium and multiscale** fluids s.t.

- *Transport and H-theorem* still need to be preserved
 - **Macroscopic** fluids
↔ Valid approximation of f from a **finite number of moments**
 - $f \approx M(f)$ is **not** a **sufficient** approximation
-

Position of the work

Focus on **nonequilibrium and multiscale** fluids s.t.

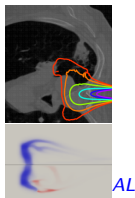
- *Transport and H-theorem* still need to be preserved
- **Macroscopic** fluids
↔ Valid approximation of f from a **finite number of moments**
- $f \approx M(f)$ is **not** a **sufficient** approximation

Contribution

- 1 *Extend this kinetic framework to various applications*^{AL,ZT,SD}
- 2 *Moment-based approximation*^{AL,ZT,SD}
- 3 *Expansion-based approximation*^{ZT}
- 4 *Numerical methods for multiscale nonequilibrium models*^{KAA,GB,SD}

Applications:

- Radiations
- Rarefied gases
- Two-phase^{AL,KAA}
- Plasma^{LR,ZT,SD}
- Biology



Supervisions (Master, PhD, postdoc): A. Loison (20-23)*, K. Ait-Ameur (20-23)[†], L. Reboul (19-24)[‡], Z. Tazakkatti (21 + 23-now)⁻, G. Brunelle (23)⁺, S. Dupuy (26-now)[±]

Funded by: *AID-X, [†]MMEED project, [‡]AID-Dim Math Innov, ⁻OpenNumDef project, ⁺CMAP, [±]ERC HiMOM (PI: AAL)

Position of the work

Focus on **nonequilibrium and multiscale** fluids s.t.

- *Transport and H-theorem* still need to be preserved
- **Macroscopic** fluids
↔ Valid approximation of f from a **finite number of moments**
- $f \approx M(f)$ is **not** a **sufficient** approximation

Contribution

- 1 *Extend this kinetic framework to various applications*^{AL,ZT,SD}
- 2 **Moment-based approximation**^{AL,ZT,SD}
- 3 **Expansion-based approximation**^{ZT}
- 4 **Numerical methods** *for multiscale nonequilibrium models*^{KAA,GB,SD}

Applications:

- Radiations
- **Rarefied gases**
- Two-phase^{AL,KAA}
- **Plasma**^{LR,ZT,SD}
- Biology

Supervisions (Master, PhD, postdoc): A. Loison (20-23)*, K. Ait-Ameur (20-23)[†], L. Reboul (19-24)[‡], Z. Tazakkatti (21 + 23-now)⁻, G. Brunelle (23)⁺, S. Dupuy (26-now)[±]

Funded by: *AID-X, [†]MMEED project, [‡]AID-Dim Math Innov, ⁻OpenNumDef project, ⁺CMAP, [±]ERC HiMOM (PI: AAL)

① Kinetic models

② Moment-based approximations

③ Expansion-based approximations

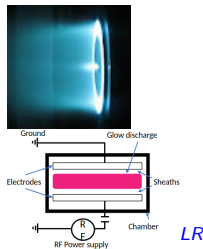
④ Numerical methods for multi-scale problems

A cold plasma model¹

Plasma: $p =$ electrons e - neutrals n - ions i

$$\partial_t f_p + \text{div}_x(v_p f_p) + q_p \text{div}_{v_p}((E + v_p \times B) f_p) = Q_p(\{f_q\}_q),$$

$$E = \nabla_x \phi, \quad \text{div}_x E = \epsilon_0^{-1} \sum_q q_q \int_{v_q} f_q, \quad B = \dots$$



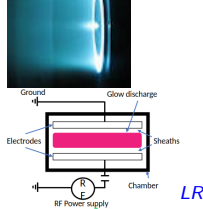
¹Collab.: A. Alvarez Laguna [A15]

A cold plasma model¹

Plasma: $p = e - n - i$

$$\partial_t f_p + \text{div}_x(v_p f_p) + q_p \text{div}_{v_p}((E + v_p \times B)f_p) = Q_p(\{f_q\}_q),$$

$$E = -\nabla_x \phi, \quad \text{div}_x E = \epsilon_0^{-1} \sum_q q_q \int_{v_q} f_q, \quad B = \dots$$



Hypothesis

- Not magnetized: $B = 0$
- Weakly ionized: $f_n \gg f_e, f_i$
- Electron scales decoupled from heavies: Study only f_e
 $f_n = M_n(f_n)$ and $f_i = M_i(f_i)$ are fixed and at equilibrium

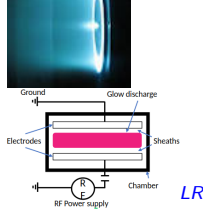
¹Collab.: A. Alvarez Laguna [A15]

A cold plasma model¹

Plasma: $p = e - n - i$

$$\partial_t f_e + \text{div}_x(v_e f_e) - q_e \text{div}_{v_e}((\nabla_x \phi) f_e) = Q_e(\{f_q\}_q),$$

$$\Delta_x \phi = \epsilon_0^{-1} \left(\rho^0 - \int_{v_e} f_e \right),$$



Hypothesis

- Not magnetized: $B = 0$
- Weakly ionized: $f_n \gg f_e, f_i$
- Electron scales decoupled from heavies: Study only f_e
 $f_n = M_n(f_n)$ and $f_i = M_i(f_i)$ are fixed and at equilibrium

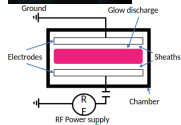
¹Collab.: A. Alvarez Laguna [A15]

A cold plasma model¹

Nondimensionalized plasma : $p = e - n - i$

$$\partial_t f + \frac{1}{\varepsilon} \left(\text{div}_x(vf) - \text{div}_v((\nabla_x \phi)f) \right) = \frac{1}{\varepsilon^2} Q_e(f),$$

$$\Delta_x \phi = \frac{1}{\chi} \left(\rho^0 - \int_v f \right)$$



Hypothesis

- *Not magnetized*: $B = 0$
- *Weakly ionized*: $f_n \approx \frac{1}{\varepsilon^2} \gg f_e = f, f_i \approx 1$
- *Electron scales decoupled from heavies*: Study only f_e
 $f_n = M_n(f_n)$ and $f_i = M_i(f_i)$ are fixed and at equilibrium

¹Collab.: A. Alvarez Laguna [A15]

A cold plasma model¹

Nondimensionalized plasma : $p = e - n - i$

$$\partial_t f + \frac{1}{\varepsilon} \left(\text{div}_x(vf) - \text{div}_v((\nabla_x \phi)f) \right) = \frac{1}{\varepsilon^2} Q_c(f),$$

$$\Delta_x \phi = \frac{1}{\chi} \left(\rho^0 - \int_v f \right)$$

Collisions (Boltzmann type):

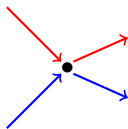
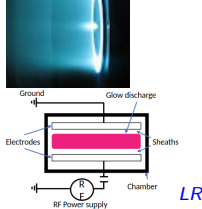
$$Q_c(f) = Q_{n,e}^{el}(f) + \varepsilon^2 (Q_{n,e}^{inel}(f) + Q_{e,e}(f) + Q_{i,n,e}^{iz}(f))$$

- *Balance equations* (collision invariants):

$$m_e \int_v \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} Q_{n,e}(f) + m_n \int_{v_n} \begin{pmatrix} 1 \\ v_n \\ v_n^2 \end{pmatrix} Q_{e,n}(f_n) = 0,$$

$$\int_v \begin{pmatrix} 1 \\ v \\ v^2 \end{pmatrix} Q_{e,e}(f) = 0, \quad \int_v \begin{pmatrix} 1 \\ v^2 \end{pmatrix} Q_{n,e}^{el}(f) = 0,$$

- *Entropy inequalities*



¹Collab.: A. Alvarez Laguna [A15]

A cold plasma model¹

Nondimensionalized plasma : $p = e - n - i$

$$\partial_t f + \frac{1}{\varepsilon} \left(\operatorname{div}_x(vf) - \operatorname{div}_v((\nabla_x \phi)f) \right) = \frac{1}{\varepsilon^2} Q_e(f),$$

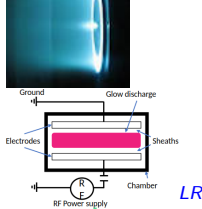
$$\Delta_x \phi = \frac{1}{\chi} \left(\rho^0 - \int_v f \right)$$

Collisions (Boltzmann type):

$$Q_e(f) = Q_{n,e}^{el}(f) + \varepsilon^2 \left(Q_{n,e}^{inel}(f) + Q_{e,e}(f) + Q_{i,n,e}^{iz}(f) \right)$$

- *Balance equations*
- *Entropy inequalities*: $\eta(f) = f(\log f - 1)$ and $\eta'(f) = \log f$

$$\int_v \eta'(f) Q_{n,e}(f) + \int_{v_n} \eta'(f_n) Q_{e,n}(f_n) \leq 0,$$
$$\int_v \eta'(f) Q_{e,e}(f) \leq 0$$



¹Collab.: A. Alvarez Laguna [A15]

A cold plasma model¹

Nondimensionalized plasma : $p = e - n - i$

$$\partial_t f + \frac{1}{\varepsilon} \left(\operatorname{div}_x(vf) - \operatorname{div}_v((\nabla_x \phi)f) \right) = \frac{1}{\varepsilon^2} Q_e(f),$$

$$\Delta_x \phi = \frac{1}{\chi} \left(\rho^0 - \int_v f \right)$$

Collisions (Boltzmann type):

$$Q_e(f) = Q_{n,e}^{el}(f) + \varepsilon^2 \left(Q_{n,e}^{inel}(f) + Q_{e,e}(f) + Q_{i,n,e}^{iz}(f) \right)$$

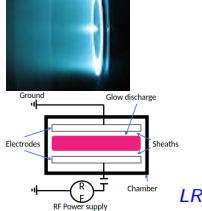
- Elastic dominant operator:

Balance :
$$\int_v \left(\frac{1}{v^2} \right) Q_{n,e}^{el}(f) = 0,$$

Expand :
$$Q_{n,e}^{el}(f) = Q_{n,e}^{0,el} + \varepsilon Q_{n,e}^{1,el} + \varepsilon^2 Q_{n,e}^{2,el} + \dots$$

with
$$Q_{n,e}^{0,el}(f)(v) = \nu(|v|) \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} f(|v|\omega) d\omega - f(v) \right), \quad Q_{n,e}^{1,el}(f) = 0 \dots$$

i.e. $Q_{n,e}^{0,el}$ is a projector onto the isotropic distributions



¹Collab.: A. Alvarez Laguna [A15]

A cold plasma model¹

Nondimensionalized plasma: $p = e - n - i$

$$\partial_t f + \frac{1}{\varepsilon} \left(\operatorname{div}_x(vf) - \operatorname{div}_v((\nabla_x \phi)f) \right) = \frac{\nu}{\varepsilon^2} \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} f - f \right) + R(f),$$
$$\Delta_x \phi = \frac{1}{\chi} \left(\rho^0 - \int_v f \right)$$

Contribution

- *Identification the dominant physical phenomena*
- **Reduced** models from a **physically relevant** dimensional analysis
- Relevant structure **adapted** to the following techniques

Difficulties

- *Modified entropic structure*
- *Remaining small parameters*

¹Collab.: A. Alvarez Laguna [A15]

- ① Kinetic models
- ② Moment-based approximations
- ③ Expansion-based approximations
- ④ Numerical methods for multi-scale problems

① Kinetic models

② Moment-based approximations

③ Expansion-based approximations

④ Numerical methods for multi-scale problems

A weak formulation² in 1D ($v \in \mathbb{R}$)

Weak formulation of the **1D toy problem**: $f(t, x, \cdot)$

$$\partial_t f + \partial_x(vf) = Q(f)$$

Definition

- Test functions

$$\mathcal{E} = \left\{ \phi(v) = \sum_{i \in \mathbb{N}} \mathbf{w}_i v^i \quad \text{s.t.} \quad |\phi(v)| < \infty \quad \forall v \in \mathbb{R} \right\}$$

such that

- $\mathcal{I} \equiv \text{Span}(1, v, v^2) \subset \mathcal{E}$
- Stable by Galilean transformation^a

$$v \mapsto \phi(v + u) \in \mathcal{E}$$

- Stable by multiplication by v

$$v \mapsto v\phi(v) \in \mathcal{E}$$

- Solution space $\mathbf{f}(t, x, \cdot) \in \mathcal{E}'$

Contribution: ²[W3]

Bibliography: ^aM. Junk, A. Unterreiter (2002)

A weak formulation² in 1D ($v \in \mathbb{R}$)

Definition

- Test functions

$$\mathcal{E} = \left\{ \phi(v) = \sum_{i \in \mathbb{N}} \mathbf{w}_i v^i \quad \text{s.t.} \quad |\phi(v)| < \infty \quad \forall v \in \mathbb{R} \right\}$$

- $\mathbf{f} \in \mathcal{E}'$
- \mathbf{f} is **positive**, if $\mathbf{f}[\phi] \geq 0$ for all $\phi(v) \geq 0$ a.e. $v \in \mathbb{R}$

Proposition (Riesz - Haviland (1935))

\mathbf{f} is **positive** iff $\exists \mu \in \mathcal{M}(\mathbb{R})_+$ such that

$$\mathbf{f}[\phi] = \int_{\mathbb{R}} \phi(v) d\mu(v) \quad \forall \phi \in \mathcal{E}$$

\Leftrightarrow A distribution function \mathbf{f} is a **positive measure with moments**

Contribution: ²[W3]

Bibliography: ^aM. Junk, A. Unterreiter (2002)

A weak formulation² in 1D ($v \in \mathbb{R}$)

Definition

- Test functions

$$\mathcal{E} = \left\{ \phi(v) = \sum_{i \in \mathbb{N}} \mathbf{w}_i v^i \quad \text{s.t.} \quad |\phi(v)| < \infty \quad \forall v \in \mathbb{R} \right\}$$

- $\mathbf{f} \in \mathcal{E}'$
- \mathbf{f} is **positive**, if $\mathbf{f}[\phi] \geq 0$ for all $\phi(v) \geq 0$ a.e. $v \in \mathbb{R}$

Proposition (Hamburger (1920) - Curto-Fialkow (1991))

Denote $\mathbf{b}_N(v) = (1, v, \dots, v^N)^T$. Then \mathbf{f} is **positive** iff:

- $\mathbf{f}[\mathbf{b}_N \mathbf{b}_N^T]$ are symmetric positive definite for all N , OR
- - $\exists J \in \mathbb{N}$ s.t. $\mathbf{f}[\mathbf{b}_J \mathbf{b}_J^T]$ is symmetric positive definite and
- $\text{Rank}(\mathbf{f}[\mathbf{b}_N \mathbf{b}_N^T]) = J$ for all $N \geq J$

$$\begin{pmatrix} f^0 & f^1 & \dots \\ f^1 & f^2 & \dots \\ \vdots & & \ddots \end{pmatrix}$$

\Leftrightarrow A distribution function \mathbf{f} is a **positive measure with moments**

\Leftrightarrow **Characterized by its sequence of moments** satisfying (HCF)

Contribution: ²[W3]

Bibliography: ^aM. Junk, A. Unterreiter (2002)

A weak formulation² in 1D ($v \in \mathbb{R}$)

Definition

- Test functions

$$\mathcal{E} = \left\{ \phi(v) = \sum_{i \in \mathbb{N}} \mathbf{w}_i v^i \quad \text{s.t.} \quad |\phi(v)| < \infty \quad \forall v \in \mathbb{R} \right\}$$

- $\mathbf{f} \in \mathcal{E}'$
- \mathbf{f} is positive, if $\mathbf{f}[\phi] \geq 0$ for all $\phi(v) \geq 0$ a.e. $v \in \mathbb{R}$

Proposition (Cramér (1938))

If $\mathbf{V} \in \mathbb{R}^{\mathbb{N}}$ satisfies *HCF criteria* and

$$\exists c \in \mathbb{R}, \quad \mathbf{V}_{2n} \leq (2n!)c^n.$$

Then, the representative measure $\mu \in \mathcal{M}(\mathbb{R})_+$ of \mathbf{V} is unique and

$$\int_v \exp(\alpha^{-1}|v|) d\mu(v) < \infty \quad \text{for all} \quad \alpha \geq \sqrt{c}.$$

- \Leftrightarrow Represented by a measure s.t. an exponential is L^1_μ
- \Leftrightarrow Represented by a sequence $\mathbf{V} \in \mathbb{R}$ satisfying (HCF-Cramér)

Contribution: ²[W3]

Bibliography: ^aM. Junk, A. Unterreiter (2002)

Reformulated kinetic framework

Classical kinetic formulation

$$\partial_t f + \partial_x(vf) = Q(f)$$

Definition

- **Transport operator:** Define

$$S : \phi \mapsto \{v \mapsto v\phi(v)\}, \quad S : \mathbf{V} \mapsto (\mathbf{V}_{i+1})_{i \in \mathbb{N}},$$

such that for $\phi = \sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{E}$, the flux is

$$\int_v v\phi(v) d\mu(v) \equiv \mathfrak{f}[S(\phi)] \equiv \sum_{i=0}^{\infty} \mathbf{W}_i S(\mathbf{V})_i$$

Reformulated kinetic framework

Classical kinetic formulation

$$\partial_t f + \partial_x(vf) = Q(f)$$

Definition

- **Transport operator:** Define $S : \mathbf{V} \mapsto (\mathbf{V}_{i+1})_{i \in \mathbb{N}}$
- **Collision operator:**

- *Collision invariants:* For all $\mu \equiv \mathbf{f} \equiv \mathbf{V}$, then $\phi(v) = \sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{I}$ s.t.

$$\int_v \phi(v) dQ(\mu)(v) \equiv Q(\mathbf{f})[\phi] \equiv \sum_{i=0}^{\infty} \mathbf{W}_i Q(\mathbf{V})_i = 0$$

Reformulated kinetic framework

Definition

- **Transport operator:** Define $S : \mathbf{V} \mapsto (\mathbf{V}_{i+1})_{i \in \mathbb{N}}$
- **Collision operator:**

- *Collision invariants:* For all $\mu \equiv \mathbf{f} \equiv \mathbf{V}$, then $\phi(v) = \sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{I}$ s.t.

$$\int_v \phi(v) dQ(\mu)(v) \equiv Q(\mathbf{f})[\phi] \equiv \sum_{i=0}^{\infty} \mathbf{W}_i Q(\mathbf{V})_i = 0$$

- *H-theorem:* Strictly convex function $H \equiv \mathbf{H} \equiv \mathbf{H}$
+ Entropy dissipation: $\forall \mu \equiv \mathbf{f} \equiv \mathbf{V}$

$$dH(\mu)(Q(\mu)) \equiv d\mathbf{H}(\mathbf{f})(Q(\mathbf{f})) \equiv d\mathbf{H}(\mathbf{V})(Q(\mathbf{V})) \leq 0$$

+ Equilibrium:

$$dH(\mu)(Q(\mu)) \equiv d\mathbf{H}(\mathbf{f})(Q(\mathbf{f})) \equiv d\mathbf{H}(\mathbf{V})(Q(\mathbf{V})) = 0 \quad \Leftrightarrow \quad \mu \equiv \mathbf{f} \equiv \mathbf{V} \equiv M$$

Reformulated kinetic framework

Definition

- **Transport operator:** Define $S : \mathbf{V} \mapsto (\mathbf{V}_{i+1})_{i \in \mathbb{N}}$
- **Collision operator:**

- *Collision invariants:* For all $\mu \equiv \mathbf{f} \equiv \mathbf{V}$, then $\phi(v) = \sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{I}$ s.t.

$$\int_{\mathbf{v}} \phi(v) dQ(\mu)(v) \equiv Q(\mathbf{f})[\phi] \equiv \sum_{i=0}^{\infty} \mathbf{W}_i Q(\mathbf{V})_i = 0$$

- *H-theorem:* Strictly convex function $H \equiv \mathbf{H} \equiv \mathbf{H}$
+ Entropy dissipation: $\forall \mu \equiv \mathbf{f} \equiv \mathbf{V}$

$$dH(\mu)(Q(\mu)) \equiv dH(\mathbf{f})(Q(\mathbf{f})) \equiv d\mathbf{H}(\mathbf{V})(Q(\mathbf{V})) \leq 0$$

+ Equilibrium:

$$dH(\mu)(Q(\mu)) \equiv dH(\mathbf{f})(Q(\mathbf{f})) \equiv d\mathbf{H}(\mathbf{V})(Q(\mathbf{V})) = 0 \quad \Leftrightarrow \quad \mu \equiv \mathbf{f} \equiv \mathbf{V} \equiv M$$

Remark: If μ has a density f , then

$$Q(\mu) \equiv Q(f), \quad M(\mu) \equiv M(f) = (\eta^*)'(\tilde{\phi})$$

$$H(\mu) \equiv \int_{\mathbf{v}} \eta(f(v)) dv \quad \text{s.t.} \quad dH(\mu)(Q(\mu)) \equiv \int_{\mathbf{v}} \eta'(f) Q(f) dv$$

Galerkin framework²

Reformulated kinetic equation: Seek μ , \mathbf{f} , \mathbf{V} s.t.

$\forall \psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$ and $\phi(v) = \sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{E}$, then:

$$0 = \int_{\mathbb{R}^+ \times \mathbb{R}} \int_v (\phi(v) \partial_t \psi + v(v) \phi(v) \partial_x \psi) d\mu(v) + \int_v \phi(v) dQ(\mu)(v) \psi dt dx$$

$$0 = \int_{\mathbb{R}^+ \times \mathbb{R}} (\mathbf{f}[\phi] \partial_t \psi + \mathbf{f}[\mathbf{S}\phi] \partial_x \psi + \mathbf{Q}(\mathbf{f})[\phi] \psi) (t, x) dt dx$$

$$0 = \int_{\mathbb{R}^+ \times \mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i (\mathbf{V}_i \partial_t \psi + \mathbf{V}_{i+1} \partial_x \psi + \mathbf{Q}(\mathbf{V})_i \psi) dt dx$$

²[W3]

Galerkin framework²

Reformulated kinetic equation: Seek \mathbf{V} s.t.

$\forall \psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$ and $\sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{E}$, then:

$$0 = \int_{\mathbb{R}^+ \times \mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i (\mathbf{V}_i \partial_t \psi + \mathbf{V}_{i+1} \partial_x \psi + \mathbf{Q}(\mathbf{V})_i \psi) dt dx$$

Galerkin framework:

- Restrict the **test function space** \mathcal{E} to \mathcal{E}_N of dimension N
- Restrict the **solution space** \mathcal{E}' to \mathcal{E}'_N of dimension N

²[W3]

Galerkin framework²

Reformulated kinetic equation: Seek \mathbf{V} s.t.

$\forall \psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$ and $\sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{E}$, then:

$$0 = \int_{\mathbb{R}^+ \times \mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i (\mathbf{V}_i \partial_t \psi + \mathbf{V}_{i+1} \partial_x \psi + \mathbf{Q}(\mathbf{V})_i \psi) dt dx$$

Galerkin framework:

- Restrict the **test function space** \mathcal{E} to $\mathbb{P}_{N-1} \supset \mathcal{I}$
- Define an **approximation** parametrized by N quantities

²[W3]

Galerkin framework²

Reformulated kinetic equation: Seek \mathbf{V} s.t.

$\forall \psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$ and $\sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{E}$, then:

$$0 = \int_{\mathbb{R}^+ \times \mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i (\mathbf{V}_i \partial_t \psi + \mathbf{V}_{i+1} \partial_x \psi + \mathbf{Q}(\mathbf{V})_i \psi) dt dx$$

Galerkin framework:

- Restrict the **test function space** \mathcal{E} to $\mathbb{P}_{N-1} \supset \mathcal{I}$
- Define an **approximation** parametrized by the N first moments \mathbf{V}^N

Moment system: Seek \mathbf{V}^N s.t. $\forall \psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$ and $i = 0, \dots, N-1$:

$$\int_{\mathbb{R}^+ \times \mathbb{R}} \mathbf{V}_i^N \partial_t \psi + \mathbf{V}_{i+1}^N \partial_x \psi + \mathbf{Q}_i^N \psi = 0$$

²[W3]

Galerkin framework²

Reformulated kinetic equation: Seek \mathbf{V} s.t.

$\forall \psi \in C_c^1(\mathbb{R}^+ \times \mathbb{R})$ and $\sum_{i=0}^{\infty} \mathbf{W}_i v^i \in \mathcal{E}$, then:

$$0 = \int_{\mathbb{R}^+ \times \mathbb{R}} \sum_{i=0}^{\infty} \mathbf{W}_i (\mathbf{V}_i \partial_t \psi + \mathbf{V}_{i+1} \partial_x \psi + \mathbf{Q}(\mathbf{V})_i \psi) dt dx$$

Galerkin framework:

- Restrict the **test function space** \mathcal{E} to $\mathbb{P}_{N-1} \supset \mathcal{I}$
- Define an **approximation** $\mathbf{F}^N, \mathbf{Q}^N$ parametrized by \mathbf{V}^N

Moment system of balance laws:

$$\partial_t \mathbf{V}^N + \partial_x \mathbf{F}^N(\mathbf{V}^N) = \mathbf{Q}^N(\mathbf{V}^N)$$

Contribution

Development of a framework² that

- *encompasses kinetic and fluid*
- *allows convergence/error analysis towards kinetic*

²[W3]

Closure problem

Moment balance laws:

$$\partial_t \mathbf{V} + \partial_x \mathbf{F}(\mathbf{V}) = \mathbf{Q}(\mathbf{V})$$

- *Transport* \rightarrow **Weak hyperbolicity:** $Sp(\mathbf{F}_V(\mathbf{V})) \subset \mathbb{R}$
 \hookrightarrow Turns $v \in \mathbb{R}$ into $v \in Sp(\mathbf{F}_V(\mathbf{V}))$
- *Collisions*

Closure problem

Moment balance laws:

$$\partial_t \mathbf{V} + \partial_x \mathbf{F}(\mathbf{V}) = \mathbf{Q}(\mathbf{V})$$

- *Transport* \rightarrow **Weak hyperbolicity:** $Sp(\mathbf{F}_V(\mathbf{V})) \subset \mathbb{R}$
 \hookrightarrow Turns $v \in \mathbb{R}$ into $v \in Sp(\mathbf{F}_V(\mathbf{V}))$
- *Collisions*
 - **Entropy dissipation:**
 - + $\exists H$ strictly convex s.t. $S(\mathbf{V}) = H_V(\mathbf{V}) \quad \mathbf{Q}(\mathbf{V}) \leq 0$
 - + Entropy flux G s.t. $H_V(\mathbf{V}) \quad \mathbf{F}_V(\mathbf{V}) = G_V(\mathbf{V})$
 - \hookrightarrow Correspondance $\int_v \eta(f)$ and $H(\mathbf{V})$?

Closure problem

Moment balance laws:

$$\partial_t \mathbf{V} + \partial_x \mathbf{F}(\mathbf{V}) = \mathbf{Q}(\mathbf{V})$$

- *Transport* \rightarrow **Weak hyperbolicity**: $Sp(\mathbf{F}_V(\mathbf{V})) \subset \mathbb{R}$
 \hookrightarrow Turns $v \in \mathbb{R}$ into $v \in Sp(\mathbf{F}_V(\mathbf{V}))$



- *Collisions*
 - **Symmetrizability**: \exists diffeomorphism $\mathbf{U}(\mathbf{V}) \equiv H_V(\mathbf{V})^T$ s.t.
 - + $\mathbf{U}(\mathbf{V})^T \mathbf{Q}(\mathbf{V}) \leq 0$
 - + $\mathbf{U}_V(\mathbf{V})$ is SPD
 - + $\mathbf{F}_V(\mathbf{V}) \mathbf{U}_V(\mathbf{V})^{-1}$ is symmetric

Closure problem

Moment balance laws:

$$\partial_t \mathbf{V} + \partial_x \mathbf{F}(\mathbf{V}) = \mathbf{Q}(\mathbf{V})$$

- *Transport* \rightarrow **Weak hyperbolicity:** $Sp(\mathbf{F}_V(\mathbf{V})) \subset \mathbb{R}$
 \hookrightarrow Turns $v \in \mathbb{R}$ into $v \in Sp(\mathbf{F}_V(\mathbf{V}))$

\Uparrow

- *Collisions*
 - **Symmetrizable:** \exists diffeomorphism $\mathbf{U}(\mathbf{V}) \equiv H_V(\mathbf{V})^T$ s.t.
 - + $\mathbf{U}(\mathbf{V})^T \mathbf{Q}(\mathbf{V}) \leq 0$
 - + $\mathbf{U}_V(\mathbf{V})$ is SPD
 - + $\mathbf{F}_V(\mathbf{V}) \mathbf{U}_V(\mathbf{V})^{-1}$ is symmetric
 - **Equilibrium:**

$$\mathbf{U}(\mathbf{V})^T \mathbf{Q}(\mathbf{V}) = 0 \quad \Leftrightarrow \quad \mathbf{V} = \mathbf{M}(\mathbf{V})$$

Remark: Correspondance $\mathbf{M}(\mathbf{V})_i$ and $\int_v v^i M(f)$?

Closure types

Flux closures³: $\mathbf{F}(\mathbf{V}) \equiv \int_{\mathcal{V}} \mathbf{v} \mathbf{b}_N(\mathbf{v}) f^R(\mathbf{v}) d\mathbf{v}$

- *Entropy-based*^{b,4}: $f^R(\mathbf{v}) = \operatorname{argmin}_{\tilde{\phi} \in \mathcal{E}_N} \int_{\mathcal{V}} \eta(\mathbf{g}) \equiv (\eta^*)'(\tilde{\phi})$ for $\tilde{\phi} \in \mathcal{E}_N$
 $\int_{\mathcal{V}} \phi(\mathbf{f} - \mathbf{g}) = 0$
 $\forall \phi \in \mathcal{E}_N$

↪ Symmetric dissipative, ill-posed, complex implementation

- *Quadrature-based*^{c,5}: $f^R(\mathbf{v}) = \sum_{i=1}^{N/2} m_i \delta_{\mathbf{v}_i}$

↪ Weakly hyperbolic, non-strictly convex entropy, efficient algorithms

- *Grad-type*^d: $f^R(\mathbf{v}) = \tilde{\phi} M(\mathbf{v}) \quad \tilde{\phi} \in \mathcal{E}$

↪ Non-positive, non-globally hyperbolic and entropic, trivial implementation

- *Realizability-based*: Interpolative⁶ or projective closures⁷

↪ Positive, various properties, complex extension to multi-D

Contributions: ³[C7], ⁴Collab. M. Abdelmalik, Z. Cai [A11,A12], ⁵Collab. F. Laurent [W5], ⁶[A3], ⁷[A7,A9]

Bibliography: ^bD. Levermore (1995), ^cR. McGraw (1997), R. Fox, F. Laurent et al, ^dH. Grad (1949), ^eP. Bhatnagar, E. Gross, M. Krook (1954)

Collision closures:

- *Consistent*: $\mathbf{Q} = \int_{\mathbf{v}} \mathbf{b}_N(\mathbf{v}) Q(f^R)(\mathbf{v}) d\mathbf{v}$
↪ No general properties, well-defined ?

- *Relaxation*^{e,5}: $\mathbf{Q} = \frac{\mathbf{M}(\mathbf{V}) - \mathbf{V}}{\tau}$
↪ Dissipative, odd approximation

Contributions: ³[C7], ⁴Collab. M. Abdelmalik, Z. Cai [A11,A12], ⁵Collab. F. Laurent [W5], ⁶[A3], ⁷[A7,A9]

Bibliography: ^bD. Levermore (1995), ^cR. McGraw (1997), R. Fox, F. Laurent et al, ^dH. Grad (1949), ^eP. Bhatnagar, E. Gross, M. Krook (1954)

- ① Kinetic models
- ② Moment-based approximations
- ③ Expansion-based approximations
- ④ Numerical methods for multi-scale problems

Generic (formal) expansions⁸

Key ingredients:

- Hilbertian structure
- Fredholm alternative

Need: Dominant linear(ized) operator L s.t. $\text{Ker}L \oplus \text{Im}L$

Construction: Expand

$$f = \sum_i \varepsilon^i f^i = \sum_i \varepsilon^i (f^{K,i} + f^{L,i})$$

⁸Further applications in Z. Tazakkati's PhD and [W1,W4,C8]

Generic (formal) expansions⁸

Key ingredients:

- Hilbertian structure
- Fredholm alternative

Need: Dominant linear(ized) operator L s.t. $\text{Ker}L \oplus \text{Im}L$

Construction: Expand $f = \sum_i \varepsilon^i f^i = \sum_i \varepsilon^i (f^{K,i} + f^{l,i})$

Hyperbolic scaling:

$$\partial_t f + \frac{1}{\varepsilon} Lf + R(f) = 0$$

Substitute and project: $f^0 = f^{K,0} \in \text{Ker}L$, $f^{l,0} = 0$

$$\begin{aligned} \Pi_{\text{Ker}L} \begin{cases} 0 & = \partial_t f^0 + R^{K,0}(f^0), \\ 0 & = \partial_t f^{K,i} + R^{K,i}(f^0, \dots, f^{l,i}, f^{K,i}) \end{cases} & i > 0, \\ \Pi_{\text{Im}L} \begin{cases} Lf^{l,1} & = -R^{l,0}(f^0), \\ Lf^{l,i+1} & = -(\partial_t f^{l,i} + R^{l,i}(f^0, \dots, f^{l,i}, f^{K,i})) \end{cases} & i > 0, \end{aligned}$$

⁸Further applications in Z. Tazakkati's PhD and [W1,W4,C8]

Generic (formal) expansions⁸

Key ingredients:

- Hilbertian structure
- Fredholm alternative

Need: Dominant linear(ized) operator L s.t. $\text{Ker}L \oplus \text{Im}L$

Construction: Expand

Parabolic scaling:

$$\partial_t f + \frac{1}{\varepsilon} Af + \frac{1}{\varepsilon^2} Lf + R(f) = 0$$

Substitute and project: $f^0 = f^{K,0} \in \text{Ker}L$, $f^{l,0} = 0$

$$\begin{cases} 0 = A^K f^0, \\ 0 = \partial_t f^0 + A^K (f^{l,1} + f^{K,1}) + R^{K,0}(f^0), \\ 0 = \partial_t f^{K,i} + A^K (f^{l,i+1} + f^{K,i+1}) + R^{K,i}(f^0, \dots, f^{l,i}, f^{K,i}) \quad i > 0, \end{cases}$$
$$\begin{cases} Lf^{l,1} = -A^l f^0, \\ Lf^{l,2} = -\left(A^l (f^{l,1} + f^{K,1}) + R^{l,0}(f^0)\right), \\ Lf^{l,i+2} = -\left(\partial_t f^{l,i} + A^l (f^{l,i+1} + f^{K,i+1}) + R^{l,i}(f^0, \dots, f^{l,i}, f^{K,i})\right) \quad i > 0 \end{cases}$$

⁸Further applications in Z. Tazakkati's PhD and [W1,W4,C8]

Application to cold plasmas¹

Parabolic scaling:

$$\partial_t f + \frac{1}{\varepsilon} (\operatorname{div}_x(vf) - \operatorname{div}_v((\nabla_x \phi)f)) = \frac{1}{\varepsilon^2} Q_{n,c}^{0,el}(f) + R(f)$$

$$Lf = Q_{n,c}^{0,el}(f) = \nu(|v|) \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} f(|v|\omega) d\omega - f(v) \right) \equiv -\nu \Pi_{\operatorname{Ker} L} f$$

s.t.

$$\operatorname{Ker} L = \left\{ f \mid f(v) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(|v|\omega) d\omega \right\} \oplus \operatorname{Im} L = \left\{ f \mid \int_{\mathbb{S}^2} f(|v|\omega) d\omega = 0 \right\}$$

¹Collab. A. Alvarez Laguna [A15]

Application to cold plasmas¹

Parabolic scaling:

$$\partial_t f + \frac{1}{\varepsilon} (\operatorname{div}_x(vf) - \operatorname{div}_v((\nabla_x \phi)f)) = \frac{1}{\varepsilon^2} Q_{n,c}^{0,el}(f) + R(f)$$

$$Lf = Q_{n,c}^{0,el}(f) = \nu(|v|) \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} f(|v|\omega) d\omega - f(v) \right) \equiv -\nu \Pi_{\operatorname{Ker} L} f$$

s.t.

$$\operatorname{Ker} L = \left\{ f \mid f(v) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(|v|\omega) d\omega \right\} \oplus \operatorname{Im} L = \left\{ f \mid \int_{\mathbb{S}^2} f(|v|\omega) d\omega = 0 \right\}$$

Expansion: $f^0(v) = f^0(|v|)$ isotropic

$$0 = \partial_t f^0 + \frac{1}{4\pi} \int_{\mathbb{S}^2} (\operatorname{div}_x(|v|\omega f^{l,1}) - \operatorname{div}_v(Ef^{l,1}) - R(f^0)),$$

$$Lf^{l,1} = - \left(\operatorname{Id} - \frac{1}{4\pi} \int_{\mathbb{S}^2} \right) (\operatorname{div}_x(|v|\omega f^0)) - \operatorname{div}_v((\nabla_x \phi) f^0)$$

Application to cold plasmas¹

Parabolic scaling:

$$\partial_t f + \frac{1}{\varepsilon} (\operatorname{div}_x(vf) - \operatorname{div}_v((\nabla_x \phi)f)) = \frac{1}{\varepsilon^2} Q_{n,c}^{0,el}(f) + R(f)$$

$$Lf = Q_{n,c}^{0,el}(f) = \nu(|v|) \left(\frac{1}{4\pi} \int_{\mathbb{S}^2} f(|v|\omega) d\omega - f(v) \right) \equiv -\nu \Pi_{\operatorname{Ker} L} f$$

s.t.

$$\operatorname{Ker} L = \left\{ f \mid f(v) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(|v|\omega) d\omega \right\} \oplus \operatorname{Im} L = \left\{ f \mid \int_{\mathbb{S}^2} f(|v|\omega) d\omega = 0 \right\}$$

$$\textbf{Moments: } p^{(2l)}(f) = \int_v |v|^{2l} f(v) dv, \quad \Gamma^{(2l+1)}(f) = \int_v |v|^{2l} v f(v) dv$$

$$\partial_t p^{(2l)}(f^0) + \operatorname{div}_x \Gamma^{(2l+1)}(f^{l,1}) - 2l \nabla_x \phi^T \Gamma^{(2l-1)}(f^{l,1}) = p^{(2l)}(R(f^0)),$$

$$\Gamma^{(2l+1)}(\nu f^{l,1}) = \frac{1}{3} \nabla_x p^{(2l+2)}(f^0) + (\nabla_x \phi) \frac{2l+3}{3} p^{(2l)}(f^0)$$

Application to cold plasmas¹

Decomposition

$$\text{Ker } L = \left\{ f \mid f(v) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(|v|\omega) d\omega \right\} \oplus \text{Im } L = \left\{ f \mid \int_{\mathbb{S}^2} f(|v|\omega) d\omega = 0 \right\}$$

$$\text{Moments: } p^{(2l)}(f) = \int_v |v|^{2l} f(v) dv, \quad \Gamma^{(2l+1)}(f) = \int_v |v|^{2l} v f(v) dv$$

$$\partial_t p^{(2l)}(f^0) + \text{div}_x \Gamma^{(2l+1)}(f^{l,1}) - 2|\nabla_x \phi|^T \Gamma^{(2l-1)}(f^{l,1}) = p^{(2l)}(R(f^0)),$$

$$\Gamma^{(2l+1)}(\nu f^{l,1}) = \frac{1}{3} \nabla_x p^{(2l+2)}(f^0) + (\nabla_x \phi) \frac{2l+3}{3} p^{(2l)}(f^0)$$

Need closure terms:

- Flux closure $\nu = \text{cste}$ s.t. $\Gamma^{(2l+1)}(\nu f^{l,1}) = \nu \Gamma^{(2l+1)}(f^{l,1})$
- Collision closure $p^{(2l)}(R(f^0)) = \text{relaxation?}$
- Diffusion (moment) closure $p^{(2L+2)}$ as a function of $(p^{(2l)})_{l=0,\dots,L}$

¹Collab. A. Alvarez Laguna [A15]

Application to cold plasmas¹

Decomposition

$$\text{Ker } L = \left\{ f \mid f(v) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(|v|\omega) d\omega \right\} \oplus \text{Im } L = \left\{ f \mid \int_{\mathbb{S}^2} f(|v|\omega) d\omega = 0 \right\}$$

$$\text{Moments: } p^{(2l)}(f) = \int_{\mathbb{V}} |v|^{2l} f(v) dv, \quad \Gamma^{(2l+1)}(f) = \int_{\mathbb{V}} |v|^{2l} v f(v) dv$$

$$\partial_t p^{(2l)}(f^0) + \text{div}_x \Gamma^{(2l+1)}(f^{l,1}) - 2|\nabla_x \phi|^T \Gamma^{(2l-1)}(f^{l,1}) = p^{(2l)}(R(f^0)),$$

$$\Gamma^{(2l+1)}(\nu f^{l,1}) = \frac{1}{3} \nabla_x p^{(2l+2)}(f^0) + (\nabla_x \phi) \frac{2l+3}{3} p^{(2l)}(f^0)$$

Need closure terms:

- Flux closure $\nu = \text{cste}$ s.t. $\Gamma^{(2l+1)}(\nu f^{l,1}) = \nu \Gamma^{(2l+1)}(f^{l,1})$
- Collision closure $p^{(2l)}(R(f^0)) = \text{relaxation?}$
- Diffusion (moment) closure $p^{(2L+2)}$ as a function of $(p^{(2l)})_{l=0,\dots,L}$

Contribution

Macroscopic model adapted to the dominant L with **flexibility** for:

- Dissipativeness (closure)
- Accuracy (higher order expansion or moments)
- 3D model but monovariate moments w.r.t. $|v| \in \mathbb{R}^+$

¹Collab. A. Alvarez Laguna [A15]

- ① Kinetic models
- ② Moment-based approximations
- ③ Expansion-based approximations
- ④ Numerical methods for multi-scale problems

Position of the problems

General systems:

$$\begin{aligned}\partial_t \mathbf{U} + \alpha \operatorname{div}_x \mathbf{F}(\mathbf{U}) + \beta \operatorname{div}_x \mathbf{G}(\nabla_x \phi, \mathbf{U}) &= \gamma \mathbf{S}(\mathbf{U}) \\ \kappa \Delta_x \phi &= T(\mathbf{U})\end{aligned}$$

Difficulties

- Definition of \mathbf{F} , \mathbf{G} , \mathbf{S} and T on $\mathbf{U} \in \mathbb{R} \not\subseteq \mathbb{R}^N$
- Stability condition depending on $\Delta x, \Delta t$ and on $\alpha, \beta, \gamma, \kappa$

Position of the problems

General systems:

$$\begin{aligned}\partial_t \mathbf{U} + \alpha \operatorname{div}_x \mathbf{F}(\mathbf{U}) + \beta \operatorname{div}_x \mathbf{G}(\nabla_x \phi, \mathbf{U}) &= \gamma \mathbf{S}(\mathbf{U}) \\ \kappa \Delta_x \phi &= T(\mathbf{U})\end{aligned}$$

Difficulties

- Definition of \mathbf{F} , \mathbf{G} , \mathbf{S} and T on $\mathbf{U} \in \mathbb{R} \not\subseteq \mathbb{R}^N$
- Stability condition depending on $\Delta x, \Delta t$ and on $\alpha, \beta, \gamma, \kappa$

Strategies:

- High-order accuracy preserving stability^{KAA,GB}
- Asymptotic-preserving schemes^{AAL,LR}

Position of the problems

General systems:

$$\begin{aligned}\partial_t \mathbf{U} + \alpha \operatorname{div}_x \mathbf{F}(\mathbf{U}) + \beta \operatorname{div}_x \mathbf{G}(\nabla_x \phi, \mathbf{U}) &= \gamma \mathbf{S}(\mathbf{U}) \\ \kappa \Delta_x \phi &= T(\mathbf{U})\end{aligned}$$

Difficulties

- Definition of \mathbf{F} , \mathbf{G} , \mathbf{S} and T on $\mathbf{U} \in \mathbb{R}^{\subsetneq \mathbb{R}^N}$
- Stability condition depending on $\Delta x, \Delta t$ and on $\alpha, \beta, \gamma, \kappa$

Strategies:

- High order accuracy preserving stability^{KAA,GB}
- Asymptotic-preserving schemes^{AAL,LR}

Notions: Degenerating problem $\mathcal{P}^\epsilon \rightarrow \mathcal{P}^0$

- AP scheme: $\mathcal{S}^\epsilon \rightarrow \mathcal{S}^0$ consistent with \mathcal{P}^0
- Uniformly AP: Consistency error uniform w.r.t. ϵ
- Uniformly stable: Stability condition does not degenerate in the limit

Difficulties

Problem-dependent *structure of the limit system*

\leftrightarrow *Problem-dependent solutions*

Key ideas:

- Make the limit equation appear in the equivalent equation
- Write the stiff terms implicit

⁹PhD of L. Reboul+[W6,C6]

Diffusive limits

Key ideas:

- Make the limit equation appear in the equivalent equation
- Write the stiff terms implicit

Toy system: $\mathbf{U} = (E, F)^T$, $\mathbf{G}(\mathbf{U}) = (F, E)^T$, $B\mathbf{U} = (0, F)^T$

$$\partial_t \mathbf{U} + \frac{1}{\varepsilon} \partial_x \mathbf{G}(\mathbf{U}) = -\frac{\sigma}{\varepsilon^2} B\mathbf{U},$$

Diffusive limit $\varepsilon \rightarrow 0$:

$$\partial_t E - \partial_x(\sigma^{-1} \partial_x E) = 0$$

⁹PhD of L. Reboul+[W6,C6]

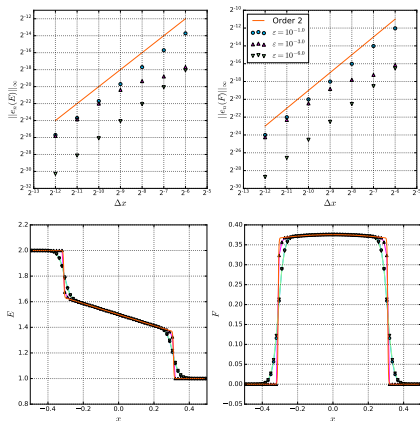
Diffusive limits

Toy system: $\mathbf{U} = (E, F)^T$,
 $\mathbf{G}(\mathbf{U}) = (F, E)^T$, $B\mathbf{U} = (0, F)^T$

$$\partial_t \mathbf{U} + \frac{1}{\varepsilon} \partial_x \mathbf{G}(\mathbf{U}) = -\frac{\sigma}{\varepsilon^2} B\mathbf{U},$$

Diffusive limit $\varepsilon \rightarrow 0$:

$$\partial_t E - \partial_x (\sigma^{-1} \partial_x E) = 0$$



Contribution

ImEx strategy⁹ for uniform 1st and 2nd order AP schemes and uniformly stable schemes, at the cost of an explicit scheme

⁹PhD of L. Reboul+[W6,C6]

Simplified system:

$$\begin{aligned} \partial_t n_e + \partial_x(n_e u_e) &= 0, \\ \partial_t(n_e u_e) + \partial_x\left(n_e u_e^2 + \frac{n_e}{\varepsilon}\right) &= -\frac{n_e}{\varepsilon} \partial_x \phi, \\ \partial_t n_i + \partial_x(n_i u_i) &= 0, \\ \partial_t(n_i u_i) + \partial_x(n_i u_i^2 + \kappa n_i) &= n_i \partial_x \phi, \\ & \partial_{xx} \phi = \frac{n_i - n_e}{\chi} \end{aligned}$$

Limits:

- $\varepsilon \rightarrow 0$: Balancing pressure with Lorentz force $\partial_x(\log n_e + \phi) = 0$
- $\chi \rightarrow 0$: Quasi-neutral limit $n_e = n_i$

¹⁰Collab. A. Alvarez Laguna [A6], extensions in the PhDs of L. Reboul and Z. Tazakkati + [C8,W4]

Simplified system:

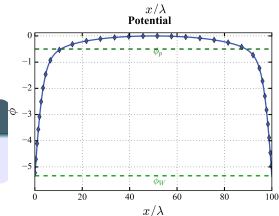
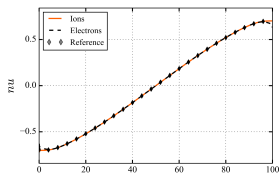
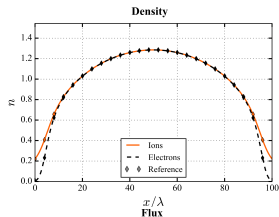
$$\begin{aligned}
 \partial_t n_e + \partial_x (n_e u_e) &= 0, \\
 \partial_t (n_e u_e) + \partial_x \left(n_e u_e^2 + \frac{n_e}{\varepsilon} \right) &= -\frac{n_e}{\varepsilon} \partial_x \phi, \\
 \partial_t n_i + \partial_x (n_i u_i) &= 0, \\
 \partial_t (n_i u_i) + \partial_x (n_i u_i^2 + \kappa n_i) &= n_i \partial_x \phi, \\
 \partial_{xx} \phi &= \frac{n_i - n_e}{\chi}
 \end{aligned}$$

Limits:

- $\varepsilon \rightarrow 0$: Balancing pressure with Lorentz force $\partial_x (\log n_e + \phi) = 0$
- $\chi \rightarrow 0$: Quasi-neutral limit $n_e = n_i$

Contribution

Development of a double AP scheme¹⁰



¹⁰Collab. A. Alvarez Laguna [A6], extensions in the PhDs of L. Reboul and Z. Tazakkati + [C8,W4]

Conclusion and perspectives

Main contributions:

- ① Kinetic
 - Derivation and reduction of kinetic equations for various applications
 - Reduction from kinetic to kinetic (gyrokinetic, isotropic ...)
- ② Moments
 - Framework for moment systems
 - Analysis of moment sets
 - Derivation of various moment-based approximations
- ③ Expansion
 - Generalize framework for expansion-based approximation
 - Derivation of various expansion-based approximations
- ④ Numerics
 - Derivation, analysis and implementation of high-order, well-balanced, asymptotic-preserving schemes adapted to specific models

Conclusion and perspectives

Key open problems:

① Kinetic

- Extensions and applications to other physics
- Rigorous derivation of 2-phase kinetic equations

② Moments

- Error estimates/convergence of moment-based approximations
- Kinetic-based boundary conditions of moment systems
- Numerical characterizations of multi-variate realizability domain
- Analysis of hyperbolic systems coupling equilibrium and nonequilibrium (similar to micro-macro)

③ Expansion

- Extension of the expansion-based analysis in a duality (moment) framework

④ Numerics

- Generalizing the AP ideas and clarifying the framework
- Well-posedness of some multi-D weakly hyperbolic moment systems