A multifractal continuous cascade model: The BDM model

From works of E. Bacry, J. Delour and J.F. Muzy

Baruch College

Thursday 17^{th} April 2014

Based on

- E. Bacry, J.F. Muzy, J. Delour, Multifractal Random Walks (2001).
- E. Bacry, J.F. Muzy, Log-infinitely divisible multifractal processes (2003).
- E.Bacry, A.Kozhemyak, J.F.Muzy, Log-Normal continuous cascades: aggregation properties and estimation. Application to financial time-series (2013).

Outline

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1. Multifractal processes
   - Data
   - Formal setting
2. Building the log normal MRM
3. A definition of more general MRM
4. The multifractal random walk

Fractals

- Something that is the same from near as from far.
- Hard to define (even for mathematicians).
- Mandelbrot: "Beautiful, damn hard, increasingly useful. That's fractals."

A first look at the data

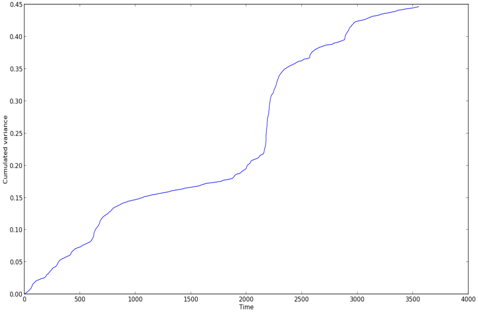


Figure : Cumulated variance: $V_t = \sum_{i=1}^{t} \sigma_i^2$ as a function of time.

A closer look

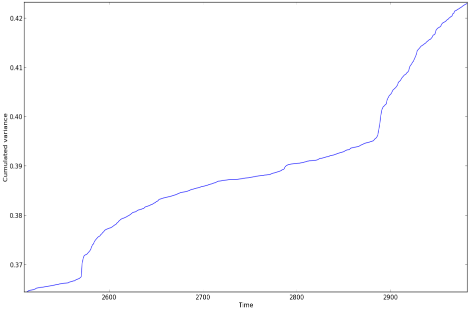


Figure : Cumulated variance: $V_t = \sum_{i=1}^{t} \sigma_i^2$ as a function of time.

An even closer look

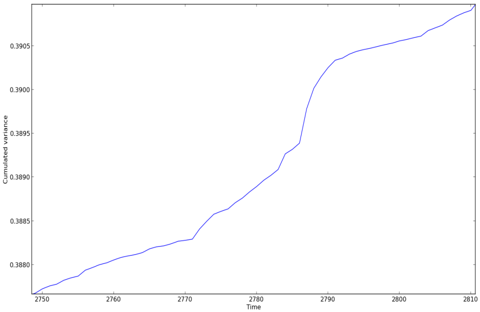


Figure : Cumulated variance: $V_t = \sum_{i=1}^{t} \sigma_i^2$ as a function of time.

Framework

We consider a process X whose increments $\delta_l X(t) = X(t+l) - X(t)$ are stationary and $X(0) = 0$.

Definition

X is stochastically scale invariant up to the time scale T (or multifractal) if for every $s \in [0,1]$:

$$\{X_{st}\}_{0 \le t \le T} =_L W_s \{X_t\}_{0 \le t \le T}$$

where W_s is a random variable independent of $\{X_t\}_{0 \le t \le T}$.

Example (Brownian motion)

$$\{B_{st}\}_{0 \le t \le T} =_L \sqrt{s} \{B_t\}_{0 \le t \le T}$$

Remarks on W_s

Proposition

W_s checks: $W_{s_1 s_2} =_L W_{s_1} W_{s_2}$ (with $W_{s_1} \perp W_{s_2}$) and thus

$$\mathbb{E}[|W_s|^q] = s^{\psi(q)}.$$

Proposition

Moreover, $\Omega_s = log(W_s)$ is infinitely divisible and thus

$$\mathbb{E}[|W_s|^q] = s^{\psi(q)}$$

where ψ is characterized by the Levy-Khintchine formula.

Moments of the increments of the cumulated variance

Let us denote the q^{th} moment of the t-increment

$$m(q,t) = \mathbb{E}[|\delta_t X(s)|^q] = \mathbb{E}[|X(t)|^q].$$

Proposition

If X is stochastically scale invariant up to the time scale T, then

$$m(q,t) = K_q t^{\zeta_q}$$

with $\zeta_q = \psi(q)$.

Empirical measures

Our measurements are coherent with this proposition:

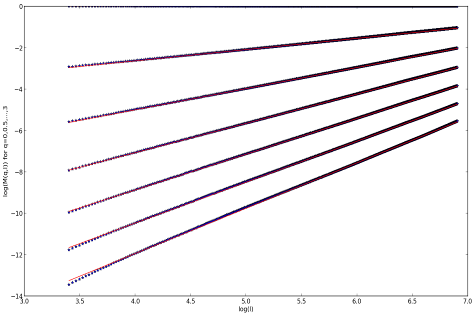


Figure : $Log(m(q,t))$ as a function of $log(t)$ for $q =$0, 0.5, ..., 3.

Empirical measures

Our measurements are coherent with this proposition:

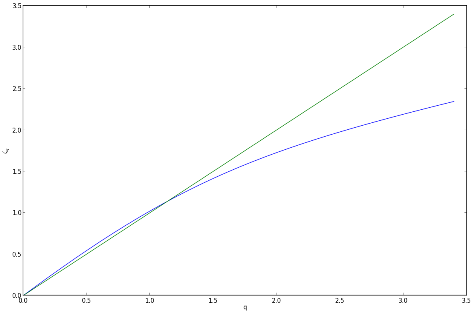


Figure : ζ_q as a function of q.

Example: The fractional Brownian motion

- The fractional Brownian motion of Hurst index H checks

$$\{B^H_{st}\}_{0\leq t} =_L s^H\{B^H_t\}$$

  and

$$\mathbb{E}[|B^H_t|^q] = K_q t^{qH}.$$

- The linearity of the scaling exponent is linked to the "deterministic scale invariance" of B^H.
- We want to have a stochastic scale invariance.
- We thus need a maximum time scale T.

Literature

Many phenomena exhibit multifractal properties

- Velocity field in turbulent flows (Kolmogorov 1962).
- Financial time series (Bouchaud Potters 2003).
- Geological shapes (Kanellopoulos and Megier 1995).
- Medical time series (West 1990).

Few models reproduce this multifractality

- Self similar processes (Taqqu and Samorodnisky 1994).
- The Mandelbrot multifractal (Mandelbrot 1974).

Outline

Intuition

The starting point of the model is the fact that the autocorrelation of the log volatility is a linear function of the logarithm of the time (Arneodo, Muzy and Sornette 1998).

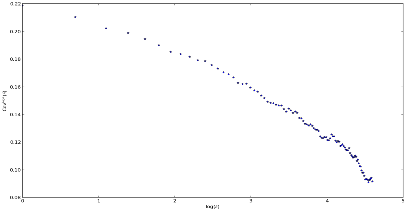


Figure : $Cov^{log(\sigma)}(\delta) = Cov(log(\sigma)(t+\delta), log(\sigma))$ as a function of $log(\delta)$.

For every $l > 0$, let us define $\{\omega_{l,T}(t)\}_t$ the Gaussian process (that will be the log volatility) such that

$$Cov(\omega_{l,T}(t), \omega_{l,T}(t+\tau)) = \begin{cases} \lambda^2 ln(\frac{T}{\tau}) & \text{if} \quad l \leq \tau \leq T \\ 0 & \text{if} \quad \tau \geq T \\ \lambda^2 (ln(\frac{T}{l}) + 1 - \frac{\tau}{l}) & \text{if} \quad l \leq \tau \leq T \end{cases}$$

and

$$\mathbb{E}[e^{2\omega_{l,T}(t)}] = 1.$$

Remark

$\omega_{l,T}$ is almost a Gaussian process of correlation function affine in $log(t)$.

Link with the fractional Bergomi model

- $\omega_{l,T}(t)$ will be the log-volatility.
- $\omega_{l,T}(t)$ can be written

$$\omega_{l,T}(t) = -\mathbb{E}[\omega_{l,T}(t)] + \int_{-\infty}^{t} K_{l,T}(t-s)dB_s$$

with $K_{l,T}(t) \sim \frac{K_0}{\sqrt{t}}$ for $l \ll t \ll T$.

- Therefore, can be seen as a fractional Bergomi model with H very small.

For every $l > 0$, let us define the random measure

$$M_{l,T}(t) = (\sigma^2) \int_0^t e^{2\omega_{l,T}(s)} ds.$$

Definition

The log-normal multi fractal random measure is defined as

$$M_T(t) = lim_{l \to 0} M_{l,T}(t).$$

Stochastic scale invariance

Theorem

M_T is stochastically scale invariant up to T

$$\{M_{st}^T\}_{0 \leq t \leq T} =_L W_s \{M_t^T\}_{0 \leq t \leq T}$$

and the renormalization random variable writes $W_s = s e^{\Omega_s}$ with Ω_s Gaussian such that

$$\mathbb{E}[\Omega_s] = -Var(\Omega_s)/2 = 2\lambda^2 log(s).$$

Proposition

$$m(q, t) = K_q t^{\zeta_q}$$

with $\zeta_q = q + 2\lambda^2(q - q^2)$.

Proof

We only need to show that

$$\{\omega_{sl}(st)\} = \{\Omega_s/2 + \omega_l(t)\}.$$

These are Gaussian processes, so we only need to show that

- $\mathbb{E}[\omega_{sl}(st)] = \mathbb{E}[\Omega_s]/2 + \mathbb{E}[\omega_l(t)]$.
- $Cov(\omega_{sl}(s(t+\tau)), \omega_{sl}(st)) = Var(\Omega_s^2)/4 + Cov(\omega_l(t), \omega_l(t+\tau))$.

Which is true.

Integral scale invariance and estimation of T

T might seem bothering. However:

Theorem

For $L \leq T$:

$$\{M_T(t)\}_{t \leq L} = W_{L/T}\{M_L(t)\}_{t \leq L}$$

Therefore, one needs to observe the process at higher time scales than T to estimate T. Setting a maximum multifractal scale T is in practice not a problem. One just needs to say than it is higher that the scale at which he observes the process.

Outline

We define:

- μ the measure on $\mathbb{R} \times \mathbb{R}_+^*$

$$\mu(dt, dl) = \frac{dtdl}{l^2}.$$

- P a Gaussian white noise on $\mathbb{R} \times \mathbb{R}_+^*$ such that for all $A \subset \mathbb{R} \times \mathbb{R}_+^*$ measurable

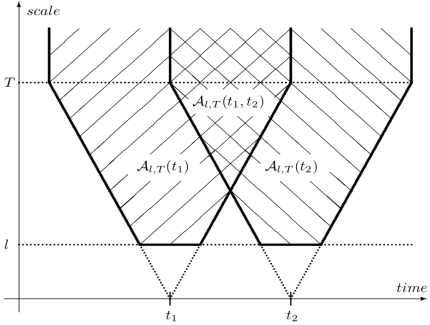
$$\mathbb{E}[e^{qP(A)}] = e^{\mu(A)\psi(q)}$$

  with $\psi(q) = 2\lambda^2(q^2 - q)$.

P should be seen as a Gaussian white noise on $\mathbb{R} \times \mathbb{R}_+^*$ of intensity λ^2/l^2 in (t, l).

Then denoting:

$$A_{l,T}(t) = \{(t', l'); l' \geq l \text{ and } |t - t'| \leq \frac{1}{2} min(l', T)\}.$$



We have the equality in law: $\omega_{l,T} =_L \frac{1}{2} P(A_{l,T})$.

More generally, if $\psi(q)$ is the characteristic exponent of an infinitely divisible random variable. Then there exists a random measure P on $\mathbb{R} \times \mathbb{R}_+^*$ such that for all $A \subset \mathbb{R} \times \mathbb{R}_+^*$ measurable

$$\mathbb{E}[e^{qP(A)}] = e^{\mu(A)\psi(q)}.$$

We define

$$\omega_{l,T} = \frac{1}{2} P(A_{l,T}),$$

$$M_{l,T}(t) = (\sigma^2) \int_0^t e^{2\omega_{l,T}(s)} ds$$

and

$$M_T(t) = lim_{l \to 0} M_{l,T}(t).$$

Properties

Proposition

M_T is stochastically scale invariant up to T and the renormalization random variable is log infinitely divisible

$$W_s = s e^{\Omega_s}$$

with Ω_s such that

$$\mathbb{E}[e^{q\Omega_s}] = K_q s^{\psi(q)}.$$

Proposition

Therefore,

$$\mathbb{E}[|M_T(t)|^q] = K_q t^{\zeta_q}$$

with $\zeta_q = q + \psi(q)$.

Outline

Definition

Definition

Once we have defined the MRM, we easily get the MRW by composing the MRM M^T and an independent Brownian motion B

$$X^T(t) = B(M^T(t))$$

Properties

Using that $\{B(at)\} =_L \sqrt{a}\{B(t)\}$, we get:

Proposition

X_T is stochastically scale invariant up to T and the renormalization random variable is log infinitely divisible

$$W_s^X = \sqrt{W_s^M}.$$

And thus:

Proposition

$\zeta^X(q) = \zeta^M(\frac{q}{2})$.

Comparison between empirical and simulated daily returns.

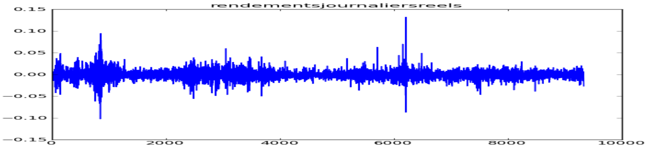


Figure : Empirical daily returns.

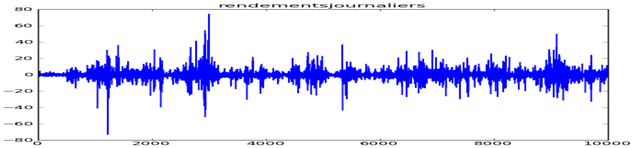


Figure : Simulated daily returns ($T = 500$).

Conclusion

- We have presented the construction of a class of stationary multifractal random measures.
- In a loose way this allows us to build any multifractal random measure.
- Applied to modelling market prices, these measures can be seen as the limit of fractional Bergomi models.
- It is possible to add skew in these models.

Thank you for your attention!