Strong error bounds for the conditional propagation of chaos for mean field systems of neurons

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1. Introduction
   - Point processes
   - Thinning
2. Model
   - Neural networks model
   - Definitions of the systems
   - Limit system
3. Propagation of chaos
   - Conditional propagation of chaos
   - First attempt of coupling
   - Formal proof

Point process : definitions

Point process (or counting process) Z :

- a random countable set of $\mathbb{R}_+$: $Z = \{T_i : i \in \mathbb{N}\}$
- a random point measure on $\mathbb{R}_+$: $Z = \sum_{i\in\mathbb{N}} \delta_{T_i}$

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A process λ is the **stochastic intensity** of Z if :

$$\forall 0 \leq a < b, \mathbb{E}\left[Z([a,b]) | \mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b \lambda_t dt \middle| \mathcal{F}_a\right]$$

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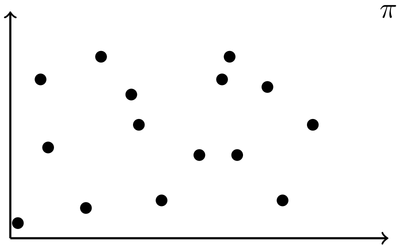
Marked point process $Z = \{(T_i, U_i) : i \in \mathbb{N}\}$ (U_i iid)

Notation abuse $U_i =: U(T_i)$

Thinning

π Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dt.dz$



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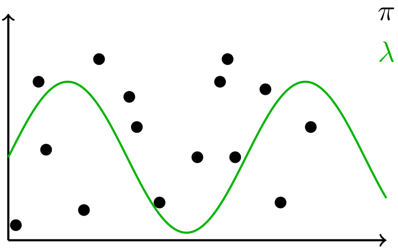

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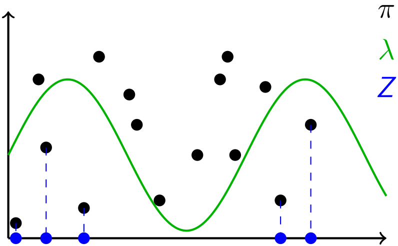

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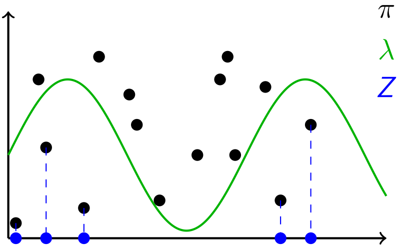

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Neural activity = Set of spike times

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Network of N neurons :

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Here, $X^{N,i}$ solves an SDE directed by $(Z^{N,j})_{1\leq j\leq N}$

Mean field limit

$N-$particle system :

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- linear scaling N^{-1} (LLN) :
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- diffusive scaling $N^{-1/2}$ (CLT) :
  [E. et al. (2022)] random and centered $u^{ji}(s)$

Diffusive scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_0^\infty \int_{\mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) - \int_0^\infty \int_{\mathbb{R}} X_{t-}^{N,i} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

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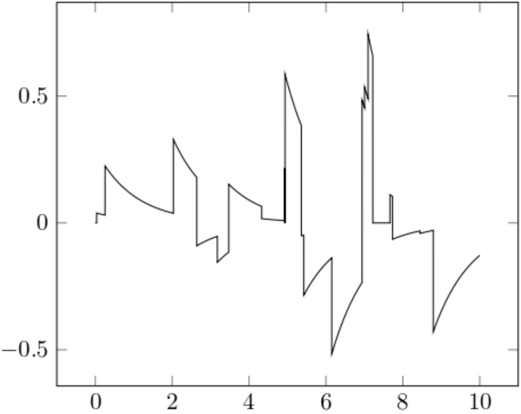
- $X_t^{N,i} = 0$ if neuron i emits a spike at t

$N-$neurons network dynamics

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} U^j(t) dZ_t^{N,j} - X_{t-}^{N,i} dZ_t^{N,i}$$



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$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + d\bar{M}_t$$

$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

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$$\langle \bar{M} \rangle_t = \lim_N \langle M^N \rangle_t = \lim_N \sigma^2 \int_0^t \frac{1}{N} \sum_{j=1}^{N} f(X_s^{N,j}) ds$$

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Then $\bar{M}$ should satisfy

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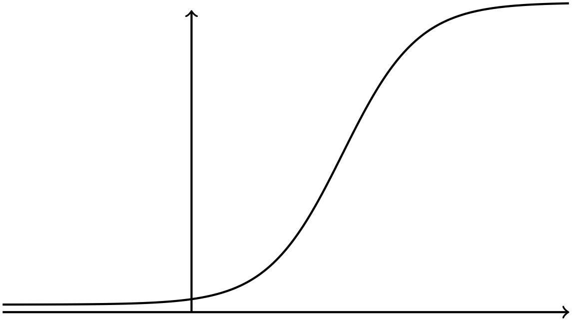

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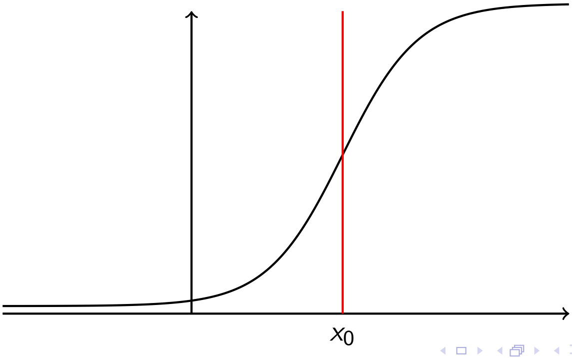

Convergence of $(X^{N,i})_{1\leq i\leq N}$

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Result [E., Löcherbach, Loukianova (2021)]

Given $N\in\mathbb{N}^*, (\pi^j)_{1\leq j\leq N}$, there exists a BM W^N such that

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_s^{N,1} - \bar{X}_s^{N,1}\right|\right] \leq C_t\frac{(\ln N)^{1/5}}{N^{1/10}}$$

Convergence of $(X^{N,i})_{1\leq i\leq N}$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_{\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\left\{z\leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$

$$- \int_{\mathbb{R}_+\times\mathbb{R}} X_{t-}^{N,i} \mathbb{1}_{\left\{z\leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

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Result [E., Löcherbach, Loukianova (2021)]

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Representation of Poisson processes

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} dP_t^N - X_{t-}^{N,i} dZ_t^{N,i}$$

with

$$P_t^N := \sum_{j=1}^N \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\left\{z \le f(X_{s-}^{N,j})\right\}} d\pi^j(s,z,u)$$

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$$A_t^N := \sum_{j=1}^{N} \int_0^t f(X_s^{N,j}) ds.$$

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Proposition 16.6.III [Daley & Vere-Jones (2008)] :

$$P_t^N = \mathbf{Z}_{A_t^N},$$

where $\mathbf{Z}$ = marked point process with rate 1

KMT coupling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} d\mathbf{Z}_{A_t^N} - X_{t-}^{N,i} dZ_t^{N,i}$$

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Corollary 7.5.5 [Ethier & Kurtz (2008)] ([Komlòs et al. (1976)]) : there exists BM B such that

$$\sup_{t \geq 0} \frac{|\mathbf{Z}_t - \sigma B_t|}{\ln(t \vee 2)} \leq E,$$

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Time-changed Brownian motion

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \sigma \frac{1}{\sqrt{N}} dB_{A_t^N} - X_{t-}^{N,i} dZ_t^{N,i} + \frac{\ln N}{\sqrt{N}}$$

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Proposition V.(3.8) [Revuz & Yor (1999)] :

$$B_{A_t^N} = \int_0^t \sqrt{\sum_{j=1}^N f(X_s^{N,j})} dW_s^N$$

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conditional LLN :

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Grönwall's lemma :

$$\mathbb{E}\left[\sup_{s\leq t}\left|X_s^{N,i} - \bar{X}_s^{N,i}\right|\right] \leq C_t \frac{\ln N}{\sqrt{N}}$$

Problems with the naïve coupling (1)

- $dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} d\mathbf{Z}_{A_t^N} - \int_{\mathbb{R}_+} X_{t-}^{N,i} \mathbb{1}_{\left\{z \le f(X_{t-}^{N,i})\right\}} d\bar{\pi}^i(t,z)$

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Problem 1 : W^N depends on B depends on $\mathbf{Z}$ depends on $(\bar{\pi}^j)_{j \geq 1}$

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Problem 1 : W^N depends on B depends on $\mathbf{Z}$ depends on $(\bar{\pi}^j)_{j\ge1}$
Solution :

- decoupling jump times and jump heights
- (jump heights,jump times) $\rightarrow$ $(B, \text{time-change})$

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Replace $\mathbf{Z}_{A_t^N}$ by $\mathbf{Z}'_{\mathbf{N}_t}$

Problems with the naïve coupling (2)

- $dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} d\mathbf{Z}'_{\mathbf{N}_t} - \int_{\mathbb{R}_+} X_{t-}^{N,i} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\bar{\pi}^i(t,z)$

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Problems with the naïve coupling (2)

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- Define B by KMT coupling : $|\mathbf{Z}'_n - \sigma B_n| \leq E \ln(n \vee 2)$
- Change of time

$$|B_{\mathbf{N}_t} - B_{A_t^N}| \leq ...$$

Problems with the naïve coupling (2)

- $dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} d\mathbf{Z}'_{\mathbf{N}_t} - \int_{\mathbb{R}_+} X_{t-}^{N,i} \mathbb{1}_{\left\{z \le f(X_{t-}^{N,i})\right\}} d\bar{\pi}^i(t,z)$
- Define B by KMT coupling : $|\mathbf{Z}'_n - \sigma B_n| \le E \ln(n \vee 2)$
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Problems :

- BMs are not Lipschitz
- **N** still depend on jump heights through its rate

Pseudo-Euler scheme (1)

Let $N \in \mathbb{N}^*, \delta = \delta(N) > 0$,

$$X_t^{N,1} = X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t]\times\mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\left\{z \leq f(X_{s-}^{N,1})\right\}} d\bar{\pi}^1(s,z)$$

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- $\mathbf{Z}^k_n = \sum_{l=1}^n U^k_l$ random walk (3rd variable of $\pi^j_{|[k\delta,(k+1)\delta[\times\mathbb{R}_+\times\mathbb{R}}$)

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$$X_t^{N,1} = X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t]\times\mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\left\{z \le f(X_{s-}^{N,1})\right\}} d\bar{\pi}^1(s,z) + \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta - 1} \mathbf{Z}^k_{\mathbf{N}^k_\delta} + R_t^1$$

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$$|R_t^1| \le \frac{1}{\sqrt{N}} \left| \sum_{j=1}^N \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u \left(\mathbb{1}_{\left\{z \le f(X^{N,j}_{\tau(s-)})\right\}} - \mathbb{1}_{\left\{z \le f(X^{N,j}_{s-})\right\}} \right) d\pi^j(s,z,u) \right|$$

with $\tau(s) = k\delta$ for $s \in [k\delta, (k+1)\delta[$

Error due to Euler scheme (1)

Lemmas

(1) For all $t \leq T$,

$$\mathbb{E}\left[|X_t^{N,1} - X_{\tau(t)}^{N,1}|\right] \leq C_T \delta^{1/2}$$

(2) For all $t \leq T$,

$$\mathbb{E}\left[|R_t^1|\right] \leq C_T \delta^{1/4}$$

Sketch of proof of (1) :

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Sketch of proof of (2) : by BDG inequality

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$$\mathbb{E}\left[|R_t^1|\right] \leq \sigma \left(\int_0^t \mathbb{E}\left[|f(X_s^{N,1}) - f(X_{\tau(s)}^{N,1})|\right] ds \right)^{1/2}$$

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KMT approximation

$$X_t^{N,1} = X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t]\times\mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\left\{z \le f(X_{s-}^{N,1})\right\}} d\bar{\pi}^1(s,z)$$
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$$\mathbb{E}\left[|R_t^2|\right] \le \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbb{E}\left[\ln(\mathbf{N}^k_\delta + 2) E^k\right]$$

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$$\leq C \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \ln(\mathbb{E}\left[\mathbf{N}^k_\delta\right] + 2)$$

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$$\le C \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \ln(\mathbb{E}\left[\mathbf{N}^k_\delta\right] + 2) \le C_t \frac{\ln(N\delta||f||_\infty)}{\delta\sqrt{N}}$$

KMT approximation

$$X_t^{N,1} = X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t]\times\mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\left\{z \le f(X_{s-}^{N,1})\right\}} d\bar{\pi}^1(s,z)$$
$$+ \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta - 1} \sigma B^k_{\mathbf{N}^k_\delta} + \delta^{1/4} + R_t^2$$

KMT approximation : let B^k BM such that

$$|\mathbf{Z}^k_n - \sigma B^k_n| \le E^k \ln(n \vee 2)$$

Remark : E^k and B^k independent of $X^{N,j}_{k\delta}, \bar{\pi}^j, \mathbf{N}^k$

$$\mathbb{E}\left[|R_t^2|\right] \le \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbb{E}\left[\ln(\mathbf{N}^k_\delta + 2)\right] \mathbb{E}\left[E^k\right]$$
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Construction of Gaussian variables

$$X_t^{N,1} = X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t]\times\mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\left\{z \le f(X_{s-}^{N,1})\right\}} d\overline{\pi}^1(s,z)$$

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Lemma

$W_\delta^{N,k} \sim \mathcal{N}(0,\delta)$ independent of $\mathbf{N}_\delta^k$

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Proof : conditionally on $\mathbf{N}_\delta^k$, $B_{\mathbf{N}_\delta^k}^k \sim \mathcal{N}(0, \mathbf{N}_\delta^k)$

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$$\Rightarrow \mathbb{E}\left[g(W_\delta^{N,k}) | \mathbf{N}_\delta^k\right] = \mathbb{E}\left[g\left(\sqrt{\frac{\delta}{\mathbf{N}_\delta^k}} B^k_{\mathbf{N}_\delta^k}\right) | \mathbf{N}_\delta^k\right]$$

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Time-changes

$$X_t^{N,1} = X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t]\times\mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\left\{z \le f(X_{s-}^{N,1})\right\}} d\bar{\pi}^1(s,z)$$

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$$\mathbb{E}\left[|R_t^3|\right] \leq \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbb{E}\left[\left|\sqrt{\frac{\mathbf{N}_\delta^k}{\delta}} - \sqrt{\sum_{j=1}^{N} f(X_{k\delta}^{N,j})}\right| \cdot \left|W_\delta^{N,k}\right|\right]$$

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Control of R_t^3

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$$\mathbb{E}\left[E_\delta^k \mathbb{1}_G \right]$$

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$$\mathbb{E}\left[E_\delta^k \mathbb{1}_G \right] \leq C(N\delta)^{-1/2} \mathbb{E}\left[\left| \tilde{\mathbf{N}}_\delta^k \right| \right]$$

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Grönwall's lemma :

$$\mathbb{E}\left[\sup_{s\leq t}\left|X_s^{N,1} - \bar{X}_s^{N,1}\right|\right] \leq C_T \left(\delta^{1/4} + \frac{\ln(N\delta)}{\delta\sqrt{N}}\right)$$

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Summary of the coupling

Step	Locale martingale	Error
Euler approximation	$\frac{1}{\sqrt{N}}\sum_{k=0}^{t/\delta-1} \mathbf{Z}^k_{\mathbf{N}^k_\delta}$	$\delta^{1/4}$
KMT approximation	$\frac{1}{\sqrt{N}}\sum_{k=0}^{t/\delta-1} B^k_{\mathbf{N}^k_\delta}$	$\frac{\ln(N\delta)}{\delta\sqrt{N}}$
Riemann sum	$\sum_{k=0}^{t/\delta-1}\sqrt{\frac{1}{N}\sum_{j=1}^{N} f(X^{N,j}_{k\delta})}\left(W^N_{(k+1)\delta}-W^N_{k\delta}\right)$	$\frac{1}{\delta\sqrt{N}}$
Stochastic integral	$\int_0^t\sqrt{\frac{1}{N}\sum_{j=1}^{N} f(X^{N,j}_s)}dW^N_s$	$\delta^{1/4}$

Bibliography (1)

- E., Löcherbach, Loukianova (2021). Strong error bounds for the convergence to its mean field limit for systems of interacting neurons in a diffusive scaling. HAL, ArXiv.
- Delattre, Fournier, Hoffman (2016). Hawkes processes on large networks. Annals of Applied Probability.
- Chevallier, Duarte, Löcherbach, Ost (2019). Mean field limits for nonlinear spatially extended Hawkes processes with exponential memory kernels. Stochastic Processes and their Applications.
- E., Löcherbach, Loukianova (2022). Mean field limits for interacting Hawkes processes in a diffusive regime. Bernoulli.
- Komlòs, Major, Tusnàdy (1976). An approximation of partial sums of independent RV's, and the sample DF. II. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete.

Bibliography (2)

- Daley, Vere-Jones (2008). An Introduction to the Theorey of Point Processes : Volume II : General Theory and Structure. Springer.
- Ethier, Kurtz (2008). Markov Processes. Characterization and Convergence. Wiley Series In Probability And Statistics.
- Revuz, Yor (1999). Continuous Martingales and Brownian Motion. Springer-Verlag Berlin.

Thank you for your attention !

Questions ?