

Strong error bounds for the conditional propagation of chaos of mean field systems of neurons

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 - Point processes
 - Thinning
- 2 Model
 - Neural networks model
 - Definitions of the systems
 - Limit system
- 3 Propagation of chaos
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 - Formal proof

Point process : definitions

Point process (or counting process) Z :

- a random countable set of \mathbb{R}_+ : $Z = \{T_i : i \in \mathbb{N}\}$
- a random point measure on \mathbb{R}_+ : $Z = \sum_{i \in \mathbb{N}} \delta_{T_i}$

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A process λ is the **stochastic intensity** of Z if :

$$\forall 0 \leq a < b, \mathbb{E} [Z([a, b]) | \mathcal{F}_a] = \mathbb{E} \left[\int_a^b \lambda_t dt \middle| \mathcal{F}_a \right]$$

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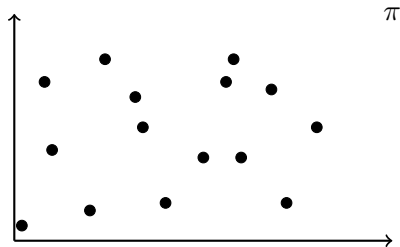
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Marked point process $Z = \{(T_i, U_i) : i \in \mathbb{N}\}$ (U_i iid)

Notation abuse $U_i =: U(T_i)$

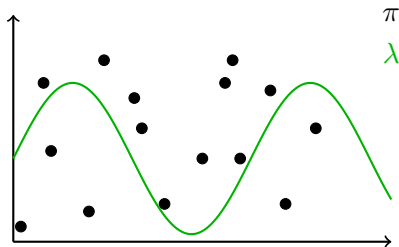
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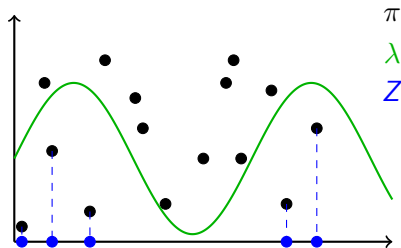


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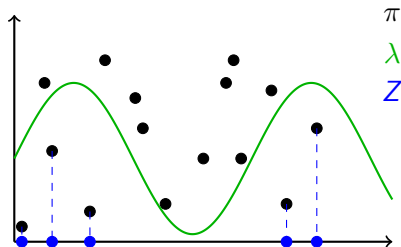
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Then : λ is the stochastic intensity of Z



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Here, $X^{N,i}$ solves an SDE directed by $(Z^{N,j})_{1 \leq j \leq N}$

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N -particle system :

- $Z_t^{N,i} = \int_0^t \int_0^\infty \mathbb{1}_{\{z \leq f(X_{s-}^{N,i})\}} d\pi^i(s, z)$
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- linear scaling N^{-1} (LLN) :
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- diffusive scaling $N^{-1/2}$ (CLT) :
[E. et al. (2022)] random and centered $u^{ji}(s)$

Diffusive scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^N \int_0^\infty \int_{\mathbb{R}} u \mathbb{1}_{\{z \leq f(X_{t-}^{N,j})\}} d\pi^j(t, z, u) \\ - \int_0^\infty \int_{\mathbb{R}} X_{t-}^{N,i} \mathbb{1}_{\{z \leq f(X_{t-}^{N,i})\}} d\pi^i(t, z, u)$$

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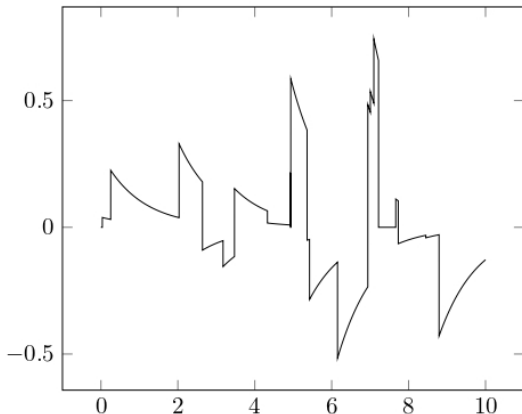
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- $X_t^{N,i} = 0$ if neuron i emits a spike at t

N -neurons network dynamics

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^N U^j(t) dZ_t^{N,j} - X_{t-}^{N,i} dZ_t^{N,i}$$



Limit system : heuristic (1)

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$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + d\bar{M}_t$$

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Then \bar{M} should satisfy

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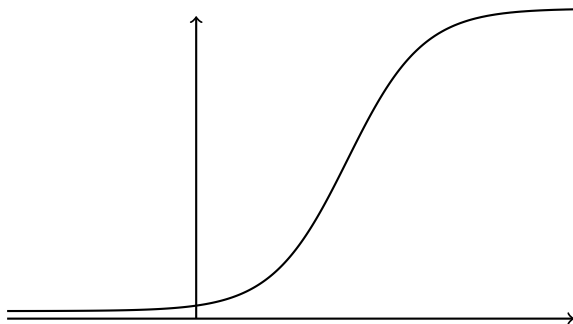
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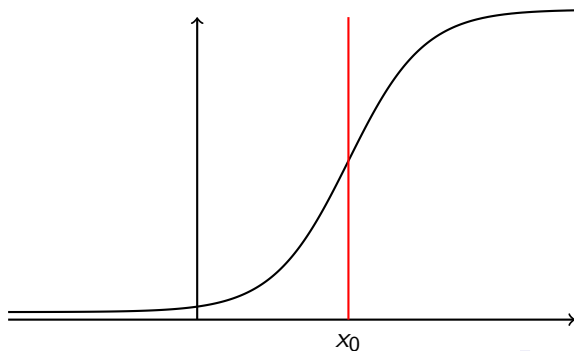


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Convergence of $(X_t^{N,i})_{1 \leq i \leq N}$

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Result [E., Löcherbach, Loukianova (2021)]

Given $N \in \mathbb{N}^*$, $(\pi^j)_{1 \leq j \leq N}$, there exists a BM W^N such that

$$\mathbb{E} \left[\sup_{0 \leq s \leq t} \left| X_s^{N,1} - \bar{X}_s^{N,1} \right| \right] \leq C_t \frac{(\ln N)^{1/5}}{N^{1/10}}$$

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$$\mathbb{E} \left[\sup_{0 \leq s \leq t} \left| X_s^{N,1} - \bar{X}_s^{N,1} \right| \right] \leq C_t \frac{(\ln N)^{1/5}}{N^{1/10}}$$

Convergence of $(X^{N,i})_{1 \leq i \leq N}$

$$\begin{aligned}dX_t^{N,i} &= -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^N \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\{z \leq f(X_{t-}^{N,j})\}} d\pi^j(t, z, u) \\ &\quad - \int_{\mathbb{R}_+} X_{t-}^{N,i} \mathbb{1}_{\{z \leq f(X_{t-}^{N,i})\}} d\bar{\pi}^i(t, z) \\ d\bar{X}_t^i &= -\alpha \bar{X}_t^i dt + \sigma \sqrt{\bar{\mu}_t(f)} dW_t - \int_{\mathbb{R}_+} \bar{X}_{t-}^i \mathbb{1}_{\{z \leq f(\bar{X}_{t-}^i)\}} d\bar{\pi}^i(t, z)\end{aligned}$$

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Representation of Poisson processes [Kurtz (1978)]

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} dP_t^N - X_{t-}^{N,i} dZ_t^{N,i}$$

with

$$P_t^N := \sum_{j=1}^N \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\{z \leq f(X_{s-}^{N,j})\}} d\pi^j(s, z, u)$$

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Proposition 16.6.III [Daley & Vere-Jones (2008)] :

$$P_t^N = \mathbf{Z}_{A_t^N},$$

where \mathbf{Z} = marked point process with rate 1

KMT coupling

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Grönwall's lemma :

$$\mathbb{E} \left[\sup_{s \leq t} \left| X_s^{N,i} - \bar{X}_s^{N,i} \right| \right] \leq C_t \frac{\ln N}{\sqrt{N}}$$

Problems with the naïve coupling (1)

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Replace $\mathbf{Z}_{A_t^N}$ by $\mathbf{Z}'_{\mathbf{N}_t}$

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- Change of time

$$|B_{\mathbf{N}_t} - B_{A_t^N}| \leq \dots$$

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- Define W^N by : $B_{A_t^N} = \int_0^t \sqrt{\sum_{j=1}^N f(X_s^{N,j})} dW_s^N$

Problems :

- BMs are not Lipschitz

Problems with the naïve coupling (2)

- $dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} d\mathbf{Z}'_{\mathbf{N}_t} - \int_{\mathbb{R}_+} X_{t-}^{N,i} \mathbb{1}_{\{z \leq f(X_{t-}^{N,i})\}} d\bar{\pi}^i(t, z)$
- Define B by KMT coupling : $|\mathbf{Z}'_n - \sigma B_n| \leq E \ln(n \vee 2)$
- Change of time

$$|B_{\mathbf{N}_t} - B_{A_t^N}| \leq \dots$$

- Define W^N by : $B_{A_t^N} = \int_0^t \sqrt{\sum_{j=1}^N f(X_s^{N,j})} dW_s^N$

Problems :

- BMs are not Lipschitz
- \mathbf{N} still depend on jump heights through its rate

Pseudo-Euler scheme (1)

Let $N \in \mathbb{N}^*$, $\delta = \delta(N) > 0$,

$$\begin{aligned} X_t^{N,1} = & X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\{z \leq f(X_{s-}^{N,1})\}} d\bar{\pi}^1(s, z) \\ & + \frac{1}{\sqrt{N}} \sum_{j=1}^N \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\{z \leq f(X_{s-}^{N,j})\}} d\pi^j(s, z, u) \end{aligned}$$

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Pseudo-Euler scheme (2)

$$X_t^{N,1} = X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\{z \leq f(X_{s-}^{N,1})\}} d\bar{\pi}^1(s, z) \\ + \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbf{z}_{N_\delta^k} + R_t^1$$

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- $\mathbf{z}_n^k = \sum_{l=1}^n U_l^k$ random walk (3rd variable of $\pi_{[k\delta, (k+1)\delta] \times \mathbb{R}_+ \times \mathbb{R}}^j$)

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Remark : \mathbf{Z}^k independent of $\bar{\pi}^j$ and $X_{k\delta}^{N,j}$ (so of \mathbf{N}^k)

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 - $\mathbf{N}_s^k = \sum_{j=1}^N \bar{\pi}^j([k\delta, k\delta + s] \times [0, f(X_{k\delta}^{N,j})])$
- Remark :** \mathbf{Z}^k independent of $\bar{\pi}^j$ and $X_{k\delta}^{N,j}$ (so of \mathbf{N}^k)

$$|R_t^1| \leq \frac{1}{\sqrt{N}} \left| \sum_{j=1}^N \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u \left(\mathbb{1}_{\{z \leq f(X_{\tau(s-)}^{N,j})\}} - \mathbb{1}_{\{z \leq f(X_{s-}^{N,j})\}} \right) d\pi^j(s, z, u) \right|$$

with $\tau(s) = k\delta$ for $s \in [k\delta, (k+1)\delta[$

Error due to Euler scheme (1)

Lemmas

(1) For all $t \leq T$,

$$\mathbb{E} \left[|X_t^{N,1} - X_{\tau(t)}^{N,1}| \right] \leq C_T \delta^{1/2}$$

(2) For all $t \leq T$,

$$\mathbb{E} [|R_t^1|] \leq C_T \delta^{1/4}$$

Sketch of proof of (1) :

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$$\mathbb{E} \left[|X_t^{N,1} - X_{\tau(t)}^{N,1}| \right] \leq C_T(t - \tau(t)) + C_T \sqrt{t - \tau(t)}$$

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$$|R_t^1| \leq \frac{1}{\sqrt{N}} \left| \sum_{j=1}^N \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u \left(\mathbb{1}_{\{z \leq f(X_{\tau(s-)}^{N,j})\}} - \mathbb{1}_{\{z \leq f(X_s^{N,j})\}} \right) d\pi^j(s, z, u) \right|$$

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Sketch of proof of (2) : by BDG inequality

$$|R_t^1| \leq \frac{1}{\sqrt{N}} \left| \sum_{j=1}^N \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u \left(\mathbb{1}_{\{z \leq f(X_{\tau(s-)}^{N,j})\}} - \mathbb{1}_{\{z \leq f(X_s^{N,j})\}} \right) d\pi^j(s, z, u) \right|$$
$$\mathbb{E} [|R_t^1|] \leq \sigma \left(\int_0^t \mathbb{E} [|f(X_s^{N,1}) - f(X_{\tau(s)}^{N,1})|] ds \right)^{1/2}$$

Error due to Euler scheme (2)

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KMT approximation

$$X_t^{N,1} = X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\{z \leq f(X_{s-}^{N,1})\}} d\bar{\pi}^1(s, z) \\ + \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbf{z}_{N_\delta^k} + \delta^{1/4}$$

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KMT approximation : let B^k BM such that

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$$\mathbb{E} [|R_t^2|] \leq \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbb{E} [\ln(\mathbf{N}_\delta^k + 2) E^k]$$

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 \mathbb{E} [|R_t^2|] & \leq \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbb{E} [\ln(\mathbf{N}_\delta^k + 2)] \mathbb{E} [E^k] \\
 & \leq C \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \ln(\mathbb{E} [\mathbf{N}_\delta^k] + 2)
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 \mathbb{E}[|R_t^2|] & \leq \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbb{E}[\ln(\mathbf{N}_\delta^k + 2)] \mathbb{E}[E^k] \\
 & \leq C \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \ln(\mathbb{E}[\mathbf{N}_\delta^k] + 2) \leq C_t \frac{\ln(N\delta \|f\|_\infty)}{\delta\sqrt{N}}
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KMT approximation

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X_t^{N,1} = & X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\{z \leq f(X_{s-}^{N,1})\}} d\bar{\pi}^1(s, z) \\
& + \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \sigma B_{\mathbf{N}_\delta^k} + \delta^{1/4} + \frac{\ln(N\delta)}{\delta\sqrt{N}}
\end{aligned}$$

KMT approximation : let B^k BM such that

$$|Z_n^k - \sigma B_n^k| \leq E^k \ln(n \vee 2)$$

Remark : E^k and B^k independent of $X_{k\delta}^{N,j}$, $\bar{\pi}^j$, \mathbf{N}^k

$$\begin{aligned}
\mathbb{E} [|R_t^2|] & \leq \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbb{E} [\ln(\mathbf{N}_\delta^k + 2)] \mathbb{E} [E^k] \\
& \leq C \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \ln(\mathbb{E} [\mathbf{N}_\delta^k] + 2) \leq C_t \frac{\ln(N\delta \|f\|_\infty)}{\delta\sqrt{N}}
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Construction of Gaussian variables

$$X_t^{N,1} = X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\{z \leq f(X_{s-}^{N,1})\}} d\bar{\pi}^1(s, z) \\ + \sigma \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} B_{\mathbf{N}_\delta^k} + \delta^{1/4} + \frac{\ln(N\delta)}{\delta\sqrt{N}}$$

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Time-changes

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$$\mathbb{E} [|R_t^3|] \leq \frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbb{E} \left[\left| \sqrt{\frac{Nk}{\delta}} - \sqrt{\sum_{j=1}^N f(X_{k\delta}^{N,j})} \right| \cdot |W_\delta^{N,k}| \right]$$

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Control of R_t^3

$$E_\delta^k := \left| \sqrt{\mathbf{N}_\delta^k} - \sqrt{\delta \sum_{j=1}^N f(X_{k\delta}^{N,j})} \right|$$

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$$\mathbb{P}(G^c) \leq \mathbb{P} \left(\mathbf{N}_\delta^k - \delta \sum_{j=1}^N f(X_{k\delta}^{N,j}) < -N\delta(\inf f)/2 \right)$$

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Construction of BM W^N

$$\begin{aligned} X_t^{N,1} = & X_0^{N,1} - \alpha \int_0^t X_s^{N,1} ds - \int_{[0,t] \times \mathbb{R}_+} X_{s-}^{N,1} \mathbb{1}_{\{z \leq f(X_{s-}^{N,1})\}} d\bar{\pi}^1(s, z) \\ & + \sigma \sum_{k=0}^{t/\delta-1} \sqrt{\frac{1}{N} \sum_{j=1}^N f(X_{k\delta}^{N,j})} W_\delta^{N,k} + \delta^{1/4} + \frac{\ln(N\delta)}{\delta\sqrt{N}} \end{aligned}$$

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Grönwall's lemma :

$$\mathbb{E} \left[\sup_{s \leq t} \left| X_s^{N,1} - \bar{X}_s^{N,1} \right| \right] \leq C_T \left(\delta^{1/4} + \frac{\ln(N\delta)}{\delta\sqrt{N}} \right)$$

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Summary of the coupling

Step	Locale martingale	Error
Euler approximation	$\frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} \mathbf{z}_{\mathbf{N}\delta}^k$	$\delta^{1/4}$
KMT approximation	$\frac{1}{\sqrt{N}} \sum_{k=0}^{t/\delta-1} B_{\mathbf{N}\delta}^k$	$\frac{\ln(N\delta)}{\delta\sqrt{N}}$
Riemann sum	$\sum_{k=0}^{t/\delta-1} \sqrt{\frac{1}{N} \sum_{j=1}^N f(X_{k\delta}^{N,j})} (W_{(k+1)\delta}^N - W_{k\delta}^N)$	$\frac{1}{\delta\sqrt{N}}$
Stochastic integral	$\int_0^t \sqrt{\frac{1}{N} \sum_{j=1}^N f(X_s^{N,j})} dW_s^N$	$\delta^{1/4}$

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Thank you for your attention !

Questions ?