Conditional propagation of chaos for mean field systems of interacting neurons

Xavier Erny [1]

Joint work with: Eva Löcherbach [2] and Dasha Loukianova [1]

[1]Université d'Evry (LaMME)
[2]Université Paris 1 Panthéon-Sorbonne (SAMM)

LaMME, 7 January 2021

Modeling in neuroscience

Neural activity $\quad = \quad$ Set of spike times

Modeling in neuroscience

$$\text{Neural activity} = \text{Set of spike times} = \text{Point process}$$

Modeling in neuroscience

$$\begin{aligned} \text{Neural activity} &= \text{Set of spike times} \\ &= \text{Point process} \end{aligned}$$

Spike rate depends on the potential of the neuron

Modeling in neuroscience

Neural activity	=	Set of spike times
	=	Point process

Spike rate depends on the potential of the neuron

Each spike modifies the potential of the neurons

Modeling in neuroscience

$$\text{Neural activity} = \text{Set of spike times} = \text{Point process}$$

Spike rate depends on the potential of the neuron

Each spike modifies the potential of the neurons

Network of N neurons :

- $$Z_t^{N,i} = \text{number of spikes of neuron } i \text{ emitted in } [0,t] = \text{point process with intensity } f(X_{t-}^{N,i})$$
- $X^{N,i}$ = potential of neuron i

Modeling in neuroscience

$$\text{Neural activity} = \text{Set of spike times} = \text{Point process}$$

Spike rate depends on the potential of the neuron

Each spike modifies the potential of the neurons

Network of N neurons :

- $Z_t^{N,i}$ = number of spikes of neuron i emitted in $[0,t]$
  = point process with intensity $f(X_{t-}^{N,i})$
- $X^{N,i}$ = potential of neuron i

Here, $X^{N,i}$ solves an SDE directed by $(Z^{N,j})_{1\leq j\leq N}$

Mean field limit

$N-$particle system :

- $Z_t^{N,i} = \int_0^t \int_0^\infty 1_{\left\{z \le f(X_{s-}^{N,i})\right\}} d\pi^i(s,z)$
- $dX_t^{N,i} = b(X_t^{N,i})dt + \sum_{j=1}^N \int_0^\infty u^{ji}(t) 1_{\left\{z \le f(X_{t-}^{N,j})\right\}} d\pi^j(t,z)$

π^j iid Poisson measures with intensity $dt \cdot dz$

Mean field limit

$N-$particle system :

- $Z_t^{N,i} = \int_0^t \int_0^\infty 1_{\left\{z \le f(X_{s-}^{N,i})\right\}} d\pi^i(s,z)$
- $dX_t^{N,i} = b(X_t^{N,i})dt + \sum_{j=1}^{N} \int_0^\infty u^{ji}(t) 1_{\left\{z \le f(X_{t-}^{N,j})\right\}} d\pi^j(t,z)$

π^j iid Poisson measures with intensity $dt \cdot dz$

Study the limit $N \to \infty \implies$ rescale the sum :

Mean field limit

$N-$particle system :

- $Z_t^{N,i} = \int_0^t \int_0^\infty 1_{\left\{z \leq f(X_{s-}^{N,i})\right\}} d\pi^i(s,z)$
- $dX_t^{N,i} = b(X_t^{N,i})dt + \sum_{j=1}^{N} \int_0^\infty u^{ji}(t) 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z)$

π^j iid Poisson measures with intensity $dt \cdot dz$

Study the limit $N \to \infty \implies$ rescale the sum :

- linear scaling N^{-1} (LLN) :
  [Delattre et al. (2016)] (Hawkes process, $u^{ji}(t) = 1$),
  [Chevallier et al. (2017)] ($u^{ji}(t) = w(j,i)$)

Mean field limit

$N-$particle system :

- $Z_t^{N,i} = \int_0^t \int_0^\infty 1_{\left\{z \leq f(X_{s-}^{N,i})\right\}} d\pi^i(s,z)$
- $dX_t^{N,i} = b(X_t^{N,i})dt + \sum_{j=1}^{N} \int_0^\infty u^{ji}(t) 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z)$

π^j iid Poisson measures with intensity $dt \cdot dz$

Study the limit $N \to \infty \implies$ rescale the sum :

- linear scaling N^{-1} (LLN) :
  [Delattre et al. (2016)] (Hawkes process, $u^{ji}(t) = 1$),
  [Chevallier et al. (2017)] ($u^{ji}(t) = w(j,i)$)
- diffusive scaling $N^{-1/2}$ (CLT) :
  [E. et al. (2019)] random and centered $u^{ji}(s)$

Linear scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_0^\infty 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z) - \int_0^\infty X_{t-}^{N,i} 1_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z)$$

π^j iid Poisson measures with intensity $dt \cdot dz$

Linear scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_0^{\infty} 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z) - \int_0^{\infty} X_{t-}^{N,i} 1_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z)$$

π^j iid Poisson measures with intensity $dt \cdot dz$

Intepretation :

- drift : $-\alpha x$ models an exponantial loss of the potential
- small jump of order N^{-1} : the effect of spike of one neuron to the potential of the others
- reset jump : the effet of spike of one neurone to its potential

Linear scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_0^{\infty} 1_{\left\{ z \leq f(X_{t-}^{N,j}) \right\}} d\pi^j(t,z) - \int_0^{\infty} X_{t-}^{N,i} 1_{\left\{ z \leq f(X_{t-}^{N,i}) \right\}} d\pi^i(t,z)$$

π^j iid Poisson measures with intensity $dt \cdot dz$

Intepretation :

- drift : $-\alpha x$ models an exponantial loss of the potential
- small jump of order N^{-1} : the effect of spike of one neuron to the potential of the others
- reset jump : the effet of spike of one neurone to its potential

[De Masi et al. (2015)] and [Fournier & Löcherbach (2016)]
Generalization to McKean-Vlasov frame [Andreis et al. (2018)]

Diffusive scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_0^\infty \int_{\mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) - \int_0^\infty \int_{\mathbb{R}} X_{t-}^{N,i} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

π^j iid Poisson measures with intensity $dt \cdot dz \cdot d\nu(u)$

ν probability measure on $\mathbb{R}$ centered with $\int_{\mathbb{R}} |u|^3 d\nu(u) < \infty$

$\sigma^2 = \int_{\mathbb{R}} u^2 d\nu(u)$

Diffusive scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_0^\infty \int_{\mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$
$$- \int_0^\infty \int_{\mathbb{R}} X_{t-}^{N,i} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

π^j iid Poisson measures with intensity $dt \cdot dz \cdot d\nu(u)$
ν probability measure on $\mathbb{R}$ centered with $\int_{\mathbb{R}} |u|^3 d\nu(u) < \infty$
$\sigma^2 = \int_{\mathbb{R}} u^2 d\nu(u)$

Dynamic of $X^{N,i}$:

- $X_t^{N,i} = X_s^{N,i} e^{-\alpha(t-s)}$ if the system does not jump in $[s,t]$
- $X_t^{N,i} = X_{t-}^{N,i} + \frac{U}{\sqrt{N}}$ if a neuron $j \neq i$ emits a spike at t
- $X_t^{N,i} = 0$ if neuron i emits a spike at t

Limit system : heuristic (1)

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{ z \leq f(X_{t-}^{N,j}) \right\}} d\pi^j(t,z,u)$$

$$- X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{ z \leq f(X_{t-}^{N,i}) \right\}} d\pi^i(t,z,u)$$

Limit system : heuristic (1)

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\left\{ z \leq f(X_{t-}^{N,j}) \right\}} d\pi^j(t,z,u)$$
$$- X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{ z \leq f(X_{t-}^{N,i}) \right\}} d\pi^i(t,z,u)$$

Limit system : heuristic (1)

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$

$$- X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

$$M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t] \times \mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{s-}^{N,j})\right\}} d\pi^j(s,z,u)$$

Limit system : heuristic (1)

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_{\mathbb{R}_+\times\mathbb{R}} u 1_{\left\{z\leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$

$$- X_{t-}^{N,i} \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

$$M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u 1_{\left\{z\leq f(X_{s-}^{N,j})\right\}} d\pi^j(s,z,u)$$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + d\bar{M}_t$$

$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

Limit system : heuristic (2)

$$M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{s-}^{N,j})\right\}} d\pi^j(s,z,u)$$

Limit system : heuristic (2)

$$M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u 1_{\left\{z \le f(X_{s-}^{N,j})\right\}} d\pi^j(s,z,u)$$

$\bar{M}$ is an integral wrt a BM W

Limit system : heuristic (2)

$$M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u1_{\left\{z\leq f(X_{s-}^{N,j})\right\}} d\pi^j(s,z,u)$$

$\bar{M}$ is an integral wrt a BM W

$$\langle \bar{M} \rangle_t = \lim_N \langle M^N \rangle_t = \lim_N \sigma^2 \int_0^t \frac{1}{N} \sum_{j=1}^{N} f(X_s^{N,j}) ds$$

Limit system : heuristic (2)

$$M_t^N := \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} u 1_{\left\{z \le f(X_{s-}^{N,j})\right\}} d\pi^j(s,z,u)$$

$\bar{M}$ is an integral wrt a BM W

$$\langle \bar{M} \rangle_t = \lim_N \langle M^N \rangle_t = \lim_N \sigma^2 \int_0^t \frac{1}{N} \sum_{j=1}^{N} f(X_s^{N,j}) ds$$

Then $\bar{M}$ should satisfy

$$\bar{M}_t = \sigma \int_0^t \sqrt{\lim_N \frac{1}{N} \sum_{j=1}^{N} f(\bar{X}_s^j)} dW_s = \sigma \int_0^t \sqrt{\lim_N \bar{\mu}_s^N(f)} dW_s$$

with $\bar{\mu}^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{\bar{X}^j}$

Limit system : heuristic (3)

$\bar{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s$ where $\mu = \lim_N \bar{\mu}^N$

Limit system : heuristic (3)

$\bar{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s$ where $\mu = \lim_N \bar{\mu}^N$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \le f(\bar{X}_{t-}^i)\}} d\pi^i(t, z, u)$$

Limit system : heuristic (3)

$\bar{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s$ where $\mu = \lim_N \bar{\mu}^N$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \leq f(\bar{X}_{t-}^i)\}} d\pi^i(t,z,u)$$

μ is the limit of empirical measures of $(\bar{X}^i)_{i \geq 1}$ exchangeable by Proposition (7.20) of [Aldous (1983)] μ is the directing measure of $(\bar{X}^i)_{i \geq 1}$ (conditionally on μ, $\bar{X}^i$ i.i.d.$\sim \mu$)

Limit system : heuristic (3)

$\bar{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s$ where $\mu = \lim_N \bar{\mu}^N$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \le f(\bar{X}_{t-}^i)\}} d\pi^i(t,z,u)$$

μ is the limit of empirical measures of $(\bar{X}^i)_{i\ge 1}$ exchangeable by Proposition (7.20) of [Aldous (1983)] μ is the directing measure of $(\bar{X}^i)_{i\ge 1}$ (conditionally on μ, $\bar{X}^i$ i.i.d.$\sim \mu$)

Conditionally on W, the $\bar{X}^i$ $(i \ge 1)$ are i.i.d.
by Lemma (2.12) of [Aldous (1983)] $\mu = \mathcal{L}(\bar{X}^1 | W) = \mathcal{L}(\bar{X}^i | W)$

Limit system : heuristic (3)

$\bar{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s$ where $\mu = \lim_N \bar{\mu}^N$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \le f(\bar{X}_{t-}^i)\}} d\pi^i(t,z,u)$$

μ is the limit of empirical measures of $(\bar{X}^i)_{i \ge 1}$ exchangeable by Proposition (7.20) of [Aldous (1983)] μ is the directing measure of $(\bar{X}^i)_{i \ge 1}$ (conditionally on μ, $\bar{X}^i$ i.i.d.$\sim \mu$)

Conditionally on W, the $\bar{X}^i$ $(i \ge 1)$ are i.i.d.
by Lemma (2.12) of [Aldous (1983)] $\mu = \mathcal{L}(\bar{X}^1 | W) = \mathcal{L}(\bar{X}^i | W)$

Limit system : heuristic (3)

$\bar{M}_t = \sigma \int_0^t \sqrt{\mu_s(f)} dW_s$ where $\mu = \lim_N \bar{\mu}^N$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mathbb{E}\left[f(\bar{X}_t^i) | W\right]} dW_t - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \le f(\bar{X}_{t-}^i)\}} d\pi^i(t,z,u)$$

μ is the limit of empirical measures of $(\bar{X}^i)_{i\geq 1}$ exchangeable
by Proposition (7.20) of [Aldous (1983)] μ is the directing measure of $(\bar{X}^i)_{i\geq 1}$ (conditionally on μ, $\bar{X}^i$ i.i.d.$\sim \mu$)

Conditionally on W, the $\bar{X}^i$ $(i \geq 1)$ are i.i.d.
by Lemma (2.12) of [Aldous (1983)] $\mu = \mathcal{L}(\bar{X}^1 | W) = \mathcal{L}(\bar{X}^i | W)$

Well-posedness of the limit equation (1)

$$d\bar{X}_t^i = -\alpha\bar{X}_t^i dt + \sigma\sqrt{\mathbb{E}\left[f(\bar{X}_t^i)|W\right]}dW_t$$
$$-\bar{X}_{t-}^i\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\{z\leq f(\bar{X}_{t-}^i)\}}d\pi^i(t,z,u)$$

Problems :

- conditional expectation in the Brownian term (McKean-Vlasov frame)
- unbounded jumps (non-Lipschitz compensator $x\mapsto -xf(x)$)
- jump term and Brownian term

Well-posedness of the limit equation (1)

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mathbb{E}\left[f(\bar{X}_t^i)|W\right]} dW_t$$
$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \le f(\bar{X}_{t-}^i)\}} d\pi^i(t,z,u)$$

Problems :

- conditional expectation in the Brownian term (McKean-Vlasov frame)
- unbounded jumps (non-Lipschitz compensator $x \mapsto -xf(x)$)
- jump term and Brownian term

Solution : consider $a : \mathbb{R} \to \mathbb{R}_+$ increasing, bounded, lower-bounded, C^2 such that

$$|a''(x) - a''(y)| + |a'(x) - a'(y)|$$
$$+ |xa'(x) - ya'(y)| + |f(x) - f(y)| \le C|a(x) - a(y)|$$

Well-posedness of the limit equation (2)

$$\begin{aligned} a(\bar{X}_t^i) =& a(\bar{X}_0^i) - \alpha \int_0^t \bar{X}_s^i a'(\bar{X}_s^i) ds + \sigma \int_0^t a'(\bar{X}_s^i) \sqrt{\mathbb{E}[f(\bar{X}_s^i)|W]} dW_s \\ &+ \frac{\sigma^2}{2} \int_0^t a''(\bar{X}_s^i) \mathbb{E}[f(\bar{X}_s^i)|W] ds \\ &+ \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} [a(0) - a(\bar{X}_{s-}^i)] 1_{\{z \le f(\bar{X}_{s-}^i)\}} d\pi^i(s,z,u) \end{aligned}$$

Well-posedness of the limit equation (2)

$$a(\bar{X}_t^i) = a(\bar{X}_0^i) - \alpha \int_0^t \bar{X}_s^i a'(\bar{X}_s^i) ds + \sigma \int_0^t a'(\bar{X}_s^i) \sqrt{\mathbb{E}[f(\bar{X}_s^i)|W]} dW_s$$
$$+ \frac{\sigma^2}{2} \int_0^t a''(\bar{X}_s^i) \mathbb{E}[f(\bar{X}_s^i)|W] ds$$
$$+ \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} [a(0) - a(\bar{X}_{s-}^i)] 1_{\{z \leq f(\bar{X}_{s-}^i)\}} d\pi^i(s,z,u)$$

To prove trajectorial uniqueness :

- $u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\check{X}_s^i)|]$ (problem with Brownian term)
- $u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\check{X}_s^i)|^2]$ (problem with jump term)

Well-posedness of the limit equation (2)

$$a(\bar{X}_t^i) = a(\bar{X}_0^i) - \alpha \int_0^t \bar{X}_s^i a'(\bar{X}_s^i) ds + \sigma \int_0^t a'(\bar{X}_s^i) \sqrt{\mathbb{E}[f(\bar{X}_s^i)|W]} dW_s$$
$$+ \frac{\sigma^2}{2} \int_0^t a''(\bar{X}_s^i) \mathbb{E}[f(\bar{X}_s^i)|W] ds$$
$$+ \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} [a(0) - a(\bar{X}_{s-}^i)] \mathbb{1}_{\{z \le f(\bar{X}_{s-}^i)\}} d\pi^i(s,z,u)$$

To prove trajectorial uniqueness :

- $u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\check{X}_s^i)|]$ (problem with Brownian term)
- $u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\check{X}_s^i)|^2]$ (problem with jump term)

Idea of [Graham (1992)] : $u(t) = \mathbb{E}\left[\sup_{0\le s\le t} |a(\hat{X}_s^i) - a(\check{X}_s^i)|\right]$

Well-posedness of the limit equation (2)

$$a(\bar{X}_t^i) = a(\bar{X}_0^i) - \alpha \int_0^t \bar{X}_s^i a'(\bar{X}_s^i) ds + \sigma \int_0^t a'(\bar{X}_s^i) \sqrt{\mathbb{E}[f(\bar{X}_s^i)|W]} dW_s$$
$$+ \frac{\sigma^2}{2} \int_0^t a''(\bar{X}_s^i) \mathbb{E}[f(\bar{X}_s^i)|W] ds$$
$$+ \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} [a(0) - a(\bar{X}_{s-}^i)] 1_{\{z \le f(\bar{X}_{s-}^i)\}} d\pi^i(s,z,u)$$

To prove trajectorial uniqueness :

- $u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\check{X}_s^i)|]$ (problem with Brownian term)
- $u(t) = \mathbb{E}[|a(\hat{X}_s^i) - a(\check{X}_s^i)|^2]$ (problem with jump term)

Idea of [Graham (1992)] : $u(t) = \mathbb{E}\left[\sup_{0\le s\le t} |a(\hat{X}_s^i) - a(\check{X}_s^i)|\right]$

$\forall t \geq 0, u(t) \leq C(t + \sqrt{t}) u(t)$

Well-posedness of the limit equation (2)

$$a(\bar{X}^i_t) = a(\bar{X}^i_0) - \alpha \int_0^t \bar{X}^i_s a'(\bar{X}^i_s) ds + \sigma \int_0^t a'(\bar{X}^i_s) \sqrt{\mathbb{E}[f(\bar{X}^i_s)|W]} dW_s$$
$$+ \frac{\sigma^2}{2} \int_0^t a''(\bar{X}^i_s) \mathbb{E}[f(\bar{X}^i_s)|W] ds$$
$$+ \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} [a(0) - a(\bar{X}^i_{s-})] 1_{\{z \le f(\bar{X}^i_{s-})\}} d\pi^i(s,z,u)$$

To prove trajectorial uniqueness :

- $u(t) = \mathbb{E}[|a(\hat{X}^i_s) - a(\check{X}^i_s)|]$ (problem with Brownian term)
- $u(t) = \mathbb{E}[|a(\hat{X}^i_s) - a(\check{X}^i_s)|^2]$ (problem with jump term)

Idea of [Graham (1992)] : $u(t) = \mathbb{E}\left[\sup_{0\le s\le t} |a(\hat{X}^i_s) - a(\check{X}^i_s)|\right]$

$\forall t \ge 0, u(t) \le C(t + \sqrt{t}) u(t) \Longrightarrow \exists t_0 > 0, u(t_0) = 0$

Well-posedness of the limit equation (2)

$$a(\bar{X}^i_t) = a(\bar{X}^i_0) - \alpha \int_0^t \bar{X}^i_s a'(\bar{X}^i_s) ds + \sigma \int_0^t a'(\bar{X}^i_s) \sqrt{\mathbb{E}[f(\bar{X}^i_s)|W]} dW_s$$
$$+ \frac{\sigma^2}{2} \int_0^t a''(\bar{X}^i_s) \mathbb{E}[f(\bar{X}^i_s)|W] ds$$
$$+ \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} [a(0) - a(\bar{X}^i_{s-})] 1_{\{z \le f(\bar{X}^i_{s-})\}} d\pi^i(s,z,u)$$

To prove trajectorial uniqueness :

- $u(t) = \mathbb{E}[|a(\hat{X}^i_s) - a(\check{X}^i_s)|]$ (problem with Brownian term)
- $u(t) = \mathbb{E}[|a(\hat{X}^i_s) - a(\check{X}^i_s)|^2]$ (problem with jump term)

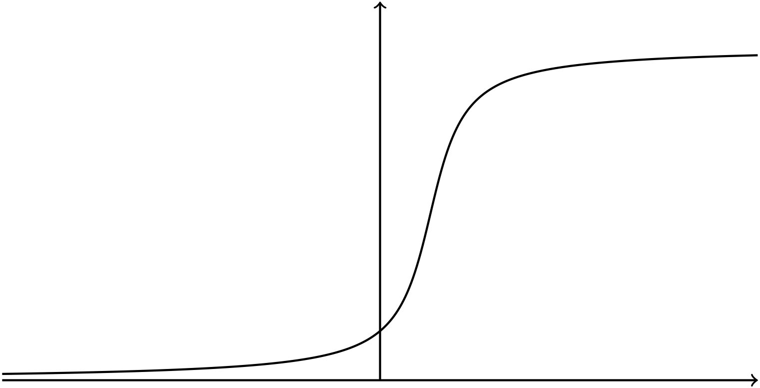
Idea of [Graham (1992)] : $u(t) = \mathbb{E}\left[\sup_{0\le s\le t} |a(\hat{X}^i_s) - a(\check{X}^i_s)|\right]$

$\forall t \ge 0, u(t) \le C(t + \sqrt{t})u(t) \Longrightarrow \exists t_0 > 0, u(t_0) = 0$

Iteratively $\forall n \in \mathbb{N}, u(nt_0) = 0$, whence $\forall t > 0, u(t) = 0$

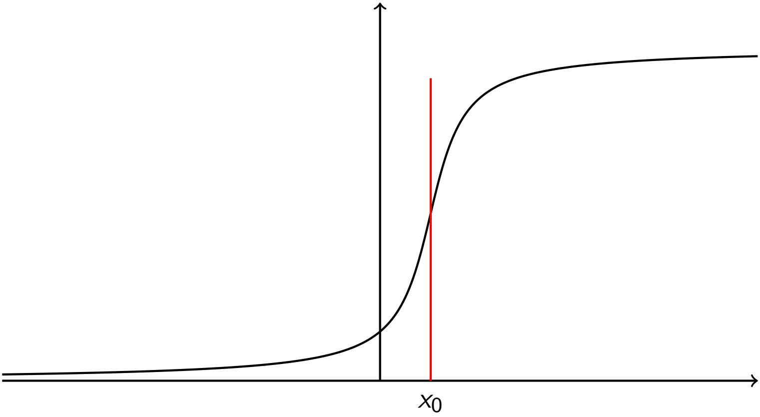
Discussion about the function a

$f(x) = a(x) = c + d \arctan(\alpha + \beta x)$ satisfy the hypothesis



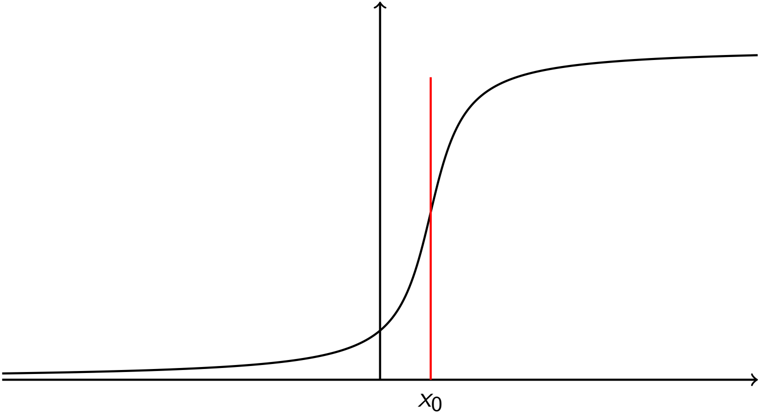
Discussion about the function a

$f(x) = a(x) = c + d\arctan(\alpha + \beta x)$ satisfy the hypothesis



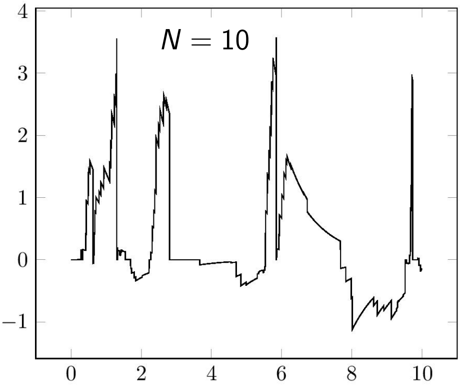
Discussion about the function a

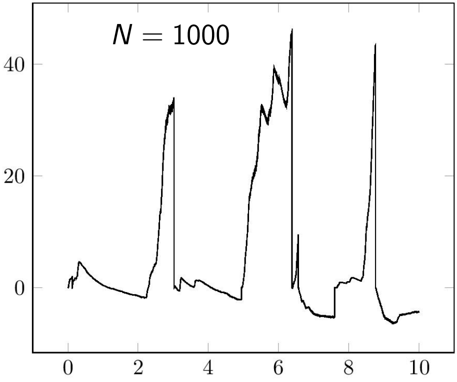
$f(x) = a(x) = c + d \arctan(\alpha + \beta x)$ satisfy the hypothesis



"Neuron i active / inactive" $\approx$ "$X^{N,i} > x_0$ / $X^{N,i} < x_0$"

Simulations of $X^{N,1}$





Another version of the limit system

The strong limit system :

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mathbb{E}\left[f(\bar{X}_t^i)|W\right]} dW_t - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\{z \leq f(\bar{X}_{t-}^i)\}} d\pi^i(t, z, u)$$

Another version of the limit system

The strong limit system :

$$d\bar{X}_t^i = -\alpha\bar{X}_t^i dt + \sigma\sqrt{\mathbb{E}\left[f(\bar{X}_t^i)|W\right]}dW_t$$
$$-\bar{X}_{t-}^i\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\{z\leq f(\bar{X}_{t-}^i)\}}d\pi^i(t,z,u)$$

The weak limit system :

$$d\bar{Y}_t^i = -\alpha\bar{Y}_t^i dt + \sigma\sqrt{\mu_t(f)}dW_t$$
$$-\bar{Y}_{t-}^i\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\{z\leq f(\bar{Y}_{t-}^i)\}}d\pi^i(t,z,u)$$

where $\mu_t = \mathcal{L}(\bar{Y}_t^1|\mu_t)$ is the directing measure of $(\bar{Y}_t^i)_{i\geq 1}$

Equivalence between the two systems

An auxiliary system :

$$d\tilde{X}_t^{N,i} = -\alpha\tilde{X}_t^{N,i}dt + \sigma\sqrt{\frac{1}{N}\sum_{j=1}^{N} f(\tilde{X}_t^{N,j})}dW_t - \tilde{X}_{t-}^{N,i}\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(\tilde{X}_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

Equivalence between the two systems

An auxiliary system :

$$d\tilde{X}_t^{N,i} = -\alpha\tilde{X}_t^{N,i}dt + \sigma\sqrt{\frac{1}{N}\sum_{j=1}^{N} f(\tilde{X}_t^{N,j})}dW_t$$
$$- \tilde{X}_{t-}^{N,i}\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(\tilde{X}_{t-}^{N,j})\right\}} d\pi^i(t,z,u)$$

Let $u_N(t) = \mathbb{E}\left[\sup_{s\leq t}\left|a(\bar{Y}_s^1) - a(\tilde{X}_s^{N,1})\right|\right]$

Equivalence between the two systems

An auxiliary system :

$$d\tilde{X}_t^{N,i} = -\alpha \tilde{X}_t^{N,i} dt + \sigma \sqrt{\frac{1}{N}\sum_{j=1}^N f(\tilde{X}_t^{N,j})} dW_t$$

$$- \tilde{X}_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{ z \leq f(\tilde{X}_{t-}^{N,j}) \right\}} d\pi^i(t,z,u)$$

Let $u_N(t) = \mathbb{E}\left[\sup_{s\leq t} \left| a(\bar{Y}_s^1) - a(\tilde{X}_s^{N,1}) \right| \right]$

$$u_N(t) \leq C(t + \sqrt{t}) u_N(t) + \underbrace{CN^{-1/2}}_{\mu_s(f) - N^{-1}\sum_{j=1}^N f(\bar{Y}_s^j)}$$

Equivalence between the two systems

An auxiliary system :

$$d\tilde{X}_t^{N,i} = -\alpha\tilde{X}_t^{N,i}dt + \sigma\sqrt{\frac{1}{N}\sum_{j=1}^{N} f(\tilde{X}_t^{N,j})}dW_t$$

$$- \tilde{X}_{t-}^{N,i}\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(\tilde{X}_{t-}^{N,j})\right\}} d\pi^i(t,z,u)$$

Let $u_N(t) = \mathbb{E}\left[\sup_{s\leq t}\left|a(\bar{Y}_s^1) - a(\tilde{X}_s^{N,1})\right|\right]$

$$u_N(t) \leq C(t+\sqrt{t})u_N(t) + \underbrace{CN^{-1/2}}_{\mu_s(f) - N^{-1}\sum_{j=1}^{N} f(\bar{Y}_s^j)}$$

For $0 \leq t \leq T$ (small enough)

$$u_N(t) \leq CN^{-1/2} \underset{N\to\infty}{\longrightarrow} 0$$

Convergence of $(X^{N,i})_{1\leq i\leq N}$

$$
\begin{aligned}
dX_t^{N,i} =& -\alpha X_t^{N,i}dt + \frac{1}{\sqrt{N}}\sum_{j\neq i}\int_{\mathbb{R}_+\times\mathbb{R}} u1_{\left\{z\leq f(X_{t-}^{N,j})\right\}}d\pi^j(t,z,u) \\
& - X_{t-}^{N,i}\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(X_{t-}^{N,i})\right\}}d\pi^i(t,z,u) \\
d\bar{X}_t^i =& -\alpha \bar{X}_t^i dt + \sigma\sqrt{\mu_t(f)}dW_t \\
& - \bar{X}_{t-}^i\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(\bar{X}_{t-}^i)\right\}}d\pi^i(t,z,u)
\end{aligned}
$$

Goal : $(X^{N,i})_{1\leq i\leq N}$ converges to $(\bar{X}^i)_{i\geq 1}$ in $D^{\mathbb{N}^*}$

Convergence of $(X^{N,i})_{1\leq i\leq N}$

$$dX_t^{N,i} = -\alpha X_t^{N,i}dt + \frac{1}{\sqrt{N}}\sum_{j\neq i}\int_{\mathbb{R}_+\times\mathbb{R}} u1_{\left\{z\leq f(X_{t-}^{N,j})\right\}}d\pi^j(t,z,u)$$
$$- X_{t-}^{N,i}\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(X_{t-}^{N,i})\right\}}d\pi^i(t,z,u)$$
$$d\bar{X}_t^i = -\alpha\bar{X}_t^i dt + \sigma\sqrt{\mu_t(f)}dW_t$$
$$- \bar{X}_{t-}^i\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(\bar{X}_{t-}^i)\right\}}d\pi^i(t,z,u)$$

Goal : $(X^{N,i})_{1\leq i\leq N}$ converges to $(\bar{X}^i)_{i\geq 1}$ in $D^{\mathbb{N}^*}$

Equivalent condition (Proposition (7.20) of [Aldous (1983)]) :
$\mu^N := \sum_{j=1}^N \delta_{X^{N,j}}$ converges to $\mu := \mathcal{L}(\bar{X}^1|W)$ in $\mathcal{P}(D)$

Outline of the proof

Step 1. $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
Equivalent condition : $(X^{N,1})_N$ is tight on D
Proof : Aldous' criterion

Step 2. Identifying the limit distribution of $(\mu^N)_N$
Proof : any limit of μ^N is solution of a martingale problem

Martingale problem

Given $Q \in \mathcal{P}(\mathcal{P}(D))$ $(Q = \mathcal{L}(\mu))$

Martingale problem

Given $Q \in \mathcal{P}(\mathcal{P}(D))$ $(Q = \mathcal{L}(\mu))$

Canonical space $\Omega := \mathcal{P}(D) \times D^2$ with $\omega = (\mu, (Y^1, Y^2))$:

Meaning : (Y^1, Y^2) mixture of iid directed by μ

Martingale problem

Given $Q \in \mathcal{P}(\mathcal{P}(D))$ $(Q = \mathcal{L}(\mu))$

Canonical space $\Omega := \mathcal{P}(D) \times D^2$ with $\omega = (\mu, (Y^1, Y^2))$:

Meaning : (Y^1, Y^2) mixture of iid directed by μ

$$P(A \times B) := \int_{\mathcal{P}(D)} 1_A(m) m \otimes m(B) dQ(m)$$

Martingale problem

Given $Q \in \mathcal{P}(\mathcal{P}(D))$ $(Q = \mathcal{L}(\mu))$

Canonical space $\Omega := \mathcal{P}(D) \times D^2$ with $\omega = (\mu, (Y^1, Y^2))$:

Meaning : (Y^1, Y^2) mixture of iid directed by μ

$$P(A \times B) := \int_{\mathcal{P}(D)} 1_A(m) m \otimes m(B) dQ(m)$$

Q is solution of $(\mathcal{M})$ if for all $g \in C_b^2(\mathbb{R}^2)$,
$g(Y_t^1, Y_t^2) - g(Y_0^1, Y_0^2) - \int_0^t Lg(\mu_s, Y_s^1, Y_s^2)ds$ is a martingale

Martingale problem

Given $Q \in \mathcal{P}(\mathcal{P}(D))$ $(Q = \mathcal{L}(\mu))$

Canonical space $\Omega := \mathcal{P}(D) \times D^2$ with $\omega = (\mu, (Y^1, Y^2))$:

Meaning : (Y^1, Y^2) mixture of iid directed by μ

$$P(A \times B) := \int_{\mathcal{P}(D)} 1_A(m) m \otimes m(B) dQ(m)$$

Q is solution of $(\mathcal{M})$ if for all $g \in C_b^2(\mathbb{R}^2)$,
$g(Y_t^1, Y_t^2) - g(Y_0^1, Y_0^2) - \int_0^t Lg(\mu_s, Y_s^1, Y_s^2) ds$ is a martingale

$$Lg(m, x^1, x^2) = -\alpha x^1 \partial_1 g(x) - \alpha x^2 \partial_2 g(x) + \frac{\sigma^2}{2} m(f) \sum_{i,j=1}^{2} \partial_{i,j}^2 g(x)$$
$$+ f(x^1)(g(0, x^2) - g(x)) + f(x^2)(g(x^1, 0) - g(x))$$

Uniqueness for the martingale problem

Let Q be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where μ is the directing measure of some exchangeable system $(\bar{Y}^i)_{i\geq 1}$

Uniqueness for the martingale problem

Let Q be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where μ is the directing measure of some exchangeable system $(\bar{Y}^i)_{i \geq 1}$

$\mathcal{L}(\mu, \bar{Y}^1, \bar{Y}^2) = P$ (from the martingale problem)

Uniqueness for the martingale problem

Let Q be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where μ is the directing measure of some exchangeable system $(\bar{Y}^i)_{i\geq 1}$

$\mathcal{L}(\mu, \bar{Y}^1, \bar{Y}^2) = P$ (from the martingale problem)

Representation theorems imply (admitted)

$$\forall i \in \{1,2\}, d\bar{Y}^i_t = -\alpha \bar{Y}^i_t dt + \sqrt{\mu_t(f)} dW_t - \bar{Y}^i_{t-} \int_{\mathbb{R}_+} 1_{\{z \leq f(\bar{Y}^i_{t-})\}} d\pi^i(t,z)$$

Uniqueness for the martingale problem

Let Q be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where μ is the directing measure of some exchangeable system $(\bar{Y}^i)_{i\geq 1}$

$\mathcal{L}(\mu, \bar{Y}^1, \bar{Y}^2) = P$ (from the martingale problem)

Representation theorems imply (admitted)

$$\begin{aligned}\forall i \in \mathbb{N}^*, d\bar{Y}^i_t =& -\alpha \bar{Y}^i_t dt + \sqrt{\mu_t(f)} dW_t \\ & - \bar{Y}^i_{t-} \int_{\mathbb{R}_+} 1_{\{z \leq f(\bar{Y}^i_{t-})\}} d\pi^i(t,z)\end{aligned}$$

Uniqueness for the martingale problem

Let Q be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where μ is the directing measure of some exchangeable system $(\bar{Y}^i)_{i\geq 1}$

$\mathcal{L}(\mu, \bar{Y}^1, \bar{Y}^2) = P$ (from the martingale problem)

Representation theorems imply (admitted)

$$\forall i \in \mathbb{N}^*, d\bar{Y}^i_t = -\alpha \bar{Y}^i_t dt + \sqrt{\mu_t(f)} dW_t - \bar{Y}^i_{t-} \int_{\mathbb{R}_+} 1_{\{z \leq f(\bar{Y}^i_{t-})\}} d\pi^i(t,z)$$

Then the law of $\mu = \mathcal{L}(\bar{Y}^1 | W)$ is uniquely determined

Convergence of μ^N to the solution of $(\mathcal{M})$

Let μ be the limit of (a subsequence of) μ^N
$\mathcal{L}(\mu)$ is solution of $(\mathcal{M})$ if

$$\mathbb{E}\left[F(\mu)\right] = 0$$

for any F of the form

$$\begin{aligned}
F(m) := \int_{D^2} m \otimes m(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s)\\
+ \alpha\int_s^t \gamma_r^1\partial_1\phi(\gamma_r)dr + \alpha\int_s^t \gamma_r^2\partial_2\phi(\gamma_r)dr\\
- \int_s^t f(\gamma_r^1)(\phi(0,\gamma_r^2) - \phi(\gamma_r))dr - \int_s^t f(\gamma_r^2)(\phi(\gamma_r^1,0) - \phi(\gamma_r))dr\\
- \frac{\sigma^2}{2}\int_s^t m_r(f)\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(\gamma_r)dr\Big]
\end{aligned}$$

The expression of $F(\mu^N)$

$$
\begin{aligned}
&F(\mu^N) := \\
&\int_{D^2} \mu^N \otimes \mu^N(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s) \\
&+ \alpha \int_s^t \gamma_r^1 \partial_1 \phi(\gamma_r)dr + \alpha \int_s^t \gamma_r^2 \partial_2 \phi(\gamma_r)dr \\
&- \int_s^t f(\gamma_r^1)(\phi(0, \gamma_r^2) - \phi(\gamma_r))dr \\
&\int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r))dr \\
&- \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1, i_2 = 1}^{2} \partial^2_{i_1, i_2} \phi(\gamma_r)dr\Big]
\end{aligned}
$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\int_{D^2} \mu^N \otimes \mu^N(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s)$$

$$+ \alpha \int_s^t \gamma_r^1 \partial_1 \phi(\gamma_r)dr + \alpha \int_s^t \gamma_r^2 \partial_2 \phi(\gamma_r)dr$$

$$- \int_s^t f(\gamma_r^1)(\phi(0, \gamma_r^2) - \phi(\gamma_r))dr$$

$$\int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r))dr$$

$$- \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1, i_2 = 1}^{2} \partial^2_{i_1, i_2} \phi(\gamma_r)dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_1(X^{N,i}_{s_1},X^{N,j}_{s_1})\dots\phi_k(X^{N,i}_{s_k},X^{N,j}_{s_k})\Big[\phi(X^{N,i}_t,X^{N,j}_t)-\phi(X^{N,i}_s,X^{N,j}_s)$$

$$+\alpha\int_s^t X^{N,i}_r\partial_1\phi(X^{N,i}_r,X^{N,j}_r)dr+\alpha\int_s^t X^{N,j}_r\partial_2\phi(X^{N,i}_r,X^{N,j}_r)dr$$

$$-\int_s^t f(X^{N,i}_r)(\phi(0,X^{N,j}_r)-\phi(X^{N,i}_r,X^{N,j}_r))dr$$

$$-\int_s^t f(X^{N,j}_r)(\phi(X^{N,i}_r,0)-\phi(X^{N,i}_r,X^{N,j}_r))dr$$

$$-\frac{\sigma^2}{2}\int_s^t \mu^N_r(f)\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X^{N,i}_r,X^{N,j}_r)dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_1(X_{s_1}^{N,i},X_{s_1}^{N,j})...\phi_k(X_{s_k}^{N,i},X_{s_k}^{N,j})\Big[\phi(X_t^{N,i},X_t^{N,j})-\phi(X_s^{N,i},X_s^{N,j})$$

$$+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i},X_r^{N,j})dr$$

$$-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j})-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0)-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\frac{\sigma^2}{2}\int_s^t \mu_r^N(f)\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_1(X^{N,i}_{s_1},X^{N,j}_{s_1})...\phi_k(X^{N,i}_{s_k},X^{N,j}_{s_k})\Big[\phi(X^{N,i}_t,X^{N,j}_t)-\phi(X^{N,i}_s,X^{N,j}_s)$$

$$+\alpha\int_s^t X^{N,i}_r\partial_1\phi(X^{N,i}_r,X^{N,j}_r)dr+\alpha\int_s^t X^{N,j}_r\partial_2\phi(X^{N,i}_r,X^{N,j}_r)dr$$

$$-\int_s^t f(X^{N,i}_r)(\phi(0,X^{N,j}_r)-\phi(X^{N,i}_r,X^{N,j}_r))dr$$

$$-\int_s^t f(X^{N,j}_r)(\phi(X^{N,i}_r,0)-\phi(X^{N,i}_r,X^{N,j}_r))dr$$

$$-\frac{\sigma^2}{2}\int_s^t \frac{1}{N}\sum_{k=1}^{N}f(X^{N,k}_r)\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X^{N,i}_r,X^{N,j}_r)dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_1(X^{N,i}_{s_1},X^{N,j}_{s_1})...\phi_k(X^{N,i}_{s_k},X^{N,j}_{s_k})\Big[\phi(X^{N,i}_t,X^{N,j}_t)-\phi(X^{N,i}_s,X^{N,j}_s)$$

$$+\alpha\int_s^t X^{N,i}_r\partial_1\phi(X^{N,i}_r,X^{N,j}_r)dr+\alpha\int_s^t X^{N,j}_r\partial_2\phi(X^{N,i}_r,X^{N,j}_r)dr$$

$$-\int_s^t f(X^{N,i}_r)(\phi(0,X^{N,j}_r)-\phi(X^{N,i}_r,X^{N,j}_r))dr$$

$$-\int_s^t f(X^{N,j}_r)(\phi(X^{N,i}_r,0)-\phi(X^{N,i}_r,X^{N,j}_r))dr$$

$$-\frac{\sigma^2}{2}\int_s^t\frac{1}{N}\sum_{k=1}^{N}f(X^{N,k}_r)\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X^{N,i}_r,X^{N,j}_r)dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^N \phi_1(X_{s_1}^{N,i},X_{s_1}^{N,j})...\phi_k(X_{s_k}^{N,i},X_{s_k}^{N,j})\Big[\phi(X_t^{N,i},X_t^{N,j}) - \phi(X_s^{N,i},X_s^{N,j})$$

$$+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i},X_r^{N,j})dr + \alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i},X_r^{N,j})dr$$

$$-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j}) - \phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0) - \phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t \int_{\mathbb{R}} \frac{u^2}{2}\frac{1}{N}\sum_{k=1}^N f(X_r^{N,k})\sum_{i_1,i_2=1}^2 \partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})d\nu(u)dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_1(X_{s_1}^{N,i},X_{s_1}^{N,j})...\phi_k(X_{s_k}^{N,i},X_{s_k}^{N,j})\Big[\phi(X_t^{N,i},X_t^{N,j})-\phi(X_s^{N,i},X_s^{N,j})$$

$$+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i},X_r^{N,j})dr$$

$$-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j})-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0)-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t\int_{\mathbb{R}}\frac{u^2}{2}\frac{1}{N}\sum_{k=1}^{N}f(X_r^{N,k})\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})d\nu(u)dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_1(X_{s_1}^{N,i},X_{s_1}^{N,j})...\phi_k(X_{s_k}^{N,i},X_{s_k}^{N,j})\Big[\phi(X_t^{N,i},X_t^{N,j})-\phi(X_s^{N,i},X_s^{N,j})$$

$$+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i},X_r^{N,j})dr$$

$$-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j})-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0)-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t\int_{\mathbb{R}}\sum_{k=1}^{N} f(X_r^{N,k})\frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})d\nu(u)dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N} \phi_1(X_{s_1}^{N,i}, X_{s_1}^{N,j})...\phi_k(X_{s_k}^{N,i}, X_{s_k}^{N,j})\Big[\phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j})$$

$$+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i}, X_r^{N,j})dr + \alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i}, X_r^{N,j})dr$$

$$-\int_s^t f(X_r^{N,i})(\phi(0, X_r^{N,j}) - \phi(X_r^{N,i}, X_r^{N,j}))dr$$

$$-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i}, 0) - \phi(X_r^{N,i}, X_r^{N,j}))dr$$

$$-\int_s^t\int_{\mathbb{R}}\sum_{k=1}^{N} f(X_r^{N,k})\frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})d\nu(u)dr\Big]$$

The expression of $\phi(X^{N,i}, X^{N,j})$

By Ito's formula,

$$\mathbb{E}\phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) =$$

$$\mathbb{E} - \alpha \int_s^t X_r^{N,i} \partial_1 \phi(X_r^{N,i}, X_r^{N,j}) dr - \alpha \int_s^t X_r^{N,j} \partial_2 \phi(X_r^{N,i}, X_r^{N,j}) dr$$

$$+ \int_s^t \int_{\mathbb{R}} f(X_r^{N,i}) (\phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j})) d\nu(u) dr$$

$$+ \int_s^t \int_{\mathbb{R}} f(X_r^{N,j}) (\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, 0) - \phi(X_r^{N,i}, X_r^{N,j})) d\nu(u) dr$$

$$+ \int_s^t \int_{\mathbb{R}} \sum_{\substack{k=1 \\ k \neq i,j}}^N f(X_r^{N,k}) (\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j})) d\nu(u) dr$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

The small jump term

$$N\left|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

The small jump term

$$N\left|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) - \frac{u}{\sqrt{N}}\sum_{i_1=1}^{2}\partial_{i_1}\phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

The small jump term

$$N\left|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j})\right.$$

$$\left. - \frac{u}{\sqrt{N}}\sum_{i_1=1}^{2}\partial_{i_1}\phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$

$$\leq CN\frac{|u|^3}{N\sqrt{N}} = CN^{-1/2}|u|^3$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

The small jump term

$$N\left|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) \right.$$
$$\left. - \frac{u}{\sqrt{N}}\sum_{i_1=1}^{2}\partial_{i_1}\phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$
$$\leq CN\frac{|u|^3}{N\sqrt{N}} = CN^{-1/2}|u|^3$$

$$CN^{-1/2} \geq \mathbb{E}\left[F(\mu^N)\right] \underset{N\to\infty}{\longrightarrow} \mathbb{E}\left[F(\mu)\right] = 0$$

Convergence of $(\mu^N)_N$

$$
\begin{aligned}
dX_t^{N,i} =& -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\left\{ z \leq f(X_{t-}^{N,j}) \right\}} d\pi^j(t,z,u) \\
& - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{ z \leq f(X_{t-}^{N,i}) \right\}} d\pi^i(t,z,u) \\
d\bar{X}_t^i =& -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\
& - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{ z \leq f(\bar{X}_{t-}^i) \right\}} d\pi^i(t,z,u)
\end{aligned}
$$

Convergence of $(\mu^N)_N$

$$
\begin{aligned}
dX_t^{N,i} =& -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) \\
& - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u) \\
d\bar{X}_t^i =& -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\
& - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)
\end{aligned}
$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$

Convergence of $(\mu^N)_N$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$

$$- X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t$$

$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
- let μ be the limit of a converging subsequence

Convergence of $(\mu^N)_N$

$$
\begin{aligned}
dX_t^{N,i} =& -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j\neq i} \int_{\mathbb{R}_+\times\mathbb{R}} u 1_{\left\{z\leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) \\
& - X_{t-}^{N,i} \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u) \\
d\bar{X}_t^i =& -\alpha \bar{X}_t^i dt + \sigma\sqrt{\mu_t(f)} dW_t \\
& - \bar{X}_{t-}^i \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)
\end{aligned}
$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
- let μ be the limit of a converging subsequence
- $\mathcal{L}(\mu)$ is the unique solution of $(\mathcal{M})$

Convergence of $(\mu^N)_N$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j\neq i} \int_{\mathbb{R}_+\times\mathbb{R}} u 1_{\left\{z\leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$
$$- X_{t-}^{N,i} \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$
$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma\sqrt{\mu_t(f)} dW_t$$
$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
- let μ be the limit of a converging subsequence
- $\mathcal{L}(\mu)$ is the unique solution of $(\mathcal{M})$
- $\mu = \mathcal{L}(\bar{X}^1 | W)$ is the only limit of $(\mu^N)_N$

McKean-Vlasov model

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \leq f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

with $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{X^{N,i}}$, and π^k has intensity $dt \cdot dz \cdot \nu(du)$ $(\nu = \nu_1^{\otimes \mathbb{N}^*})$

McKean-Vlasov model

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$
$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \le f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

with $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{X^{N,i}}$, and π^k has intensity $dt \cdot dz \cdot \nu(du)$ $(\nu = \nu_1^{\otimes \mathbb{N}^*})$

Dynamic of $X^{N,i}$:

- while there is no jump, the dynamic is given by the drift and Brownian terms
- if there is a jump at time t, created by neuron k, each neuron i creates a r.v. U^i (the U^i are i.i.d.),

$$X_t^{N,i} = X_{t-}^{N,i} + \frac{1}{\sqrt{N}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, U^k, U^i)$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \le f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \le f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t, z, u)$$

$$J_t^{N,i} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int \Psi(X_{s-}^{N,k}, X_{s-}^{N,i}, \mu_{s-}^N, u^k, u^i) 1_{\left\{z \le f(X_{s-}^{N,k}, \mu_{s-}^N)\right\}} d\pi^k(s, z, u)$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$
$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \le f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

$$J_t^{N,i} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int \Psi(X_{s-}^{N,k}, X_{s-}^{N,i}, \mu_{s-}^N, u^k, u^i) 1_{\left\{z \le f(X_{s-}^{N,k}, \mu_{s-}^N)\right\}} d\pi^k(s,z,u)$$

$$\langle J^{N,i}, J^{N,j} \rangle_t = \frac{1}{N} \sum_{k=1}^{N} \int_0^t \int_{\mathbb{R}^{\mathbb{N}^*}}$$
$$\Psi(X_s^{N,k}, X_s^{N,i}, \mu_s^N, u^k, u^i) \Psi(X_s^{N,k}, X_s^{N,j}, \mu_s^N, u^k, u^j) f(X_s^{N,k}, \mu_s^N) \nu(du) ds$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \le f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

$$J_t^{N,i} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int \Psi(X_{s-}^{N,k}, X_{s-}^{N,i}, \mu_{s-}^N, u^k, u^i) 1_{\left\{z \le f(X_{s-}^{N,k}, \mu_{s-}^N)\right\}} d\pi^k(s,z,u)$$

$$\langle J^{N,i}, J^{N,j} \rangle_t = \frac{1}{N} \sum_{k=1}^{N} \int_0^t \int_{\mathbb{R}^{\mathbb{N}^*}}$$

$$\Psi(X_s^{N,k}, X_s^{N,i}, \mu_s^N, u^k, u^i) \Psi(X_s^{N,k}, X_s^{N,j}, \mu_s^N, u^k, u^j) f(X_s^{N,k}, \mu_s^N) \nu(du) ds$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \leq f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

$$J_t^{N,i} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int \Psi(X_{s-}^{N,k}, X_{s-}^{N,i}, \mu_{s-}^N, u^k, u^i) 1_{\left\{z \leq f(X_{s-}^{N,k}, \mu_{s-}^N)\right\}} d\pi^k(s,z,u)$$

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}}$$

$$\Psi(x, X_s^{N,i}, \mu_s^N, u^k, u^i)\Psi(x, X_s^{N,j}, \mu_s^N, u^k, u^j) f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \le f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

$$J_t^{N,i} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int \Psi(X_{s-}^{N,k}, X_{s-}^{N,i}, \mu_{s-}^N, u^k, u^i) 1_{\left\{z \le f(X_{s-}^{N,k}, \mu_{s-}^N)\right\}} d\pi^k(s,z,u)$$

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}}$$

$$\Psi(x, X_s^{N,i}, \mu_s^N, u^k, u^i) \Psi(x, X_s^{N,j}, \mu_s^N, u^k, u^j) f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \le f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

$$J_t^{N,i} = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int \Psi(X_{s-}^{N,k}, X_{s-}^{N,i}, \mu_{s-}^N, u^k, u^i) 1_{\left\{z \le f(X_{s-}^{N,k}, \mu_{s-}^N)\right\}} d\pi^k(s,z,u)$$

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}}$$

$$\Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i) \Psi(x, X_s^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds$$

Particular case

$$\Psi(x, y, m, u, v) = \Psi(u, v)$$

Particular case

$$\Psi(x, y, m, u, v) = \Psi(u, v)$$

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(u^1, u^i)\Psi(u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx)ds$$

Particular case

$$\Psi(x, y, m, u, v) = \Psi(u, v)$$

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(u^1, u^i)\Psi(u^1, u^j) f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds$$

$$= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(u^1, u^i)^2 f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds \quad \text{if } i = j$$

Particular case

$$\Psi(x, y, m, u, v) = \Psi(u, v)$$

$$\langle J^{N,i}, J^{N,j}\rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(u^1, u^i)\Psi(u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx)ds$$
$$= \varsigma^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N)\mu_s^N(dx)ds \quad \text{if } i = j$$

with $\varsigma^2 = \int \Psi(u^1, u^2)^2 \nu(du)$

Particular case

$$\Psi(x, y, m, u, v) = \Psi(u, v)$$

$$\begin{aligned}
\langle J^{N,i}, J^{N,j} \rangle_t &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(u^1, u^i)\Psi(u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx)ds \\
&= \varsigma^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N)\mu_s^N(dx)ds \quad \text{if } i = j \\
&= \kappa^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N)\mu_s^N(dx)ds \quad \text{if } i \neq j
\end{aligned}$$

with $\varsigma^2 = \int \Psi(u^1, u^2)^2 \nu(du)$ and $\kappa^2 = \int \Psi(u^1, u^2)\Psi(u^1, u^3)\nu(du)$

Particular case

$$\Psi(x,y,m,u,v)=\Psi(u,v)$$

$$\begin{aligned}\langle J^{N,i},J^{N,j}\rangle_t &= \int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}^{\mathbb{N}^*}}\Psi(u^1,u^i)\Psi(u^1,u^j)f(x,\mu_s^N)\nu(du)\mu_s^N(dx)ds\\ &=\varsigma^2\int_0^t\int_{\mathbb{R}}f(x,\mu_s^N)\mu_s^N(dx)ds\quad \text{if } i=j\\ &=\kappa^2\int_0^t\int_{\mathbb{R}}f(x,\mu_s^N)\mu_s^N(dx)ds\quad \text{if } i\neq j\end{aligned}$$

with $\varsigma^2=\int\Psi(u^1,u^2)^2\nu(du)$ and $\kappa^2=\int\Psi(u^1,u^2)\Psi(u^1,u^3)\nu(du)$

$$\bar{J}_t^i=\kappa\int_0^t\sqrt{\int_{\mathbb{R}}f(x,\mu_s)\mu_s(dx)}dW_s+\sqrt{\varsigma^2-\kappa^2}\int_0^t\sqrt{\int_{\mathbb{R}}f(x,\mu_s)\mu_s(dx)}dW_s^i$$

with W,W^i i.i.d. Brownian motions and $\mu=\mathcal{L}(\bar{X}^i|W)$

General case

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}}$$

$$\Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i) \Psi(x, X^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds$$

General case

$$\langle J^{N,i}, J^{N,j}\rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i)\Psi(x, X^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx)ds$$

General case

$$\langle J^{N,i}, J^{N,j}\rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}}$$
$$\Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i)\Psi(x, X^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx) ds$$

Problem : the blue term is not a product, but an integral of a product

General case

$$\langle J^{N,i}, J^{N,j}\rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i)\Psi(x, X^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N)\nu(du)\mu_s^N(dx)ds$$

Problem : the blue term is not a product, but an integral of a product

Solution : let $M(dt, dz) = M_t(dz)$ be a martingale measure on $\mathbb{R}_+ \times E$ with intensity $dt \cdot m_t(dz)$,

$$\langle M_\cdot(A), M_\cdot(B)\rangle_t = \int_0^t \int_E 1_A(z) \cdot 1_B(z) m_s(dz) ds$$

General case

$$\langle J^{N,i}, J^{N,j} \rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(x, X_s^{N,i}, \mu_s^N, u^1, u^i) \Psi(x, X^{N,j}, \mu_s^N, u^1, u^j) f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds$$

Problem : the blue term is not a product, but an integral of a product

Solution : let $M(dt, dz) = M_t(dz)$ be a martingale measure on $\mathbb{R}_+ \times E$ with intensity $dt \cdot m_t(dz)$,

$$\langle M.(A), M.(B) \rangle_t = \int_0^t \int_E 1_A(z) \cdot 1_B(z) m_s(dz) ds$$

Here : $E = \mathbb{R}^{\mathbb{N}^*} \times \mathbb{R}$ and $m_s(du, dx) = \nu(du) \cdot \mu_s(dx)$

Limit system (1)

$$d\bar{X}_t^i = b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i + \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) + \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)$$

with

$$\tilde{\Psi}(x, y, m, v) = \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(x, y, m, v, u^1)\nu(du)$$

$$\kappa(x, y, m)^2 = \int \Psi(x, y, m, u^1, u^2)^2\nu(du) - \int \tilde{\Psi}(x, y, m, u^1)^2\nu(du)$$

Limit system (1)

$$
\begin{aligned}
d\bar{X}_t^i =& b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\
&+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\
&+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)
\end{aligned}
$$

with

$$
\tilde{\Psi}(x, y, m, v) = \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(x, y, m, v, u^1)\nu(du)
$$

$$
\kappa(x, y, m)^2 = \int \Psi(x, y, m, u^1, u^2)^2\nu(du) - \int \tilde{\Psi}(x, y, m, u^1)^2\nu(du)
$$

Interpretation of $\tilde{\Psi}$:

$$
\int \tilde{\Psi}(x, y, m, u^1)^2\nu(du) = \int \Psi(x, y, m, u^1, u^2)\Psi(x, y, m, u^1, u^3)\nu(du)
$$

Limit system (2)

$$
\begin{aligned}
d\bar{X}^i_t =& b(\bar{X}^i_t, \mu_t)dt + \sigma(\bar{X}^i_t, \mu_t)d\beta^i_t \\
&+ \int \tilde{\Psi}(x, \bar{X}^i_t, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\
&+ \int \kappa(x, \bar{X}^i_t, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)
\end{aligned}
$$

Limit system (2)

$$\begin{aligned} d\bar{X}^i_t =& b(\bar{X}^i_t, \mu_t)dt + \sigma(\bar{X}^i_t, \mu_t)d\beta^i_t \\ &+ \int \tilde{\Psi}(x, \bar{X}^i_t, \mu_t, v)\sqrt{f(x,\mu_t)}dM(t,x,v) \\ &+ \int \kappa(x, \bar{X}^i_t, \mu_t)\sqrt{f(x,\mu_t)}dM^i(t,x) \end{aligned}$$

M and M^i are orthogonal (not independent) :

$$M^i_t(A) = \int_0^t \int_0^1 1_A(F_s^{-1}(p))dW^i(s,p)$$

$$M_t(A \times B) = \int_0^t \int_0^1 \int_{\mathbb{R}} 1_A(F_s^{-1}(p))1_B(v)dW(s,p,v)$$

Limit system (2)

$$\begin{aligned} d\bar{X}_t^i =& b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\ &+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\ &+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x) \end{aligned}$$

M and M^i are orthogonal (not independent) :

$$M_t^i(A) = \int_0^t \int_0^1 1_A(F_s^{-1}(p))dW^i(s, p)$$

$$M_t(A \times B) = \int_0^t \int_0^1 \int_{\mathbb{R}} 1_A(F_s^{-1}(p))1_B(v)dW(s, p, v)$$

with :

- W^i, W independent WN with intensities $dtdp$ and $dtdp\nu_1(dv)$

Limit system (2)

$$
\begin{aligned}
d\bar{X}_t^i =& b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\
&+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\
&+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)
\end{aligned}
$$

M and M^i are orthogonal (not independent) :

$$
M_t^i(A) = \int_0^t \int_0^1 1_A(F_s^{-1}(p))dW^i(s, p)
$$

$$
M_t(A \times B) = \int_0^t \int_0^1 \int_{\mathbb{R}} 1_A(F_s^{-1}(p))1_B(v)dW(s, p, v)
$$

with :

- W^i, W independent WN with intensities $dtdp$ and $dtdp\nu_1(dv)$
- F_s^{-1} = generalized inverse distribution function of μ_s

Limit system (2)

$$d\bar{X}^i_t = b(\bar{X}^i_t, \mu_t)dt + \sigma(\bar{X}^i_t, \mu_t)d\beta^i_t$$
$$+ \int \tilde{\Psi}(x, \bar{X}^i_t, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v)$$
$$+ \int \kappa(x, \bar{X}^i_t, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)$$

M and M^i are orthogonal (not independent) :

$$M^i_t(A) = \int_0^t \int_0^1 1_A(F_s^{-1}(p))dW^i(s, p)$$
$$M_t(A \times B) = \int_0^t \int_0^1 \int_{\mathbb{R}} 1_A(F_s^{-1}(p))1_B(v)dW(s, p, v)$$

with :

- W^i, W independent WN with intensities $dtdp$ and $dtdp\nu_1(dv)$
- F_s^{-1} = generalized inverse distribution function of μ_s
- $\mu_s = \mathcal{L}(\bar{X}^i_s | W)$

Bibliography (1)

- E., Löcherbach, Loukianova (2020). Conditional propagation of chaos for mean field systems of interacting neurons. HAL, arXiv.
- Delattre, Fournier, Hoffman (2016). Hawkes processes on large networks. Annals of Applied Probability.
- Chevallier, Duarte, Löcherbach, Ost (2019). Mean field limits for nonlinear spatially extended Hawkes processes with exponential memory kernels. Stochastic Processes and their Applications.
- E., Löcherbach, Loukianova (2019). Mean field limits for interacting Hawkes processes in a diffusive regime. HAL, arXiv.
- De Masi, Galves, Löcherbach, Presutti (2015). Hydrodynamic limit for interacting Hawkes processes. Journal of Statistical Physics.
- Fournier, Löcherbach (2016). On a toy model of interacting neurons. Annales de l'institut Henri Poincaré.
- Andreis, Dai Pra, Fischer (2017). McKean-Vlasov limit for interacting systems with simultaneous jumps. arXiv.

Bibliography (2)

- Aldous (1983). Exchangeability and related topics. Ecoles d'Eté de Probabilités de Saint-Flour : XIII.
- Graham (1992).McKean-Vlasov Ito-Skorohod equations, and nonlinear diffusions with discrete jump sets. Stochastic Processes and their Applications.
- El Karoui, Méléard (1990). Martingale measures and stochastic calculus. Probability theory and related fields.

Thank you for your attention !

Questions ?