Conditional propagation of chaos for mean field systems of interacting neurons

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Joint work with: Eva Löcherbach [2] and Dasha Loukianova [1]

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Point process : definitions

A point process Z is :

- a random countable set of $\mathbb{R}_+$: $Z = \{T_i : i \in \mathbb{N}\}$
- a random point measure on $\mathbb{R}_+$: $Z = \sum_{i\in\mathbb{N}} \delta_{T_i}$

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A process λ is the stochastic intensity of Z if :

$$\forall 0 \leq a < b, \mathbb{E}\left[Z([a,b]) | \mathcal{F}_a\right] = \mathbb{E}\left[\int_a^b \lambda_t dt \middle| \mathcal{F}_a\right]$$

Poisson measure

E measurable space

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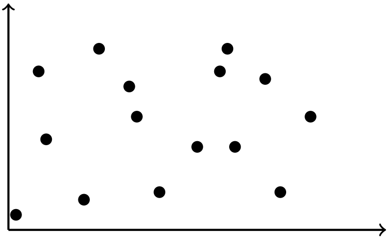
π Poisson measure on E : random point measure that satisfies



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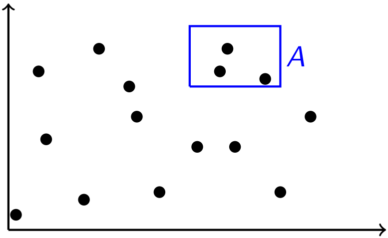

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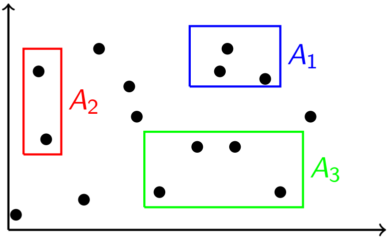

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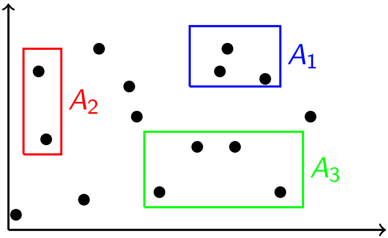

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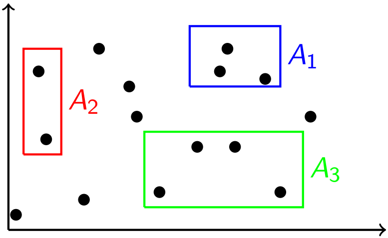
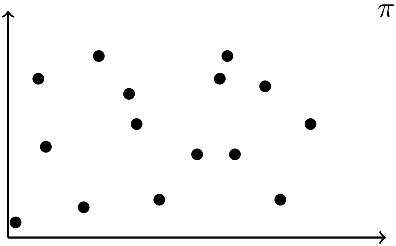


Intensity of π : $\mu(A) = \mathbb{E}[\pi(A)]$

μ characterizes the law of π

Thinning

π Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $dt.dz$



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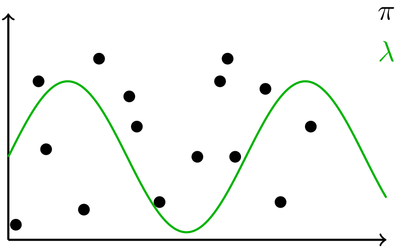

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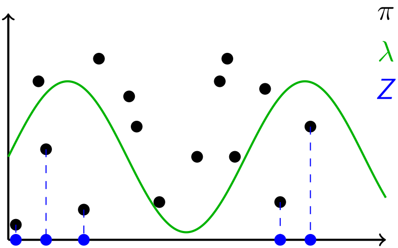
$$Z(A) = \int_{A\times\mathbb{R}_+} 1_{\{z\leq\lambda(t)\}} d\pi(t,z)$$



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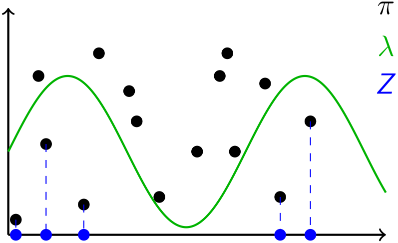
Then : λ is the stochastic intensity of Z



Exchangeable system

Definition

A system of r.v. $(X_i)_{i\in I}$ is exchangeable if :
for all finite permutation σ, $\mathcal{L}((X_i)_{i\in I}) = \mathcal{L}((X_{\sigma(i)})_{i\in I})$

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- μ is unique a.s.
- μ is the directing measure of $(X_i)_{i\in I}$

Modeling in neuroscience

Neural activity $=$ Set of spike times

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Network of N neurons :

- $$\begin{aligned}Z^{N,i} &= \text{set of spike times of neuron } i\\ &= \text{point process with intensity } f(X^{N,i}_{t-})\end{aligned}$$
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Here, $X^{N,i}$ solves an SDE directed by $(Z^{N,j})_{1\leq j\leq N}$

Mean field limit

$N-$particle system :

- $$Z_t^{N,i} = \int_0^t \int_0^\infty 1_{\left\{z \le f(X_{s-}^{N,i})\right\}} d\pi^i(s,z)$$
- $$dX_t^{N,i} = b(X_t^{N,i})dt + \sum_{j=1}^N \int_0^\infty u^{ji}(t) 1_{\left\{z \le f(X_{t-}^{N,j})\right\}} d\pi^j(t,z)$$

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- linear scaling N^{-1} (LLN) :
  [Delattre et al. (2016)] (Hawkes process, $u^{ji}(t) = 1$),
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- diffusive scaling $N^{-1/2}$ (CLT) :
  [E. et al. (2019)] random and centered $u^{ji}(s)$

Linear scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_0^\infty 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z) - \int_0^\infty X_{t-}^{N,i} 1_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z)$$

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Intepretation :

- drift : $-\alpha x$ models an exponantial loss of the potential
- small jump of order N^{-1} : the effect of spike of one neuron to the potential of the others
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[De Masi et al. (2015)] and [Fournier & Löcherbach (2016)]
Generalization to McKean-Vlasov frame [Andreis et al. (2018)]

Diffusive scaling

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_0^\infty \int_{\mathbb{R}} u 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$
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π^j iid Poisson measures with intensity $dt \cdot dz \cdot d\nu(u)$

ν probability measure on $\mathbb{R}$ centered with $\int_{\mathbb{R}} |u|^3 d\nu(u) < \infty$

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Dynamic of $X^{N,i}$:

- $X_t^{N,i} = X_s^{N,i} e^{-\alpha(t-s)}$ if the system does not jump in $[s,t]$
- $X_t^{N,i} = X_{t-}^{N,i} + \frac{U}{\sqrt{N}}$ if a neuron $j \neq i$ emits a spike at t
- $X_t^{N,i} = 0$ if neuron i emits a spike at t

Limit system : heuristic (1)

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\left\{ z \leq f(X_{t-}^{N,j}) \right\}} d\pi^j(t,z,u)$$

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$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + d\bar{M}_t$$

$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z \le f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

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Then $\bar{M}$ should satisfy

$$\bar{M}_t = \sigma \int_0^t \sqrt{\lim_N \frac{1}{N} \sum_{j=1}^{N} f(\bar{X}_s^j)} dW_s = \sigma \int_0^t \sqrt{\lim_N \bar{\mu}_s^N(f)} dW_s$$

with $\bar{\mu}^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{\bar{X}^j}$

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μ is the limit of empirical measures of $(\bar{X}^i)_{i \ge 1}$ exchangeable by Proposition (7.20) of [Aldous (1983)] μ is the directing measure of $(\bar{X}^i)_{i \ge 1}$ (conditionally on μ, $\bar{X}^i$ i.i.d.$\sim \mu$)

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Well-posedness of the limit equation (1)

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$$-\bar{X}^i_{t-}\int_{\mathbb{R}_+\times\mathbb{R}} 1_{\{z\leq f(\bar{X}^i_{t-})\}}d\pi^i(t,z,u)$$

Problems :

- conditional expectation in the Brownian term (McKean-Vlasov frame)
- unbounded jumps (non-Lipschitz compensator $x \mapsto -xf(x)$)
- jump term and Brownian term

Well-posedness of the limit equation (1)

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mathbb{E}\left[f(\bar{X}_t^i)|W\right]} dW_t$$
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Problems :

- conditional expectation in the Brownian term (McKean-Vlasov frame)
- unbounded jumps (non-Lipschitz compensator $x \mapsto -xf(x)$)
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Solution : consider $a : \mathbb{R} \to \mathbb{R}_+$ increasing, bounded, lower-bounded, C^2 such that

$$|a''(x) - a''(y)| + |a'(x) - a'(y)|$$
$$+|xa'(x) - ya'(y)| + |f(x) - f(y)| \leq C|a(x) - a(y)|$$

Well-posedness of the limit equation (2)

$$
\begin{aligned}
a(\bar{X}_t^i) =& a(\bar{X}_0^i) - \alpha \int_0^t \bar{X}_s^i a'(\bar{X}_s^i) ds + \sigma \int_0^t a'(\bar{X}_s^i) \sqrt{\mathbb{E}[f(\bar{X}_s^i) | W]} dW_s \\
&+ \frac{\sigma^2}{2} \int_0^t a''(\bar{X}_s^i) \mathbb{E}[f(\bar{X}_s^i) | W] ds \\
&+ \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} [a(0) - a(\bar{X}_{s-}^i)] 1_{\{z \le f(\bar{X}_{s-}^i)\}} d\pi^i(s,z,u)
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$$\begin{aligned} a(\bar{X}^i_t) = & a(\bar{X}^i_0) - \alpha \int_0^t \bar{X}^i_s a'(\bar{X}^i_s) ds + \sigma \int_0^t a'(\bar{X}^i_s) \sqrt{\mathbb{E}[f(\bar{X}^i_s)|W]} dW_s \\ & + \frac{\sigma^2}{2} \int_0^t a''(\bar{X}^i_s) \mathbb{E}[f(\bar{X}^i_s)|W] ds \\ & + \int_{[0,t]\times\mathbb{R}_+\times\mathbb{R}} [a(0) - a(\bar{X}^i_{s-})] 1_{\{z \le f(\bar{X}^i_{s-})\}} d\pi^i(s,z,u) \end{aligned}$$

To prove trajectorial uniqueness :

- $u(t) = \mathbb{E}[|a(\hat{X}^i_s) - a(\check{X}^i_s)|]$ (problem with Brownian term)
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Iteratively $\forall n \in \mathbb{N}, u(nt_0) = 0$, whence $\forall t > 0, u(t) = 0$

Discussion about the function f

Any $f \in C_b^1(\mathbb{R}, \mathbb{R}_+)$ satisfying $f'(x) \leq C(1+|x|)^{-(1+\varepsilon)}$ $(\varepsilon > 0)$

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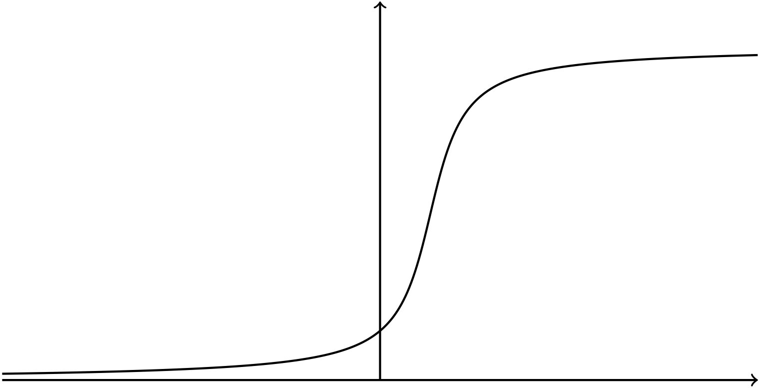
$f(x) = c + d \arctan(\alpha + \beta x)$ satisfy the hypothesis



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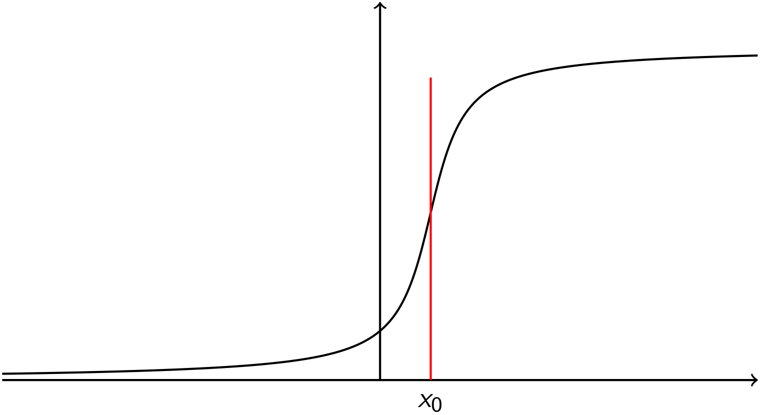

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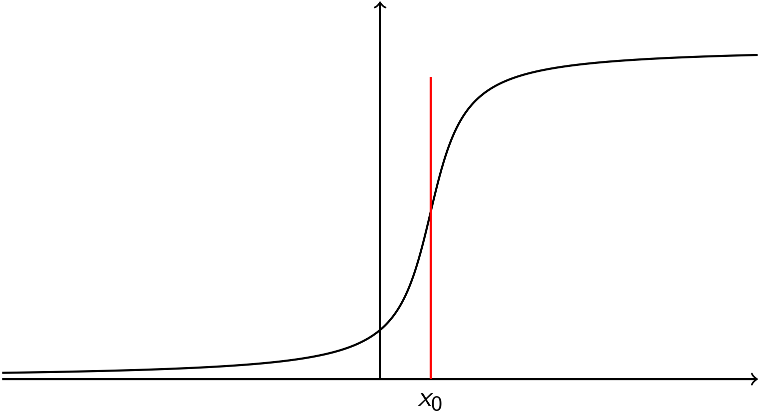
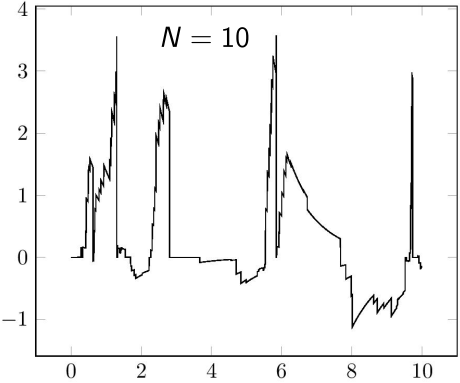
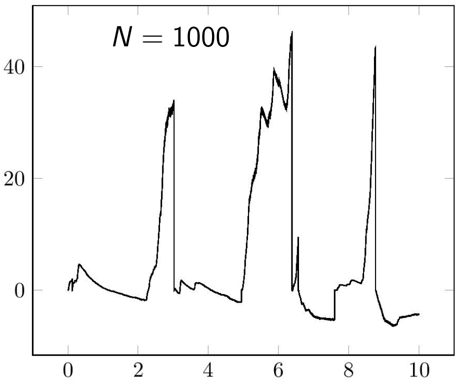


"Neuron i active / inactive" $\approx$ "$X^{N,i} > x_0$ / $X^{N,i} < x_0$"

Simulations of $X^{N,1}$





Another version of the limit system

The strong limit system :

$$d\bar{X}^i_t = -\alpha\bar{X}^i_t dt + \sigma\sqrt{\mathbb{E}\left[f(\bar{X}^i_t)|W\right]}dW_t$$
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The weak limit system :

$$d\bar{Y}_t^i = -\alpha\bar{Y}_t^i dt + \sigma\sqrt{\mu_t(f)}dW_t - \bar{Y}_{t-}^i \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\{z\le f(\bar{Y}_{t-}^i)\}} d\pi^i(t,z,u)$$

where $\mu_t = \mathcal{L}(\bar{Y}_t^1|\mu_t)$ is the directing measure of $(\bar{Y}_t^i)_{i\ge 1}$

Equivalence between the two systems

An auxiliary system :

$$
\begin{aligned}
d\tilde{X}_t^{N,i} = & -\alpha \tilde{X}_t^{N,i} dt + \sigma \sqrt{\frac{1}{N} \sum_{j=1}^{N} f(\tilde{X}_t^{N,j})} dW_t \\
& - \tilde{X}_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{ z \leq f(\tilde{X}_{t-}^{N,i}) \right\}} d\pi^i(t,z,u)
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For $0 \leq t \leq T$ (small enough)

$$u_N(t) \leq CN^{-1/2} \underset{N\to\infty}{\longrightarrow} 0$$

Convergence of $(X^{N,i})_{1\leq i\leq N}$

$$\begin{aligned} dX_t^{N,i} =& -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j\neq i} \int_{\mathbb{R}_+\times\mathbb{R}} u 1_{\left\{z\leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) \\ & - X_{t-}^{N,i} \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u) \\ d\bar{X}_t^i =& -\alpha \bar{X}_t^i dt + \sigma\sqrt{\mu_t(f)} dW_t \\ & - \bar{X}_{t-}^i \int_{\mathbb{R}_+\times\mathbb{R}} 1_{\left\{z\leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u) \end{aligned}$$

Goal : $(X^{N,i})_{1\leq i\leq N}$ converges to $(\bar{X}^i)_{i\geq 1}$ in $D^{\mathbb{N}^*}$

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Equivalent condition (Proposition (7.20) of [Aldous (1983)]) :
$\mu^N := \sum_{j=1}^N \delta_{X^{N,j}}$ converges to $\mu := \mathcal{L}(\bar{X}^1|W)$ in $\mathcal{P}(D)$

Outline of the proof

Step 1. $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
Equivalent condition : $(X^{N,1})_N$ is tight on D
Proof : Aldous' criterion

Step 2. Identifying the limit distribution of $(\mu^N)_N$
Proof : any limit of μ^N is solution of a martingale problem

Martingale problem : Principle

SDE :

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_{\mathbb{R}_+ \times E} \Phi(X_{t-}, u) 1_{\{z \leq f(X_{t-})\}} d\pi(t, z, u)$$

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Martingale problem : for g smooth

$$g(Y_t) - g(Y_0) - \int_0^t Lg(Y_s)ds \text{ is a local martingale,}$$

$$Lg(x) = b(x)g'(x) + \frac{1}{2}\sigma(x)^2 g''(x) + f(x)\int_E \left(g(x+\Phi(x,u)) - g(x)\right) d\nu(u)$$

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Martingale problem $\Rightarrow$ SDE : representation theorems

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$$\begin{aligned} Lg(m, x^1, x^2) = & -\alpha x^1 \partial_1 g(x) - \alpha x^2 \partial_2 g(x) + \frac{\sigma^2}{2} m(f) \sum_{i,j=1}^{2} \partial_{i,j}^2 g(x) \\ & + f(x^1)(g(0, x^2) - g(x)) + f(x^2)(g(x^1, 0) - g(x)) \end{aligned}$$

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$\mathcal{L}(\mu, \bar{Y}^1, \bar{Y}^2) = P$ (from the martingale problem)

Representation theorems imply (admitted)

$$\begin{aligned}\forall i \in \mathbb{N}^*, d\bar{Y}^i_t =& -\alpha \bar{Y}^i_t dt + \sqrt{\mu_t(f)} dW_t \\ & - \bar{Y}^i_{t-} \int_{\mathbb{R}_+} 1_{\{z \leq f(\bar{Y}^i_{t-})\}} d\pi^i(t,z)\end{aligned}$$

Then the law of $\mu = \mathcal{L}(\bar{Y}^1 | W)$ is uniquely determined

Convergence of μ^N to the solution of $(\mathcal{M})$

Let μ be the limit of (a subsequence of) μ^N
$\mathcal{L}(\mu)$ is solution of $(\mathcal{M})$ if

$$\mathbb{E}\left[F(\mu)\right] = 0$$

for any F of the form

$$F(m) := \int_{D^2} m \otimes m(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s) - \int_s^t L\phi(m_r, \gamma_r)dr\Big]$$

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$$F(m) := \int_{D^2} m \otimes m(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s)$$
$$+\alpha \int_s^t \gamma_r^1 \partial_1 \phi(\gamma_r)dr + \alpha \int_s^t \gamma_r^2 \partial_2 \phi(\gamma_r)dr$$
$$-\int_s^t f(\gamma_r^1)(\phi(0,\gamma_r^2) - \phi(\gamma_r))dr - \int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r))dr$$
$$-\frac{\sigma^2}{2}\int_s^t m_r(f) \sum_{i_1,i_2=1}^{2} \partial^2_{i_1,i_2}\phi(\gamma_r)dr\Big]$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\int_{D^2} \mu^N \otimes \mu^N(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s)$$

$$+ \alpha \int_s^t \gamma_r^1 \partial_1 \phi(\gamma_r) dr + \alpha \int_s^t \gamma_r^2 \partial_2 \phi(\gamma_r) dr$$

$$- \int_s^t f(\gamma_r^1)(\phi(0, \gamma_r^2) - \phi(\gamma_r)) dr$$

$$\int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r)) dr$$

$$- \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1, i_2 = 1}^{2} \partial^2_{i_1, i_2} \phi(\gamma_r) dr\Big]$$

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$$
\begin{aligned}
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&+ \alpha \int_s^t \gamma_r^1 \partial_1 \phi(\gamma_r)dr + \alpha \int_s^t \gamma_r^2 \partial_2 \phi(\gamma_r)dr \\
&- \int_s^t f(\gamma_r^1)(\phi(0,\gamma_r^2) - \phi(\gamma_r))dr \\
&\int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r))dr \\
&- \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1,i_2=1}^2 \partial^2_{i_1,i_2}\phi(\gamma_r)dr\Big]
\end{aligned}
$$

The expression of $F(\mu^N)$

$$
\begin{aligned}
&F(\mu^N) := \\
&\frac{1}{N^2}\sum_{i,j=1}^{N} \phi_1(X_{s_1}^{N,i}, X_{s_1}^{N,j})\ldots\phi_k(X_{s_k}^{N,i}, X_{s_k}^{N,j})\Big[\phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) \\
&+ \alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i}, X_r^{N,j})dr + \alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i}, X_r^{N,j})dr \\
&- \int_s^t f(X_r^{N,i})(\phi(0, X_r^{N,j}) - \phi(X_r^{N,i}, X_r^{N,j}))dr \\
&- \int_s^t f(X_r^{N,j})(\phi(X_r^{N,i}, 0) - \phi(X_r^{N,i}, X_r^{N,j}))dr \\
&- \frac{\sigma^2}{2}\int_s^t \mu_r^N(f)\sum_{i_1,i_2=1}^{2} \partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})dr\Big]
\end{aligned}
$$

The expression of $F(\mu^N)$

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\begin{aligned}
&F(\mu^N) := \\
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&+\alpha\int_s^t X^{N,i}_r\partial_1\phi(X^{N,i}_r,X^{N,j}_r)dr+\alpha\int_s^t X^{N,j}_r\partial_2\phi(X^{N,i}_r,X^{N,j}_r)dr\\
&-\int_s^t f(X^{N,i}_r)(\phi(0,X^{N,j}_r)-\phi(X^{N,i}_r,X^{N,j}_r))dr\\
&-\int_s^t f(X^{N,j}_r)(\phi(X^{N,i}_r,0)-\phi(X^{N,i}_r,X^{N,j}_r))dr\\
&-\frac{\sigma^2}{2}\int_s^t \mu^N_r(f)\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X^{N,i}_r,X^{N,j}_r)dr\Big]
\end{aligned}
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&+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i},X_r^{N,j})dr \\
&-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j})-\phi(X_r^{N,i},X_r^{N,j}))dr \\
&-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0)-\phi(X_r^{N,i},X_r^{N,j}))dr \\
&-\frac{\sigma^2}{2}\int_s^t \frac{1}{N}\sum_{k=1}^{N} f(X_r^{N,k})\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})dr\Big]
\end{aligned}
$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^N \phi_1(X_{s_1}^{N,i},X_{s_1}^{N,j})...\phi_k(X_{s_k}^{N,i},X_{s_k}^{N,j})\Big[\phi(X_t^{N,i},X_t^{N,j})-\phi(X_s^{N,i},X_s^{N,j})$$

$$+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i},X_r^{N,j})dr$$

$$-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j})-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0)-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\frac{\sigma^2}{2}\int_s^t \frac{1}{N}\sum_{k=1}^N f(X_r^{N,k})\sum_{i_1,i_2=1}^2 \partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})dr\Big]$$

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&+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i},X_r^{N,j})dr\\
&-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j})-\phi(X_r^{N,i},X_r^{N,j}))dr\\
&-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0)-\phi(X_r^{N,i},X_r^{N,j}))dr\\
&-\int_s^t\int_{\mathbb{R}}\frac{u^2}{2}\frac{1}{N}\sum_{k=1}^{N}f(X_r^{N,k})\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})d\nu(u)dr\Big]
\end{aligned}
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The expression of $F(\mu^N)$

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\begin{aligned}
&F(\mu^N) := \\
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&+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i}, X_r^{N,j})dr + \alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i}, X_r^{N,j})dr \\
&-\int_s^t f(X_r^{N,i})(\phi(0, X_r^{N,j}) - \phi(X_r^{N,i}, X_r^{N,j}))dr \\
&-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i}, 0) - \phi(X_r^{N,i}, X_r^{N,j}))dr \\
&-\int_s^t\int_{\mathbb{R}} \frac{u^2}{2}\frac{1}{N}\sum_{k=1}^{N} f(X_r^{N,k})\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})d\nu(u)dr\Big]
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&+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i},X_r^{N,j})dr \\
&-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j})-\phi(X_r^{N,i},X_r^{N,j}))dr \\
&-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0)-\phi(X_r^{N,i},X_r^{N,j}))dr \\
&-\int_s^t\int_{\mathbb{R}}\sum_{k=1}^{N} f(X_r^{N,k})\frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})d\nu(u)dr\Big]
\end{aligned}
$$

The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_1(X_{s_1}^{N,i},X_{s_1}^{N,j})...\phi_k(X_{s_k}^{N,i},X_{s_k}^{N,j})\Big[\phi(X_t^{N,i},X_t^{N,j})-\phi(X_s^{N,i},X_s^{N,j})$$

$$+\alpha\int_s^t X_r^{N,i}\partial_1\phi(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}\partial_2\phi(X_r^{N,i},X_r^{N,j})dr$$

$$-\int_s^t f(X_r^{N,i})(\phi(0,X_r^{N,j})-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0)-\phi(X_r^{N,i},X_r^{N,j}))dr$$

$$-\int_s^t\int_{\mathbb{R}}\sum_{k=1}^{N}f(X_r^{N,k})\frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})d\nu(u)dr\Big]$$

The expression of $\phi(X^{N,i}, X^{N,j})$

By Ito's formula,

$$\mathbb{E}\phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) =$$

$$\mathbb{E} - \alpha \int_s^t X_r^{N,i} \partial_1 \phi(X_r^{N,i}, X_r^{N,j}) dr - \alpha \int_s^t X_r^{N,j} \partial_2 \phi(X_r^{N,i}, X_r^{N,j}) dr$$

$$+ \int_s^t \int_{\mathbb{R}} f(X_r^{N,i}) \left(\phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) \right) d\nu(u) dr$$

$$+ \int_s^t \int_{\mathbb{R}} f(X_r^{N,j}) \left(\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, 0) - \phi(X_r^{N,i}, X_r^{N,j}) \right) d\nu(u) dr$$

$$+ \int_s^t \int_{\mathbb{R}} \sum_{\substack{k=1 \\ k \neq i,j}}^{N} f(X_r^{N,k}) \left(\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) \right) d\nu(u) dr$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

The small jump term

$$N\left|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

The small jump term

$$N\left|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) \right.$$
$$\left. - \frac{u}{\sqrt{N}}\sum_{i_1=1}^{2}\partial_{i_1}\phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

The small jump term

$$N\Bigg|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j})$$
$$- \frac{u}{\sqrt{N}}\sum_{i_1=1}^{2}\partial_{i_1}\phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\Bigg|$$
$$\leq CN\frac{|u|^3}{N\sqrt{N}} = CN^{-1/2}|u|^3$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

The small jump term

$$N\left|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j})\right.$$
$$\left. - \frac{u}{\sqrt{N}}\sum_{i_1=1}^{2}\partial_{i_1}\phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$
$$\leq CN\frac{|u|^3}{N\sqrt{N}} = CN^{-1/2}|u|^3$$

$$CN^{-1/2} \geq \mathbb{E}\left[F(\mu^N)\right] \underset{N\to\infty}{\longrightarrow} \mathbb{E}\left[F(\mu)\right] = 0$$

Convergence of $(\mu^N)_N$

$$
\begin{aligned}
dX_t^{N,i} =& -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) \\
& - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u) \\
d\bar{X}_t^i =& -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\
& - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)
\end{aligned}
$$

Convergence of $(\mu^N)_N$

$$
\begin{aligned}
dX_t^{N,i} =& -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) \\
& - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u) \\
d\bar{X}_t^i =& -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\
& - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)
\end{aligned}
$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$

Convergence of $(\mu^N)_N$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$
$$- X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$
$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t$$
$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
- let μ be the limit of a converging subsequence

Convergence of $(\mu^N)_N$

$$
\begin{aligned}
dX_t^{N,i} =& -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) \\
& - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u) \\
d\bar{X}_t^i =& -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\
& - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)
\end{aligned}
$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
- let μ be the limit of a converging subsequence
- $\mathcal{L}(\mu)$ is the unique solution of $(\mathcal{M})$

Convergence of $(\mu^N)_N$

$$
\begin{aligned}
dX_t^{N,i} =& -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u) \\
& - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u) \\
d\bar{X}_t^i =& -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\
& - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)
\end{aligned}
$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
- let μ be the limit of a converging subsequence
- $\mathcal{L}(\mu)$ is the unique solution of $(\mathcal{M})$
- $\mu = \mathcal{L}(\bar{X}^1 | W)$ is the only limit of $(\mu^N)_N$

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Thank you for your attention !

Questions ?