Limite de grande échelle de systèmes de particules en interaction avec sauts simultanés en régime diffusif

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grand amphi, bâtiment IBGBI, Evry

Contents

Interacting particle systems

Our frame :

- Particle system = system of point processes $(Z^{N,i})_{1 \leq i \leq N}$
- Interactions : the intensity of each $Z^{N,i}$ $(1 \leq i \leq N)$ is solution to a SDE directed by the $Z^{N,j}$ $(1 \leq j \leq N)$

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Modelization : consider a $N-$particle system

- each particle triggers events
- $Z^{N,i}_t$ = number of events triggered by particule i before time t
- $\Rightarrow$ the particles excite or inhibite each others

Neuroscience

Neural network :

- Network of N neurons (neuron $=$ particle)

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- $X^{N,i}$ is the potential of neuron i : $X^{N,i}$ is a càdlàg process that jumps when neuron i receives a spike

Particle systems with diffusive mean field interactions

For each $N \in \mathbb{N}^*$, we consider $(Z^{N,1}, ..., Z^{N,N})$

$$dX_t^{N,i} = b^i\left(X_t^{N,i}\right)dt + \sum_{j=1}^{N} dZ_t^{N,j}$$

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Convergence of X^N

$$dX_t^N = -\alpha X_t^N dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{(z,u)\in\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\{z \leq f(X_{t-}^N)\}} d\pi^j(t,z,u)$$

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Consequence of Theorem IX.4.15 of [Jacod & Shiryaev (2003)]

X^N converges in distribution to $\bar{X}$ in $D(\mathbb{R}_+,\mathbb{R})$

$$d\bar{X}_t = -\alpha \bar{X}_t dt + \sigma \sqrt{f(\bar{X}_t)} dW_t$$

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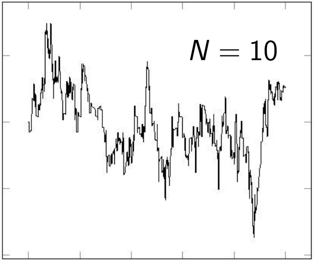

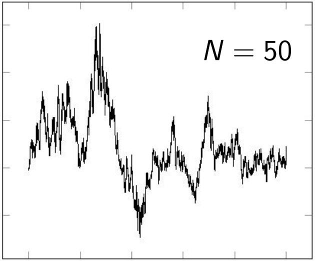

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Dynamic of a neural network

System of N neurons :

$$dX_t^N = -\alpha X_t^N \, dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u \mathbb{1}_{\left\{ z \leq f(X_{t-}^N) \right\}} d\pi^j(t, z, u)$$

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Dynamic of $X^{N,i}$:

- $X_t^{N,i} = X_s^{N,i} e^{-\alpha(t-s)}$ if the system does not jump in $]s,t]$
- $X_t^{N,i} = X_{t-}^{N,i} + \frac{U}{\sqrt{N}}$ if a neuron $j \neq i$ emits a spike at t
- $X_t^{N,i} = 0$ if neuron i emits a spike at t

Limit system : heuristic (1)

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{\substack{j=1 \\ j \neq i}}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$
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$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + d\bar{M}_t$$
$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+\times\mathbb{R}} \mathbb{1}_{\left\{z\leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

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$$\langle \bar{M} \rangle_t = \lim_N \langle M^N \rangle_t = \lim_N \sigma^2 \int_0^t \frac{1}{N} \sum_{j=1}^{N} f(X_s^{N,j}) ds$$

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Then $\bar{M}$ should satisfy

$$\bar{M}_t = \sigma \int_0^t \sqrt{\lim_N \frac{1}{N} \sum_{j=1}^{N} f(\bar{X}_s^j)} dW_s = \sigma \int_0^t \sqrt{\lim_N \bar{\mu}_s^N(f)} dW_s$$

with $\bar{\mu}^N := \frac{1}{N} \sum_{j=1}^{N} \delta_{\bar{X}^j}$

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Well-posedness of the limit equation (1)

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Problems :

- conditional expectation in the Brownian term (McKean-Vlasov frame)
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Solution : consider $a : \mathbb{R} \to \mathbb{R}_+$ increasing, bounded, lower-bounded, C^2 such that

$$|a''(x) - a''(y)| + |a'(x) - a'(y)| + |xa'(x) - ya'(y)| + |f(x) - f(y)| \le C|a(x) - a(y)|$$

Well-posedness of the limit equation (2)

$$a(\bar{X}^i_t) = a(\bar{X}^i_0) - \alpha \int_0^t \bar{X}^i_s a'(\bar{X}^i_s) ds + \sigma \int_0^t a'(\bar{X}^i_s) \sqrt{\mathbb{E}[f(\bar{X}^i_s)|W]} dW_s$$
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To prove trajectorial uniqueness :

- $u(t) = \mathbb{E}[|a(\hat{X}^i_s) - a(\check{X}^i_s)|]$ (problem with Brownian term)
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Iteratively $\forall n \in \mathbb{N}, u(nt_0) = 0$, whence $\forall t > 0, u(t) = 0$

Discussion about the function f

Any $f \in C_b^1(\mathbb{R}, \mathbb{R}_+)$ satisfying $f'(x) \leq C(1 + |x|)^{-(1+\varepsilon)}$ $(\varepsilon > 0)$

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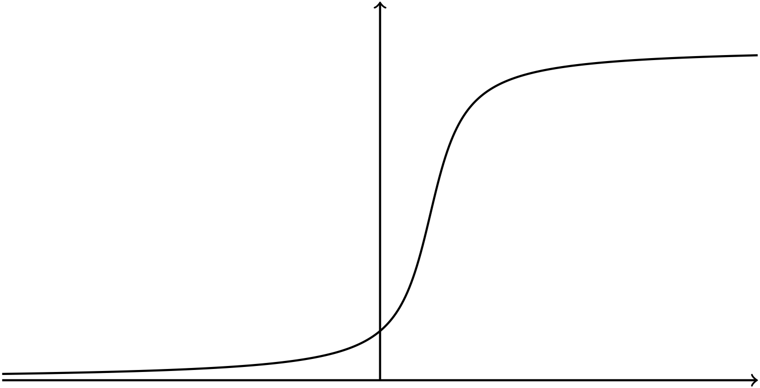

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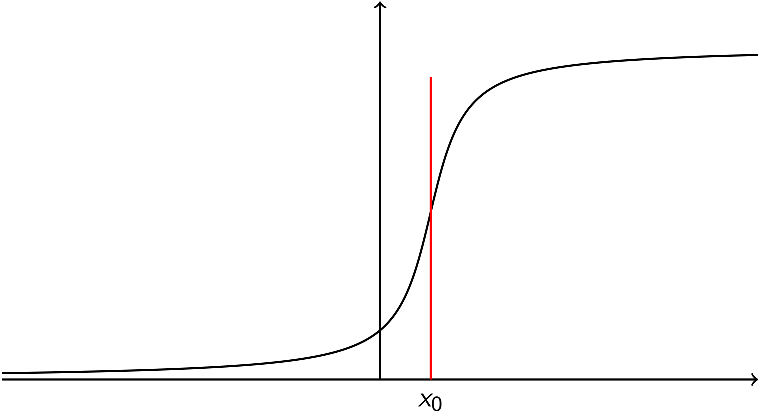

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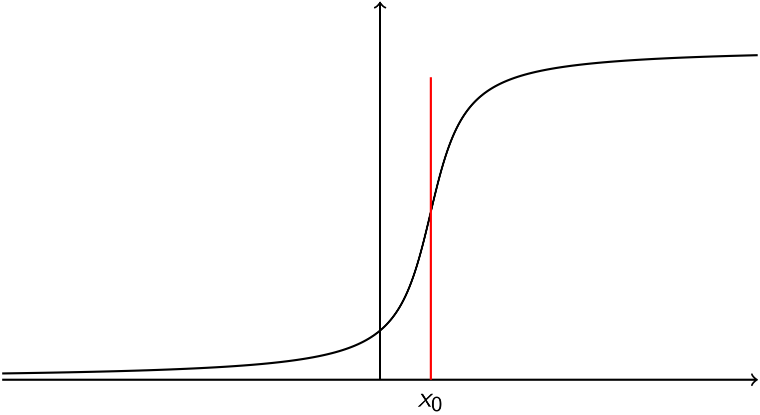
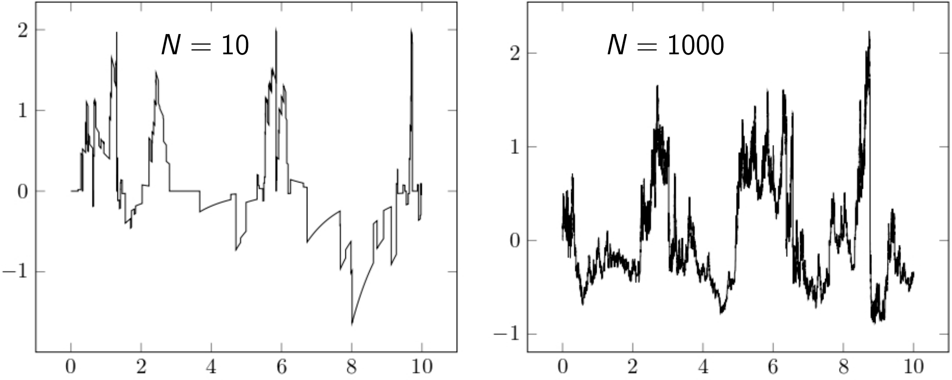


"Neuron i active / inactive" $\approx$ "$X^{N,i} > x_0$ / $X^{N,i} \leq x_0$"

Simulations of $X^{N,1}$



Another version of the limit system

The strong limit system :

$$d\bar{X}^i_t = -\alpha \bar{X}^i_t dt + \sigma \sqrt{\mathbb{E}\left[f(\bar{X}^i_t) | W \right]} dW_t$$
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The weak limit system :

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where $\mu_t = \mathcal{L}(\bar{Y}^1_t|\mu_t)$ is the directing measure of $(\bar{Y}^i_t)_{i\ge1}$

Equivalence between the two systems

An auxiliary system :

$$d\tilde{X}_t^{N,i} = -\alpha\tilde{X}_t^{N,i}dt + \sigma\sqrt{\frac{1}{N}\sum_{j=1}^{N} f(\tilde{X}_t^{N,j})}dW_t$$
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For $0 \le t \le T$ (small enough)

$$u_N(t) \le CN^{-1/2} \underset{N\to\infty}{\longrightarrow} 0$$

Convergence of $(X^{N,i})_{1\leq i\leq N}$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j\neq i} \int_{\mathbb{R}_+\times\mathbb{R}} u \mathbb{1}_{\left\{z\leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$
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Result

$(X^{N,i})_{1\leq i\leq N}$ converges to $(\bar{X}^i)_{i\geq 1}$ in $D(\mathbb{R}_+,\mathbb{R})^{\mathbb{N}^*}$

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NSC (Proposition (7.20) of [Aldous (1983)]) :
$\mu^N := \sum_{j=1}^N \delta_{X^{N,j}}$ converges to $\mu := \mathcal{L}(\bar{X}^1|W)$ in $\mathcal{P}(D(\mathbb{R}_+,\mathbb{R}))$

Outline of the proof

Step 1. $(\mu^N)_N$ is tight on $\mathcal{P}(D(\mathbb{R}_+, \mathbb{R}))$
Equivalent condition : $(X^{N,1})_N$ is tight on $D(\mathbb{R}_+, \mathbb{R})$
Proof : Aldous' criterion

Step 2. Identifying the limit distribution of $(\mu^N)_N$
Proof : any limit of μ^N is solution of a martingale problem

Martingale problem

Given $Q \in \mathcal{P}(\mathcal{P}(D(\mathbb{R}_+, \mathbb{R}))) = \mathcal{P}(\mathcal{P}(D))$ $(Q = \mathcal{L}(\mu))$

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Meaning : (Y^1, Y^2) mixture of iid directed by μ

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Q is solution of $(\mathcal{M})$ if for all $g \in C_b^2(\mathbb{R}^2)$,
$g(Y_t^1, Y_t^2) - g(Y_0^1, Y_0^2) - \int_0^t Lg(\mu_s, Y_s^1, Y_s^2) ds$ is a martingale

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Canonical space $\Omega := \mathcal{P}(D) \times D^2$ with $\omega = (\mu, (Y^1, Y^2))$:

Meaning : (Y^1, Y^2) mixture of iid directed by μ

$$P(A \times B) := \int_{\mathcal{P}(D)} 1_A(m) m \otimes m(B) dQ(m)$$

Q is solution of $(\mathcal{M})$ if for all $g \in C_b^2(\mathbb{R}^2)$,

$g(Y_t^1, Y_t^2) - g(Y_0^1, Y_0^2) - \int_0^t Lg(\mu_s, Y_s^1, Y_s^2) ds$ is a martingale

$$Lg(m, x^1, x^2) = -\alpha x^1 \partial_1 g(x) - \alpha x^2 \partial_2 g(x) + \frac{\sigma^2}{2} m(f) \sum_{i,j=1}^{2} \partial^2_{i,j} g(x)$$
$$+ f(x^1)(g(0, x^2) - g(x)) + f(x^2)(g(x^1, 0) - g(x))$$

Uniqueness for the martingale problem

Let Q be a solution of $(\mathcal{M})$. Write $Q = \mathcal{L}(\mu)$ where μ is the directing measure of some exchangeable system $(\bar{Y}^i)_{i \geq 1}$

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Representation theorems imply (admitted)

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Then the law of $\mu = \mathcal{L}(\bar{Y}^1|W)$ is uniquely determined

Convergence of μ^N to the solution of $(\mathcal{M})$

Let μ be the limit of (a subsequence of) μ^N
$\mathcal{L}(\mu)$ is solution of $(\mathcal{M})$ if

$$\mathbb{E}\left[F(\mu)\right] = 0$$

for any F of the form

$$F(m) := \int_{D^2} m \otimes m(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s) - \int_s^t L\phi(m_r, \gamma_r)dr\Big]$$

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$$-\int_s^t f(\gamma_r^1)(\phi(0,\gamma_r^2) - \phi(\gamma_r))dr - \int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r))dr$$
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The expression of $F(\mu^N)$

$$F(\mu^N) :=$$

$$\int_{D^2} \mu^N \otimes \mu^N(d\gamma)\phi_1(\gamma_{s_1})...\phi_k(\gamma_{s_k})\Big[\phi(\gamma_t) - \phi(\gamma_s)$$

$$+ \alpha \int_s^t \gamma_r^1 \partial_1(\gamma_r)dr + \alpha \int_s^t \gamma_r^2 \partial_2(\gamma_r)dr$$

$$- \int_s^t f(\gamma_r^1)(\phi(0, \gamma_r^2) - \phi(\gamma_r))dr$$

$$\int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r))dr$$

$$- \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1, i_2 = 1}^{2} \partial^2_{i_1, i_2}\phi(\gamma_r)dr\Big]$$

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&+ \alpha \int_s^t \gamma_r^1 \partial_1(\gamma_r) dr + \alpha \int_s^t \gamma_r^2 \partial_2(\gamma_r) dr \\
&- \int_s^t f(\gamma_r^1)(\phi(0, \gamma_r^2) - \phi(\gamma_r)) dr \\
&\int_s^t f(\gamma_r^2)(\phi(\gamma_r^1, 0) - \phi(\gamma_r)) dr \\
&- \frac{\sigma^2}{2} \int_s^t \mu_r^N(f) \sum_{i_1, i_2 = 1}^{2} \partial^2_{i_1, i_2} \phi(\gamma_r) dr \Big]
\end{aligned}
$$

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$$\frac{1}{N^2}\sum_{i,j=1}^{N}\phi_1(X_{s_1}^{N,i},X_{s_1}^{N,j})...\phi_k(X_{s_k}^{N,i},X_{s_k}^{N,j})\Big[\phi(X_t^{N,i},X_t^{N,j})-\phi(X_s^{N,i},X_s^{N,j})$$

$$+\alpha\int_s^t X_r^{N,i}\partial_1(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}{}_r\partial_2(X_r^{N,i},X_r^{N,j})dr$$

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&+\alpha\int_s^t X_r^{N,i}\partial_1(X_r^{N,i},X_r^{N,j})dr+\alpha\int_s^t X_r^{N,j}{}_r\partial_2(X_r^{N,i},X_r^{N,j})dr \\
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&-\int_s^t f(X_r^{N,j})(\phi(X_r^{N,i},0)-\phi(X_r^{N,i},X_r^{N,j}))dr \\
&-\frac{\sigma^2}{2}\int_s^t\frac{1}{N}\sum_{k=1}^{N}f(X_r^{N,k})\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})dr\Big]
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$$+\alpha\int_s^t X^{N,i}_r\partial_1(X^{N,i}_r,X^{N,j}_r)dr+\alpha\int_s^t X^{N,j}_r{}_r\partial_2(X^{N,i}_r,X^{N,j}_r)dr$$

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$$-\int_s^t\int_{\mathbb{R}}\frac{u^2}{2}\frac{1}{N}\sum_{k=1}^{N}f(X^{N,k}_r)\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X^{N,i}_r,X^{N,j}_r)d\nu(u)dr\Big]$$

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$$-\int_s^t\int_{\mathbb{R}}\sum_{k=1}^{N}f(X_r^{N,k})\frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i},X_r^{N,j})d\nu(u)dr\Big]$$

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The expression of $\phi(X^{N,i}, X^{N,j})$

By Ito's formula,

$$\mathbb{E}\phi(X_t^{N,i}, X_t^{N,j}) - \phi(X_s^{N,i}, X_s^{N,j}) =$$

$$\mathbb{E} - \alpha \int_s^t X_r^{N,i} \partial_1(X_r^{N,i}, X_r^{N,j}) dr - \alpha \int_s^t X_r^{N,j}{}_r \partial_2(X_r^{N,i}, X_r^{N,j}) dr$$

$$+ \int_s^t \int_{\mathbb{R}} f(X_r^{N,i}) (\phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j})) d\nu(u) dr$$

$$+ \int_s^t \int_{\mathbb{R}} f(X_r^{N,j}) (\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, 0) - \phi(X_r^{N,i}, X_r^{N,j})) d\nu(u) dr$$

$$+ \int_s^t \int_{\mathbb{R}} \sum_{\substack{k=1 \\ k \neq i,j}}^N f(X_r^{N,k}) (\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j})) d\nu(u) dr$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

The small jump term

$$N\left|\phi(X_r^{N,i} + \frac{u}{\sqrt{N}}, X_r^{N,j} + \frac{u}{\sqrt{N}}) - \phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2} \partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

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$$\left. - \frac{u}{\sqrt{N}}\sum_{i_1=1}^{2}\partial_{i_1}\phi(X_r^{N,i}, X_r^{N,j}) - \frac{u^2}{2N}\sum_{i_1,i_2=1}^{2}\partial^2_{i_1,i_2}\phi(X_r^{N,i}, X_r^{N,j})\right|$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

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$$\leq CN\frac{|u|^3}{N\sqrt{N}} = CN^{-1/2}|u|^3$$

Vanishing of $\mathbb{E}\left[F(\mu^N)\right]$

The reset jump term

$$\left|\phi(0, X_r^{N,j}) - \phi(0, X_r^{N,j} + \frac{u}{\sqrt{N}})\right| \leq C\frac{|u|}{\sqrt{N}}$$

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$$\leq CN\frac{|u|^3}{N\sqrt{N}} = CN^{-1/2}|u|^3$$

$$CN^{-1/2} \geq \mathbb{E}\left[F(\mu^N)\right] \underset{N\to\infty}{\longrightarrow} \mathbb{E}\left[F(\mu)\right] = 0$$

Convergence of $(\mu^N)_N$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$

$$- X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$

$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t$$

$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

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$$- X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$
$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t$$
$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{1}_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$

Convergence of $(\mu^N)_N$

$$
\begin{aligned}
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& - X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u) \\
d\bar{X}_t^i = & -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t \\
& - \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)
\end{aligned}
$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
- let μ be the limit of a converging subsequence

Convergence of $(\mu^N)_N$

$$dX_t^{N,i} = -\alpha X_t^{N,i} dt + \frac{1}{\sqrt{N}} \sum_{j \neq i} \int_{\mathbb{R}_+ \times \mathbb{R}} u 1_{\left\{z \leq f(X_{t-}^{N,j})\right\}} d\pi^j(t,z,u)$$
$$- X_{t-}^{N,i} \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(X_{t-}^{N,i})\right\}} d\pi^i(t,z,u)$$
$$d\bar{X}_t^i = -\alpha \bar{X}_t^i dt + \sigma \sqrt{\mu_t(f)} dW_t$$
$$- \bar{X}_{t-}^i \int_{\mathbb{R}_+ \times \mathbb{R}} 1_{\left\{z \leq f(\bar{X}_{t-}^i)\right\}} d\pi^i(t,z,u)$$

- $(\mu^N)_N$ is tight on $\mathcal{P}(D)$
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McKean-Vlasov model

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

$$+ \frac{1}{\sqrt{N}} \sum_{k=1}^{N} \int_{\mathbb{R}_+ \times \mathbb{R}^{\mathbb{N}^*}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, u^k, u^i) 1_{\left\{z \le f(X_{t-}^{N,k}, \mu_{t-}^N)\right\}} d\pi^k(t,z,u)$$

with $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{X^{N,i}}$, and π^k has intensity $dt \cdot dz \cdot \nu(du)$ $(\nu = \nu_1^{\otimes \mathbb{N}^*})$

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Dynamic of $X^{N,i}$:

- while there is no jump, the dynamic is given by the drift and Brownian terms
- if there is a jump at time t, created by particle k, each particle i creates a r.v. U^i (the U^i are i.i.d.),

$$X_t^{N,i} = X_{t-}^{N,i} + \frac{1}{\sqrt{N}} \Psi(X_{t-}^{N,k}, X_{t-}^{N,i}, \mu_{t-}^N, U^k, U^i)$$

Heuristics for the limit system

$$dX_t^{N,i} = b(X_t^{N,i}, \mu_t^N)dt + \sigma(X_t^{N,i}, \mu_t^N)d\beta_t^i$$

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$$= \varsigma^2 \int_0^t \int_{\mathbb{R}} f(x,\mu_s^N)\mu_s^N(dx)ds \quad \text{if } i = j$$

with $\varsigma^2 = \int \Psi(u^1,u^2)^2 \nu(du)$

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$$\begin{aligned}
\langle J^{N,i}, J^{N,j} \rangle_t &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(u^1, u^i)\Psi(u^1, u^j) f(x, \mu_s^N) \nu(du) \mu_s^N(dx) ds \\
&= \varsigma^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N) \mu_s^N(dx) ds \quad \text{if } i = j \\
&= \kappa^2 \int_0^t \int_{\mathbb{R}} f(x, \mu_s^N) \mu_s^N(dx) ds \quad \text{if } i \neq j
\end{aligned}$$

with $\varsigma^2 = \int \Psi(u^1, u^2)^2 \nu(du)$ and $\kappa^2 = \int \Psi(u^1, u^2)\Psi(u^1, u^3)\nu(du)$

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$$\bar{J}_t^i = \kappa \int_0^t \sqrt{\int_{\mathbb{R}} f(x,\mu_s)\mu_s(dx)} dW_s + \sqrt{\varsigma^2-\kappa^2} \int_0^t \sqrt{\int_{\mathbb{R}} f(x,\mu_s)\mu_s(dx)} dW_s^i$$

with W, W^i i.i.d. Brownian motions and $\mu = \mathcal{L}(\bar{X}^i|W)$

General case

$$\langle J^{N,i}, J^{N,j}\rangle_t = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}^{\mathbb{N}^*}}$$
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Problem : the blue term is not a product, but an integral of a product

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Solution : let $M(dt, dz) = M_t(dz)$ be a martingale measure on $\mathbb{R}_+ \times E$ with intensity $dt \cdot m_t(dz)$,

$$\langle M_.(A), M_.(B)\rangle_t = \int_0^t \int_E 1_A(z) \cdot 1_B(z) m_s(dz) ds$$

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Here : $E = \mathbb{R}^{\mathbb{N}^*} \times \mathbb{R}$ and $m_s(du, dx) = \nu(du) \cdot \mu_s(dx)$

Limit system (1)

$$
\begin{aligned}
d\bar{X}_t^i =& b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\
&+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\
&+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)
\end{aligned}
$$

with

$$
\tilde{\Psi}(x, y, m, v) = \int_{\mathbb{R}^{\mathbb{N}^*}} \Psi(x, y, m, v, u^1)\nu(du)
$$

$$
\kappa(x, y, m)^2 = \int \Psi(x, y, m, u^1, u^2)^2\nu(du) - \int \tilde{\Psi}(x, y, m, u^1)^2\nu(du)
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Interpretation of $\tilde{\Psi}$:

$$\int \tilde{\Psi}(x, y, m, u^1)^2\nu(du) = \int \Psi(x, y, m, u^1, u^2)\Psi(x, y, m, u^1, u^3)\nu(du)$$

Limit system (2)

$$
\begin{aligned}
d\bar{X}_t^i =& b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\
&+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\
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M and M^i are orthogonal (not independent) :

$$M_t^i(A) = \int_0^t \int_0^1 1_A(F_s^{-1}(p))dW^i(s,p)$$

$$M_t(A \times B) = \int_0^t \int_0^1 \int_{\mathbb{R}} 1_A(F_s^{-1}(p))1_B(u)dW(s,p,u)$$

Limit system (2)

$$d\bar{X}_t^i = b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i + \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) + \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)$$

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with :

- W^i, W independent WN with intensities $dtdp$ and $dtdp\nu_1(du)$

Limit system (2)

$$
\begin{aligned}
d\bar{X}^i_t = & b(\bar{X}^i_t, \mu_t)dt + \sigma(\bar{X}^i_t, \mu_t)d\beta^i_t \\
& + \int \tilde{\Psi}(x, \bar{X}^i_t, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\
& + \int \kappa(x, \bar{X}^i_t, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x)
\end{aligned}
$$

M and M^i are orthogonal (not independent) :

$$
M^i_t(A) = \int_0^t \int_0^1 1_A(F_s^{-1}(p))dW^i(s, p)
$$

$$
M_t(A \times B) = \int_0^t \int_0^1 \int_{\mathbb{R}} 1_A(F_s^{-1}(p))1_B(u)dW(s, p, u)
$$

with :

- W^i, W independent WN with intensities $dtdp$ and $dtdp\nu_1(du)$
- F_s^{-1} = generalized inverse distribution function of μ_s

Limit system (2)

$$\begin{aligned} d\bar{X}_t^i =& b(\bar{X}_t^i, \mu_t)dt + \sigma(\bar{X}_t^i, \mu_t)d\beta_t^i \\ &+ \int \tilde{\Psi}(x, \bar{X}_t^i, \mu_t, v)\sqrt{f(x, \mu_t)}dM(t, x, v) \\ &+ \int \kappa(x, \bar{X}_t^i, \mu_t)\sqrt{f(x, \mu_t)}dM^i(t, x) \end{aligned}$$

M and M^i are orthogonal (not independent) :

$$M_t^i(A) = \int_0^t \int_0^1 1_A(F_s^{-1}(p))dW^i(s, p)$$

$$M_t(A \times B) = \int_0^t \int_0^1 \int_{\mathbb{R}} 1_A(F_s^{-1}(p))1_B(u)dW(s, p, u)$$

with :

- W^i, W independent WN with intensities $dtdp$ and $dtdp\nu_1(du)$
- F_s^{-1} = generalized inverse distribution function of μ_s
- $\mu_s = \mathcal{L}(\bar{X}_s^i | W)$

Main technical difficulty

$$Lg(y,m,x,v) = b(y^1,m)\partial_{y^1}g(y) + b(y^2,m)\partial_{y^2}g(y)$$
$$+ \frac{1}{2}\sigma(y^1,m)^2\partial^2_{y^1}g(y) + \frac{1}{2}\sigma(y^2,m)^2\partial^2_{y^2}g(y)$$
$$+ \frac{1}{2}f(x,m)\kappa(x,y^1,m)^2\partial^2_{y^1}g(y) + \frac{1}{2}f(x,m)\kappa(x,y^2,m)^2\partial^2_{y^2}g(y)$$
$$+ \frac{1}{2}f(x,m)\sum_{i,j=1}^{2}\tilde{\Psi}(x,y^i,m,v)\tilde{\Psi}(x,y^j,m,v)\partial^2_{y^iy^j}g(y)$$

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Consider

$$F(\mu) := \int_{D^2}\mu\otimes\mu(d\gamma)\left[\phi(\gamma_t) - \phi(\gamma_s) - \int_s^t\int_{\mathbb{R}}\int_{\mathbb{R}} L\phi(\gamma_r,\mu_r,x,v)\mu_r(dx)\nu_1(dv)\right]$$

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$\mu^N \xrightarrow{\mathcal{L}} \mu$ in $\mathcal{P}(D)$ (Prohorov topology) $\Rightarrow \mathbb{E}\left[F(\mu^N)\right] \to \mathbb{E}\left[F(\mu)\right]$
Problem : regularity of $L\phi$ w.r.t. μ_r is given for Wasserstein topology

References (1)

- Aldous (1983). Exchangeability and related topics. Ecoles d'Eté de Probabilités de Saint-Flour : XIII.
- El Karoui, Méléard (1990). Martingale measures and stochastic calculus. Probability theory and related fields.
- Graham (1992).McKean-Vlasov Ito-Skorohod equations, and nonlinear diffusions with discrete jump sets. Stochastic Processes and their Applications.
- Jacod, Shiryaev (2003).Limit Theorems for Stochastic Processes. Springer-Verlag BerlinHeidelberg NewYork.

References (2)

- E., Löcherbach, Loukianova (2019). Mean field limits for interacting Hawkes processes in a diffusive regime. Accepted at Bernoulli.
- E., Löcherbach, Loukianova (2021). Conditional propagation of chaos for mean field systems of interacting neurons. EJP.
- E., Löcherbach, Loukianova (2021). White-noise driven conditional McKean-Vlasov limits for systems of particles with simultaneous and random jumps. ArXiv, HAL.
- E. (2021). A convergence criterion for systems of point processes from the convergence of their stochastic intensities. ECP.
- E. (2021). Well-posedness and propagation of chaos for McKean-Vlasov equations with jumps and locally Lipschitz coefficients. ArXiv, HAL.
- E. (2021). Mean field system of a two-layers neural model in a diffusive regime. ArXiv, HAL.

Thank you for your attention !

Questions ?